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Comparison between regularity of small symbolic powers and ordinary powers of an edge ideal

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A B S T R A C T

Let $I = I(G)$ be the edge ideal of a simple graph G. We prove that

$$\operatorname{reg}(I^{(s)}) = \operatorname{reg}(I^s)$$

for $s = 2, 3$, where $I^{(s)}$ is the s-th symbolic power of I. As a consequence, we prove the following bounds

$$\operatorname{reg} I^s \le \operatorname{reg} I + 2s - 2, \text{ for } s = 2, 3,$$

$$\operatorname{reg} I^{(s)} \le \operatorname{reg} I + 2s - 2, \text{ for } s = 2, 3, 4.$$

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1. Introduction

The Castelnuovo-Mumford regularity (or regularity for short) is an important invariant of graded algebras. It bounds the maximum degree of the syzygies and the maximum non-vanishing degree of the local cohomology modules. It is a celebrated result that the regularity of I^s is asymptotically a linear function for any homogeneous ideal I in a polynomial ring S over a field (see [8,25]). It is a natural question then to ask whether a similar result holds for symbolic powers of I. In general, Cutkosky [6] gave an example of a homogeneous ideal I such that $\lim_{t\longrightarrow\infty} \dfrac{\operatorname{reg}(I^{(t)})}{t}$ is not rational. So $\operatorname{reg}(I^{(t)})$ is in general far from being asymptotically a linear function. For a monomial ideal Herzog, Hoa, and N. V. Trung [20] showed that the regularity of symbolic powers is bounded by a linear function. In recent work, Dung, Hien, Nguyen, and T. N. Trung [10] have constructed a class of squarefree monomial ideals for which $\operatorname{reg}(I^{(t)})$ is not asymptotically a linear function. On the other hand, when I is the Stanley-Reisner ideal of a simplicial complex of dimension one or a matroid then $\operatorname{reg}(I^{(t)})$ is a linear function of t (see [22,28]). It is not known whether the regularity of symbolic powers of edge ideals of graphs is asymptotically a linear function. More exactly, in this case, the first author raised the following conjecture.

Conjecture A. *Let $I(G)$ be the edge ideal of a simple graph G. Then for all $s \geq 1$,*

$$\operatorname{reg}(I(G)^{(s)}) = \operatorname{reg}(I(G)^s).$$

By [35, Theorem 5.9], the graph G is a bipartite graph if and only if $I(G)^{(s)} = I(G)^s$ for all $s \geq 1$. Thus, the above conjecture is trivially true in this case. If G is not bipartite, then it must contain an odd cycle. Gu, Ha, O'Rourke, and Skelton [17] took the first step in verifying this conjecture for odd cycle graphs. Subsequently, the conjecture is verified for several other classes of graphs in [23,12–14,26,27,30]. The equality $\operatorname{reg} I^{(2)} = \operatorname{reg} I^2$ is known in some cases [2].

In this paper, we prove

Theorem 1.1. *Let $I(G)$ be the edge ideal of a simple graph G. Then*

$$\operatorname{reg} I(G)^{(s)} = \operatorname{reg}(I(G)^s)$$

for $s = 2, 3$.

In other words, we establish Conjecture A for $s = 2, 3$. Note that in most cases where $\operatorname{reg} I^{(s)}$ was computed, the main technical step was to bound the regularity of certain colon ideals. We do not know of any direct comparison between the regularity of powers and symbolic powers of ideals when the regularity of the corresponding symbolic/ordinary power is unknown. We now outline the idea of proof of Theorem 1.1.

(1) We reduce the problem of comparing the regularity of two monomial ideals to the problem of comparing radicals of the colon of these ideals by certain monomials, see Lemma 2.18 and Lemma 2.19.
(2) By Lemma 2.23 and induction, we further reduce to studying degree complexes of symbolic powers/ordinary powers of edge ideals of special exponents. We then analyse these degree complexes in detail via the Stanley-Reisner correspondence.

This procedure for comparing the regularity of monomial ideals is especially useful when the ideals are closely related; for example, an ideal versus its integral closure, various types of powers of an ideal. Furthermore, our study of degree complexes of symbolic/ordinary powers reveals interesting information on the extremal exponents of powers of edge ideals which will be exploited further in subsequent work to study the regularity of powers of edge ideals themselves.

The main obstructions to proceed the comparison further with higher powers are:

(1) Explicit description of symbolic powers of higher powers is unknown.
(2) Even in the case where an explicit description of symbolic powers is known, e.g. the case of perfect graphs, the number of critical exponents grows and the radical ideals of colon ideals of powers with respect to these critical exponents are difficult to compute.

By combining a recent result of Fakhari [15, Theorem 3.6], we establish a conjecture of Alilooee, Banerjee, Beyarslan, and Ha [1, Conjecture 1] for the second and third powers of edge ideals extending work of Banerjee and Nevo [5].

Theorem 1.2. *Let $I(G)$ be the edge ideal of a simple graph G. Then*

$$\operatorname{reg}(I(G)^s) \leq 2s - 2 + \operatorname{reg}(I(G)),$$

for $s = 2, 3$.

For symbolic powers, we extend [15, Corollary 3.9] to prove

Theorem 1.3. *Let $I(G)$ be the edge ideal of a simple graph G. Then*

$$\operatorname{reg}(I(G)^{(s)}) \leq 2s - 2 + \operatorname{reg}(I(G)),$$

for $s = 2, 3, 4$.

Finally, we obtain explicit values of the regularity of small symbolic powers of $I(G)$ for some new classes of graphs.

Now we explain the organization of the paper. In Section 2, we recall some notation and basic facts about the symbolic powers of a squarefree monomial ideal, the degree complexes, and Castelnuovo-Mumford regularity. In Section 3, we prove Theorem 1.1 for $s = 2$. In Section 4, we prove Theorem 1.1 for $s = 3$. Finally, Section 5 contains some applications of the main results.

2. Castelnuovo-Mumford regularity, symbolic powers and degree complexes

In this section, we recall some definitions and properties concerning Castelnuovo-Mumford regularity, the symbolic powers of a squarefree monomial ideal, and the degree complexes of a monomial ideal. The interested readers are referred to ([3,9,11,34]) for more details.

2.1. Simplicial complexes and Stanley-Reisner correspondence

Let Δ be a simplicial complex on $[n] = \{1, \dots, n\}$ that is a collection of subsets of $[n]$ closed under taking subsets. We put $\dim F = |F| - 1$, where $|F|$ is the cardinality of F. The dimension of Δ is $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$. The set of its maximal elements under inclusion, called by facets, is denoted by $\mathfrak{F}(\Delta)$.

A simplicial complex Δ is called a cone over $x \in [n]$ if $x \in B$ for any $B \in \mathfrak{F}(\Delta)$.

For a face $F \in \Delta$, the link of F in Δ is the subsimplicial complex of Δ defined by

$$\operatorname{lk}_\Delta F = \{G \in \Delta \mid F \cup G \in \Delta, F \cap G = \emptyset\}.$$

For each subset F of $[n]$, let $x_F = \prod_{i \in F} x_i$ be a squarefree monomial in S. We now recall the Stanley-Reisner correspondence.

Definition 2.1. For a squarefree monomial ideal I, the Stanley-Reisner complex of I is defined by

$$\Delta(I) = \{F \subseteq [n] \mid x_F \notin I\}.$$

For a simplicial complex Δ, the Stanley-Reisner ideal of Δ is defined by

$$I_\Delta = (x_F \mid F \notin \Delta).$$

The Stanley-Reisner ring of Δ is $K[\Delta] = S/I_\Delta$.

From the definition, it is easy to see the following:

Lemma 2.2. *Let I, J be squarefree monomial ideals of $S = K[x_1, \dots, x_n]$. Then*

(1) *$\Delta(I)$ is a cone over $t \in [n]$ if and only if x_t does not divide any minimal generator of I.*
(2) *$\Delta(I + J) = \Delta(I) \cap \Delta(J)$.*
(3) *$\Delta(I \cap J) = \Delta(I) \cup \Delta(J)$.*

2.2. Simplicial homology of simplicial complexes

Let Δ be a simplicial complex. An oriented q-simplex of Δ is a face $F \in \Delta$, $|F| = q+1$, with an ordering of the vertices, with the rule that two orderings define the same orientation if and only if they differ by an even permutation. Let $C_q(\Delta)$ be the K-vector space with basis consisting of the oriented q-simplices of Δ. We define the homomorphisms $\partial_q : C_q(\Delta) \to C_{q-1}(\Delta)$ for $q \geq 1$ by defining them on the basis elements by

$$\partial_q[v_0, \dots, v_q] = \sum_{i=0}^{q} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_q],$$

where $\hat{v}_i$ denotes that v_i is missing. It is easily verified that $\partial_q \partial_{q+1} = 0$. The chain complex $C(\Delta) = \{C_q(\Delta), \partial_q\}$ is the *oriented chain complex* of Δ. Let $C_{-1}(\Delta)$ be the K-vector space with basis $\{\emptyset\}$, and define an augmentation $\epsilon : C_0(\Delta) \to C_{-1}(\Delta)$ by $\epsilon(x) = \emptyset$ for every $x \in [n]$. The augmented chain complex $(C(\Delta), \epsilon)$ is the *augmented oriented chain complex* of Δ.

Definition 2.3. The q-th reduced homology group of Δ with coefficients K, denoted $\widetilde{H}_q(\Delta; K)$ is defined to be the q-th homology group of the augmented oriented chain complex of Δ over K.

A simplicial complex Δ is called *acyclic* if $\widetilde{H}_i(\Delta; K) = 0$ for all i.

Remark 2.4.

(1) If Δ is the empty complex (i.e., $\Delta = \{\emptyset\}$), then $\widetilde{H}_i(\Delta; K) \neq 0$ if and only if $i = -1$.
(2) If Δ is a cone over some $t \in [n]$ or Δ is the void complex (i.e., $\Delta = \emptyset$), then it is acyclic.

The following lemma will be useful later on.

Lemma 2.5. *Let Δ be a simplicial complex on $[n]$ with $\widetilde{H}_{i-1}(\Delta; K) \neq 0$ for some $i \geq 0$. Assume that $\Delta = \Gamma_1 \cup \Gamma_2$ is a decomposition of Δ as the union of two subsimplicial complexes. Then at least one of the homology groups $\widetilde{H}_{i-1}(\Gamma_1; K)$, $\widetilde{H}_{i-1}(\Gamma_2; K)$, $\widetilde{H}_{i-2}(\Gamma_1 \cap \Gamma_2; K)$ is non-zero.*

Proof. Applying the Mayer-Vietoris sequence for the decomposition $\Delta = \Gamma_1 \cup \Gamma_2$, we have the following long exact sequence of homology groups

$$\cdots \to \widetilde{H}_{i-1}(\Gamma_1; K) \oplus \widetilde{H}_{i-1}(\Gamma_2; K) \to \widetilde{H}_{i-1}(\Delta; K) \to \widetilde{H}_{i-2}(\Gamma_1 \cap \Gamma_2; K) \to \cdots.$$

The conclusion follows as the middle term is non-zero. $\square$

2.3. Castelnuovo-Mumford regularity

Let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the maximal homogeneous ideal of $S = K[x_1, \ldots, x_n]$ a polynomial ring over a field K. For a finitely generated graded S-module L, let

$$a_i(L) = \begin{cases} \max\{j \in \mathbb{Z} \mid H^i_{\mathfrak{m}}(L)_j \neq 0\} & \text{if } H^i_{\mathfrak{m}}(L) \neq 0 \\ -\infty & \text{otherwise,} \end{cases}$$

where $H^i_{\mathfrak{m}}(L)$ denotes the i-th local cohomology module of L with respect to $\mathfrak{m}$. Then, the Castelnuovo-Mumford regularity (or regularity for short) of L is defined to be

$$\operatorname{reg}(L) = \max\{a_i(L) + i \mid i = 0, \ldots, \dim L\}.$$

The regularity of L can also be defined via the minimal graded free resolution. Assume that the minimal graded free resolution of L is

$$0 \longleftarrow L \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_p \longleftarrow 0.$$

Let $t_i(L)$ be the maximal degree of graded generators of F_i. Then,

$$\operatorname{reg}(L) = \max\{t_i(L) - i \mid i = 0, \ldots, p\}.$$

From the minimal graded free resolution of S/J, we obtain $\operatorname{reg}(J) = \operatorname{reg}(S/J) + 1$ for a non-zero and proper homogeneous ideal J of S.

