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Universally Koszul and initially Koszul properties of Orlik–Solomon algebras

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Let K be a field with $\mathrm{char}(K) = 0$ and $E = K\langle e_1, \ldots, e_n\rangle$ an exterior algebra over K with a standard grading $\deg e_i = 1$. Let $R = E/J$ be a graded algebra, where J is a graded ideal in E. In this paper, we study universally Koszul and initially Koszul properties of R and find classes of ideals J which characterize such properties of R. As applications, we classify arrangements whose Orlik–Solomon algebras are universally Koszul or initially Koszul. These results are related to a long-standing question of Shelton–Yuzvinsky [B. Shelton and S. Yuzvinsky, Koszul algebras from graphs and hyperplane arrangements, *J. London Math. Soc.* **56** (1997) 477–490].

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1. Introduction

Let K be a field with $\mathrm{char}(K) = 0$ and let R be a standard graded K-algebra. R is called a *Koszul algebra* over K if K has a linear resolution over R. To prove Koszulness of a graded algebra, one usually shows that the defining ideal of it has a quadratic Gröbner basis with respect to some term order. In this case, one says the graded algebra is *G-quadratic*. For recent surveys of Koszul algebras, we refer the reader to [8, 16]. Another powerful tool to deduce Koszulness, namely Koszul filtrations, is introduced by Conca, Trung and Valla in [10]. A *Koszul filtration* of R is a family $\mathcal{F}$ of linear ideals (generated by linear forms) of R such that: $\mathcal{F}$ contains the ideal 0 and the maximal graded ideal $\mathfrak{m}$ of R; and for $0 \neq I \in \mathcal{F}$, there exists $J \in \mathcal{F}$ such that $J \subset I$, I/J is cyclic and $J :_R I \in \mathcal{F}$.

In this paper, we study variations of the Koszul property, namely universally Koszul and initially Koszul properties. Let R be a graded K-algebra. R is said to be *universally Koszul* if the family of all linear ideals of R is a Koszul filtration; R is said to be *initially Koszul* with respect to a sequence $u_1, \dots, u_n$ of R_1 if the family $\mathcal{F} = \{0, (u_1, \dots, u_j) \text{ for } 1 \leq j \leq n\}$ is a Koszul filtration of R. See, e.g. [3, 6, 7, 9, 18] for more details of these properties.

Let $\mathcal{A}$ be an essential central hyperplane arrangement in $\mathbb{C}^l$ with the complement $X(\mathcal{A}) = \mathbb{C}^l \setminus \bigcup_{H \in \mathcal{A}} H$. A well-known result of arrangements theory, proved by Orlik and Solomon [23, Theorem 5.2], is that the singular cohomology $H^\bullet(X(\mathcal{A}); K)$ of $X(\mathcal{A})$ with coefficients in K is isomorphic to the *Orlik–Solomon algebra* A of $\mathcal{A}$ which is the quotient of E by the *Orlik–Solomon ideal* J of $\mathcal{A}$ generated by the "differentials" of dependent sets of hyperplanes in $\mathcal{A}$. See Orlik and Solomon [23, Sec. 5] for details.

Concerning the Koszul property of Orlik–Solomon algebras, Shelton and Yuzvinsky [29, Theorem 4.6] proved that the Orlik–Solomon A of an essential central hyperplane arrangement $\mathcal{A}$ is a Koszul algebra whenever $\mathcal{A}$ is supersolvable. Here, supersolvable arrangements are some of the best understood arrangements which are defined based on a purely combinatorial property of the intersection lattice. See Orlik–Terao [24, Sec. 2.1] and Yuzvinsky [31, Sec. 6.3] for details. Whether the converse holds remains the following famous question.

Question 1.1 (Shelton–Yuzvinsky [29, Sec. 5]). *A is Koszul if and only if $\mathcal{A}$ is supersolvable*?

This question has been answered affirmatively for certain classes of arrangements such as hypersolvable arrangements [20], graphic arrangements [27], ideal arrangements [19] and arrangements with disjoint minimal broken circuits [30].

Our goal is to consider Shelton–Yuzvinsky's question for universally Koszul and initially Koszul properties. More precisely, we aim to classify completely essential central hyperplane arrangements whose Orlik–Solomon algebras are universally Koszul or initially Koszul. First, we get a combinatorial characterization of universally Koszul Orlik–Solomon algebra as in the following first main result.

Theorem 1.1 (see Theorem 5.1). *Let $\mathcal{A}$ be a central essential hyperplane arrangement with Orlik–Solomon algebra A. The following statements are equivalent*:

(i) *A is universally Koszul.*
(ii) *The underlying matroid of $\mathcal{A}$ is $M(\mathcal{A}) = U_{2,s} \oplus U_{n-s,n-s}$ for some $2 \leq s \leq n$.*

This result implies that if A is universally Koszul then $\mathcal{A}$ is supersolvable; see Remark 5.1. For the initially Koszul property, our second main result gives us an algebraic characterization of supersolvable arrangements as follows.

Theorem 1.2 (see Theorem 5.2). *$\mathcal{A}$ is supersolvable if and only if A is initially Koszul.*

The paper is organized as follows. In Sec. 2, we recall some basic facts about Koszul algebras and Orlik–Solomon algebras. After that, we consider in Sec. 3

the universally Koszul property of algebras defined by edge ideals in the exterior algebra. It turns out that the classification of universally Koszul algebras defined by monomial ideals in the polynomial ring [7, Theorem 5] still holds for the exterior algebra case. For the convenience of the reader and for the proofs in Sec. 5 we reproduce this in Theorem 3.1.

In Sec. 4, we study the initially Koszul property of graded algebras over the exterior algebra. It is known that if an algebra is initially Koszul then it has a quadratic Gröbner basis, thus it is Koszul (see [3, Proposition 2.3] or [9, Proposition 2.5]). This also holds for graded algebras over the exterior algebra (Proposition 4.2).

The last section is devoted to the proofs of Theorem 1.1 (see Theorem 5.1) and Theorem 1.2 (see Theorem 5.2).

2. Preliminaries

2.1. *Koszul algebras*

We present in this section basis facts about Koszul algebras over the exterior algebra. For more details, we refer to the surveys by Conca *et al.* [8] or Fröberg [16] and the book by Ene and Herzog [12, Sec. 6.1].

Definition 2.1. A standard graded K-algebra R over E is said to be *Koszul* if the R-module $K = R/\mathfrak{m}$ has a linear free resolution over R.

Example 2.1. (i) The exterior algebra E is Koszul since the Cartan complex is the linear free resolution of K over E (see [2, Sec. 4]).
(ii) The K-algebra E/J defined by a quadratic monomial ideal $J \subset E$ is Koszul (see, e.g. Theorem 2.1).

We collect some well-known facts in the following lemma whose proof is trivial.

Lemma 2.1 ([8, Sec. 2]). *Let R be a standard graded K-algebra over E. The following statements are equivalent*:

(i) *R is a Koszul algebra.*
(ii) $\operatorname{reg}_R(K) = 0$.
(iii) $\operatorname{Tor}_i^R(K, K)_j = 0$ *for all $i \neq j$.*

Remark 2.1. From Lemma 2.1, one can deduce a well-known necessary condition for the Koszulness of the K-algebra E/J, where $J \subset E$ is a graded ideal: If E/J is Koszul, then J is generated in degrees ≤ 2. From now on, we always assume that J does not contain linear forms. In other words, we consider the Koszul property of $R = E/J$ only in the case that J is generated in degree 2. We also identify $e_i \in E$ with $[e_i] \in R$ for $i = 1, \ldots, n$.

A well-known sufficient condition is the following result (see, e.g. [16]).

Theorem 2.1. *Let $J \subset E$ be a graded ideal which has a quadratic Gröbner basis with respect to some monomial order on E. Then E/J is a Koszul algebra.*

Note that, the converse of Theorem 2.1 is false in the polynomial ring case. However, no counter example is known over an exterior algebra. We propose a possible counter example in Example 3.1.

2.2. *Matroids*

In this section, we collect some of necessary matroid notions that are used in this paper. They can be found in many books on matroids.

Let M be a collection $\mathcal{C}$ of subsets of $[n]$, called *circuits* (or *minimal dependent sets*). Then M is called a *matroid* on the ground set $[n]$ if the following conditions hold:

(i) $\emptyset \notin \mathcal{C}$.
(ii) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$ then $C_1 = C_2$.
(iii) Let C_1, C_2 be distinct members of $\mathcal{C}$. If $i \in C_1 \cap C_2 \neq \emptyset$, then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \backslash \{i\}$.

