Optical and Quantum Electronics (2023) 55:246
https://doi.org/10.1007/s11082-022-04521-x

Slow light amplification in a three-level cascade-type system via spontaneously generated coherence and incoherent pumping

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Received: 16 October 2022 / Accepted: 21 December 2022

Abstract
By analytically solving the density matrix equations of a three-level atomic system interacting with two coherent laser fields and an incoherent pumping field under steady-state condition, we have derived the expressions for the absorption and dispersion (and thus group index and group velocity), and population difference as a function of the intensity, frequency, polarization and phase parameters of the laser fields. We have demonstrated that the appearance of spontaneously generated coherence—SGC (characterized by the polarization and non-orthogonality of the electric dipole moments) fundamentally changes the atomic optical responses. By adjusting the strength of SGC and/or incoherent pumping rate, the medium can switch between absorption and amplification regimes, between normal dispersion and anomalous dispersion, and thus light propagation in the medium can switch from slow light to fast light and vice versa. In particular, we have also found that in amplification regime the light is slowed down considerably. Amplifying and slowing light also become more efficient as incoherent pumping rate increases. In addition, the presence of the SGC and incoherent pumping field leads to a sensitive dependence of the atomic responses on the relative phase of the laser fields. Namely, the absorption, dispersion, and group index vary periodically with relative phase.

Keywords Slow light · Light amplification · Spontaneously generated coherence

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1 Introduction

As we known that, in dispersive medium the monochromatic waves propagate with different velocities. Each monochromatic wave travels in medium with phase velocity v_p, while a light pulse propagates with group velocity v_g. That is, the group velocity is the speed of energy transfer (carrying information) and it is related to Poynting vector $\vec{S}$, whereas the phase velocity is related to the wave vector $\vec{k}$. Mathematically, the phase velocity is defined by $v_p = c/n(\omega)$, where ω is angular frequency of light and $n(\omega)$ is often called as the phase index. Similarly, the group velocity is defined by $v_g = c/n_g$, with $n_g = n(\omega) + \omega dn(\omega)/d\omega$ called as the group index. Thus, the group velocity depends on the dispersion $dn(\omega)/d\omega$ of the medium. And hence, slow light can be achieved in normal dispersion region of the refractive index ($v_g < c$), while fast light occurs in anomalous dispersion region ($v_g > c$ or $v_g < 0$). Currently, the search for solutions to slow light with pulses that are not absorbed and/or especially amplified in the atomic medium is of great interest to researchers due to its potential applications in optical buffers, low-power optical switches, quantum memories, radars, spectrometers, magnetometers and enhanced magneto-optical rotation (Boyd 2009; Sultan et al. 2022).

In a two-level atomic medium, slow light can be achieved in the resonant far regions with a very small and unchangeable group index (in resonance region is the presence of fast light). In the last decades, a good knowledge of light-atom interaction that can lead to interesting quantum interference effects in multi-level atomic systems such as electromagnetically induced transparency (EIT) (Boller et al. 1991), spontaneously generated coherence (SGC) (Javanainen 1992). Such quantum interference effects can significantly change the linear and nonlinear optical properties of the atomic medium. The EIT effect is generated from the destructive quantum interference of the transition amplitudes that significantly reduces resonant absorption and enhances dispersion of the medium (Fleischhauer et al. 2005). Therefore, the EIT was used to slow down the group velocity of light (Hau et al. 1999), even stop and store the light pulse (Phillips et al. 2001) in the atomic medium. On the other hands, the SGC can create from quantum interference between spontaneously emission paths in an atomic/molecular system. In EIT medium, the SGC does not destroy EIT, however, the absorption peak on both sides of zero detuning and the linewidth of the absorption line become larger and narrower than those in the case of the SGC absents and hence the steeper dispersion (Ma et al. 2006). This leads to the group velocity can be further slowed down or speeded up under SGC (Bang et al. 2019). And then, the influence of SGC on several optical effects has also been investigated, such as Kerr nonlinearity (Niu and Gong 2006), optical bitability (Joshi et al. 2003), pulse propagation and optical switching (Dong et al. 2018, 2022). In particular, under the EIT and SGC conditions, the medium easily achieves light amplification without inversion population (Bai et al. 2004; Tian et al. 2014; Karimi et al. 2015; Al-Nashy et al. 2017; Hazra and Hossain 2021). Besides, these studies also show that the atomic medium under SGC is sensitively dependent on relative phase of the applied laser fields, whose optical properties can vary periodically with relative phase (Dong et al. 2018, 2022; Li et al. 2007). In addition, to increase the strength of SGC, some research models have proposed to introduce incoherent pumping field to increase the atomic population in the excited levels and thus enhance the intensity of the spontaneous emissions (Dong et al. 2018, 2022; Dutta and Mahapatra 2008; Fan et al. 2010). For example, in Ref. (Fan et al. 2010), Fan et al. studied the SGC on the optical response of a three-level ladder system with or without of incoherent pumping field. It showed that, when incoherent pumping is absent, no gain (with or without inversion) can appear for any value of SGC, however, it can be realized with the presence of incoherent pumping field.