2.4. Symbolic powers

Let I be a non-zero and proper homogeneous ideal of S. Let $\{P_1, \ldots, P_r\}$ be the set of the minimal prime ideals of I. Given a positive integer s, the s-th symbolic power of I is defined by

$$I^{(s)} = \bigcap_{i=1}^{r} I^s S_{P_i} \cap S.$$

For $f \in S$ and $x^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$, we denote $\frac{\partial(f)}{\partial(x^{\mathbf{a}})}$ the partial derivative of f with respect to $x^{\mathbf{a}}$. For each s, we denote

$$I^{\langle s\rangle} = \left(f \in S \mid \frac{\partial f}{\partial x^{\mathbf{a}}} \in I, \text{ for all } x^{\mathbf{a}} \text{ with } |\mathbf{a}| \leq s-1 \right),$$

the s-th differential power of I. When K is a field of characteristic 0 and I is a radical ideal, it is a well-known theorem of Nagata-Zariski that $I^{(s)} = I^{\langle s\rangle}$.

When f is a monomial, we denote $\frac{\partial^*(f)}{\partial^*(x^{\mathbf{a}})}$ the $*$-partial derivative of f with respect to $x^{\mathbf{a}}$, which is derivative without coefficients. In general, $\partial f/\partial x^{\mathbf{a}} = c\partial^*(f)/\partial^*(x^{\mathbf{a}})$ for some constant c. Similarly, we define

$$I^{[s]} = \left(f \in S \mid \frac{\partial^* f}{\partial^* x^{\mathbf{a}}} \in I, \text{ for all } x^{\mathbf{a}} \text{ with } |\mathbf{a}| \le s-1 \right),$$

the s-th $*$-differential power of I. When the characteristic of K is equal to 0, $I^{\langle s \rangle} = I^{[s]}$. In general, we only have $I^{\langle s \rangle} \subseteq I^{[s]}$. When I is a squarefree monomial ideal, we first prove that the symbolic powers of I are equal to the $*$-differential powers of I.

Lemma 2.6. *Let I be a squarefree monomial ideal. Then $I^{(s)} = I^{[s]}$.*

Proof. This result is folkloric, though we could not find a reference. So we give a simple proof here. Let $P_1, \dots, P_r$ be the minimal prime ideals of I. By [18, Proposition 1.4.4], $I^{(s)} = P_1^s \cap \cdots \cap P_r^s$. Let f be a monomial in $I^{(s)}$ and $\mathbf{a} \in \mathbb{N}^n$ such that $|\mathbf{a}| \le s-1$. Since $f \in P_i^s$ for all $i = 1, \dots, r$, $\partial^* f/\partial^* x^{\mathbf{a}} \in P_i$. Thus, $\partial^* f/\partial^* x^{\mathbf{a}} \in P_1 \cap \cdots \cap P_r = I$. Therefore, $f \in I^{[s]}$.

Conversely, assume by contradiction that $I^{[s]}$ strictly contains $I^{(s)}$. Let $f = x^{\mathbf{b}}$ be a minimal generator of $I^{[s]}$ of smallest degree that is not in $I^{(s)}$. In particular, $f \notin P^s$ for some minimal prime P of I. Since I is a squarefree monomial ideal, P is generated by variables. Without loss of generality, we assume that $P = (x_1, \dots, x_t)$. Since $f \notin P^s$, $b_1 + \cdots + b_t < s$. Take $\mathbf{a} = (b_1, \dots, b_t, 0, \dots, 0)$, then $|\mathbf{a}| \le s-1$. Now, we have $\partial^*(f)/\partial^*(x^{\mathbf{a}}) \mid x_{t+1}^{b_{t+1}} \cdots x_n^{b_n}$, which is not contained in P. Thus, $\partial^*(f)/\partial^*(x^{\mathbf{a}}) \notin I$, a contradiction. $\square$

For a monomial f in S, we denote the support of f by $\operatorname{supp}(f) = \{i \in [n] \mid x_i \text{ divides } f\}$. For an exponent $\mathbf{a} \in \mathbb{Z}^n$, we denote the support of $\mathbf{a}$ by $\operatorname{supp}(\mathbf{a}) = \{i \in [n] \mid a_i \neq 0\}$. For a subset $V \subseteq [n]$, the restriction of I to V is defined by

$$I_V = (f \mid f \text{ is a monomial in } I \text{ with } \operatorname{supp}(f) \subseteq V).$$

We have

Corollary 2.7. *Let I be a squarefree monomial ideal and f be a monomial in S. Denote $V = \operatorname{supp}(f)$. Then, $f \in I^{(s)}$ if and only if $f \in I_V^{(s)}$.*

Proof. Since $I_V \subseteq I$, if $f \in I_V^{(s)}$ then $f \in I^{(s)}$. Conversely, assume that $f \in I^{(s)}$. By Lemma 2.6, for any exponent $\mathbf{a}$ in $\mathbb{N}^n$ such that $\operatorname{supp} \mathbf{a} \subseteq V$ and $|\mathbf{a}| \le s-1$, we have $\partial^*(f)/\partial^*(x^{\mathbf{a}}) \in I$. But $\operatorname{supp}(\partial^*(f)/\partial^*(x^{\mathbf{a}})) \subseteq \operatorname{supp}(f) = V$. By definition, $\partial^*(f)/\partial^*(x^{\mathbf{a}}) \in I_V$. By Lemma 2.6, $f \in I_V^{(s)}$. $\square$

As a consequence of Corollary 2.7, we deduce a generalization of [17, Corollary 4.5] for squarefree monomial ideals.

Corollary 2.8. *Let I be a squarefree monomial ideal in S. Let V be a subset of $[n]$ and I_V be the restriction of I to V. Then for all $s \geq 1$,*

$$\operatorname{reg} I_V^{(s)} \leq \operatorname{reg} I^{(s)}.$$

Proof. By Corollary 2.7, $I_V^{(s)}$ is the restriction of $I^{(s)}$ to V. The conclusion follows from Corollary 2.22. $\square$

2.5. Edge ideals of graphs and their symbolic powers

Let G denote a finite simple graph over the vertex set $V(G) = [n] = \{1, 2, \ldots, n\}$ and the edge set $E(G)$. For a vertex $x \in V(G)$, let the neighbours of x be the subset $N_G(x) = \{y \in V(G) \mid \{x, y\} \in E(G)\}$, and set $N_G[x] = N_G(x) \cup \{x\}$. For a subset U of the vertex set $V(G)$, $N_G(U)$ and $N_G[U]$ are defined by $N_G(U) = \cup_{u \in U} N_G(u)$ and $N_G[U] = \cup_{u \in U} N_G[u]$. If G is fixed, we shall use $N(U)$ or $N[U]$ for short.

A subgraph H is called an induced subgraph of G if for any vertices $u, v \in V(H) \subseteq V(G)$ we have $\{u, v\} \in E(H)$ if and only if $\{u, v\} \in E(G)$.

An induced matching is a subset of the edges that do not share any vertices and it is an induced subgraph. The induced matching number of G, denoted by $\mu(G)$, is the largest size of an induced matching in G.

An m-cycle in G is a sequence of m distinct vertices $1, \ldots, m \in V(G)$ such that $\{1, 2\}, \ldots, \{m-1, m\}, \{m, 1\}$ are edges of G. We also use $C = 12 \ldots m$ to denote the m-cycle whose sequence of vertices is $1, \ldots, m$.

A clique in G is a complete subgraph of G. We also call a clique of size 3 a triangle.

The edge ideal of G is defined to be

$$I(G) = (x_i x_j \mid \{i, j\} \in E(G)) \subseteq S.$$

Let $J_1(G)$ be the ideal generated by all squarefree monomials $x_i x_j x_r$ where $\{i, j, r\}$ forms a triangle in G. Let $J_2(G)$ be the ideal generated by all squarefree monomials $x_i x_j x_r x_s$ where $\{i, j, r, s\}$ forms a clique of size 4 in G and all squarefree monomials x_C where C is a 5-cycle of G.

We have the following expansion formulae of the second and third symbolic powers of an edge ideal. Note that the first formula is [36, Corollary 3.12].

Theorem 2.9. *Let I be the edge ideal of a simple graph G. Then*

$$I^{(2)} = I^2 + J_1(G).$$

Proof. Using Lemma 2.6, it is easy to see that the left hand side contains the right hand side. Conversely, let $f \in I^{(2)}$ be a monomial generator. By Corollary 2.7, we may assume that $\operatorname{supp} f = V(G)$. If G contains a triangle, then $f \in J_1(G)$. If G contains a cycle of length ≥ 4, then $f \in I^2$. Thus we may assume that G does not contain any cycles. The conclusion follows from [35, Theorem 5.9]. $\square$

Theorem 2.10. *Let I be the edge ideal of a simple graph G. Then*

$$I^{(3)} = I^3 + IJ_1(G) + J_2(G).$$

Proof. Using Lemma 2.6, it is easy to see that the left hand side contains the right hand side. Conversely, let $f \in I^{(3)}$ be a monomial generator. By Corollary 2.7, we may assume that $\operatorname{supp}(f) = V(G)$. If G contains a 5-cycle, then $f \in J_2(G)$. If the matching number of G is at least 3, then $f \in I^3$. Thus, we may assume that G does not contain a cycle of length ≥ 5. If G does not contain a triangle, then G is bipartite, and the conclusion follows from [35, Theorem 5.9]. Thus, we may assume that G contains a triangle, say 123. Let $f = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ where $\alpha_i \geq 1$. If two of the exponents $\alpha_1, \alpha_2, \alpha_3$ is at least 2, then $f \in IJ_1(G)$. Thus, we may assume that $\alpha_2 = \alpha_3 = 1$. By Lemma 2.6, $\partial^* f/\partial^*(x_2x_3) = x_1^{\alpha_1}x_4^{\alpha_4} \cdots x_n^{\alpha_n} \in I$. If $x_4^{\alpha_4} \cdots x_n^{\alpha_n} \in I$ then $f \in IJ_1(G)$. Thus, we may assume that $x_4^{\alpha_4} \cdots x_n^{\alpha_n} \notin I$, and $x_1x_4 \in I$. If $\alpha_1 > 1$, then $f \in IJ_1(G)$ as $(x_1x_4) \cdot (x_1x_2x_3) | f$. Thus, we may assume that $\alpha_1 = 1$. Similarly, we deduce that x_2x_i and $x_3x_j \in I$ for some $i, j \geq 4$. If $|\{4, i, j\}| = 3$, then $f \in I^3$. If $|\{4, i, j\}| = 1$, then $f \in J_2(G)$, as $\{1, 2, 3, 4\}$ forms a clique of size 4. Now, assume that $i = 4$, and $j \neq 4$. In this case $x_1x_2x_4 \in J_1(G)$; hence, $(x_1x_2x_4)(x_3x_j) \in IJ_1(G)$. This concludes our proof. $\square$

2.6. Degree complexes

For a monomial ideal I in S, Takayama in [37] found a combinatorial formula for $\dim_K H^i_{\mathfrak{m}}(S/I)_{\mathbf{a}}$ for all $\mathbf{a} \in \mathbb{Z}^n$ in terms of certain simplicial complexes which are called degree complexes. For every $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ we set $G_{\mathbf{a}} = \{i \mid \; a_i < 0\}$ and write $x^{\mathbf{a}} = \Pi_{j=1}^n x_j^{a_j}$. Thus, $G_{\mathbf{a}} = \emptyset$ whenever $\mathbf{a} \in \mathbb{N}^n$. The degree complex $\Delta_{\mathbf{a}}(I)$ is the simplicial complex whose faces are $F \setminus G_{\mathbf{a}}$, where $G_{\mathbf{a}} \subseteq F \subseteq [n]$, so that for every minimal generator $x^{\mathbf{b}}$ of I there exists an index $i \notin F$ with $a_i < b_i$. It is noted that $\Delta_{\mathbf{a}}(I)$ may be either the empty set or $\{\emptyset\}$ and its vertex set may be a proper subset of $[n]$.

Example 2.11. Let $I = (x_1x_2, x_1x_3, x_2x_3, x_3x_4, x_4x_5, x_5x_2)$ be an edge ideal in $S = K[x_1, \ldots, x_5]$. Let $\mathbf{a} = (2, 2, 0, -1, 0)$. Then, $\Delta_{\mathbf{a}}(I^3)$ is the simplicial complex with facets $\{1\}, \{2\}$.