Example 2.2. A *uniform matroid* $U_{p,q}$ with $p \leq q$ is a matroid on the ground set $[q]$ such that every subset of $p+1$ elements is a circuit. For example,

$$U_{2,4} = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}.$$

Let M be a matroid with the collection $\mathcal{C}$ of circuits. Let $i \in [n]$. If $\{i\}$ is a circuit of M, then i is called a *loop* of M. If i is contained in no circuit of M, then i is called a *coloop* of M. In this case, the matroid on $\{i\}$ is $U_{1,1}$, so we can write $M = M' \oplus U_{1,1}$, where M' is the restricted matroid of M on $[n]\backslash\{i\}$ with the collection of circuits are the same to the one of M, write $M' = M|_{\{n\}\backslash\{i\}}$. Note that $U_{1,1} \oplus U_{r,r} = U_{r+1,r+1}$. So we can represent a matroid M in the form $M = M' \oplus U_{p,p}$, where M' is a matroid which has no coloop and p is the number of coloops of M.

2.3. *Orlik–Solomon algebra*

We review in this section some algebraic and combinatorial aspects of arrangements theory with particular attention to the Orlik–Solomon algebra of an arrangement. For more details of this theory, we refer to the book by Orlik and Terao [24] or [26].

We always assume that $\mathcal{A} = \{H_1, \ldots, H_n\}$ is an essential central hyperplane arrangement in $\mathbb{C}^l$ with the complement $\mathcal{X}(\mathcal{A}) = \mathbb{C}^l \backslash \bigcup_{H \in \mathcal{A}} H$. We say that a set of hyperplanes $\{H_{i_1}, \ldots, H_{i_t}\}$ is *dependent* if the set of their defining linear forms is linearly dependent. Let $E = K\langle e_1, \ldots, e_n\rangle$ be the standard graded exterior algebra over a field K with $\deg e_i = 1$, $i = 1, \ldots, n$ and $\operatorname{char} K = 0$. Let $\partial : E \to E$ be the

K-linear map on E defined by $\partial e_i = 1$ for $i = 1, \dots, n$ and

$$\partial e_F = \sum_{j=1}^{t} (-1)^{j-1} e_{i_1} \wedge \cdots \widehat{e_{i_j}} \cdots \wedge e_{i_t} \quad \text{for } F = \{i_1, \dots, i_t\} \subseteq [n], \tag{2.1}$$

where $1 \le i_1 < i_2 < \cdots < i_t \le n$, $t \ge 2$ and $e_F = \prod_{i \in F} e_i$. One can show that ∂ is a differential map on E satisfying the graded Leibniz formula. Moreover, for a set of indices $F = \{i_1, \dots, i_t\} \subseteq [n]$ with $1 \notin F$ and $1 \le i_1 < i_2 < \cdots < i_t \le n$, one can also check by formula (2.1) that

$$\partial e_F = (e_{i_2} - e_{i_1}) \cdots (e_{i_t} - e_{i_1}) = \sum_{j=1}^{t} (-1)^{j-1} \, \partial \, e_{F \setminus \{i_j\} \cup \{1\}}. \tag{2.2}$$

In the last decades, many properties of hyperplane arrangements have been studied using the so-called the *Orlik–Solomon algebra* of $\mathcal{A}$. This algebra is the quotient ring E/J, where J is the *Orlik–Solomon ideal* of $\mathcal{A}$ given by

$$J = (\partial \, e_F : \{H_i : i \in F\} \text{ is dependent}).$$

Orlik and Solomon [23, Theorem 5.2] showed that the cohomology ring of $\mathcal{X}(\mathcal{A})$ is entirely determined by the intersection lattice $L(\mathcal{A}) = \{\bigcap_{H \in \mathcal{A}'} H \mid \mathcal{A}' \subseteq \mathcal{A}\}$ of $\mathcal{A}$. More precisely, the singular cohomology $H^{\bullet}(\mathcal{X}(\mathcal{A}); K)$ of $\mathcal{X}(\mathcal{A})$ with coefficients in K is isomorphic to the Orlik–Solomon algebra of $\mathcal{A}$. See Orlik–Terao [24, Chap. 3] and Yuzvinsky [31, Sec. 2] for details.

For a central arrangement $\mathcal{A}$, the intersection lattice $L(\mathcal{A})$ is a ranked poset, i.e. it has a unique minimal element $\hat{0} = \mathbb{C}^{\ell}$ and for any two comparable elements $X, Y \in L(\mathcal{A})$, all maximal linearly ordered chains $X = X_0 < X_1 < \cdots < X_s = Y$ have the same cardinality. The *rank* of $X \in L(\mathcal{A})$, write $\operatorname{rank}(X)$, is the length of a maximal chain from $\hat{0}$ to X. Note that $\operatorname{rank}(X) = \operatorname{codim}(\bigcap_{H \supset X} H)$ and the maximal element $\hat{1} = \bigcap_{H \in \mathcal{A}} H$ of $L(\mathcal{A})$ is the element with maximal rank. An element $X \in L(\mathcal{A})$ is *modular* if for all $Y \in L(\mathcal{A})$ and all $Z < Y$, $Z \vee (X \wedge Y) = (Z \vee X) \wedge Y$. A central hyperplane arrangement $\mathcal{A}$ is called *supersolvable* if there exists a maximal chain $\hat{0} = X_0 < X_1 < \cdots < X_n = \hat{1}$ of modular elements in $L(\mathcal{A})$.

For an element X in the intersection lattice $L(\mathcal{A})$ of $\mathcal{A}$, we denote $|X| = s$ if X can be expressed as an intersection of maximally s hyperplanes and in this section we identify X with the sequence of indices of s hyperplanes whose intersection equals to X, i.e. if $|X| = s$ and $X = H_{i_1} \cap \cdots \cap H_{i_s}$ then we write $X = \{i_1, \dots, i_s\}$ up to a permutation of indices. We always have $|X| \ge \operatorname{rank}(X)$. We say that X is a *dependent element* in $L(\mathcal{A})$ if $|X| > \operatorname{rank}(X)$. Note that, this is equivalents to the fact that $|X| > \operatorname{codim} \bigcap_{j=1}^{s} H_{i_j}$. In other word, X is dependent if and only if $\{H_i : i \in X\}$ is a dependent set. We also denote by $\overline{L_i}$ the subset of $L(\mathcal{A})$ containing all dependent elements of rank i in $L(\mathcal{A})$. A set $\{j_1, \dots, j_t\} \subseteq [n]$ is said to be a *circuit* of $\mathcal{A}$ if $\{H_{j_1}, \dots, H_{j_t}\}$ is a minimal dependent set of hyperplanes. The collection of all circuits of $\mathcal{A}$ forms a matroid which is called the *underlying* *matroid of* $\mathcal{A}$ and denoted by $M(\mathcal{A})$. See [1, 11, 13, 21, 24, 27, 28, 31] for the study of Orlik–Solomon algebras via exterior algebra methods.

3. Universally Koszul Property

In this section, analogously to the polynomial ring case (see [6]), we present the universally Koszul property for standard graded K-algebras over the exterior algebra. At first, we recall the notion of Koszul filtrations as follows.

Definition 3.1 (Conca *et al.* [10]). Let R be a standard graded K-algebra. A *Koszul filtration* of R is a family $\mathcal{F}$ of ideals of R such that

(i) Every ideal $0 \neq I \in \mathcal{F}$ is generated by linear forms.
(ii) $\mathcal{F}$ contains the ideal 0 and the maximal graded ideal $\mathfrak{m}$ of R.
(iii) For $0 \neq I \in \mathcal{F}$, there exists $J \in \mathcal{F}$ such that $J \subset I$, I/J is cyclic and $J :_R I \in \mathcal{F}$.

Example 3.1. (i) Let $R = K\langle e_1, e_2, e_3\rangle/(e_{12})$. Then the collection

$$\mathcal{F} = \{0, (e_1), (e_1, e_2), (e_1, e_2, e_3)\}$$

is a Koszul filtration of R since $0 :_R (e_1) = (e_1, e_2)$, $(e_1) :_R (e_1, e_2) = (e_1, e_2)$ and $(e_1, e_2) :_R (e_1, e_2, e_3) = (e_1, e_2, e_3)$.