Recently, several studies have combined the EIT effect with other effects to further enhance the amplitude of the group index or change its sign and thus can control the light propagation between fast and slow light modes. For example, Ali and Ahmend (2013) used the Kerr field in four-level atomic system to control over the group velocity of the light propagating through the medium and shown that the group index could be more positive and more negative by increasing the Kerr field. They then developed this model to include SGC and obtained the result that the collective effect of SGC and Kerr field enhanced the positive group index (Ali and Ahmed 2014). This leads to a much slower group velocity inside the medium.

Although the influences of the SGC and the relative phase on the group velocity in three-level atomic systems have investigated (Bai et al. 2005; Mahmoudi et al. 2006; Carreño et al. 2005; Han et al. 2009). However, these studies have not yet shown how slowing light in amplified regime. Recently, this topic has also been interested in some EIT systems (Kim et al. 2003; Caruso et al. 2005; Liu et al. 2017). In this work, we develop an analytic model to study slow light in amplification conditions via spontaneously generated coherence in a three-level cascade-type system with incoherent pumping field. In steady-regime, the expressions for absorption, amplification, and group index have derived according to the parameters of laser fields. The influences of the strength of SGC, the incoherent pumping rate and the relative phase of the laser fields on absorption, amplification and group index have investigated. The analytical model can be useful for experimental observations and relative applications.

2 Theoretical model

The three-level cascade-type atomic system interacting with two coherent laser fields and an incoherent pumping field is shown in Fig. 1a. The transition $|1\rangle \leftrightarrow |2\rangle$ is driven by a probe laser field of frequency ω_p, while the transition $|2\rangle \leftrightarrow |3\rangle$ is excited by a coupling laser field of frequency ω_c. The incoherent pumping field with a pumping rate 2R is applied to the transition $|1\rangle \rightarrow |3\rangle$. The spontaneously emission rates from the state $|3\rangle$ to state $|2\rangle$ and from the state $|2\rangle$ to state $|1\rangle$ are Γ_2 and Γ_1, respectively. We define the frequency detuning of the probe and coupling lasers from the relevant atomic transitions are respectively as:

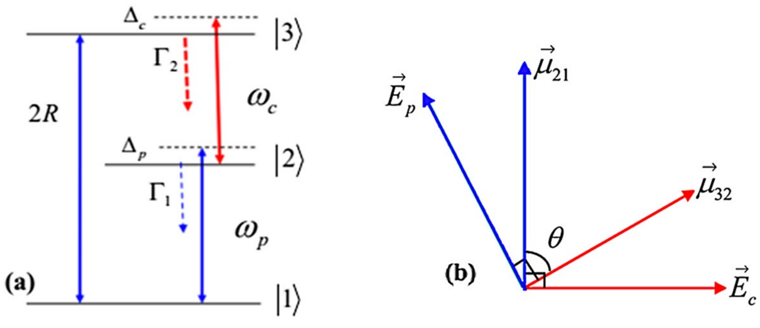


Fig. 1 **a** The energy scheme of the three-level cascade-type atomic system, **b** light polarizations are chosen such as only one light field drives one transition

$$\Delta_{\mathrm{p}} = \mathrm{w}_{\mathrm{p}} - \mathrm{w}_{21} \text{ and } \Delta_{\mathrm{c}} = \mathrm{w}_{\mathrm{c}} - \mathrm{w}_{32}. \tag{1}$$

The Rabi frequencies of probe and coupling fields are respectively defined by:

$$\Omega_p = 2\vec{\mu}_{21} \cdot \vec{E}_p/\hbar \text{ and } \Omega_c = 2\vec{\mu}_{32} \cdot \vec{E}_c/\hbar, \tag{2}$$

where $\vec{\mu}_{21}$ and $\vec{\mu}_{32}$ are the electric dipole matrix elements. To ensure that one field acts only one transition we have chosen $\vec{E}_c \perp \vec{\mu}_{21}$ and $\vec{E}_p \perp \vec{\mu}_{32}$ as illustrated in Fig. 1b. We denote θ is the angle between the two induced dipole moments and set $p = \cos\theta = \frac{\vec{\mu}_{21}\cdot\vec{\mu}_{32}}{|\vec{\mu}_{21}||\vec{\mu}_{32}|}$ is called as quantum interference parameter arising from the cross-coupling between spontaneously emission paths of the transitions $|2\rangle \rightarrow |1\rangle$ and $|3\rangle \rightarrow |2\rangle$. Thus, the value of parameter p depends on the non-orthogonality of the dipole moments $\vec{\mu}_{21}$ and $\vec{\mu}_{32}$, that is, if the two dipole moments are orthogonal to each other than $p=0$ and there is no quantum interference due to spontaneously emissions. When the two dipole moments are parallel to each other than the quantum interference is maximal and $p=1$.