The regularity of a monomial ideal can be computed in terms of its degree complexes as follows.

Lemma 2.12. *Let I be a monomial ideal in S. Then*

$$\operatorname{reg}(S/I) = \max\{|\mathbf{a}| + i \mid \mathbf{a} \in \mathbb{N}^n, i \geq 0, \widetilde{H}_{i-1}(\operatorname{lk}_{\Delta_{\mathbf{a}}(I)} F; K) \neq 0 \text{ for some } F \in \Delta_{\mathbf{a}}(I) \text{ with } F \cap \operatorname{supp} \mathbf{a} = \emptyset\}.$$

In particular, if $I = I_\Delta$ is the Stanley-Reisner ideal of a simplicial complex Δ then $\operatorname{reg}(K[\Delta]) = \max\{i \mid i \geq 0, \widetilde{H}_{i-1}(\operatorname{lk}_\Delta F; K) \neq 0$ *for some* $F \in \Delta\}$.

Proof. By [7, Proposition 2.5],

$$\operatorname{reg}(S/I) = \max\left\{|\mathbf{a}| + |G_{\mathbf{a}}| + i \mid \mathbf{a} \in \mathbb{Z}^n, i \geq 0, \widetilde{H}_{i-1}(\Delta_{\mathbf{a}}(I); K) \neq 0\right\}.$$

Assume that $\operatorname{reg}(S/I) = |\mathbf{a}| + |G_{\mathbf{a}}| + i$ for some $\mathbf{a} \in \mathbb{Z}^n$. By maximality, we may assume that $a_j = -1$ for all $j \in G_{\mathbf{a}}$. Let $\mathbf{a}^+ \in \mathbb{N}^n$ by $a_j^+ = a_j$ if $j \notin G_{\mathbf{a}}$ and $a_j^+ = 0$ otherwise. Then $\Delta_{\mathbf{a}}(I) = \operatorname{lk}_{\Delta_{\mathbf{a}^+}(I)} G_{\mathbf{a}}$; $\operatorname{reg}(S/I) = |\mathbf{a}^+| + i$ and $\widetilde{H}_{i-1}(\operatorname{lk}_{\Delta_{\mathbf{a}^+}(I)} G_{\mathbf{a}}; K) \neq 0$. It implies that

$$\operatorname{reg}(S/I) \leq \max\{|\mathbf{a}| + i \mid \mathbf{a} \in \mathbb{N}^n, i \geq 0, \widetilde{H}_{i-1}(\operatorname{lk}_{\Delta_{\mathbf{a}}(I)} F; K) \neq 0 \text{ for some } F \in \Delta_{\mathbf{a}}(I) \text{ with } F \cap \operatorname{supp} \mathbf{a} = \emptyset\}.$$

Conversely, assume that $\operatorname{reg} S/I = |\mathbf{a}| + i$ for some $\mathbf{a} \in \mathbb{N}^n$, $i \geq 0$ such that $\widetilde{H}_{i-1}(\operatorname{lk}_{\Delta_{\mathbf{a}}(I)} F; K) \neq 0$ for some $F \in \Delta_{\mathbf{a}}(I)$ with $F \cap \operatorname{supp} \mathbf{a} = \emptyset$. Let $\mathbf{b} \in \mathbb{Z}^n$ by $b_j = -1$ if $j \in F$ and $b_j = a_j$ otherwise. Then $\widetilde{H}_{i-1}(\Delta_{\mathbf{b}}(I); K) \neq 0$ and $|\mathbf{b}| + |G_{\mathbf{b}}| + i = |\mathbf{a}| + i$. Using [7, Proposition 2.5] again, we obtain the reverse inequality.

Now assume that $I = I_\Delta$ is the Stanley-Reisner ideal of a simplicial complex Δ. By the proof of [37, Theorem 1], for each $\mathbf{a} \in \mathbb{N}^n$, if $a_j \geq 1$, then $\Delta_{\mathbf{a}}(I)$ is either a cone over j or the void complex. Hence, $\mathbf{a} = \mathbf{0}$. Since $\Delta_{\mathbf{0}}(I) = \Delta$, this completes our proof. □

Remark 2.13. Let I be a monomial ideal in S and $\mathbf{a}$ be a vector in $\mathbb{Z}^n$. In the proof of Theorem 1 in [37], Takayama showed that if there exists $j \in [n] \setminus G_{\mathbf{a}}$ such that $a_j \geq \rho_j = \max\{\deg_{x_j}(u) \mid u$ is a minimal monomial generator of $I\}$ then $\Delta_{\mathbf{a}}(I)$ is either a cone over j or the void complex. Thus, we only consider exponents $\mathbf{a}$ belonging to the finite set

$$\Gamma(I) = \{\mathbf{a} \in \mathbb{N}^n \mid a_j < \rho_j \text{ for all } j = 1, \ldots, n\}.$$

By Lemma 2.12 and Remark 2.13, we obtain an upper bound for the regularity of a monomial ideal in terms of its degree complexes.

Corollary 2.14. *Let I be a monomial ideal in S. Then*

$$\operatorname{reg}(S/I) \leq \max\{|\mathbf{a}| + \operatorname{reg}(K[\Delta_{\mathbf{a}}(I)]) \mid \mathbf{a} \in \Gamma(I)\}.$$

One might expect that this inequality becomes equality. Unfortunately, this is not the case.

Example 2.15. Let $I = x_1(x_2, x_3, x_4, x_5)$ be an edge ideal in $S = K[x_1, \dots, x_5]$. For each $s \geq 2$, let $x^{\mathbf{a}} = (x_2x_3x_4x_5)^{s-1}$. Then, $\mathbf{a} \in \Gamma(I^s)$. Furthermore, we have $\operatorname{reg}(S/I^s) = 2s - 1$, $|\mathbf{a}| = 4s - 4$, and $\operatorname{reg}(K[\Delta_{\mathbf{a}}(I^s)]) = 0$. Thus, $\operatorname{reg}(S/I^s) < \max\{|\mathbf{a}| + \operatorname{reg}(K[\Delta_{\mathbf{a}}(I^s)]) \mid \mathbf{a} \in \Gamma(I^s)\}$.

Definition 2.16. Let I be a monomial ideal in S. A pair $(\mathbf{a}, i) \in \mathbb{N}^n \times \mathbb{N}$ is called *an extremal exponent of* I, if there exists a face $F \in \Delta_{\mathbf{a}}(I)$ with $F \cap \operatorname{supp} \mathbf{a} = \emptyset$ such that $\widetilde{H}_{i-1}(\operatorname{lk}_{\Delta_{\mathbf{a}}(I)} F; K) \neq 0$ and $\operatorname{reg}(S/I) = |\mathbf{a}| + i$.

Remark 2.17. We sometimes call $\mathbf{a}$ instead of $(\mathbf{a}, i)$ an extremal exponent of I. Let $\mathbf{a}$ be an extremal exponent of I. By Remark 2.13, $\mathbf{a}$ belongs to $\Gamma(I)$ and $\Delta_{\mathbf{a}}(I)$ is not a cone over t with $t \in \operatorname{supp} \mathbf{a}$. By Lemma 2.2, for each $t \in \operatorname{supp} \mathbf{a}$, there exists a minimal generator g of $\sqrt{I : x^{\mathbf{a}}}$ such that $x_t \mid g$.

From the definition, it is easy to see the following

Lemma 2.18. *Let I, J be proper monomial ideals of S. Let $(\mathbf{a}, i)$ be an extremal exponent of I. If $\Delta_{\mathbf{a}}(I) = \Delta_{\mathbf{a}}(J)$, then $\operatorname{reg} I \leq \operatorname{reg} J$. In particular, if $J \subseteq I$ and $\Delta_{\mathbf{a}}(I) = \Delta_{\mathbf{a}}(J)$ for all exponent $\mathbf{a} \in \mathbb{N}^n$ such that $x^{\mathbf{a}} \notin I$ then $\operatorname{reg} I \leq \operatorname{reg} J$.*

Proof. By definition, there exists a face $F \in \Delta_{\mathbf{a}}(I)$ such that $F \cap \operatorname{supp} \mathbf{a} = \emptyset$, $\widetilde{H}_{i-1}(\operatorname{lk}_{\Delta_{\mathbf{a}}(I)} F; K) \neq 0$, and $\operatorname{reg} S/I = |\mathbf{a}| + i$. By Lemma 2.12 and the assumption that $\Delta_{\mathbf{a}}(I) = \Delta_{\mathbf{a}}(J)$, we have $\operatorname{reg} S/J \geq |\mathbf{a}| + i$ as required. $\square$

The degree complexes of a monomial ideal can be computed via the Stanley-Reisner correspondence as follows.

Lemma 2.19. *Let I be a monomial ideal in S and $\mathbf{a} \in \mathbb{N}^n$. Then*

$$I_{\Delta_{\mathbf{a}}(I)} = \sqrt{I : x^{\mathbf{a}}}.$$

In particular, $x^{\mathbf{a}} \in I$ if and only if $\Delta_{\mathbf{a}}(I)$ is the void complex.

Proof. This lemma appeared in [29], we include an argument here for completeness. Let $G(I)$ be the set of minimal monomial generators of I. For any $F \subseteq [n]$, we have

$$\begin{aligned} x_F \in I_{\Delta_{\mathbf{a}}(I)} &\iff F \notin \Delta_{\mathbf{a}}(I) \iff \exists\ x^{\mathbf{b}} \in G(I) \text{ such that } \forall\ i \notin F, b_i \leq a_i \\ &\iff \exists\ t \in \mathbb{N} \setminus \{0\}, (x_F)^t x^{\mathbf{a}} \in I \iff x_F \in \sqrt{I : x^{\mathbf{a}}}. \quad \square \end{aligned}$$

We first deduce the following inequality on the regularity of restriction of a monomial ideal.

Lemma 2.20. *Let I be a monomial ideal and x_j be a variable. Then*

$$\operatorname{reg}(I, x_j) \le \operatorname{reg} I.$$

Proof. This is [7, Corollary 4.8]. We give an alternative proof here. Let $J = (I, x_j)$ and $(\mathbf{a}, i)$ be an extremal exponent of J. Then $j \notin \operatorname{supp}(\mathbf{a})$. By Lemma 2.24,

$$\sqrt{J : x^{\mathbf{a}}} = \sqrt{I : x^{\mathbf{a}}} + (x_j).$$

In other words, $\Delta_{\mathbf{a}}(J)$ is the restriction of $\Delta_{\mathbf{a}}(I)$ to $[n] \setminus \{j\}$. Let F be a face of $\Delta_{\mathbf{a}}(I)$ such that $\widetilde{H}_{i-1}(\operatorname{lk}_{\Delta_{\mathbf{a}}(J)} F; K) \neq 0$. Denote $\Delta = \operatorname{lk}_{\Delta_{\mathbf{a}}(I)} F$, $\Gamma = \operatorname{lk}_{\Delta_{\mathbf{a}}(J)} F$, and $\operatorname{st}_\Delta\{j\} = \operatorname{lk}_\Delta\{j\} * \{j\}$ the star of $\{j\}$ in Δ. We have

$$\Delta = \Gamma \cup \operatorname{st}_\Delta\{j\} \text{ and } \operatorname{lk}_\Delta\{j\} = \Gamma \cap \operatorname{st}_\Delta\{j\}.$$

Applying the Mayer-Vietoris sequence, we get the following long exact sequence of homology groups

$$\cdots \longrightarrow \widetilde{H}_{i-1}(\operatorname{lk}_\Delta\{j\}) \longrightarrow \widetilde{H}_{i-1}(\Gamma) \oplus \widetilde{H}_i(\operatorname{st}_\Delta\{j\}) \longrightarrow \widetilde{H}_{i-1}(\Delta) \longrightarrow \cdots$$

Since the middle term is non-zero, either the term on the left or the term on the right is non-zero. The conclusion follows from Lemma 2.12. □

Remark 2.21. As in the proof of [4, Lemma 4.2], we may use the upper Koszul complexes to deduce a stronger inequality on Betti numbers.