(ii) Let $J = (e_{12} - e_{34}, e_{13} - e_{24}) \subset E = K\langle e_1, \ldots, e_4\rangle$ and $R = E/J$. At first, we see that R has a Koszul filtration so R is Koszul by Remark‘3.1. Indeed, the following family of ideals is a Koszul filtration of R:

$$\mathcal{F} = \{(0_R), (e_1 + e_4), (e_1 + e_4, e_2 + e_3), (e_1 + e_4, e_2 + e_3, e_3),$$
$$(e_1 + e_4, e_2 + e_3, e_3, e_4)\},$$

since

$$0 :_R (e_1 + e_4) = (e_1 + e_4, e_2 + e_3),$$
$$(e_1 + e_4) :_R (e_1 + e_4, e_2 + e_3) = (e_1 + e_4, e_2 + e_3),$$
$$(e_1 + e_4, e_2 + e_3) :_R (e_1 + e_4, e_2 + e_3, e_3) = (e_1 + e_4, e_2 + e_3, e_3, e_4),$$
$$(e_1 + e_4, e_2 + e_3, e_3) :_R (e_1 + e_4, e_2 + e_3, e_3, e_4) = (e_1 + e_4, e_2 + e_3, e_3, e_4).$$

Then $\mathcal{F}$ is a Koszul filtration of R.

Next, we claim that J does not have a quadratic Gröbner basis with respect to the coordinates $e_1, \ldots, e_4$ and any monomial order on E. Indeed, assume the contrary that J has a quadratic Gröbner basis with respect to some order $<$ on E. Then the initial ideal $\mathrm{in}_<(J)$ of J is one of the following monomial ideals: (e_{12}, e_{13}), (e_{12}, e_{24}), (e_{34}, e_{13}) and (e_{34}, e_{24}). Observe that none of these four ideals contain all monomials of degree 3. This contradicts to the fact that J contains all monomials of degree 3.

So J does not have a quadratic Gröbner basic with respect to the natural coordinates of E, but $R = E/J$ is Koszul.

Following [10, Proposition 1.2], one has

Proposition 3.1. *Let $\mathcal{F}$ be a Koszul filtration of R. Then for every $I \in \mathcal{F}$, the quotient R/I has a linear R-free resolution.*

Denote by
$\mathcal{L}(R) = \{I \subset R : I$ is an ideal generated by lnear forms$\}$.

Definition 3.2. A standard graded K-algebra R over E is called *universally Koszul* if $\mathcal{L}(R)$ is a Koszul filtration of R.

The universally Koszul property over the exterior algebra has the following characterizations which can be proved analogously to [6, Proposition 1.4].

Proposition 3.2. *Let R be a standard graded K-algebra over E. The following statements are equivalent*:

(i) *R is universally Koszul.*
(ii) *For every ideal $I \in \mathcal{L}(R)$ the quotient R/I has a linear R-free resolution.*
(iii) *For every $I \in \mathcal{L}(R)$ one has $\operatorname{Tor}_2^R(R/I, K)_j = 0$ for $j > 2$.*
(iv) *For every $I \in \mathcal{L}(R)$ and $x \in R_1 \backslash I$ one has $I :_R (x) \in \mathcal{L}(R)$.*

Remark 3.1. Observe the following:

(i) Since every Koszul filtration contains the maximal graded ideal $\mathfrak{m}_R$ of R, by Proposition 3.1 we observe that if R has a Koszul filtration then $\mathfrak{m}$ has a linear R-free resolution. Hence, R is Koszul. Thus, the universally Koszul property implies the Koszul property of R.
(ii) If R is universally Koszul and $J \subset R$ is a graded ideal generated by linear forms in R_1 then R/J is universally Koszul (see [6, Lemma 1.6] for the polynomial ring case).

Example 3.2. (i) The exterior algebra $E = K\langle e_1, \ldots, e_n\rangle$ is universally Koszul since one can check that the condition (iv) in Proposition 3.2 is fulfilled. Indeed, let $I \in \mathcal{L}(E)$ and $x \in E_1 \backslash I$. By changing the coordinates, we may assume that $x = e_s$ and $I = (e_1, \ldots, e_{s-1})$ for some $1 \leq s \leq n$. Then $I :_E (e_s) = (e_1, \ldots, e_s) \in \mathcal{L}(E)$.

(ii) Let $f \in E_2$. If f is a product of two linear forms, then $R = E/(f)$ is universally Koszul. Indeed, after a suitable change of coordinates, f is a quadratic monomial. Thus, we may assume that $f = e_{12}$. Now, let $I \in \mathcal{L}(R)$ and $x \in R_1 \backslash I$. Let $J \in E$ be the corresponding linear ideal to I, i.e. $I = (J + (e_{12}))/(e_{12})$.

If $e_{12} \in J$, then $(J + (e_{12})) :_E (x) = J :_E (x)$ is generated by linear forms. So $I :_R (x)$ is also generated by linear forms. If $e_{12} \notin J$, then we may assume that $J = (e_3, \ldots, e_s)$ for some $3 \leq s \leq n$. If $(x) \in (e_1, \ldots, e_s)$, then $(J + (e_{12})) :_E (x) = (e_1, \ldots, e_s)$. Otherwise, we may assume that $x = e_{s+1}$. Then

$$(J + (e_{12})) :_E (x) = (e_{12}, e_3, \ldots, e_s) :_E (e_{s+1}) = (e_{12}, e_3, \ldots, e_s, e_{s+1}).$$

Thus, $I :_R (x) = I + (x)$ is generated by linear forms. By Proposition 3.2, we conclude that R is universally Koszul.

(iii) In the polynomial ring case, following [6, Proposition 3.1] we have that a quadratic hypersurface ring defined by an irreducible quadric is universally Koszul. But this is not true in the exterior algebra case. For example, let $f = e_{12} + e_{34}$ in $E = K\langle e_1, \ldots, e_4\rangle$. Then $R = E/(f)$ is not Koszul. More precisely, using Macaulay2 [17] we get that $\beta^R_{3,4}(K) = 5 \neq 0$. Thus, K does not have a linear free resolution over R and then R is not Koszul.

Using the same method as in [7], we can classify all graphs such that the algebras defined by their edge ideals are universally Koszul over the exterior algebra. We recall first some facts. If $R = K\langle e_1, \ldots, e_n\rangle/I$, we set $R\langle e\rangle = K\langle e_1, \ldots, e_n, e\rangle/I$ and consider this with its natural grading. Let $A = K\langle e_1, \ldots, e_n\rangle/I$ and $B = K\langle f_1, \ldots, f_m\rangle/J$. The *fiber product* of A and B is $K\langle e_1, \ldots, e_n, f_1, \ldots, f_m\rangle/P$, where

$$P = I + J + (e_i f_j : i = 1, \ldots, n \text{ and } j = 1, \ldots, m).$$

Analogously to [6, Lemma 1.6], one has

Lemma 3.1. *Let R, A and B be standard graded algebras over E. One has*

(i) *The extension $R\langle e\rangle$ of R is universally Koszul if and only if R is universally Koszul.*
(ii) *The fiber product of A and B is universally Koszul if and only if both A and B are universally Koszul.*

Proof. (i) Let $R' = R\langle e\rangle$. We need to prove that $I :_{R'} x \in \mathcal{L}(R')$ for every $I \in \mathcal{L}(R')$ and $x \in R'_1 \backslash I$. We have following cases:

Case 1. The generators of I belong to R. Then $I = JR' = J + Je$, where $J \in \mathcal{L}(R)$.

If $x \in R_1$, we claim that $I :_{R'} x = (J :_R x)R' \in \mathcal{L}(R')$ since $J :_R x \in \mathcal{L}(R)$. Indeed, it is clear that $(J :_R x)R' \subseteq I :_{R'} x$. Let $f = f_1 + f_2 e \in I :_{R'} x$, where $f_1, f_2 \in R$. Then $f_1 x + f_2 x e \in J + Je$. Thus, $f_1 x, f_2 x \in J$. Hence, $f \in (J :_R x)R'$. So we have $I :_{R'} x = (J :_R x)R' \in \mathcal{L}(R')$.

If $x \notin R_1$, we may assume that $x = z + e$, where $z \in R_1$. We claim that $I :_{R'} x = I + (x) \in \mathcal{L}(R')$. Indeed, it is clear that $I + (x) \subseteq I :_{R'} x$. Let $f = f_1 + f_2 e \in I :_{R'} x$, where $f_1, f_2 \in R$. Then $f = f_1 - f_2 z + f_2 x$. Since $x \in I :_{R'} x$, we may assume that $f \in R$. Then $fz + fe \in I = J + Je$. Thus, $f \in J \subset I$. Hence, $I :_{R'} x = I + (x) \in \mathcal{L}(R')$.

Case 2. Some of the generators of I do not belong to R. Then we may decompose I as $JR' + (y + e)$, where $J \in \mathcal{L}(R)$ and $y \in R_1$. One can check that $I \cap R = J$.