The evolution of the atomic system in light fields can be determined by the Liouville equation as follows:

$$\frac{\partial\rho}{\partial t} = -\frac{i}{\hbar}[H,\rho] + \Lambda\rho. \tag{3}$$

where H is the total Hamiltonian and in interaction picture it can be written as,

$$H = \hbar\begin{bmatrix} \Delta_p & -\Omega_p & 0 \\ -\Omega_p & 0 & -\Omega_c \\ 0 & -\Omega_c & \Delta_c \end{bmatrix}, \tag{4}$$

and $\Lambda\rho$ is the decay part which is given by Ficek and Swain (2005):

$$\Lambda\rho = -\sum_{i,j=1}^{2}\Gamma_{ij}\left(S_i^+S_j^-\rho + \rho S_i^+S_j^- - 2S_j^-\rho S_i^+\right), \tag{5}$$

where $S_i^- = |i\rangle\langle 2|$, $S_i^+ = |2\rangle\langle i|$, $S_j^- = |3\rangle\langle j|$ and $S_j^+ = |j\rangle\langle 3|$ represent respectively symmetric and antisymmetric superpositions of the dipole moments of the two bare systems, Γ_{ij} presents cross-damping rates between coherence superpositions.

From Eqs. (3)–(5), the density matrix equations of motion of this system in the rotating-wave and electric dipole approximations can be derived as follows:

$$\dot{\rho}_{11} = -2R\rho_{11} + 2\Gamma_1\rho_{22} + i\Omega_p\rho_{21} - i\Omega_p\rho_{12}, \tag{6a}$$

$$\dot{\rho}_{22} = -2\Gamma_1\rho_{22} + 2\Gamma_2\rho_{33} + i\Omega_p\rho_{12} - i\Omega_p\rho_{21} + i\Omega_c\rho_{32} - i\Omega_c\rho_{23}, \tag{6b}$$

$$\dot{\rho}_{33} = 2R\rho_{11} - 2\Gamma_2\rho_{33} - i\Omega_c\rho_{32} + i\Omega_c\rho_{23}, \tag{6c}$$

$$\dot{\rho}_{21} = -(R - i\Delta_p + \Gamma_1)\rho_{21} - i\Omega_p(\rho_{22} - \rho_{11}) + i\Omega_c\rho_{31} + 2p\sqrt{\Gamma_1\Gamma_2}\rho_{32}, \tag{6d}$$

$$\dot{\rho}_{31} = [i(\Delta_p + \Delta_c) - \Gamma_2 - R]\rho_{31} - i\Omega_P\rho_{32} + i\Omega_C\rho_{21}, \tag{6e}$$

$$\dot{\rho}_{32} = -\left(\Gamma_1 + \Gamma_2 - i\Delta_c\right)\rho_{32} - i\Omega_c(\rho_{33} - \rho_{22}) - i\Omega_p\rho_{31}, \tag{6f}$$

In the above equations, the term $2p\sqrt{\Gamma_1\Gamma_2}\rho_{23}$ represents spontaneously generated coherence (SGC) arising from the quantum interference of spontaneously emissions, it depends on the parameter p. In the presence of SGC, the optical response of the atomic system depend on not only intensity and frequency but also the phase of the probe and coupling fields, and hence we treat Rabi frequencies Ω_p and Ω_c as complex parameters and can be written by

$$\Omega_p = G_p \exp\left(i\varphi_p\right) \text{ and } \Omega_c = G_c \exp\left(i\varphi_c\right), \tag{7}$$

where G_p and G_c are the real parameters, φ_p and φ_c are the phases of the probe and coupling fields, respectively.

We let $\sigma_{ii} = \rho_{ii}$, $\sigma_{21} = \rho_{21}\exp(-i\varphi_p)$, $\sigma_{32} = \rho_{32}\exp(-i\varphi_c)$, therefore, from Eqs. (6a)–(6f) we obtain:

$$\dot{\sigma}_{11} = -2R\sigma_{11} + 2\Gamma_1\sigma_{22} + iG_p(\sigma_{21} - \sigma_{12}), \tag{8a}$$

$$\dot{\sigma}_{22} = -2\Gamma_1\sigma_{22} + 2\Gamma_2\sigma_{33} + iG_p(\sigma_{12} - \sigma_{21}) + iG_c(\sigma_{32} - \sigma_{23}), \tag{8b}$$

$$\dot{\sigma}_{33} = 2R\sigma_{11} - 2\Gamma_2\sigma_{33} + iG_c(\sigma_{23} - \sigma_{32}), \tag{8c}$$

$$\dot{\sigma}_{21} = \gamma_{21}\sigma_{21} - iG_p(\sigma_{22} - \sigma_{11}) + iG_c\sigma_{31} + 2p\sqrt{\Gamma_1\Gamma_2}e^{-i\varphi}\sigma_{32}, \tag{8d}$$

$$\dot{\sigma}_{31} = \gamma_{31}\sigma_{31} - iG_p\sigma_{32} + iG_c\sigma_{21}. \tag{8e}$$

$$\dot{\sigma}_{32} = \gamma_{32}\sigma_{32} - iG_p\sigma_{31} - iG_c(\sigma_{33} - \sigma_{22}), \tag{8f}$$

here $\varphi = \varphi_p - \varphi_c$ is relative phase between the probe and coupling fields.

The above equations satisfy the condition of population conservation $\sigma_{11} + \sigma_{22} + \sigma_{33} = 1$ and $\sigma_{ji} = \sigma_{ij}^*$. Here, we set, $\gamma_{31} = i(\Delta_p + \Delta_c) - \Gamma_2 - R$, $\gamma_{21} = i\Delta_p - \Gamma_1 - R$, $\gamma_{32} = i\Delta_c - \Gamma_1 - \Gamma_2$.