As a consequence, we have

Corollary 2.22. *Let J be a monomial ideal in S. Let $V \subseteq [n]$. We have*

$$\operatorname{reg}(J_V) \le \operatorname{reg}(J).$$

Proof. Let $\{t, \dots, n\} = [n] \setminus V$. Then, $J_V + (x_t, \dots, x_n) = J + (x_t, \dots, x_n)$. The conclusion follows from Lemma 2.20 and the fact that $x_t, \dots, x_n$ is a regular sequence with respect to S/J_V. □

The following lemma is essential to using the induction method in studying the regularity of a monomial ideal.

Lemma 2.23. *Let I be a monomial ideal and $(\mathbf{a}, i) \in \mathbb{N}^n \times \mathbb{N}$ be an extremal exponent of I. Assume that x_j is a variable that appears in $\sqrt{I : x^{\mathbf{a}}}$ and $j \notin \operatorname{supp} \mathbf{a}$. Then*

$$\operatorname{reg}(I) = \operatorname{reg}(I, x_j) = \operatorname{reg} I_V,$$

where $V = [n] \setminus \{j\}$.

Proof. Let $J = (I, x_j) = I + (x_j)$. Since $I_V + (x_j) = I + (x_j)$ and x_j is a regular element of S/I_V, we have $\operatorname{reg} S/I_V = \operatorname{reg} S/J$. By Lemma 2.20, it suffices to prove that $\operatorname{reg} I \le \operatorname{reg} J$. Since $j \notin \operatorname{supp} \mathbf{a}$, by Lemma 2.24, we have

$$\sqrt{J : x^{\mathbf{a}}} = \sqrt{I : x^{\mathbf{a}}} + (x_j) = \sqrt{I : x^{\mathbf{a}}}.$$

By Lemma 2.19, $\Delta_{\mathbf{a}}(I) = \Delta_{\mathbf{a}}(J)$. The conclusion follows from Lemma 2.18. $\square$

2.7. Radicals of colon ideals

For a monomial f in S, the radical of f is defined by $\sqrt{f} = \prod_{i \in \operatorname{supp} f} x_i$. We start with a simple observation.

Lemma 2.24. *Let I be a monomial ideal in S generated by the monomials $f_1, \ldots, f_r$ and $\mathbf{a} \in \mathbb{N}^n$. Then $\sqrt{I : x^{\mathbf{a}}}$ is generated by $\sqrt{f_1 / \gcd(f_1, x^{\mathbf{a}})}, \ldots, \sqrt{f_r / \gcd(f_r, x^{\mathbf{a}})}$.*

Proof. Let g be a minimal generator of $\sqrt{I : x^{\mathbf{a}}}$. Then there exists a natural number $t > 0$ such that $g^t x^{\mathbf{a}} \in I$. We may assume that $f_1 \mid g^t x^{\mathbf{a}}$. Thus, $f_1 / \gcd(f_1, x^{\mathbf{a}}) \mid g^t$. Taking radicals, we deduce that $\sqrt{f_1 / \gcd(f_1, x^{\mathbf{a}})} \mid g$. This concludes our proof. $\square$

Assume that $I = I(G)$ is the edge ideal of a simple graph G. We now recall the following description of generators of $\sqrt{I^s : x^{\mathbf{a}}}$ given in [30, Lemma 2.18]. This helps to simplify our arguments later on. The I-order of f is defined by

$$\operatorname{ord}_I(f) = \max(t \mid f \in I^t).$$

From the definition, it is clear that if $g \mid f$, then $\operatorname{ord}_I(g) \le \operatorname{ord}_I(f)$. Let f be a monomial in S and $\mathbf{a}$ be an exponent in $\mathbb{N}^n$. The $\mathbf{a}$-term excluding f is defined by

$$X_{\mathbf{a}}(f) = \prod_{u \notin N[\operatorname{supp} f]} x_u^{a_u}.$$

When $f = x_F$, we also use $X_{\mathbf{a}}(F)$ for $X_{\mathbf{a}}(x_F)$.

Lemma 2.25. *Let F be an independent set of G and $\mathbf{a}$ be an exponent in $\mathbb{N}^n$. Assume that*

$$\sum_{j \in N(F)} a_j + \operatorname{ord}_I(X_{\mathbf{a}}(F)) \ge s, \tag{2.1}$$

then $x_F \in \sqrt{I^s : x^{\mathbf{a}}}$. Conversely, if x_F is a minimal generator of $\sqrt{I^s : x^{\mathbf{a}}}$ then (2.1) holds.

The following lemma is also useful for comparing radicals of two colon ideals.

Lemma 2.26. *Let $J \subseteq L$ be two monomial ideals in S and $\mathbf{a}$ be an exponent such that $x^{\mathbf{a}} \notin L$. Let P be a minimal generator of L and $f = \sqrt{P/\gcd(P, x^{\mathbf{a}})}$. Assume that f belongs to $\sqrt{J : P}$. Then f belongs to $\sqrt{J : x^{\mathbf{a}}}$.*

Proof. Since $f = \sqrt{P/\gcd(P, x^{\mathbf{a}})}$, there exists $u > 0$ such that $P \mid f^u x^{\mathbf{a}}$. Since $f \in \sqrt{J : P}$, there exists $v > 0$ such that $f^v P \in J$. Thus, $f^{u+v} x^{\mathbf{a}}$ which is divisible by $f^u P$ belongs to J. Hence, $f \in \sqrt{J : x^{\mathbf{a}}}$. $\square$

3. Proof of Theorem 1.1 for $s = 2$

Let $I = I(G)$ be the edge ideal of a simple graph G with vertex set $[n]$. In this section, we prove Theorem 1.1 for $s = 2$. First, we give a property of the degree complexes of the second symbolic/ordinary power of I.

Lemma 3.1. *Let $I = I(G)$ and $\mathbf{a} \in \mathbb{N}^n$ such that $x^{\mathbf{a}} \notin I^{(2)}$. Then,*

$$\sqrt{I^{(2)} : x^{\mathbf{a}}} = \sqrt{I^2 : x^{\mathbf{a}}}.$$

In particular, $\Delta_{\mathbf{a}}(I^{(2)}) = \Delta_{\mathbf{a}}(I^2)$.

Proof. Since $I^2 \subseteq I^{(2)}$, it suffices to prove that if f is a minimal generator of $\sqrt{I^{(2)} : x^{\mathbf{a}}}$ then $f \in \sqrt{I^2 : x^{\mathbf{a}}}$. By Theorem 2.9 and Lemma 2.24, we may assume that $f = \sqrt{x_C/\gcd(x_C, x^{\mathbf{a}})}$ where C is a triangle of G. Since $f = \sqrt{x_C/\gcd(x_C, x^{\mathbf{a}})}$ and $x^{\mathbf{a}} \notin I^{(2)}$, we have $\emptyset \neq \operatorname{supp} f \subseteq \operatorname{supp} C$. Since $x_j x_C \in I^2$ for all $j \in \operatorname{supp} C$, $f \cdot x_C \in I^2$. By Lemma 2.26, $f \in \sqrt{I^2 : x^{\mathbf{a}}}$, as required.

The last part follows from Lemma 2.19. $\square$

To avoid the repetition of arguments in the latter sections, we record the following situation, which occurs quite frequently.

Lemma 3.2. *Assume that $s \geq 2$ and $\operatorname{reg} I(H)^s \leq \operatorname{reg} I(H)^{(s)}$ for all simple graphs H on at most $n-1$ vertices. Let $I = I(G)$ be the edge ideal of a simple graph G on n vertices. Let $\mathbf{a}$ be an extremal exponent of I^s. Assume that there exists an $r \in [n]$ such that $x_r \in \sqrt{I^s : x^{\mathbf{a}}}$ and $r \notin \operatorname{supp} \mathbf{a}$. Then $\operatorname{reg} I(G)^s \leq \operatorname{reg} I(G)^{(s)}$.*

Proof. Let $V = [n] \setminus \{r\}$ and $J = I_V$. By Lemma 2.23, $\operatorname{reg} I^s = \operatorname{reg} J^s$. By assumption, $\operatorname{reg} J^s \leq \operatorname{reg} J^{(s)}$. By Corollary 2.7 and Corollary 2.22, $\operatorname{reg} J^{(s)} \leq \operatorname{reg} I^{(s)}$. The conclusion follows. $\square$

We are now in a position to prove the main result of this section.

Theorem 3.3. *Let $I = I(G)$ be the edge ideal of a simple graph G. Then*

$$\operatorname{reg}(I^{(2)}) = \operatorname{reg}(I^2).$$

Proof. By Lemma 2.18 and Lemma 3.1, $\operatorname{reg} I^{(2)} \le \operatorname{reg} I^2$.

Conversely, we prove by induction on $n = |V(G)|$ that $\operatorname{reg}(S/I^2) \le \operatorname{reg}(S/I^{(2)})$. The base case $n = 2$ is clear. By results of [21,33], we may assume that G is connected and has no isolated vertices.

Let $(\mathbf{a}, i) \in \mathbb{N}^n \times \mathbb{N}$ be an extremal exponent of I^2. By Lemma 3.1 and Lemma 2.18, we may assume that there exists a triangle of G, say $C = 123$ such that $x^{\mathbf{a}}$ is divisible by x_C.

If $n = 3$, then $I = (x_1x_2, x_2x_3, x_1x_3)$. In this case, we have $\operatorname{reg} I^2 = \operatorname{reg} I^{(2)} = 4$. Thus, we may assume that $n \ge 4$. Since G is connected and has no isolated vertices, there exists $r \in N(\{1,2,3\}) \setminus \{1,2,3\}$. Since $x^{\mathbf{a}} \notin I^2$, we must have $r \notin \operatorname{supp} \mathbf{a}$. Since $r \in N(\{1,2,3\})$, $x_r \in \sqrt{I^2 : x^{\mathbf{a}}}$. By Lemma 3.2, $\operatorname{reg} I^2 \le \operatorname{reg} I^{(2)}$. The conclusion follows. $\square$

4. Proof of Theorem 1.1 for $s = 3$

Let $I = I(G)$ be the edge ideal of a simple graph G with vertex set $[n]$. In this section, we prove Theorem 1.1 for $s = 3$. First, we prove a technical lemma for degree complexes of the third symbolic/ordinary power of I.

Lemma 4.1. *Let $\mathbf{a} \in \mathbb{N}^n$ be an exponent such that $x^{\mathbf{a}} \notin I^{(3)}$. Assume that $\sqrt{I^{(3)} : x^{\mathbf{a}}} \neq \sqrt{I^3 : x^{\mathbf{a}}}$. Let f be a minimal monomial generator of $\sqrt{I^{(3)} : x^{\mathbf{a}}}$ such that $f \notin \sqrt{I^3 : x^{\mathbf{a}}}$. Then*

(1) *there exist a triangle* 123 *of G and $v \notin \{1,2,3\}$ such that $x_1x_2x_3x_v \mid x^{\mathbf{a}}$,*
(2) $\deg f = 1$ *and* $\operatorname{supp} f \nsubseteq \operatorname{supp} \mathbf{a}$.

Proof. By Lemma 2.24, there exists a minimal generator P of $I^{(3)}$ such that $f = \sqrt{P/\gcd(P, x^{\mathbf{a}})}$. Since $x^{\mathbf{a}} \notin I^{(3)}$, we have

$$\emptyset \neq \operatorname{supp} f \subseteq \operatorname{supp} P \subseteq \operatorname{supp} f \cup \operatorname{supp} \mathbf{a}. \tag{4.1}$$

By Theorem 2.10, there are three cases as follows.

Case 1. $P = x_C$, where C is a clique of size 4 of G. By Eq. (4.1) and the fact that $x_j^2 x_C \in I^3$ for any $j \in \operatorname{supp} C$, we have $f \in \sqrt{I^3 : x_C}$. By Lemma 2.26, $f \in \sqrt{I^3 : x^{\mathbf{a}}}$, a contradiction.

Case 2. $P = x_C$, where C is a 5-cycle of G. By Eq. (4.1) and the fact that $x_j x_C \in I^3$ for any $j \in \operatorname{supp} C$, we have $f \in \sqrt{I^3 : x_C}$. By Lemma 2.26, $f \in \sqrt{I^3 : x^{\mathbf{a}}}$, a contradiction.