If $x \in R_1$, we claim that $I :_{R'} x = (J :_R x)R' + (y + e) \in \mathcal{L}(R')$. Indeed, it is clear that $(J :_R x)R' + (y + e) \subseteq I :_{R'} x$. Let $f \in I :_{R'} x$. We may assume that $f \in R$ since $(y + e) \in I :_{R'} x$. Then $fx \in I \cap R = J$. Thus, $f \in (J :_R x)$. Hence,

$$I :_{R'} x = (J :_R x)R' + (y + e) \in \mathcal{L}(R').$$

It remains to consider $x = z + e$, where $z \in R_1$. Since $x \notin I$, we have $z - y \notin J$. We claim that

$$I :_{R'} x = (J :_R (z - y))R' + (x) + (y + e) \in \mathcal{L}(R').$$

Indeed, one can check that $(J :_R (z-y))R' + (x) + (y+e) \subseteq I :_{R'} x$. Let $f \in I :_{R'} x$. Then $fx \in I = JR' + (y+e)$. We may assume that $f \in R$ since $(y+e) \in I :_{R'} x$. Observe that $fx = fz + fe = fz - fy + f(y + e)$, so we get $f(z - y) = fx - f(y + e) \in I \cap R = J$. Since $z - y \notin J$ and R is universally Koszul, we have $f \in J :_R (z - y) \in \mathcal{L}(R)$. This concludes the proof.

(ii) The proof is verbatim the same as in [6, Lemma 1.6]. □

We also have an exterior algebra version of [7, Lemma 4] as follows.

Lemma 3.2. *Let $J \subset E = K\langle x, y, z, t\rangle$ be a quadratic monomial ideal. Then $R = E/J$ is not universally Koszul if J is one of the following ideals*:

(i) (xy, zt),
(ii) (xy, yz, zt).

Proof. In both cases, we claim that $0 :_R (y+z)$ has a minimal generator of degree 2. Indeed, it is clear that $xt \in 0 :_R (y + z)$. Since $\mathrm{span}_K\{y, z\}$ is the vector space containing all elements of degree 1 in $(xy, zt) :_E (y + z)$, $(xy, yz, zt) :_E (y + z)$, and $xt \notin (y, z)$, we get that xt is a minimal generator of $0 :_R (y + z)$. Thus, E/J is not universally Koszul. □

Let $J \subset E$ be a quadratic monomial ideal in a set of variables X. The *restriction* of J to a subset of variables $Y \subset X$ is the ideal I generated by those monomial generators of J which involve only elements of the set Y. Using the above lemmas, one can classify universally Koszul algebras defined by monomial ideals over the exterior algebra. For the convenience of the reader we reproduce here an exterior algebra version of [7, Theorem 5].

Theorem 3.1. *Let $R = E/J$, where $J \subset E$ is a quadratic monomial ideal. The following statements are equivalent*:

(i) *R is universally Koszul.*
(ii) *The restriction of J to any subset of 4 variables is not one of the types in Lemma 3.2.*

Proof. (i) ⇒ (ii): The assertion follows directly from Remark 3.1(ii) and Lemma 3.2.

(ii) ⇒ (i): Let $E = K\langle e_1, \dots, e_n\rangle$. We prove the statement by induction on n. The case $n = 1$ is trivial. Consider the case $n > 1$. Let $U = \{e_1, \dots, e_n\}$ and let $V = \{v_1, \dots, v_r\}$ be a maximal subset of U such that for all $v_i, v_j \in V$ with $i \neq j$ one has $v_iv_j \notin J$. Let $W = U\backslash V$ and $G_i = \{x \in W : xv_i \in J\}$. Then we have

$W = \bigcup_{i=1}^{r} G_i$. We claim that for $1 \le i < j \le r$ then either $G_i \subseteq G_j$ or $G_j \subseteq G_i$. Indeed, if there exist $x \in G_i \backslash G_j$ and $y \in G_j \backslash G_i$ then one has $xv_i, yv_j \in J$. Moreover, by the definition of the sets V, G_i, G_j, we note that J does not contain v_iv_j, xv_j and yv_i. This is a contradiction since the restriction of J to $\{x, y, v_i, v_j\}$ would be either (xv_i, yv_j) or (xv_i, yv_j, xy) which are one of the types (i), (ii) in Lemma 3.2. Hence, by a suitable renumbering if needed, we may assume that $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_r = W$.

We also note that if $x \in G_i$ and $y \in W \backslash G_i$ then $xy \in J$ for $i = 1, \dots, r$. Indeed, since $y \in W$, $y \notin V$. So there exists v_j such that $yv_j \in J$. This means $y \in G_j$. Since $y \notin G_i$ and either $G_i \subseteq G_j$ or $G_j \subseteq G_i$, we have $G_i \subset G_j$. Thus, $xv_j \in J$. Since $(yv_j, v_jx, xv_i) \subset J$ is of the type (ii) in Lemma 3.2 and $v_iv_j, yv_i \notin J$, one has $xy \in J$.

If $G_1 = \emptyset$ then v_1 does not appear in the minimal set of generators of J. Let J' be the ideal in $E' = E/(v_1)$ generated by the same minimal set of generators of J. Then by the induction hypothesis, we have $R' = E'/J'$ is universally Koszul. Therefore, $R = R'\langle v_1 \rangle$ is universally Koszul by Lemma 3.1(i).

If $G_1 = U$ then $V = \emptyset$. Thus, $xy \in J$ for every $x, y \in U$. This implies that $J = (e_1, \dots, e_n)^2$ and obviously $R = E/J$ is universally Koszul.

If G_1 is a proper non-empty subset of U then for $x \in G_1$ and $y \in U \backslash G_1$ we have $xy \in J$. Indeed, if $y \in W \backslash G_1$ then by the above argument we have $xy \in J$. Since $G_1 \subseteq G_i$ for $i = 1, \dots, r$, $xv_i \in J$ for all $v_i \in V$. So if $y \in U \backslash W = V$ then $xy \in J$. Now, let J_1, J_2 denote the restriction of J to G_1 and $\overline{G_1} = U \backslash G_1$, respectively. Set $A = K\langle G_1 \rangle / J_1$ and $B = K\langle \overline{G_1} \rangle / J_2$. Observe that, R is the fiber product of A and B. By the induction hypothesis, we note that A and B are universally Koszul algebras. Hence, by Lemma 3.1(ii), R is also a universally Koszul algebra. □

Recall that for a graph G with the vertex set $V(G)$, the edge ideal $J(G)$ of G is defined by

$$J(G) = (e_{ij} : i, j \in V(G) \text{ and } (i, j) \text{ is an edge of } G).$$

As direct consequences of Theorem 3.1, we have the following corollaries.

Corollary 3.1. *Let G be a graph with the edge ideal $J(G) \subset E$. Then the algebra $E/J(G)$ is universally Koszul if and only if every subgraph of* 4 *vertices in G is not one of the forms in Fig.* 1

Fig. 1. Subgraphs of 4 vertices.

Corollary 3.2. *Let $J \subset E$ be a monomial ideal and $I \subset K[x_1, \ldots, x_n] = S$ the corresponding squarefree monomial ideal. Then*

$$S/I \text{ is universally Koszul} \Leftrightarrow E/J \text{ is universally Koszul}.$$

Proof. The assertion follows directly from [7, Theorem 5] and Theorem 3.1. □

4. Initially Koszul Property

We study in this section standard graded K-algebras over the exterior algebra with the initially Koszul property. This is an analogue to the work of Blum in [3] and Conca *et al.* in [9] for standard graded K-algebras over the polynomial ring.

Definition 4.1. Let R be a standard graded K-algebra over E and let

$$F : V_0 = 0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n = R_1$$

be a complete flag of R_1, where V_i is a subspace of dimension i for $i = 1, \ldots, n$. We say that F is a *Gröbner flag* of R if the ideals (V_i) form a Koszul filtration of R, i.e. for $i = 1, \ldots, n$, there exists j_i such that $(V_{i-1}) :_R (V_i) = (V_{j_i})$. If R has a Gröbner flag, following [3], R is said to be an *initially Koszul* algebra.

Remark 4.1. Note that the universally Koszul property is equivalent to the existence of a Koszul filtration which is as large as possible, and the existence of a Gröbner flag is equivalent to the existence of a Koszul filtration which is as small as possible. More precisely, if R has a Gröbner flag then there exists an ordered system of generators $u_1, \ldots, u_n$ of R_1 such that $\{0, (u_1, \ldots, u_j) \text{ for } 1 \leq j \leq n\}$ is a Koszul filtration of R, i.e. for every $i = 1, \ldots, n$, we have

$$(u_1, \ldots, u_{i-1}) :_R u_i = (u_1, u_2, \ldots, u_{j_i}) \quad \text{for some } j_i \leq n.$$

Note that

$$(u_1, \ldots, u_{i-1}) :_R u_i \supseteq (u_1, u_2, \ldots, u_i).$$

Thus, $j_i \geq i$ for $i = 1, \ldots, n$. We denote by $j(F)$ the sequence of numbers $j_1, j_2, \ldots, j_n$.