Now, we analytically solve the density matrix equations under the steady-state condition by setting the time derivatives to zero. From Eq. (8e) we have:

$$\sigma_{32} - \sigma_{23} = \frac{\gamma_{31}\sigma_{31} + \gamma_{13}\sigma_{13} + iG_c(\sigma_{21} - \sigma_{12})}{iG_p} \tag{9}$$

From Eq. (8b) combining with Eq. (8c), we obtain:

$$\sigma_{22} - \sigma_{33} = \frac{iG_p^2\Gamma_2(\sigma_{12} - \sigma_{21}) + \Gamma_1 G_c[\gamma_{31}\sigma_{31} + \gamma_{13}\sigma_{13} + iG_c(\sigma_{21} - \sigma_{12})] + 2RG_p(\Gamma_2 - \Gamma_1)}{2\Gamma_1\Gamma_2 G_p} \tag{10}$$

Substituting Eqs. (8e), (10) into Eq. (8f), we find:

$$\begin{aligned} 0 =& (2\Gamma_1\Gamma_2\gamma_{32}\gamma_{31} + 2\Gamma_1\Gamma_2 G_p^2 - \Gamma_1 G_c^2\gamma_{31})\sigma_{31} + i(2\Gamma_1\Gamma_2\gamma_{32}G_c + G_p^2 G_c\Gamma_2 - \Gamma_1 G_c^3)\sigma_{21} \\ &+ i(\Gamma_1 G_c^3 - G_p^2 G_c\Gamma_2)\sigma_{12} - \gamma_{13}\Gamma_1 G_c^2\sigma_{13} - 2RG_pG_c(\Gamma_2 - \Gamma_1), \end{aligned} \tag{11}$$

From Eq. (11), we get:

$$\sigma_{31} = \frac{i(A_{12}^*A_{13} - A_{21}A_{31}^*)\sigma_{21} + i(A_{21}^*A_{13} - A_{12}A_{31}^*)\sigma_{12} - A^*A_{13} - AA_{31}^*}{A_{31}A_{31}^* - A_{13}^*A_{13}}, \tag{12}$$

where

$$A_{31} = 2\Gamma_1\Gamma_2\gamma_{32}\gamma_{31} + 2\Gamma_1\Gamma_2 G_p^2 - \Gamma_1 G_c^2\gamma_{31}, \tag{13}$$

$$A_{21} = 2\Gamma_1\Gamma_2\gamma_{32}G_c + G_p^2 G_c\Gamma_2 - \Gamma_1 G_c^3, \tag{14}$$

$$A_{12} = \Gamma_1 G_c^3 - G_p^2 G_c\Gamma_2, \tag{15}$$

$$A_{13} = \gamma_{13}\Gamma_1 G_c^2, \tag{16}$$

$$A = -2RG_pG_c(\Gamma_2 - \Gamma_1). \tag{17}$$

and (*) denotes the complex conjugation.

By putting Eq. (12) in Eq. (8f) we derive the density matrix element σ_{32}:

$$\sigma_{32} = \frac{i[(A_{12}^*A_{13} - A_{21}A_{31}^*)\gamma_{31} + G_c(A_{31}A_{31}^* - A_{13}^*A_{13})]\sigma_{21} + i(A_{21}^*A_{13} - A_{12}A_{31}^*)\gamma_{31}\sigma_{12} - \gamma_{31}(A^*A_{13} + AA_{31}^*)]}{iG_p(A_{31}A_{31}^* - A_{13}^*A_{13})}, \tag{18}$$

Substituting Eqs. (8b), (12) and (18) into Eq. (8d) and using the initial conditions: $\sigma_{11} \approx 1$, $\sigma_{22} \approx \sigma_{33} \approx 0$, we find the density matrix element σ_{21} for the probe response as:

$$\begin{aligned} 0 =& i[(2\Gamma_1 G_p\gamma_{21} - G_p^3 + 4\Gamma_1\Gamma_{12}e^{i\varphi}G_c)(A_{31}A_{31}^* - A_{13}^*A_{13}) \\ &+ 2\Gamma_1(2\gamma_{31}\Gamma_{12}e^{i\varphi} - G_pG_c)(A_{12}^*A_{13} - A_{21}A_{31}^*)]\sigma_{21} \\ &+ i[G_p^3(A_{31}A_{31}^* - A_{13}^*A_{13}) + 2\Gamma_1(2\gamma_{31}\Gamma_{12}e^{i\varphi} - G_pG_c)(A_{21}^*A_{13} - A_{12}A_{31}^*)]\sigma_{12} \\ &+ 2G_p^2(A_{31}A_{31}^* - A_{13}^*A_{13})(R - \Gamma_1) + 2\Gamma_1(G_pG_c - 2\Gamma_{12}e^{i\varphi}\gamma_{31})(A^*A_{13} + AA_{31}^*), \end{aligned} \tag{19}$$