Case 3. $P \in IJ_1(G)$. We may assume that $P = x_1x_2x_3x_ux_v$ where 123 is a triangle and uv is an edge of G. Note that u, v might belong to $\{1,2,3\}$. By Lemma 2.26 and the fact that $x_j P \in I^3$ for any $j \in N[\{1,2,3\}]$, we must have

$$\operatorname{supp} f \cap N[\{1,2,3\}] = \emptyset. \tag{4.2}$$

By Eq. (4.1) and Eq. (4.2), we have $\operatorname{supp} f \subseteq \{u, v\}$. Since $f \notin \sqrt{I^3 : x^{\mathbf{a}}}$, $f \notin I$. Thus, $\deg f = 1$. We may assume that $f = x_u$. By Eq. (4.1), Eq. (4.2), and the assumption that $f = \sqrt{P/\gcd(P, x^{\mathbf{a}})}$, we deduce that

$$v \notin \{1,2,3\} \text{ and } x_v x_1 x_2 x_3 \mid x^{\mathbf{a}}.$$

Since $x^{\mathbf{a}} \notin I^{(3)}$, $u \notin \operatorname{supp} \mathbf{a}$. The conclusion follows. $\square$

Example 4.2. One might hope that in general we have for $\mathbf{a} \in \mathbb{N}^n$ and $x^{\mathbf{a}} \notin I^{(s)}$ then

$$\sqrt{I^{(s)} : x^{\mathbf{a}}} = \sqrt{I^s : x^{\mathbf{a}}} + (\text{variables}).$$

Unfortunately, this is not the case for $s \geq 4$. Indeed, let

$$I = (x_1x_2, x_2x_3, x_3x_1, x_1x_4, x_4x_5, x_2x_6, x_6x_7) \text{ and } x^{\mathbf{a}} = x_1x_2x_3x_4x_6,$$

then x_5x_7 is a minimal generator of $\sqrt{I^{(4)} : x^{\mathbf{a}}}$ but does not belong to $\sqrt{I^4 : x^{\mathbf{a}}}$.

We are now in a position to prove the first inequality of the main result of this section.

Theorem 4.3. *Let $I = I(G)$ be the edge ideal of a simple graph G. Then*

$$\operatorname{reg}(I^{(3)}) \leq \operatorname{reg}(I^3).$$

Proof. We prove by induction on $n = |V(G)|$. The base case $n = 2$ is obvious. Let $(\mathbf{a}, i)$ be an extremal exponent of $I^{(3)}$. By Lemma 2.18, we may assume that $\Delta_{\mathbf{a}}(I^{(3)}) \neq \Delta_{\mathbf{a}}(I^3)$. By Lemma 2.19 and Lemma 4.1, there exists a variable x_t such that $x_t \in \sqrt{I^{(3)} : x^{\mathbf{a}}}$ and $t \notin \operatorname{supp} \mathbf{a}$. Let $V = [n] \setminus \{t\}$ and $J = I_V$. By Lemma 2.23, $\operatorname{reg} I^{(3)} = \operatorname{reg} J^{(3)}$. By induction, $\operatorname{reg} J^{(3)} \leq \operatorname{reg} J^3$. By Corollary 2.22 and the fact that J^3 is the restriction of I^3 to V, $\operatorname{reg} J^3 \leq \operatorname{reg} I^3$. The conclusion follows. $\square$

To prove the reverse inequality $\operatorname{reg}(S/I^3) \leq \operatorname{reg}(S/I^{(3)})$, we also use induction on $n = |V(G)|$. The base case $n = 2$ is obvious. By results of [21,33], we may assume that G is connected and has no isolated vertices. Throughout the rest of this section, we always assume that $(\mathbf{a}, i) \in \mathbb{N}^n \times \mathbb{N}$ is an extremal exponent of I^3, where $I = I(G)$. It is clear that $x^{\mathbf{a}} \notin I^3$. Then, we fix a face $F \in \Delta_{\mathbf{a}}(I^3)$ such that $F \cap \operatorname{supp}(\mathbf{a}) = \emptyset$ and $\widetilde{H}_{i-1}(\operatorname{lk}_{\Delta_{\mathbf{a}}(I^3)} F; K) \neq 0$. By Lemma 2.18, it suffices to consider the cases $x^{\mathbf{a}} \notin I^{(3)}$ with $\Delta_{\mathbf{a}}(I^{(3)}) \neq \Delta_{\mathbf{a}}(I^3)$ or $x^{\mathbf{a}} \in I^{(3)}$. We now proceed to each form of $\mathbf{a}$.

Lemma 4.4. *Assume that $x^{\mathbf{a}} = x_C \cdot f$, where C is a 5-cycle of G and f is a monomial in S. Then $\operatorname{reg} I^3 \leq \operatorname{reg} I^{(3)}$.*

Proof. Since $x^{\mathbf{a}} \notin I^3$, $\operatorname{supp}(f) \cap N(C) = \emptyset$ and $\operatorname{supp}(f)$ is an independent set of G. First, assume that $[n]$ properly contains $\operatorname{supp} \mathbf{a}$. Since G is connected and has no isolated vertices, there exists an r such that $r \in N(\operatorname{supp} \mathbf{a}) \setminus \operatorname{supp} \mathbf{a}$. It is clear that $x_r \in \sqrt{I^3 : x^{\mathbf{a}}}$. By Lemma 3.2, we have $\operatorname{reg} I^3 \le \operatorname{reg} I^{(3)}$. Thus, we may assume that $[n] = \operatorname{supp} \mathbf{a} = \operatorname{supp} f \cup \operatorname{supp} C$. Since $\operatorname{supp} f \cap N(C) = \emptyset$ and G is connected and has no isolated vertices, we must have $f = 1$, $n = 5$, and $x^{\mathbf{a}} = x_C$. In this case, $\Delta_{\mathbf{a}}(I^3) = \{\emptyset\}$. Thus, $i = 0$ and $\operatorname{reg} S/I^3 = |\mathbf{a}| + i = 5 \le \operatorname{reg} S/I^{(3)}$. $\square$

Lemma 4.5. *Assume that* $x^{\mathbf{a}} \in IJ_1(G)$. *Then* $\operatorname{reg}(I^3) \le \operatorname{reg}(I^{(3)})$.

Proof. Without loss of generality, we may assume that $x^{\mathbf{a}} = x_1x_2x_3x_ux_v \cdot f$, where $C = 123$ is a 3-cycle, uv is an edge of G, and f is a monomial in S. Since $x^{\mathbf{a}} \notin I^3$,

$$\operatorname{supp} f \text{ is an independent set and } \operatorname{supp}(f) \cap N(\{1,2,3\}) = \emptyset. \tag{4.3}$$

First, assume that there exists an r in $N(\{1,2,3\} \cup \operatorname{supp} f) \setminus \operatorname{supp} \mathbf{a}$. It is clear that $x_r \in \sqrt{I^3 : x^{\mathbf{a}}}$. By Lemma 3.2, $\operatorname{reg} I^3 \le \operatorname{reg} I^{(3)}$. Thus, we may assume that

$$N(\{1,2,3\} \cup \operatorname{supp} f) \subseteq \operatorname{supp} \mathbf{a} = \{1,2,3\} \cup \{u,v\} \cup \operatorname{supp} f. \tag{4.4}$$

By Eq. (4.3), Eq. (4.4), and the assumption that G is connected and has no isolated vertices, we must have $N(\{1,2,3\}) \cap \{u,v\} \ne \emptyset$. We may assume that $v \in N(\{1,2,3\})$. In particular, $x_1x_2x_3x_v \in I^2$. Thus,

$$x_j \in \sqrt{I^3 : x^{\mathbf{a}}} \text{ for all } j \in N(\{1,2,3\} \cup \{u\} \cup \operatorname{supp} f). \tag{4.5}$$

Since $x^{\mathbf{a}} \notin I^3$,

$$f x_u \notin I \implies N(\operatorname{supp} f) \cap \{u\} = \emptyset. \tag{4.6}$$

We now claim that $u \in N(\{1,2,3\})$. By Remark 2.17, there exists a minimal generator g of $\sqrt{I^3 : x^{\mathbf{a}}}$ such that $x_u \mid g$. By Lemma 2.24, there exists a minimal generator P of I^3 such that $g = \sqrt{P/\gcd(P, x^{\mathbf{a}})}$. Since $x_u \mid g$ and $u \in \operatorname{supp} \mathbf{a}$, $x_u^2 \mid P$. Thus, there exists $r, t \in N(u)$ such that $x_u^2 x_r x_t \mid P$. By Eq. (4.5) and the assumption that $g = \sqrt{P/\gcd(P, x^{\mathbf{a}})}$ is a minimal generator of $\sqrt{I^3 : x^{\mathbf{a}}}$, we must have $x_r x_t \mid x^{\mathbf{a}}$. Since $x^{\mathbf{a}} \notin I^3$, we deduce that $\{r,t\} \cap \operatorname{supp} f = \emptyset$. Hence, $x_r x_t \mid x_1x_2x_3x_v$. Thus, $\{r,t\} \cap \{1,2,3\} \ne \emptyset$. Since $r, t \in N(u)$, we deduce that $u \in N(\{1,2,3\})$.

Since $x^{\mathbf{a}} \notin I^3$, we have

$$N(\operatorname{supp} f) \cap (\{1,2,3\} \cup \{u,v\}) = \emptyset. \tag{4.7}$$

By Eq. (4.4), Eq. (4.7), and the assumption that G is connected and has no isolated vertices, we must have $f = 1$, $x^{\mathbf{a}} = x_1x_2x_3x_ux_v$, and $[n] = \{1,2,3\} \cup \{u,v\}$. Thus, $\Delta_{\mathbf{a}}(I^3) = \{\emptyset\}$. Hence, $i = 0$ and $\operatorname{reg} S/I^3 = |\mathbf{a}| + i = 5 \le \operatorname{reg} S/I^{(3)}$. $\square$

In the remainder of this section, we use the following notation. For a subset $V \subseteq [n]$, (V) denotes the ideal $(x_j \mid j \in V)$. When $V = \emptyset$, by convention, (V) denotes the zero ideal. The notation $(V) \cdot (W)$ means the product of two ideals (V) and (W). For a monomial ideal M, we denote by $G(M)$ the set of minimal monomial generators of M and

$$\operatorname{supp} G(M) = \cup_{f \in G(M)} \operatorname{supp} f.$$

We have the following simple observation.

Lemma 4.6. *Let L, M, N be monomial ideals of S. Assume that* $\operatorname{supp} G(M) \cap \operatorname{supp} G(N) = \emptyset$. *Then*

$$L + M \cdot N = (L + M) \cap (L + N).$$

Proof. Since $\operatorname{supp} G(M) \cap \operatorname{supp} G(N) = \emptyset$, we have $M \cdot N = M \cap N$. The lemma follows from the fact that $L + (M \cap N) = (L + M) \cap (L + N)$. $\square$

Lemma 4.7. *Assume that $x^{\mathbf{a}} = x_C \cdot f$, where C is a clique of size 4 of G and f is a monomial in S. Then* $\operatorname{reg}(I^3) \le \operatorname{reg}(I^{(3)})$.

Proof. Without loss of generality, we may assume that $x^{\mathbf{a}} = x_1x_2x_3x_4 \cdot f$, where 1234 is a clique of size 4 of G and $\operatorname{supp} f$ is an independent set. Using Lemma 4.5, we may assume that $N(\operatorname{supp}(f)) \cap \{1,2,3,4\} = \emptyset$.

First, assume that there exists an r in $N(\operatorname{supp} f) \setminus \operatorname{supp} \mathbf{a}$. It is clear that $x_r \in \sqrt{I^3 : x^{\mathbf{a}}}$. By Lemma 3.2, $\operatorname{reg} I^3 \le \operatorname{reg} I^{(3)}$. Thus, we may assume that $N(\operatorname{supp} f) \subseteq \operatorname{supp} \mathbf{a} = \{1,2,3,4\} \cup \operatorname{supp} f$. Since $N(\operatorname{supp} f) \cap \{1,2,3,4\} = \emptyset$ and the assumption that G is connected and has no isolated vertices, we have $f = 1$ and $x^{\mathbf{a}} = x_1x_2x_3x_4$.