Similarly, to results of Conca *et al.* in [9], we present next some properties of the standard graded algebra $R = E/J$ with a Gröbner flag F, where $J \subset E$ is a graded ideal. At first we have the following proposition.

Proposition 4.1. *Let $J \subset E$ be a graded ideal such that $R = E/J$ is initially Koszul with a Gröbner flag F. Then for $i = 0, \ldots, n$, the Hilbert series of $R/(V_i)$ depends only on $j(F)$.*

Proof. Let $F : V_0 = 0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n = R_1$. For $i = 1, \ldots, n$ we have short exact sequences

$$0 \longrightarrow R/(V_{j_i})[-1] \xrightarrow{u_i} R/(V_{i-1}) \longrightarrow R/(V_i) \longrightarrow 0.$$

Thus,

$$H_{R/(V_{i-1})}(t) = H_{R/(V_i)}(t) + tH_{R/(V_{j_i})}(t) \quad \text{for } i = 1, \dots, n. \tag{4.1}$$

Note that $j_i \geq i$ for $i = 1, \dots, n$, and $H_{R/(V_n)}(t) = 1$. Hence,

$$j_n = n \quad \text{and} \quad H_{R/(V_{n-1})}(t) = 1 + t.$$

By induction on i using (4.1), we get that for every i, the Hilbert series of $R/(V_i)$ depends only on $j(F)$. This concludes the proof. □

Example 4.1. Let $J = (e_{12}, e_{13}, e_{14}, e_{23}) \subset K\langle e_1, \dots, e_5\rangle$. Then R/J is initially Koszul. Indeed, the flag $F: V_0 = 0 \subset V_1 \subset \cdots \subset V_5 = R_1$, where $V_i = \text{span}_K\{e_1, \dots, e_i\}$ for $i = 1, \dots, 5$, is a Gröbner flag of R since $0 :_R (V_1) = (V_4)$, $(V_1) :_R (V_2) = (V_3)$, $(V_2) :_R (V_3) = (V_3)$, $(V_3) :_R (V_4) = (V_4)$ and $(V_4) :_R (V_5) = (V_5)$. Thus, $j(F) = (4, 3, 3, 4, 5)$. By Proposition 4.1 we get that

$$\begin{aligned}
H_{R/(V_5)}(t) &= H_K(t) = 1,\\
H_{R/(V_4)}(t) &= H_{R/(V_5)}(t) + tH_{R/(V_5)}(t) = 1 + t,\\
H_{R/(V_3)}(t) &= H_{R/(V_4)}(t) + tH_{R/(V_4)}(t) = 1 + 2t + t^2,\\
H_{R/(V_2)}(t) &= H_{R/(V_3)}(t) + tH_{R/(V_3)}(t) = 1 + 3t + 3t^2 + t^3,\\
H_{R/(V_1)}(t) &= H_{R/(V_2)}(t) + tH_{R/(V_3)}(t) = 1 + 4t + 5t^2 + 2t^3,\\
H_{R/(V_0)}(t) &= H_R(t) = H_{R/(V_1)}(t) + tH_{R/(V_4)}(t) = 1 + 5t + 6t^2 + 2t^3.
\end{aligned}$$

Recall that a graded algebra $R = E/J$, where $J \subset E$ is a graded ideal, is *G-quadratic* if J has a quadratic Gröbner basis with respect to some coordinate system of E_1 and some monomial order $<$ on E. Conca *et al.* in [9] and Blum in [3] obtained a characterization of the algebras which have Gröbner flags. We sketch an exterior algebra version of [3, Proposition 2.3], [9, Proposition 2.5] as follows.

Proposition 4.2. *Let R be a standard graded K-algebra over E. The following statements are equivalent*:

(i) *R has a Gröbner flag.*
(ii) *There exists a presentation of R, say $R \cong E/J$, such that if $<$ is the reverse lexicographic order induced by the total order $e_1 > e_2 > \cdots > e_n$, then $\text{in}_<(J)$ is a quadratic monomial ideal and if $e_ie_j \in \text{in}_<(J)$ with $i < j$ then $e_ke_j \in \text{in}_<(J)$ for all $i < k < j$.*

In particular, if R has a Gröbner flag, then R is G-quadratic.

5. Arrangements with Universally Koszul and Initially Koszul Properties

In this section, by applying previous results we classify hyperplane arrangements whose Orlik–Solomon algebras are universally Koszul or initially Koszul. Note that the universally Koszul, initially Koszul properties imply the Koszul property. Thus, a necessary condition for an Orlik–Solomon algebra A to be universally Koszul or initially Koszul is that A is quadratic, i.e. the corresponding Orlik–Solomon ideal is generated by quadrics.

Let $X \in L(\mathcal{A})$ with $\operatorname{rank}(X) = r$ and $|X| = s \geq r+1$. Observe that every subset of $(r+1)$ elements of X is dependent. Let J_X be the ideal defined by

$$J_X = (\partial\, e_F : F \subseteq X, |F| = r+1).$$

We present first some facts about arrangements and their intersection lattices.

Lemma 5.1. *Let $X = \{i_1, \dots, i_m\}$ be a dependent element of rank r in $L(\mathcal{A})$, where $i_1 = \min X$. Then J_X has a system of minimal generators of the form*

$$G(J_X) = \{\partial\, e_F : F \subset X, |F| = r+1 \text{ and } i_1 \in F\}.$$

Proof. At first we prove that $G(J_X)$ is a linearly independent set over K. This follows from the fact that $e_{i_1} G(J_X)$ is a linearly independent set of monomials since $e_{i_1}\, \partial\, e_F = e_F$ because F contains i_1.

Moreover, for $T \subset X, |T| = r+1, i_1 \notin T$, by Eq. (2.2) we get that $\partial\, e_T \in (G(J_X))$. Since

$$J_X = (\partial\, e_T : T \subset X, |T| = r+1)$$

we have $G(J_X)$ is a set of generators. This concludes the proof. □

For a graded ideal $J \subset E$ and an integer number i, we denote by $J_{\langle i \rangle}$ the ideal generated by all homogeneous elements of degree i in J. If J is the Orlik–Solomon ideal of an essential central hyperplane arrangement $\mathcal{A}$ then every set of two hyperplanes is independent. Therefore, J is generated in degree ≥ 2 and

$$J_{\langle 2 \rangle} = (\partial\, e_F : |F| = 3 \text{ and } F \text{ is dependent}).$$

Lemma 5.2. *Let $\mathcal{A}$ be an essential central hyperplane arrangement with intersection lattice $L(\mathcal{A})$ and Orlik–Solomon ideal J. Suppose that elements in $\overline{L_2}$ are disjoint. Then there exists a change of coordinates $\varphi \in \mathrm{GL}_n(K)$ such that $\varphi(J_{\langle 2 \rangle})$ is a monomial ideal.*

Proof. Assume that $\overline{L_2} = \{X_1, \dots, X_r\}$, where $X_i = \{i_1, \dots, i_{s_i}\}$ are dependent elements of rank 2 in $L(\mathcal{A})$ such that $X_i \cap X_j = \emptyset$ for $i, j \in \{1, \dots, r\}$, $i \neq j$. Consider the change of coordinates $\varphi : E_1 \to E_1$ defined by: $\varphi(e_{i_1}) = e_{i_1}$, $\varphi(e_{i_k}) = e_{i_k} + e_{i_1}$ for $k = 2, \dots, s_i$ and $i = 1, \dots, r$, $\varphi(e) = e$ for $e \in E_1 \setminus \bigcup_{i=1}^{r} X_i$. Then

$$\varphi(\partial\, e_{i_1 i_k i_t}) = \varphi((e_{i_k} - e_{i_1})(e_{i_t} - e_{i_1})) = e_{i_k} e_{i_t} \quad \text{for } k, t = 2, \dots, s_i \text{ and } i = 1, \dots, r.$$

By Lemma 5.1 we get that $\varphi(J_{X_i}) = (e_{i_k} e_{i_t} : k, t = 2, \dots, s_i)$.

Since $J_{\langle 2 \rangle} = J_{X_1} + \cdots + J_{X_r}$, we get that $\varphi(J_{\langle 2 \rangle})$ is a monomial ideal. □

We illustrate the above lemma by the following example.