From Eq. (19), we get

$$\sigma_{21} = \frac{i(F_1F^* + FF_2^*)}{F_2F_2^* - F_1F_1^*}, \tag{20}$$

By taking the complex conjugation of Eq. (20) we obtain σ_{12} as follows:

$$\sigma_{12} = \frac{i(F_1^*F + F^*F_2)}{F_1F_1^* - F_2F_2^*}, \tag{21}$$

where

$$F = 2G_p^2(A_{31}A_{31}^* - A_{13}^*A_{13})(R - \Gamma_1) + 2\Gamma_1(G_pG_c - 2\Gamma_{12}e^{i\varphi}\gamma_{31})(A^*A_{13} + AA_{31}^*), \tag{22}$$

$$F_1 = G_p^3(A_{31}A_{31}^* - A_{13}^*A_{13}) + 2\Gamma_1(2\gamma_{31}\Gamma_{12}e^{i\varphi} - G_pG_c)(A_{21}^*A_{13} - A_{12}A_{31}^*), \tag{23}$$

$$F_2 = (2\Gamma_1 G_p \gamma_{21} - G_p^3 + 4\Gamma_1\Gamma_{12}e^{i\varphi}G_c)(A_{31}A_{31}^* - A_{13}^*A_{13}) + 2\Gamma_1(2\gamma_{31}\Gamma_{12}e^{i\varphi} - G_pG_c)(A_{12}^*A_{13} - A_{21}A_{31}^*). \tag{24}$$

We can find the expression for the population difference between the level |2⟩ and the level |1⟩ by substituting Eq. (20) and Eq. (21) into Eq. (8a) as:

$$W_{12} = \sigma_{22} - \sigma_{11} = \frac{G_p(F_1^*F + F^*F_2 + F_1F^* + FF_2^*) + 2(R - \Gamma_1)(F_2F_2^* - F_1F_1^*)}{2\Gamma_1(F_2F_2^* - F_1F_1^*)}. \tag{25}$$

The optical response of the atomic system to the probe field is determined by susceptibility χ which can be expressed in terms of the coherence σ_{21} by

$$\chi = 2\frac{N\mu_{21}}{\varepsilon_0 E_p}\sigma_{21}, \tag{26}$$

where N is the atomic density and ε_0 is the permittivity in a vacuum. The absorption and dispersion coefficients for the probe light are related to imaginary χ'' and real χ' parts of the susceptibility and can be determined via the following relations:

$$\alpha = \frac{\omega_p \chi''}{c}, \tag{27}$$

$$n = 1 + \frac{\chi'}{2}. \tag{28}$$

The group index n_g and group velocity v_g of the probe light can be calculated by using the expression:

$$v_g = \frac{c}{n_g} = \frac{c}{1 + 2\pi\chi'(\omega_p) + 2\pi\omega_p\frac{\partial\chi'(\omega_p)}{\partial\omega_p}}. \tag{29}$$

3 Results and discussion

In this section, we apply the analytical model to ^{85}Rb atomic system with the states |1⟩, |2⟩ and |3⟩ are $5S_{1/2}(F=1)$, $5P_{1/2}(F'=2)$ and $5D_{3/2}(F''=2)$, respectively. The atomic parameters are (http://steck.us/alkalidata): $\Gamma_1 = 6$ MHz, $\Gamma_2 = 0.64$ MHz, $\mu_{21} = 2.53\times10^{-29}$ C.m, $\mu_{32} = 1.6\times10^{-29}$ C.m, $\omega_p = 3.77\times10^{12}$ Hz and $N = 10^{19}$ atoms/m^3. For simplicity, all the parameters related to frequency are given in units of $\gamma = 1$ MHz.

First of all, we plot the variations of probe light amplification [Im(σ_{12})] and population difference ($W_{21} = \sigma_{22} - \sigma_{11}$) versus probe detuning for different values of quantum parameter p in the presence of incoherent pumping with $R = 3\gamma$ as shown in Fig. 2. Other parameters used in Fig. 2 as $\varphi = 0$, $\Delta_c = 0$ and $G_c = 15\gamma$. We notice that, if Im(σ_{12}) > 0, the system exhibits gain for the probe field; if Im(σ_{12}) < 0, the probe field is attenuated. From the solid line in Fig. 2a we can see that when the SGC absents ($p = 0$), the atomic medium represents an EIT phenomenon for the probe light (i.e., absorption coefficient has vanished at the resonant frequency) and the amplification coefficient has a negative value (i.e., there is no amplification of the probe light). At the same time, the population difference (W_{21})