If there exists an r in $N(1) \cap N(2) \setminus \{3,4\}$, then $x_r \in \sqrt{I^3 : x^{\mathbf{a}}}$ and $r \notin \operatorname{supp}(\mathbf{a})$. By Lemma 3.2, $\operatorname{reg} I^3 \le \operatorname{reg} I^{(3)}$. Hence, we may assume that

$$N(j) \cap N(k) = \{1,2,3,4\} \setminus \{j,k\} \text{ for all } 1 \le j \ne k \le 4. \tag{4.8}$$

For each $V \subseteq \{1,2,3,4\}$, let $N^*(V) = N(V) \setminus \{1,2,3,4\}$. We claim

$$\sqrt{I^3 : x^{\mathbf{a}}} = I + \sum_{1 \le j < k \le 4} (N^*(j)) \cdot (N^*(k)) + (x_1, x_2, x_3, x_4). \tag{4.9}$$

Proof of Eq. (4.9). It is clear that the left hand side contains the right hand side. Now assume that $x_U \notin I$ is a minimal generator of $\sqrt{I^3 : x^{\mathbf{a}}}$ with $U \cap \{1,2,3,4\} = \emptyset$. By Lemma 2.25, we have

$$|N(U) \cap \operatorname{supp} \mathbf{a}| + \operatorname{ord}_I(X_{\mathbf{a}}(U)) \ge 3.$$

Since $X_{\mathbf{a}}(U) \mid x^{\mathbf{a}}$ has order at most 2, $|N(U) \cap \operatorname{supp} \mathbf{a}| \geq 1$. We may assume that $1 \in N(U) \cap \operatorname{supp} \mathbf{a}$. Then $X_{\mathbf{a}}(U) \mid x_2x_3x_4$ has order at most 1. Thus $|N(U) \cap \operatorname{supp} \mathbf{a}| \geq 2$. By Eq. (4.8), the conclusion follows. $\square$

Let $L = I + (N^*(\{1,2\})) \cdot (N^*(3)) + (N^*(1)) \cdot (N^*(2)) + (x_1, x_2, x_3, x_4)$. By Eq. (4.8), Eq. (4.9), and Lemma 4.6,

$$\sqrt{I^3 : x^{\mathbf{a}}} = (L + (N^*(\{1,2,3\}))) \cap (L + (N^*(4))).$$

Let $\Delta = \Delta_{\mathbf{a}}(I^3)$, $\Gamma_1 = \Delta(L + (N^*(\{1,2,3\})))$, and $\Gamma_2 = \Delta(L + (N^*(4)))$. By Lemma 2.2, $\Delta = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2$ is the simplicial complex of

$$L + (N^*(\{1,2,3\})) + L + (N^*(4)) = I + (x_j \mid j \in N(\{1,2,3,4\})).$$

Note that $\operatorname{lk}_\Delta F = \operatorname{lk}_{\Gamma_1} F \cup \operatorname{lk}_{\Gamma_2} F$ and $\operatorname{lk}_{\Gamma_1} F \cap \operatorname{lk}_{\Gamma_2} F = \operatorname{lk}_{\Gamma_1 \cap \Gamma_2} F$. By Lemma 2.5, there are three cases.

Case 1. $\widetilde{H}_{i-2}(\operatorname{lk}_{\Gamma_1 \cap \Gamma_2} F; K) \neq 0$. We have

$$\sqrt{I^3 : x^{\mathbf{b}}} = I + (x_j \mid j \in N(\{1,2,3,4\})),$$

where $x^{\mathbf{b}} = (x_1x_2)(x_1x_3x_4)$. By Lemma 2.19, $\Delta_{\mathbf{b}}(I^3) = \Gamma_1 \cap \Gamma_2$. Thus, $|\mathbf{b}| + i - 1 \leq \operatorname{reg}(S/I^3) = |\mathbf{a}| + i = |\mathbf{b}| + i - 1$. It implies that $(\mathbf{b}; i-1)$ is also an extremal exponent of I^3. By Lemma 4.5, $\operatorname{reg} I^3 \leq \operatorname{reg} I^{(3)}$.

Case 2. $\widetilde{H}_{i-1}(\operatorname{lk}_{\Gamma_1} F; K) \neq 0$. We have

$$\sqrt{I^3 : (x_1^2x_2^2x_3)} = I + (x_j \mid j \in N(\{1,2,3\})) = L + (N^*(\{1,2,3\})).$$

By Lemma 2.19, $\Gamma_1 = \Delta_{x_1^2x_2^2x_3}(I^3)$. By Lemma 2.12, $\operatorname{reg}(S/I^3) \geq |\mathbf{b}| + i = 5 + i > |\mathbf{a}| + i$, a contradiction.

Case 3. $\widetilde{H}_{i-1}(\operatorname{lk}_{\Gamma_2} F; K) \neq 0$. We have

$$L + (N^*(4)) = I + (N^*(\{1,2\})) \cdot (N^*(3)) + (N^*(1)) \cdot (N^*(2)) + (x_j \mid j \in N[4]).$$

Let $H = I + (N^*(1)) \cdot (N^*(2)) + (x_j \mid j \in N[4])$. By Lemma 4.6 and Eq. (4.8),

$$L + (N^*(4)) = (H + (N^*(\{1,2\}))) \cap (H + (N^*(3))).$$

Let $\gamma_1 = \Delta(H + (N^*(\{1,2\}))) = \Delta(I + (N(\{1,2,4\})))$ and $\gamma_2 = \Delta(H + (N^*(3))) = \Delta(I + N^*(1) \cdot N^*(2) + (N(\{3,4\})))$. By Lemma 2.2, $\Gamma_2 = \gamma_1 \cup \gamma_2$ and $\gamma_1 \cap \gamma_2$ is the simplicial complex of

$$I + (N(\{1,2,4\})) + I + (N^*(1)) \cdot (N^*(2)) + (N(\{3,4\})) = I + (x_j \mid j \in N(\{1,2,3,4\})).$$

Note that $\operatorname{lk}_{\Gamma_2} F = \operatorname{lk}_{\gamma_1} F \cup \operatorname{lk}_{\gamma_2} F$ and $\operatorname{lk}_{\gamma_1} F \cap \operatorname{lk}_{\gamma_2} F = \operatorname{lk}_{\gamma_1 \cap \gamma_2} F$. By Lemma 2.5, there are three subcases. The cases $\widetilde{H}_{i-2}(\operatorname{lk}_{\gamma_1} F \cap \operatorname{lk}_{\gamma_2} F; K) \neq 0$ and $\widetilde{H}_{i-1}(\operatorname{lk}_{\gamma_1} F; K) \neq 0$ can be done similarly to Case 1 and Case 2, respectively. It remains to consider the case $\widetilde{H}_{i-1}(\operatorname{lk}_{\gamma_2} F; K) \neq 0$. By Lemma 4.6 and Eq. (4.8),

$$I + (N^*(1)) \cdot (N^*(2)) + (N(\{3,4\})) = (I + (N(\{1,3,4\}))) \cap (I + (N(\{2,3,4\}))).$$

Let $\delta_1 = \Delta(I + (N(\{1,3,4\})))$ and $\delta_2 = \Delta(I + (N(\{2,3,4\})))$. By Lemma 2.2, $\gamma_2 = \delta_1 \cup \delta_2$ and $\delta_1 \cap \delta_2$ is the simplicial complex of $I + (x_j \mid j \in N(\{1,2,3,4\}))$. By Lemma 2.5, one of the homology groups $\widetilde{H}_{i-1}(\operatorname{lk}_{\delta_1} F; K)$, $\widetilde{H}_{i-1}(\operatorname{lk}_{\delta_2} F; K)$, $\widetilde{H}_{i-2}(\operatorname{lk}_{\delta_1} F \cap \operatorname{lk}_{\delta_2} F; K)$ is non-zero. The cases $\widetilde{H}_{i-1}(\operatorname{lk}_{\delta_1} F; K) \neq 0$, $\widetilde{H}_{i-1}(\operatorname{lk}_{\delta_2} F; K) \neq 0$ can be done similarly to Case 2. The case $\widetilde{H}_{i-2}(\operatorname{lk}_{\delta_1} F \cap \operatorname{lk}_{\delta_2} F; K) \neq 0$ can be done similarly to Case 1. $\square$

Lemma 4.8. *Assume that $x^{\mathbf{a}} \notin I^{(3)}$ and $\Delta_{\mathbf{a}}(I^{(3)}) \neq \Delta_{\mathbf{a}}(I^3)$. Then* $\operatorname{reg}(I^3) \leq \operatorname{reg}(I^{(3)})$.

Proof. We first have

Claim B. *$x^{\mathbf{a}}$ is squarefree.*

Proof of Claim B. By Lemma 4.1, we may assume that $x^{\mathbf{a}} = x_1^{a_1} \cdots x_t^{a_t}$ for some $t \geq 4$. Fix $j \in \{1, \ldots, t\}$. By Remark 2.17 there exists a minimal generator x_U of $\sqrt{I^3 : x^{\mathbf{a}}}$ such that $x_j \mid x_U$. By Lemma 2.24, there exists a minimal generator P of I^3 such that $x_U = \sqrt{P / \gcd(P, x^{\mathbf{a}})}$. Assume by contradiction that $x_j^2 \mid x^{\mathbf{a}}$. Then

$$x_r \in \sqrt{I^3 : x^{\mathbf{a}}} \text{ for all } r \in N(j). \tag{4.10}$$

Since $x_j \mid x_U$ and $x_j^2 \mid x^{\mathbf{a}}$, it implies that $x_j^3 \mid P$. In other words, there exists $u, v, w \in N(j)$ such that $P = x_j^3 x_u x_v x_w$. By Eq. (4.10) and the minimality of x_U, we must have $x_U = x_j$ and $x_u x_v x_w \mid x^{\mathbf{a}}$. First, assume that $\{u, v, w\} \setminus \{1,2,3\} \neq \emptyset$, say $u \notin \{1,2,3\}$. Then $(x_1 x_2 x_3)(x_j x_u) \mid x^{\mathbf{a}}$. Hence, $x^{\mathbf{a}} \in I^{(3)}$, a contradiction. Thus, we may assume that $u, v, w \in \{1,2,3\}$. There are two cases.

B. 1. $j \in \{1,2,3\}$. We may assume that $j = 1$. Since $u, v, w \neq 1$, we must have either x_2^2 or x_3^2 divides $x^{\mathbf{a}}$. In either cases, $x^{\mathbf{a}} \in I^{(3)}$, a contradiction.

B. 2. $j \notin \{1,2,3\}$. First, assume that $\{u, v, w\}$ is a proper subset of $\{1,2,3\}$. We may assume that $u = v = 1$. Then $(x_j^2 x_1^2)(x_2 x_3) \mid x^{\mathbf{a}}$. Hence, $x^{\mathbf{a}} \in I^{(3)}$, a contradiction. Now, assume that $\{u, v, w\} = \{1,2,3\}$ then $\{1,2,3,j\}$ forms a clique of size 4. Hence, $x^{\mathbf{a}} \in I^{(3)}$, a contradiction. $\square$

By Lemma 4.1 and Claim B, we may assume that $x^{\mathbf{a}} = x_1x_2x_3 \cdot f$ where 123 is a triangle of G and $f = x_4 \cdots x_t$ for some $t \geq 4$. Since $x^{\mathbf{a}} \notin I^{(3)}$, $\operatorname{supp} f$ is an independent set.

First, assume that there exist $r \in \operatorname{supp} f \cap N(\{1,2,3\})$ and $s \in \operatorname{supp} f \setminus N(\{1,2,3\})$. Since G is connected and has no isolated vertices, there exists a neighbour of s, say u. Since $\operatorname{supp} f$ is an independent set, $u \notin \operatorname{supp} \mathbf{a}$. Furthermore, $x_u \in \sqrt{I^3 : x^{\mathbf{a}}}$ as $x_u x_s x_1 x_2 x_3 x_r \in I^3$. By Lemma 3.2, $\operatorname{reg} I^3 \leq \operatorname{reg} I^{(3)}$. Thus, we may assume that either $\{4,\dots,t\} \cap N(\{1,2,3\}) = \emptyset$ or $\{4,\dots,t\} \subseteq N(\{1,2,3\})$.