Example 5.1. Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^3$ defined by the equation:

$$Q = xy(x+y)(x+3y+z)(x+4y+2z)(y+z).$$

Let $E = K\langle e_1,\dots,e_6\rangle$ where e_i responds to ith factor in the equation of $\mathcal{A}$ for $i = 1,\dots,6$. Then $\overline{L_2} = \{\{1,2,3\},\{4,5,6\}\}$ and $J_{\langle 2\rangle} = (\partial\, e_{123}, \partial\, e_{456})$. Consider the change of coordinates $\varphi\colon E_1 \to E_1$ defined by $\varphi(e_1) = e_1$, $\varphi(e_2) = e_2 + e_1$, $\varphi(e_3) = e_3 + e_1$, $\varphi(e_4) = e_4$, $\varphi(e_5) = e_5 + e_4$, $\varphi(e_6) = e_6 + e_4$. Then we have

$$\varphi(J_{\langle 2\rangle}) = (e_2e_3, e_4e_5).$$

Recall that a uniform matroid $U_{p,q}$ with $p \le q$ is a matroid over the ground set $[q]$ whose circuits (i.e. minimal dependent sets) are all subsets of $[q]$ of cardinality $p+1$. Now, we classify classes of Orlik–Solomon algebras which are universally Koszul.

Theorem 5.1. *Let A be the Orlik–Solomon algebra of an essential central hyperplane arrangement $\mathcal{A}$. The following statements are equivalent*:

(i) *A is universally Koszul.*
(ii) *The underlying matroid of $\mathcal{A}$ is $M(\mathcal{A}) = U_{2,s} \oplus U_{n-s,n-s}$ for some $2 \le s \le n$.*

Proof. (ii) $\Rightarrow$ (i): Suppose that the underlying matroid of $\mathcal{A}$ is $M(\mathcal{A}) = U_{2,s} \oplus U_{n-s,n-s}$ for some $2 \le s \le n$. Without loss of generality, we may assume that the indices $i = 1,\dots,s$ of hyperplanes H_i in $\mathcal{A}$ constitute the ground set of the uniform matroid $U_{2,s}$. Then every set of three hyperplanes H_i, H_j, H_k, where $1 \le i,j,k \le s$ is dependent. Therefore,

$$\begin{aligned} J &= (\partial e_F : F \subset [s] \text{ is circuit}) \\ &= ((e_i - e_k)(e_j - e_k) : \{i,j,k\} \text{ is dependent for } 1 \le i < j < k \le s) \\ &= ((e_i - e_1)(e_j - e_1) : \{1,i,j\} \text{ is dependent for } 2 \le i < j \le s) \\ &= (e_2 - e_1, \dots, e_s - e_1)^2. \end{aligned}$$

J is of the form

$$J = (e_2 - e_1, \dots, e_s - e_1)^2 \quad \text{for some } 3 \le s \le n.$$

Then by changing the coordinates:

$$e_i \mapsto \begin{cases} e_i + e_1 & \text{if } 2 \le i \le s, \\ e_i, & \text{otherwise,} \end{cases}$$

the ideal J becomes a monomial ideal of the form $(e_2,\dots,e_s)^2$. Since $E/(e_2,\dots,e_s)^2$ is universally Koszul by Theorem 3.1, $A = E/J$ is also universally Koszul.

(i) $\Rightarrow$ (ii): Suppose that A is universally Koszul. If $J = 0$ then there is no circuit in $M(\mathcal{A})$. This implies that $M(\mathcal{A}) = U_{n,n} = U_{2,2} \oplus U_{n-2,n-2}$. We consider the case

$J \neq 0$. Since A is universally Koszul, A is Koszul. Therefore, $0 \neq J$ is generated in degree 2. So we have $\overline{L_2} \neq \emptyset$. We claim that the set $\overline{L_2}$ of dependent elements of rank 2 in $L(\mathcal{A})$ has only one element. Assume the contrary, i.e. $|\overline{L_2}| \geq 2$. Then we consider two cases as follows:

Case 1. All elements of $\overline{L_2}$ are disjoint. Let $\overline{L_2} = \{X_1, \ldots, X_r\}$, where $r \geq 2$ and $X_i \cap X_j = \emptyset$ for $1 \leq i, j \leq r$. Then after the change of coordinates φ as in Lemma 5.2, all $\varphi(J_{X_i})$ are monomial ideals and so is $\varphi(J)$. Moreover, since $X_i \cap X_j = \emptyset$ for $1 \leq i, j \leq r$, we get that the restriction of $\varphi(J)$ to the set of variables $\{e_{i_2}, e_{i_3}, e_{j_2}, e_{j_3}\}$ is the monomial ideal $(e_{i_2}e_{i_3}, e_{j_2}e_{j_3})$ for $i, j \in \{1, \ldots, r\}$, $i \neq j$. Observe that, $(e_{i_2}e_{i_3}, e_{j_2}e_{j_3})$ is a monomial ideal of the type as in Lemma 3.2 (i). By Theorem 3.1, E/J is not universally Koszul which is a contradiction to our assumption (i).

Case 2. There exist $X_1, X_2 \in \overline{L_2}$ such that $X_1 \neq X_2$ and $X_1 \cap X_2 \neq \emptyset$. Since $X_1 \neq X_2$, we see that $X_1 \cap X_2$ is a set of one element, say $X_1 \cap X_2 = \{1\}$. Note that $|X_1|, |X_2| \geq 3$ since they are dependent elements of $L(\mathcal{A})$. Without loss of generality, we may assume that $2, 3 \in X_1$ and $4, 5 \in X_2$. Then $C_1 = \{1, 2, 3\}$ and $C_2 = \{1, 4, 5\}$ are two 3-circuits of $\mathcal{A}$.

Observe that we have no more 3-circuits in $[5] = \{1, \ldots, 5\}$. Otherwise, let $C \subset [5]$ be a 3-circuit and $C \neq C_1, C_2$. Then $|C \cap C_1| = 2$ or $|C \cap C_2| = 2$. If, say $|C \cap C_1| = 2$, then $|C \cap X_1| \geq 2$. Thus, $C \subset X_1$ and $X_1 \cap X_2 \supseteq (C \cup C_1) \cap C_2$. This is a contradiction since $(C \cup C_1) \cap C_2$ has at least two elements but $|X_1 \cap X_2| = 1$. Hence, there exist only the 3-circuits C_1, C_2 in $[5]$.

Write $J = (\partial\, e_{123}, \partial\, e_{145}, \partial\, e_{F_1}, \ldots, \partial\, e_{F_r})$ where F_i are 3-circuits of $\mathcal{A}$ for $1 \leq i \leq r$. Let $u = e_2 - e_1 + e_4 - e_1 = \partial\, e_{12} + \partial\, e_{14}$. We claim that

$$J :_E (u) = J + (u).$$

Since A is universally Koszul, $0 :_A (u)$ is generated by linear forms in A. We only need to prove that $(J :_E (u))_1 = (J + (u))_1$.

For a linear form $w = \sum_{i=1}^{n} \alpha_i e_i \in E_1$ and a set of indices $F \subset [n]$, we denote by $w_F = \sum_{i \in F} \alpha_i e_i$. It is obvious that $(J + (u))_1 \subset (J :_E (u))_1$ since $u \in J :_E (u)$. Assume that there exists $v \in E_1$ such that $0 \neq uv \in J$. Then by [14, Corollary 3.2] (see also [26, Corollary 4.9]), we have that for every $X \in \overline{L_2}$, either $\partial\, u_X = \partial\, v_X = 0$, or $v_X = \alpha u_X$, where $\alpha \in K$ since we have already $|X| \geq 3$. Since $\partial\, u_{X_1} = \partial\, u_{X_2} = -1$ and $e_1 \in \operatorname{supp}(u_{X_1}) \cap \operatorname{supp}(u_{X_2})$, we get that

$$v_{X_1} = cu_{X_1} = ce_2 - 2ce_1 \quad \text{and} \quad v_{X_2} = cu_{X_2} = ce_4 - 2ce_1, \quad \text{where } c \in K.$$

This implies that $v = ce_2 + ce_4 - 2ce_1 + v' = cu + v'$, where $v' \in E_1$ such that $e_1, e_2, e_4 \notin \operatorname{supp}(v')$. Note that $\operatorname{supp}(v') \neq \emptyset$ since $uv \neq 0$. Let $k \in \operatorname{supp}(v')$. Since $uv \in J$ and $e_1e_k \in \operatorname{supp}(uv)$, there exists $X \in \overline{L_2}$ with $|X| \geq 3$ such that $\{1, k\} \subset X$. If $\partial\, u_X \neq 0$, then $v_X = \alpha u_X$, where $\alpha \in K$, by [14, Corollary 3.2]. This is impossible since $k \in \operatorname{supp}(v) \backslash \operatorname{supp}(u)$. So $\partial\, u_X = 0$. This implies that $\{1, 2, 4\} \subset X$. Thus, $\{1, 2, 4\} \subset [5]$ is a 3-circuit. This contradicts the fact that we have only two 3-circuits $C_1, C_2 \subset [5]$. Hence, such a v chosen as above cannot exist and this implies $(J :_E (u))_1 = (J + (u))_1$. So we have $J :_E (u) = J + (u)$.