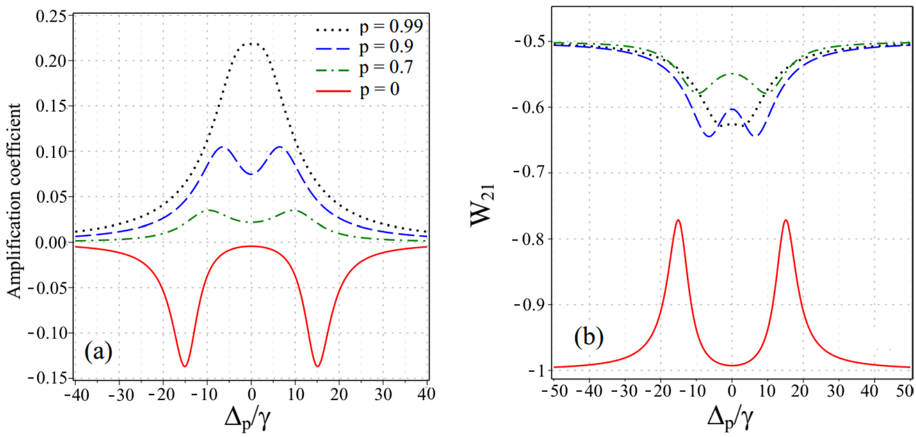


Fig. 2 Variations of amplification (**a**) and population difference (**b**) versus probe detuning Δ_p for different values of $p=0$ (solid line), $p=0.7$ (dotted-dash line), $p=0.9$ (dashed line) and $p=0.99$ (dotted line) and other parameters are $\Delta_c=0$, $\varphi=0$, $R=3\gamma$, and $G_c=15\gamma$

between the level $|2\rangle$ and level $|1\rangle$ is approximately -1 (i.e., $\sigma_{11}\approx 1$ while $\sigma_{22}\approx 0$), as we can see from the solid line in Fig. 2b. However, the situation is changed when the SGC presents with incoherent pump rate $R=3\gamma$, the medium exhibits the probe light amplification. In particular, from Fig. 2a we see that as the parameter p increases, the amplification also increases gradually and becomes significantly larger when $p=0.99$. At this moment, the absolute value of W_{21} is also increased, but W_{21} still takes a negative value (see Fig. 2b). This means that the probe light is amplified without population inversion. This can be explained by the ratio between the rates of absorption and stimulated emission of light by the atomic medium (Mompart and Corbalan 2000), $\frac{\text{absorption rate}}{\text{stimulated emission rate}}=\frac{B_{abs}}{B_{st\,em}}\cdot\frac{\sigma_{11}}{\sigma_{22}}$, that light amplification can be achieved if this ratio is less than 1. Of course, this is not possible in a two-level atomic system because $B_{abs}=B_{st\ em}$ and $\sigma_{22}<\sigma_{11}$. However, for the EIT medium the absorption of light is reduced or even cancelled, and light amplification is possible with a small fraction of the atoms in the upper level. In particular, the presence of SGC and incoherent pumping field not only further reduces the absorption of light by the medium, but also increases the atomic population in the upper level, so light amplification can easily occur. At the same time, for this reason the light amplification is also increased with the increase of incoherent pumping rate.

Consequently, in amplified light regime the dispersion becomes larger and very steeper as the parameter p increases, as described in Fig. 3a. For example, the dotted line in Fig. 3a represents a very high and steep dispersion with $p=0.99$. This leads to the group index becoming very large and positive, so that the light is slowed down significantly in the amplified regime (see the dotted line in Fig. 3b) compared to the solid line in the absence of SGC. To see this more clearly, in Fig. 4 we plot the variations of amplification (solid line) and group index (dashed line) versus the parameter p when $\Delta_p=\Delta_c=0$, $\varphi=0$, $R=3\gamma$ and $G_c=15\gamma$. It is obvious that when $p>0.6$ there is a sudden increase in both amplification and index group. Especially, in the range of p values from 0.9 to 0.99 both amplification and index group have large and positive values, i.e., slow light amplification occurs.

Next, in Fig. 5 we investigate the influence of the incoherent pumping rate on the amplification and group index by plotting the amplification (a) and group index (b) versus the

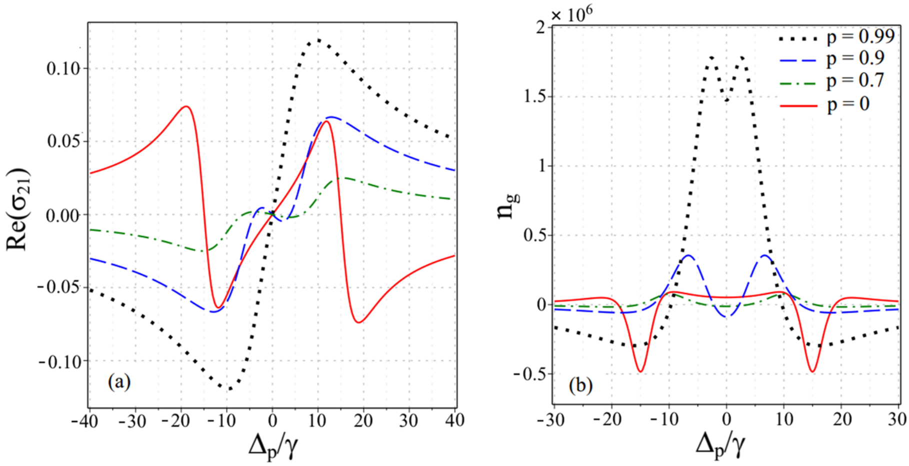
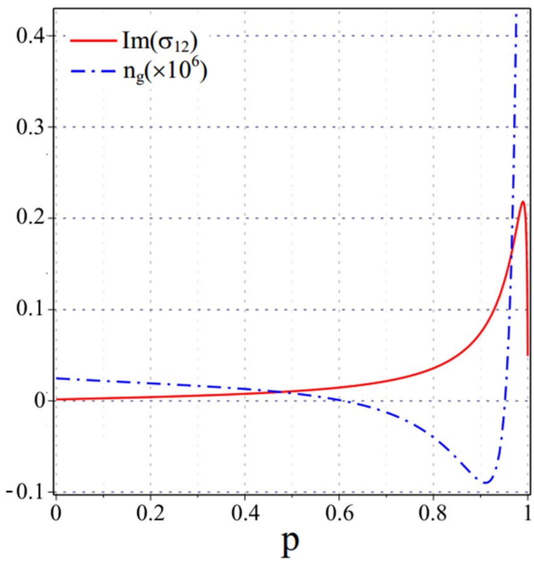