Second, assume that there exists an r in one of the following sets $(N(j) \cap N(k)) \setminus \{1,2,3\}$ for some $4 \leq j < k \leq t$, $(N(\{1,2,3\}) \cap N(\{4,\dots,t\})) \setminus \{1,2,3\}$, $N(1) \cap N(2) \cap N(3)$. Then, $x_r \in \sqrt{I^3 : x^{\mathbf{a}}}$ and $r \notin \operatorname{supp} \mathbf{a}$. By Lemma 3.2, $\operatorname{reg} I^3 \leq \operatorname{reg} I^{(3)}$. Thus, we may assume that

$$N(j) \cap N(k) \subseteq \{1,2,3\} \text{ for all } 4 \leq j < k \leq t. \tag{4.11}$$

$$N(\{1,2,3\}) \cap N(\{4,\dots,t\}) \subseteq \{1,2,3\}. \tag{4.12}$$

$$N(1) \cap N(2) \cap N(3) = \emptyset. \tag{4.13}$$

Case 1. $\{4,\dots,t\} \cap N(\{1,2,3\}) = \emptyset$. In this case, Eq. (4.11) and Eq. (4.12) become

$$N(j) \cap N(k) = \emptyset \text{ for all } 4 \leq j < k \leq t. \tag{4.14}$$

$$N(\{1,2,3\}) \cap N(\{4,\dots,t\}) = \emptyset. \tag{4.15}$$

For each $j \in \{1,2,3\}$, let $N^*(j) = N(j) \setminus \{1,2,3\}$. We have

Claim C.

$$\begin{aligned} \sqrt{I^3 : x^{\mathbf{a}}} = I + \sum_{4 \leq j < k \leq t} (N(j)) \cdot (N(k)) + (N(\{1,2,3\})) \cdot (N(\{4,\dots,t\})) \\ + (N^*(1)) \cdot (N^*(2)) \cdot (N^*(3)). \end{aligned}$$

Proof of Claim C. It is clear that the left hand side contains the right hand side. Let U be an independent set of G such that x_U is a minimal monomial generator of $\sqrt{I^3 : x^{\mathbf{a}}}$. It suffices to prove that x_U belongs to the right hand side. By Lemma 2.25,

$$|N(U) \cap \operatorname{supp} \mathbf{a}| + \operatorname{ord}_I(X_{\mathbf{a}}(U)) \geq 3.$$

Since $\operatorname{ord}_I(X_{\mathbf{a}}(U)) \leq 1$, $|N(U) \cap \operatorname{supp} \mathbf{a}| \geq 2$. By Eq. (4.14) and Eq. (4.15), if $N(U) \cap \{4,\dots,t\} \neq \emptyset$ then x_U belongs to $\sum_{4 \leq j < k \leq t} (N^*(j)) \cdot (N^*(k)) + (N(\{1,2,3\})) \cdot (N^*(\{4,\dots,t\}))$. Thus, we may assume that $N(U) \cap \operatorname{supp} \mathbf{a} \subseteq \{1,2,3\}$. Hence, $X_{\mathbf{a}}(U)$ has order zero and $|N(U) \cap \operatorname{supp} \mathbf{a}| = 3$. By Eq. (4.13), $x_U \in (N^*(1)) \cdot (N^*(2)) \cdot (N^*(3))$. $\square$

First, assume that $t \geq 5$. Let

$$H = I + \sum_{4 \leq j < k \leq t, (j,k) \neq (t-1,t)} (N(j)) \cdot (N(k)) + (N(\{1,2,3\})) \cdot (N(\{4,\dots,t\})) \\ + N^*(1) \cdot N^*(2) \cdot N^*(3).$$

By Lemma 4.6, Claim C, and Eq. (4.14),

$$\sqrt{I^3 : x^{\mathbf{a}}} = (H + (N(t-1))) \cap (H + (N(t))).$$

Let $\Gamma_1 = \Delta(H + (N(t-1)))$ and $\Gamma_2 = \Delta(H + (N(t)))$. By Lemma 2.2, $\Delta_{\mathbf{a}}(I^3) = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \Delta(H + (N(\{t-1,t\})))$. By Lemma 2.2, Γ_1 is a cone over $t-1$, Γ_2 and $\Gamma_1 \cap \Gamma_2$ are cones over t. Since $F \cap \operatorname{supp}(\mathbf{a}) = \emptyset$, we have $t-1, t \notin F$. Thus, $\operatorname{lk}_{\Gamma_1} F$ is a cone over $t-1$, $\operatorname{lk}_{\Gamma_2} F$ and $\operatorname{lk}_{\Gamma_1} F \cap \operatorname{lk}_{\Gamma_2} F$ are cones over t. By Lemma 2.5, this is a contradiction.

Thus, we must have $t = 4$. Let $H = I + N^*(1) \cdot N^*(2) \cdot N^*(3)$. By Lemma 4.6, Claim C, and Eq. (4.15),

$$\sqrt{I^3 : x^{\mathbf{a}}} = (H + (N(\{1,2,3\}))) \cap (H + (N(4))).$$

Let $\Gamma_1 = \Delta(H + (N(\{1,2,3\})))$ and $\Gamma_2 = \Delta(H + (N(4)))$. By Lemma 2.2, $\Delta_{\mathbf{a}}(I^3) = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \Delta(H + (x_j \mid j \in N(\{1,2,3,4\})))$. By Lemma 2.2 and Eq. (4.15), Γ_2 and $\Gamma_1 \cap \Gamma_2$ are cones over 4. Since $F \cap \operatorname{supp}(\mathbf{a}) = \emptyset$, we have $4 \notin F$. Thus, $\operatorname{lk}_{\Gamma_2} F$ and $\operatorname{lk}_{\Gamma_1} F \cap \operatorname{lk}_{\Gamma_2} F$ are cones over 4. By Lemma 2.5, $\widetilde{H}_{i-1}(\operatorname{lk}_{\Gamma_1} F; K) \neq 0$. On the other hand,

$$H + (x_j \mid j \in N(\{1,2,3\}) = I + (x_j \mid j \in N(\{1,2,3\}).$$

Let $\mathbf{b} = x_1^2 x_2^2 x_3$. Then $\sqrt{I^3 : x^{\mathbf{b}}} = I + (x_j \mid j \in N(\{1,2,3\}))$. By Lemma 2.12 and Lemma 2.19,

$$5 + i = |\mathbf{b}| + i \leq \operatorname{reg}(S/I^3) = |\mathbf{a}| + i = 4 + i,$$

which is a contradiction.

Case 2. $\{4,\dots,t\} \subseteq N(\{1,2,3\})$. If $t \geq 5$, we have a contradiction by

Claim D. $\Delta_{\mathbf{a}}(I^3)$ *is a cone over* 4.

Proof of Claim D. Since $x^{\mathbf{a}} \notin I^{(3)}$, $|N(4) \cap \{1,2,3\}| \leq 2$. Assume by contradiction that there exists a minimal generator g of $\sqrt{I^3 : x^{\mathbf{a}}}$ such that $x_4 \mid g$. By Lemma 2.24, there exists a minimal generator P of I^3 such that $g = \sqrt{P/\gcd(P, x^{\mathbf{a}})}$. Since $x_4 \mid g$, $x_4^2 \mid P$. Hence, we may write $P = x_4^2 x_s x_u x_v x_w$ where $s, u \in N(4)$ and $x_v x_w \in I$. Since $t \geq$

5 and $t \in N(\{1,2,3\})$, we have $x_s, x_u \in \sqrt{I^3 : x^{\mathbf{a}}}$. By the minimality of g, we must have $x_s x_u \mid x^{\mathbf{a}}$. Since $x^{\mathbf{a}} = x_1 \cdots x_t$ and $|N(4) \cap \{1,2,3\}| \leq 2$, we may assume that $s = 1, u = 2$ and $x_4 x_3 \notin I$. Since $x^{\mathbf{a}} \notin I^{(3)}$ and 412 is a triangle of G, we must have $x_3 x_5 \cdots x_t \notin I$. Furthermore, $g = x_4 \sqrt{x_v x_w / \gcd(x_v x_w, x_3 x_5 \cdots x_t)}$. Since g is minimal, $|\{v,w\} \cap \{3,5,\dots,t\}| = 1$. We may assume that $v \in \{3,5,\dots,t\}$. Then $g = x_4 x_w$. There are two cases.

D. 1. $v = 3$. Since $x_3 x_5 \notin I$ and $5 \in N(\{1,2,3\})$, we may assume that $x_5 x_1 \in I$. Since $(x_1 x_5)(x_2 x_4)(x_3 x_w) \in I^3$, $x_w \in \sqrt{I^3 : x^{\mathbf{a}}}$.

D. 2. $v \in \{5,\dots,t\}$. Since $(x_1 x_4)(x_2 x_3)(x_v x_w) \in I^3$, $x_w \in \sqrt{I^3 : x^{\mathbf{a}}}$.
Hence, g is not minimal, a contradiction. $\square$

Thus, $t = 4$. Since $x^{\mathbf{a}} \notin I^{(3)}$, $|N(4) \cap \{1,2,3\}| \leq 2$. There are two subcases:

Subcase 2.1. $|N(4) \cap \{1,2,3\}| = 1$. Assume that $x_1 x_4 \in I$ and $x_2 x_4, x_3 x_4 \notin I$. For each $V \subseteq \{1,2,3,4\}$, let $N^*(V) = N(V) \setminus \{1\}$. First, assume that there exists $r \in N^*(2) \cap N^*(3)$. Then $x_r \in \sqrt{I^3 : x^{\mathbf{a}}}$ as $x_r^2 x_2 x_3 (x_1 x_4) \in I^3$. Furthermore, $r \notin \operatorname{supp} \mathbf{a}$. By Lemma 3.2, $\operatorname{reg} I^3 \leq \operatorname{reg} I^{(3)}$. Thus, we may assume that

$$N^*(2) \cap N^*(3) = \emptyset. \tag{4.16}$$

Since $2, 3 \notin N(4)$, Eq. (4.12) becomes

$$N^*(\{1,2,3\}) \cap N^*(4) = \emptyset. \tag{4.17}$$

We have

$$\sqrt{I^3 : x^{\mathbf{a}}} = I + (N^*(2)) \cdot (N^*(3)) + (N^*(\{1,2,3\})) \cdot (N^*(4)) + (x_1). \tag{4.18}$$

Proof of Eq. (4.18). It is clear that the left hand side contains the right hand side. It suffices to prove that if U is an independent set of G with $1 \notin U$ and x_U is a minimal generator of $\sqrt{I^3 : x^{\mathbf{a}}}$ then x_U belongs to the right hand side. By Lemma 2.25,

$$|N(U) \cap \operatorname{supp} \mathbf{a}| + \operatorname{ord}_I(X_{\mathbf{a}}(U)) \geq 3.$$

Since $X_{\mathbf{a}}(U) \mid x^{\mathbf{a}}$ has order at most 2, $|N(U) \cap \operatorname{supp} \mathbf{a}| \geq 1$. Let j be an element of $N(U) \cap \operatorname{supp} \mathbf{a}$. Then $X_{\mathbf{a}}(U)$, which is a divisor of $x^{\mathbf{a}}/x_j$ has order at most 1. Thus, $|N(U) \cap \operatorname{supp} \mathbf{a}| \geq 2$. If $N(U) \cap \operatorname{supp} \mathbf{a} = \{1,2\}$ or $\{1,3\}$, then $X_{\mathbf{a}}(U)$ has order zero. Hence, $|N(U) \cap \operatorname{supp} \mathbf{a}| \geq 3$. Therefore, $N(U) \cap \operatorname{supp} \mathbf{a}$ must contain $j \neq k$, $j,k \in \{1,2,3,4\}$ such that $\{j,k\} \neq \{1,2\}, \{1,3\}$. By Eq. (4.16) and Eq. (4.17), x_U belongs to the right hand side. $\square$

Let $J = I + (x_1) + (N^*(2)) \cdot (N^*(3))$. By Lemma 4.6, Eq. (4.18), and Eq. (4.17),

$$\sqrt{I^3 : x^{\mathbf{a}}} = (J + (N^*(4))) \cap (J + (N^*(\{1,2,3\}))).$$

Let $\Gamma_1 = \Delta(J + (N^*(4)))$ and $\Gamma_2 = \Delta(J + (N^*(\{1,2,3\})))$. By Lemma 2.2, $\Delta_{\mathbf{a}}(I^3) = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \Delta(J + (N^*(\{1,2,3,4\}))) = \Delta(I + (N(\{1,2,3,4\})))$. By Lemma 2.2, Γ_1 is a cone over 4. Since $F \cap \operatorname{supp} \mathbf{a} = \emptyset$, we have $4 \notin F$. Thus, $\operatorname{lk}_{\Gamma_1} F$ is a cone over 4. By Lemma 2.5, either $\widetilde{H}_{i-1}(\operatorname{lk}_{\Gamma_2} F; K) \neq 0$ or $\widetilde{H}_{i-2}(\operatorname{lk}_{\Gamma_1 \cap \Gamma_2} F; K) \neq 0$.