Let $f = \partial e_{135} = e_{13} - e_{15} + e_{35}$. By formula (2.2), one can check that

$$\partial e_{T_1} \partial e_{T_2} = \partial e_{T_1 \cup T_2} \quad \text{for } T_1, T_2 \subset [n] \text{ and } T_1 \cap T_2 = \{1\}.$$

Using this equation, we have

$$uf = (\partial e_{12} + \partial e_{14}) \partial e_{135} = \partial e_{1235} + \partial e_{1345} = \partial e_{15} \partial e_{123} + \partial e_{13} \partial e_{145} \in J.$$

Thus, $f \in J :_E (u) = J + (u)$. Hence, there is a representation

$$\partial e_{135} = g + uh = g + (\partial e_{12} + \partial e_{14})h, \quad \text{where } g \in J \text{ and } h \in E_1. \tag{5.1}$$

Since $e_{35} \in \operatorname{supp}(\partial e_{135})$ and e_{35} does not occur in $(\partial e_{12} + \partial e_{14})h$ for every $h \in E_1$, we get that $e_{35} \in \operatorname{supp}(g)$. Thus, there exists F_i such that e_{35} occurs in ∂e_{F_i}. This implies that $\{3, 5\} \subset F_i$. Let $X \in \overline{L_2}$ with $F_i \subset X$. Note that the sets $\{1, 3, 5\}, \{2, 3, 5\}, \{3, 4, 5\}$ are not circuits. Hence, $1, 2, 4 \notin X$.

Let $I = (e_j : j \notin X)$ and $P = (e_{i_1} e_{i_2} : i_1, i_2 \in X)$. Then $e_1, e_2, e_4 \in I$, $e_{35} \in P_2$ and $J_X \subset P$. Observe that $I_2 \cap (J_X)_2 = \{0\}$ since $I_2 \cap P_2 = \{0\}$. Moreover, $J \subset J_X + I$ since $\partial e_{F_j} \in I$ for every $F_j \not\subset X$. Recall from Eq. (5.1) that $f = \partial e_{135} = g + uh$. Write $g = g_1 + g_2$, where $g_1 \in (J_X)_2$ and $g_2 \in I_2$. It follows that

$$e_{13} - e_{15} + e_{35} = g_1 + g_2 + (\partial e_{12} + \partial e_{14})h.$$

Thus,

$$e_{35} - g_1 = g_2 - e_{13} + e_{15} + (\partial e_{12} + \partial e_{14})h \in I_2.$$

Since $e_{35} - g_1 \in P_2$ and $g_2 - e_{13} + e_{15} + (\partial e_{12} + \partial e_{14})h \in I_2$, we get that $e_{35} = g_1 \in (J_X)_2$ because $I_2 \cap P_2 = \{0\}$. So $e_{135} \in J_X$. Assume that $X = \{i_1, \ldots, i_s\}$, where $i_1 \neq 3, 5$. Then by the change of coordinates φ as in Lemma 5.2 defined by $\varphi(e_{i_1}) = e_{i_1}, \varphi(e_{i_t}) = e_{i_t} + e_{i_1}$ for $t = 2, \ldots, s$ and $\varphi(e_j) = e_j$ for $j \notin X$, we have $\varphi(J_X) = (e_{i_2}, \ldots, e_{i_s})^2$. Note that $3, 5 \in X$ and $\varphi(e_{135}) \in \varphi(J_X)$. If $1 \notin X$ then

$$\varphi(e_{135}) = e_1(e_3 + e_{i_1})(e_5 + e_{i_1}) = e_{135} + e_1 e_{i_1}(e_5 - e_3) \in (e_{i_2}, \ldots, e_{i_s})^2.$$

Since $e_{35} \in (e_{i_2}, \ldots, e_{i_s})^2$, $e_1 e_{i_1}(e_5 - e_3) \in (e_{i_2}, \ldots, e_{i_s})^2$. So $e_1 e_{i_1}(e_5 - e_3) \in E_1(e_{i_2}, \ldots, e_{i_s})^2$. This is impossible since $e_3, e_5 \in \{e_{i_2}, \ldots, e_{i_s}\}$ but $e_1, e_{i_1} \notin \{e_{i_2}, \ldots, e_{i_s}\}$. Therefore, $1 \in X$. So $\{1, 3, 5\} \subset X$, i.e. $\{1, 3, 5\}$ is a circuit. This is a contradiction since there exist only the 3-circuits $C_1, C_2 \subset [5]$.

Combining Cases 1 and 2 we get that $\overline{L_2}$ has indeed only one element, say $\overline{L_2} = \{X\}$, where $X = \{i_1, \ldots, i_s\}$. Since J is generated in degree 2, we have $J = J_X$. Thus, every 3-circuit of $M(\mathcal{A})$ is a subset of X. Assume that F is a m-circuit with $m \geq 4$. Then $|F \cap X| \leq 2$. If $1 \leq |F \cap X| \leq 2$ then without loss of generality, we can assume that $i_1 \in F \cap X$. By using the change of coordinates ψ defined by $\psi(e_{i_1}) = e_{i_1}$ and $\psi(e_j) = e_j + e_{i_1}$ for $j \neq i_1$, we have that $\psi(J) = \psi(J_X) = (e_{i_2}, \ldots, e_{i_s})^2$ and $\psi(\partial e_F) = e_{F \backslash \{i_1\}}$. Note that $e_{F \backslash \{i_1\}} \notin (e_{i_2}, \ldots, e_{i_s})^2$ since $|(F \backslash \{i_1\}) \cap \{i_2, \ldots, i_s\}| \leq 1$. So $\psi(\partial e_F) \notin \psi(J)$. This is a contradiction since

$\partial e_F \in J$. If $F \cap X = \emptyset$ then $\psi(\partial e_F) = \partial e_F$ which does not contain any monomial in $(e_{i_2}, \ldots, e_{i_s})^2$ since $F \cap \{i_2, \ldots, i_s\} = \emptyset$. Hence, $\psi(\partial e_F) \notin \psi(J)$. This is also a contradiction. Therefore, $M(\mathcal{A})$ has no m-circuit with $m \geq 4$. Moreover, since $\mathcal{A}$ is an essential central hyperplane arrangement, $M(\mathcal{A})$ does not have any circuit of ≤ 2 elements. Hence, we get that $M(\mathcal{A}) = U_{2,s} \oplus U_{n-s,n-s}$, as desired. □

Remark 5.1. Let $<$ be the reverse lexicographic order on E with $e_1 > e_2 > \cdots > e_n$. Let $\mathcal{A}$ be an essential central hyperplane arrangement with the underlying matroid $M(\mathcal{A})$. A *broken circuit* with respect to $<$ of $M(\mathcal{A})$ is the set received from a circuit by deleting the largest index in the circuit. If the underlying matroid of $\mathcal{A}$ is $M(\mathcal{A}) = U_{2,s} \oplus U_{n-s,n-s}$ for some $2 \leq s \leq n$, then the minimal broken circuits have size two. This property characterizes supersolvable arrangements; see [5, Theorem 2.8]. Therefore, by Theorem 5.1 we have that if the Orlik–Solomon algebra A of $\mathcal{A}$ is universally Koszul then $\mathcal{A}$ is supersolvable.

Let $\mathcal{A}$ be an essential central hyperplane arrangement with the complement $\mathcal{X}(\mathcal{A})$ and its fundamental group $\pi_1(\mathcal{X}(\mathcal{A}))$. Let

$$Z = Z_1 = \pi_1(\mathcal{X}(\mathcal{A})), \quad Z_2 = [Z_1, Z], \ldots, Z_{i+1} = [Z_i, Z], \ldots$$

be the *lower central series* (LCS for short) and set $\varphi_i = \operatorname{rank}(Z_i/Z_{i+1})$. There is a lot of attention in [15, 22, 25, 27–29] to a special formula, called *LCS formula*, which states that

$$\prod_{j=1}^{\infty} (1 - t^j)^{\varphi_j} = H_A(-t).$$

It was proved by Shelton and Yuzvinsky in [29] that the formula holds if and only if the algebra A is Koszul. From the classification of Orlik–Solomon algebras satisfying the universally Koszul property, we compute the LCS formula in this case as follows.