Fig. 3 Variations of dispersion (**a**) and group index (**b**) versus probe detuning Δ_p for different values of $p=0$ (solid line), $p=0.7$ (dotted-dash line), $p=0.9$ (dashed line) and $p=0.99$ (dotted line) and $\Delta_c=0$, $\varphi=0$, $R=3\gamma$ and $G_c=15\gamma$

Fig. 4 Variations of amplification (solid line) and group index (dashed line) versus the parameter p when $\Delta_p=\Delta_c=0$, $\varphi=0$, $R=3\gamma$ and $G_c=15\gamma$



probe detuning for different values of pumping rate R when the parameter $p=0.99$ and other parameters as $\varphi=0$, $\Delta_c=0$, and $G_c=15\gamma$. From Fig. 5 we see that the incoherent pumping rate plays an important role in generating light amplification. Indeed, as we can see in Fig. 5a, when $R=0$ the medium does not exhibit light amplification even with the parameter $p=0.99$. In the presence of incoherent pumping field, however the amplification of the probe light is achieved. Especially, when the pumping rate R increases gradually the amplification is increased considerably. This leads to an increase in the group index and hence a remarkable decrease in group velocity of light, as shown in Fig. 5b. For example,

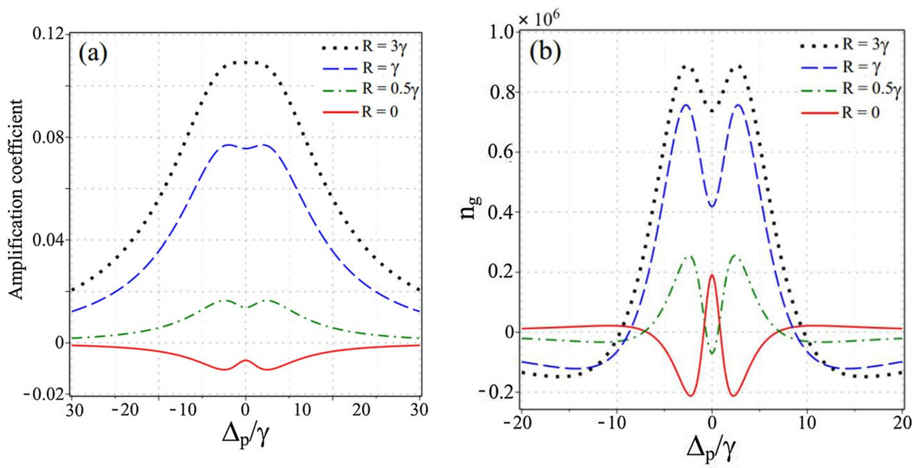
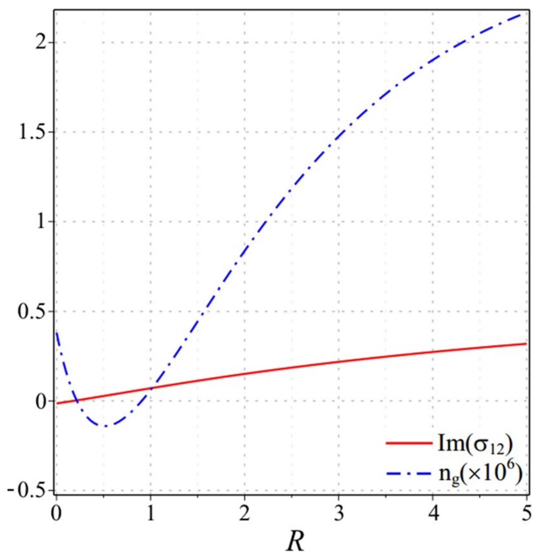


Fig. 5 Variations of amplification (**a**) and group index (**b**) versus probe detuning Δ_p for different values of $R=0$ (solid line), $R=0.5\gamma$ (dotted-dash line), $R=1\gamma$ (dashed line) and $R=3\gamma$ (dotted line) and $\Delta_c=0$, $\varphi=0$, $p=0.99$ and $G_c=15\gamma$

when $R=3\gamma$ the group index (at $\Delta_p=0$) is about four times larger than the case $R=0$. The variations of amplification (solid line) and group index (dashed line) versus the pumping rate R also plotted in Fig. 6. It is shown that when $R>1$ then the group index n_g increases rapidly, at the same time, the amplification also increases with growing the pumping rate R. This again shows that the incoherent pumping field plays an important role in the slow light amplification which is more effective for slowing light in the amplified mode.