Corollary 5.1. *Let A be the Orlik–Solomon algebra of an essential central hyperplane arrangement $\mathcal{A}$ such that A is universally Koszul. Then the underlying matroid $M(\mathcal{A})$ of $\mathcal{A}$ is $U_{2,n-f} \oplus U_{f,f}$ for some $0 \leq f \leq n-2$ and we have*

$$\prod_{j=1}^{\infty} (1 - t^j)^{\varphi_j} = 1 - nt + \cdots + (-1)^k \left(\binom{f+1}{k} + (n-f-1)\binom{f+1}{k-1} \right) t^k + \cdots$$

Proof. By Theorem 5.1, we have already that $M(\mathcal{A}) = U_{2,n-f} \oplus U_{f,f}$ for some $0 \leq f \leq n-2$. Therefore, $J = (\partial e_{ijk} : 1 \leq i,j,k \leq n-f)$. Let $<$ be the lexicographic order on E with $e_1 > e_2 > \cdots > e_n$. We only need to compute the Hilbert function of A, i.e. the numbers of monomials of each degree in E which do not belong to $\operatorname{in}_<(J)$. Note that the broken circuits of $M(\mathcal{A})$ generate the initial ideal of J with respect to $<$ (see, e.g. [4], [25, Theorem 4.1]). So we have $\operatorname{in}_<(J) = (e_{ij} : 1 \leq i,j \leq n-f-1)$. Thus, $e_F \notin \operatorname{in}_<(J)$ if and only if $F \cap \{1, \ldots, n-f-1\}$ has at most 1 element. Therefore,

$$H(A,k) = \binom{f+1}{k} + (n-f-1)\binom{f+1}{k-1} \quad \text{for } k = 1,\ldots,n.$$

This concludes the proof. □

In the hyperplane arrangement theory, there is an important characterization of supersolvable arrangement as follows: let $\mathcal{A}$ be an arrangement with its Orlik–Solomon algebra A. Then we have

$$\mathcal{A} \text{ is supersolvable } \Leftrightarrow A \text{ is } G\text{-quadratic.}$$

As an application of Sec. 4, we get another characterization of supersolvable arrangements. We have the following theorem.

Theorem 5.2. *Let $\mathcal{A}$ be an arrangement with Orlik–Solomon algebra A. The following statements are equivalent*:

(i) *$\mathcal{A}$ is supersolvable.*
(ii) *A is G-quadratic.*
(iii) *A is initially Koszul.*

Proof. (i) $\Leftrightarrow$ (ii) see, e.g. [25, Theorem 4.3]. (iii) $\Rightarrow$ (ii) follows from Proposition 4.2.

(i) $\Rightarrow$ (iii): Suppose that $\mathcal{A}$ is supersolvable with the Orlik–Solomon ideal J. By [5, Theorem 2.8(5)], there exists a partition $[n] = F_1 \cup \cdots \cup F_r$ such that for any two distinct indices $x, y \in F_i$, there is $z \in F_j$ with $j < i$ such that $\{x, y, z\}$ is a circuit. By a suitable change of indices, we may assume that for $s < r$ and $i \in F_s$, $j \in F_r$ we have $i > j$. Moreover, let $M_i = \cup_{j \le i} F_j$. [5, Theorem 2.8(5)] also implies that $\widehat{0} = M_0 < M_1 < \cdots < M_r = \widehat{1}$ is a maximal chain of modular elements of the supersolvable lattice $L(\mathcal{A})$. Note that we identify M_i with the element $\cap_{j \in M_i} H_j$ in $L(\mathcal{A})$.

Let $<$ be the reverse lexicographic order on E with $e_1 > e_2 > \cdots > e_n$. Recall that a broken circuit with respect to $<$ of $M(\mathcal{A})$ is the set received from a circuit by deleting the largest index in the circuit. We claim that if $\{x, z\}$ is a broken circuit with $x < z$ then x, z belong to the same F_i. Let $\{w, x, z\}$ be the circuit containing $\{x, z\}$, where $x < z < w$ and $x \in F_i$. Note that if $z \in F_j$, then $j \le i$. If $w \in F_k$ and $z \in F_j$ with $k \le j < i$, then $H_z \cap H_w \supset M_j$. Since $\{w, x, z\}$ is a circuit, we have $H_x \supset H_z \cap H_w$. Thus $H_x \supset M_j$. This implies that $x \in F_t$ with $t \le j$. This contradicts the fact that $x \in F_i$ and $x \notin \cup_{k \le j} F_k$ since $i > j$. Thus, $z \in F_i$.

Next, we claim that the condition (ii) of Proposition 4.2 holds for J. By [25, Proposition 4.2, Theorems 4.1 and 4.3], we have that $\text{in}_<(J)$ is quadratic. More precisely, in $_<(J)$ is generated by squarefree monomials corresponding to broken circuits of 2 indices of $M(\mathcal{A})$. Now, for a set of indices $\{x, y, z\}$ with $x < y < z$, we have that if $e_x e_z \in \text{in}(J)$ then $\{x, z\}$ is a broken circuit of $M(\mathcal{A})$. By the above argument, there exists $1 \le i \le r$ such that $\{x, z\} \subset F_i$. By the assumption for

F_i, we also have that $y \in F_i$ because $x < y < z$. Since $\{y, z\} \in F_i$, there exists $t \in F_j$, where $j < i$ and $t > y, z$ such that $\{t, y, z\}$ is a circuit of $\mathcal{A}$. Thus, $\{y, z\}$ is also a broken circuit and $e_y e_z \in$ in $_<(J)$. Hence, the condition (ii) of Proposition 4.2 is fulfilled. So A has a Gröbner flag by Proposition 4.2, i.e. A is initially Koszul. □

Remark 5.2. By Theorem 5.2, one has that supersolvable arrangements could be characterized as the initially Koszul property of their Orlik–Solomon algebras. So Shelton–Yuzvinsky's question can be formulated in a purely algebraic terms: Is it true that a Koszul Orlik–Solomon algebra is always initially Koszul?

To conclude this section, we illustrate the proof of Theorem 5.2 by the following example.

Example 5.2. Let $\mathcal{A}_3$ be the rank-three braid arrangement in $\mathbb{C}^4$ which is defined by the equation

$$Q = (x - y)(x - z)(y - z)(x - t)(y - t)(z - t).$$

It is well-known that $\mathcal{A}_3$ is a supersolvable arrangement; see, e.g. [24, Example 2.33]. From the underlying matroid of $\mathcal{A}_3$ (see Fig. 2), we get that the Orlik–Solomon ideal of $\mathcal{A}_3$ is

$$J = (\partial\, e_{125}, \partial\, e_{134}, \partial\, e_{236}, \partial\, e_{456}) \subset E = K\langle e_1, \ldots, e_6\rangle.$$

The partition of [6] satisfying the condition in [5, Theorem 2.8(5)] is: $(6|5, 4|3, 2, 1)$, i.e. $F_1 = \{6\}$, $F_2 = \{5, 4\}$, $F_3 = \{3, 2, 1\}$. The broken circuits of size 2 of $\mathcal{A}_3$ are $\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}$. We see that two elements of every broken circuit are in one F_i. Moreover, we can check directly that

$$0 \subset \operatorname{span}_K\{e_6\} \subset \operatorname{span}_K\{e_6, e_5\} \subset \cdots \subset \operatorname{span}_K\{e_6, \ldots, e_1\}$$

is a Gröbner flag of the Orlik–Solomon algebra $A = E/J$ of $\mathcal{A}_3$ since

$$0 :_A e_6 = (e_1), \quad (e_6) :_A e_5 = (e_6, e_5, e_4), \quad (e_6, e_5) :_A e_4 = (e_6, e_5, e_4)$$

and

$$(e_6, e_5, e_4) :_A e_3 = (e_6, \ldots, e_3) :_A e_2 = (e_6, \ldots, e_2) :_A e_1 = (e_6, \ldots, e_1).$$

Thus, A is initially Koszul.

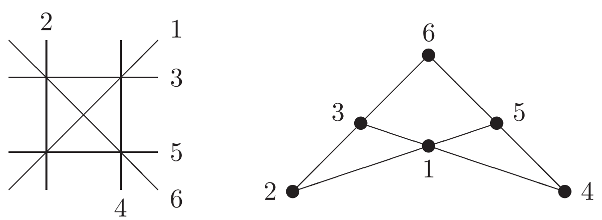

Fig. 2. The braid arrangement and its matroid.

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