Finally, in Fig. 7 we consider the influence of the relative phase on amplification and group index by plotting amplification (a) and group index (b) versus probe detuning for

Fig. 6 Variations of amplification (solid line) and group index (dashed line) versus the pumping rate R when $\Delta_p=\Delta_c=0$, $\varphi=0$, $p=0.99$ and $G_c=15\gamma$



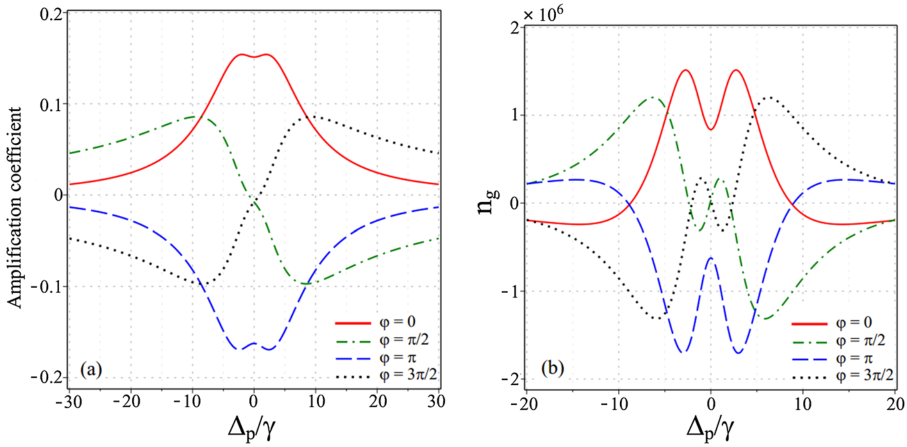


Fig. 7 Variations of amplification (**a**) and group index (**b**) versus probe detuning Δ_p for different values of the relative phase $\varphi=0$ (solid line), $\varphi=\pi/2$ (dot-dashed line) and $\varphi=\pi$ (dash-line) and other parameters are $p=0.99$, $R=3\gamma$, $\Delta_c=0$ and $G_c=15\gamma$

different values of the relative phase when $p=0.99$, $R=3\gamma$, $\Delta_c=0$, and $G_c=15\gamma$. It is shown that both the amplification and group index vary periodically versus the relative phase with a period of 2π. Indeed, the variations of amplification and group index at $\varphi=0$ (or $\varphi=\pi/2$) is opposite to that at $\varphi=\pi$ (or $\varphi=3\pi/2$), respectively. Moreover, the group index also becomes very large (i.e., the group velocity has been much slower) when the amplification increases (see at $\varphi=0$ or $\pm 2\pi$), as displayed in Fig. 8.

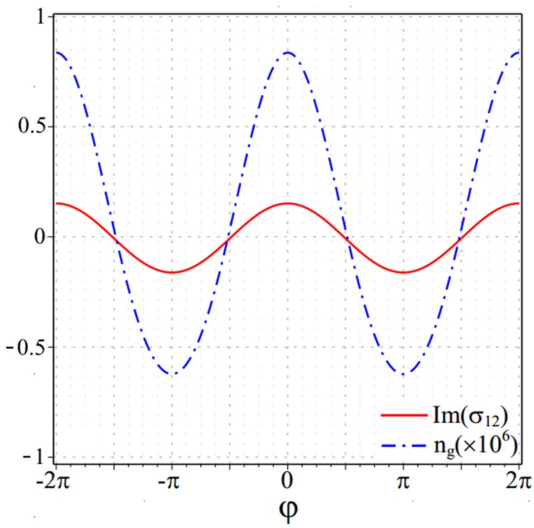


Fig. 8 Variations of amplification (solid line) and group index (dashed line) versus the relative phase φ when $\Delta_p=0$, $\Delta_c=0$, $\varphi=0$, $p=0.99$, $R=3\gamma$ and $G_c=15\gamma$

4 Conclusion

We derived the analytical expressions for amplification, population difference and group index in a three-level cascade-type atomic system under SGC and incoherent pumping. We found that in the presence of SGC and incoherent pumping field, the atomic medium is easily achieved probe light amplification without inversion population. When increasing the strength of SGC and/or the incoherent pumping rate, the amplification also increases gradually. In amplification regime, the group refractive index is greatly increased and thus the light becomes super slow with increasing amplification. In addition, both the amplification and group index vary periodically versus the relative phase with a period of 2π. The variations of amplification and group index at $\varphi=0$ (or $\varphi=\pi/2$) is opposite to that at $\varphi=\pi$ (or $\varphi=3\pi/2$), respectively. The analytical model can be useful for experimental observations and relative applications.

Author contributions LNMA, NHB, NVP, HMD, NTTH and LVD conceived the presented idea, developed the theory and performed the analytic calculations. All authors co-wrote the paper, discussed the results and contributed to the final manuscript.

Funding This work is supported by Vietnam's Ministry of Education and Training under Grant No. B2022-TDV-05, and Nong Lam University Ho Chi Minh City under Grant code CS-CB21-KH-02.

Data availability The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declarations

Competing interests The authors declare no competing interests.

Conflict of interest The authors have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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