Subcase 2.1.1. $\widetilde{H}_{i-1}(\operatorname{lk}_{\Gamma_2} F; K) \neq 0$. Let $\mathbf{b} = x_1^2 x_2^2 x_3$. Then

$$\sqrt{I^3 : x^{\mathbf{b}}} = I + (x_j \mid j \in N(\{1,2,3\})) = J + (x_j \mid j \in N^*(\{1,2,3\})).$$

By Lemma 2.12 and Lemma 2.19, $5 + i = |\mathbf{b}| + i \leq \operatorname{reg}(S/I^3) = |\mathbf{a}| + i = 4 + i$, which is a contradiction.

Subcase 2.1.2. $\widetilde{H}_{i-2}(\operatorname{lk}_{\Gamma_1 \cap \Gamma_2} F; K) \neq 0$. Let $\mathbf{b} = x_1^2 x_2 x_3 x_4 = (x_1 x_4)(x_1 x_2 x_3)$. Then

$$\sqrt{I^3 : x^{\mathbf{b}}} = I + (x_j \mid j \in N(\{1,2,3,4\})) = J + (x_j \mid j \in N^*(\{1,2,3,4\})).$$

By Lemma 2.12 and Lemma 2.19, $|\mathbf{b}| + i - 1 \leq \operatorname{reg}(S/I^3) = |\mathbf{a}| + i = |\mathbf{b}| + i - 1$. Hence, $(\mathbf{b}, i-1)$ is also an extremal exponent of I^3. By Lemma 4.5, $\operatorname{reg} I^3 \leq \operatorname{reg} I^{(3)}$.

Subcase 2.2. $|N(4) \cap \{1,2,3\}| = 2$. Assume that $x_1 x_4, x_4 x_2 \in I$ and $x_3 x_4 \notin I$. For each $V \subseteq \{1,2,3,4\}$, let $N^*(V) = N(V) \setminus \{1,2\}$. First, assume that there exists $r \in N^*(3) \cap N^*(\{1,2\})$. Then $x_r \in \sqrt{I^3 : x^{\mathbf{a}}}$ and $r \notin \operatorname{supp} \mathbf{a}$. By Lemma 3.2, $\operatorname{reg} I^3 \leq \operatorname{reg} I^{(3)}$. Thus, we may assume that

$$N^*(3) \cap N^*(\{1,2\}) = \emptyset. \tag{4.19}$$

Since $3 \notin N(4)$, Eq. (4.12) becomes

$$N^*(\{1,2,3\}) \cap N^*(4) = \emptyset. \tag{4.20}$$

We have

$$\sqrt{I^3 : x^{\mathbf{a}}} = I + (N^*(4)) \cdot (N^*(\{1,2,3\})) + (N^*(3)) \cdot (N^*(\{1,2\})) + (x_1, x_2). \tag{4.21}$$

Proof of Eq. (4.21). It is clear that the left hand side contains the right hand side. Now assume that U is an independent set of G with $1, 2 \notin \operatorname{supp} U$ and x_U is a minimal generator of $\sqrt{I^3 : x^{\mathbf{a}}}$. By Lemma 2.25,

$$|N(U) \cap \operatorname{supp} \mathbf{a}| + \operatorname{ord}_I(X_{\mathbf{a}}(U)) \geq 3.$$

Since $X_{\mathbf{a}}(U) \mid x^{\mathbf{a}}$ has order at most 2, $|N(U) \cap \operatorname{supp} \mathbf{a}| \geq 1$. Let j be an element of $N(U) \cap \operatorname{supp} \mathbf{a}$. Then $X_{\mathbf{a}}(U)$, which is a divisor of $x^{\mathbf{a}}/x_j$ has order at most 1. Thus,

$|N(U) \cap \operatorname{supp} \mathbf{a}| \geq 2$. If $N(U) \cap \operatorname{supp} \mathbf{a} = \{1,2\}$, then $X_{\mathbf{a}}(U)$ has order zero. Hence $|N(U) \cap \operatorname{supp} \mathbf{a}| \geq 3$. Therefore, $N(U) \cap \operatorname{supp} \mathbf{a}$ must contain $j \neq k$, $j,k \in \{1,2,3,4\}$ such that $\{j,k\} \neq \{1,2\}$. By Eq. (4.19) and Eq. (4.20), x_U belongs to the right hand side. $\square$

Let $H = I + (x_1, x_2) + (N^*(3)) \cdot (N^*(\{1,2\}))$. By Lemma 4.6, Eq. (4.21), and Eq. (4.20), we have

$$\sqrt{I^3 : x^{\mathbf{a}}} = (H + (N^*(4))) \cap (H + (N^*(\{1,2,3\}))).$$

Let $\delta_1 = \Delta(H + (N^*(4)))$ and $\delta_2 = \Delta(H + (N^*(\{1,2,3\})))$. By Lemma 2.2, $\Delta_{\mathbf{a}}(I^3) = \delta_1 \cup \delta_2$ and $\delta_1 \cap \delta_2 = \Delta(H + (N^*(\{1,2,3,4\})))$. By Lemma 2.5, there are three subcases. Since $H + (N^*(\{1,2,3\})) = I + (x_j \mid j \in N(\{1,2,3\}))$, and $H + (N^*(\{1,2,3,4\})) = I + (x_j \mid j \in N(\{1,2,3,4\}))$, these cases can be done similarly to cases 2.1.1 and 2.1.2, respectively.

Thus, we assume that $\widetilde{H}_{i-1}(\operatorname{lk}_{\delta_1} F; K) \neq 0$. We have $H + (N^*(4)) = I + (N(4)) + (N^*(3)) \cdot (N^*(\{1,2\}))$. Let $L = I + (N(4))$. By Lemma 4.6 and Eq. (4.19),

$$H + (N(4)) = L + (N^*(3)) \cdot (N^*(\{1,2\})) = (L + (N^*(3))) \cap (L + (N^*(\{1,2\}))).$$

Note that $L + N^*(3) = I + (N(\{3,4\}))$ and $L + N^*(\{1,2\}) = I + (N(\{1,2,4\}))$. Let $\gamma_1 = \Delta(I + (N(\{3,4\})))$ and $\gamma_2 = \Delta(I + (N(\{1,2,4\})))$. By Lemma 2.2, $\delta_1 = \gamma_1 \cup \gamma_2$ and $\gamma_1 \cap \gamma_2 = \Delta(I + (N(\{1,2,3,4\})))$. By Lemma 2.2 and the assumption that $x_3x_4 \notin I$, γ_1 is a cone over 4. Since $F \cap \operatorname{supp} \mathbf{a} = \emptyset$, we have $4 \notin F$. Thus, $\operatorname{lk}_{\gamma_1} F$ is a cone over 4. By Lemma 2.5, either $\widetilde{H}_{i-1}(\operatorname{lk}_{\gamma_2} F; K) \neq 0$ or $\widetilde{H}_{i-2}(\operatorname{lk}_{\gamma_1 \cap \gamma_2} F; K) \neq 0$. These cases can be done similarly to cases 2.1.1 and 2.1.2, respectively. $\square$

We are now ready for the main result of this section.

Theorem 4.9. *Let $I = I(G)$ be the edge ideal of a simple graph G. Then*

$$\operatorname{reg} I^{(3)} = \operatorname{reg} I^3.$$

Proof. By Theorem 4.3, it suffices to prove that $\operatorname{reg} I^3 \leq \operatorname{reg} I^{(3)}$. Let $(\mathbf{a}, i)$ be an extremal exponent of I^3. Then $x^{\mathbf{a}} \notin I^3$. We have the following cases.

Case 1. $x^{\mathbf{a}} \notin I^{(3)}$ and $\Delta_{\mathbf{a}}(I^3) = \Delta_{\mathbf{a}}(I^{(3)})$. The conclusion follows from Lemma 2.18.

Case 2. $x^{\mathbf{a}} \notin I^{(3)}$ and $\Delta_{\mathbf{a}}(I^3) \neq \Delta_{\mathbf{a}}(I^{(3)})$. The conclusion follows from Lemma 4.8.

Case 3. $x^{\mathbf{a}} \in I^{(3)}$. The conclusion follows from Theorem 2.10, and Lemmas 4.4, 4.5, and 4.7. $\square$

5. Applications

In this section, we give some applications of our main results. First, we establish Alilooee-Banerjee-Beyarslan-Ha Conjecture [1, Conjecture 1] for $s = 2$ and $s = 3$, which extends work of Banerjee and Nevo [5].

Theorem 5.1. *Let $I = I(G)$ be the edge ideal of a simple graph G. Then*

$$\operatorname{reg} I^s \leq \operatorname{reg} I + 2s - 2$$

for $s = 2, 3$.

Proof. By [15, Theorem 3.6], we have

$$\operatorname{reg} I^{(s)} \leq \max\{\operatorname{reg}(I^{(s)} + I^{s-1}), \operatorname{reg} I + 2s - 2\}.$$

For $s = 2, 3$, note that $I^{(s)} \subseteq I^{s-1}$. Thus, for $s = 2$, by Theorem 3.3,

$$\operatorname{reg} I^2 = \operatorname{reg} I^{(2)} \leq \max\{\operatorname{reg} I, \operatorname{reg} I + 2\} = \operatorname{reg} I + 2.$$

For $s = 3$, by Theorem 4.9,

$$\operatorname{reg} I^3 = \operatorname{reg} I^{(3)} \leq \max\{\operatorname{reg} I^2, \operatorname{reg} I + 4\} = \operatorname{reg} I + 4.$$

This completes our proof. $\square$

Remark 5.2. Theorem 5.1 shows that bounds for regularity of edge ideals generalize to bounds for regularity of second and third powers of edge ideals. For example, combinatorial bound given by Woodroofe [38] carries over to bounds for regularity of the second/third powers of an edge ideal.

For symbolic powers, we prove

Theorem 5.3. *Let $I = I(G)$ be the edge ideal of a simple graph G. Then*

$$\operatorname{reg} I^{(s)} \leq \operatorname{reg} I + 2s - 2$$

for $s = 2, 3, 4$.

The case $s \leq 3$ was already proved in Theorem 5.1. Let $s = 4$. Using [15, Theorem 3.6], we need to bound $\operatorname{reg}(I^{(4)} + I^3)$. First, we have

Lemma 5.4. *Let $J_3(G)$ be the ideal of S generated by all squarefree monomials x_C where C is a clique of size 5 of G. Then*

$$I^{(4)} + I^3 = I^3 + J_1(G)J_1(G) + J_3(G).$$

Proof. Using Lemma 2.6, it is easy to see that the left hand side contains the right hand side. Conversely, let f be a minimal generator of $I^{(4)}$. By Corollary 2.7, we may assume that $\operatorname{supp} f = V(G) = [n]$ and G has no isolated vertices. If the matching number of G is at least 3, then $f \in I^3$. Thus, we may assume that the matching number of G is at most 2. There are two cases:

Case 1. G contains an induced 5-cycle, say 12345. Write $f = x_1x_2x_3x_4x_5g$. By Lemma 2.6, $\partial^*(f)/\partial^*(x_2x_4x_5) = x_1x_3g \in I$. Since 12345 is an induced 5-cycle, $x_1x_3 \notin I$. Thus, either x_1g or x_3g belongs to I. We may assume that $x_1g \in I$. This implies that $f = (x_1g)(x_2x_3)(x_4x_5) \in I^3$.

Case 2. G does not contain any induced 5-cycle. Since the matching number of G is at most 2, by [9, Theorem 5.5.3], G is a perfect graph. The conclusion follows from [36, Theorem 3.10]. □

We are now ready for

Proof of Theorem 5.3. By Theorem 5.1, we may assume that $s = 4$. By [15, Theorem 3.6], we have

$$\operatorname{reg} I^{(4)} \leq \max\{\operatorname{reg}(I^{(4)} + I^3), \operatorname{reg} I + 6\}.$$

Let $H = I^{(4)} + I^3$. By Lemma 5.4, $H = I^3 + J_1(G)J_1(G) + J_3(G)$. Fix $\mathbf{a} \in \mathbb{N}^n$ such that $x^{\mathbf{a}} \notin H$. We first prove that $\Delta_{\mathbf{a}}(H) = \Delta_{\mathbf{a}}(I^3)$. Since $I^3 \subseteq H$, by Lemma 2.19, it suffices to prove that if f is a minimal monomial generator of $\sqrt{H : x^{\mathbf{a}}}$ then $f \in \sqrt{I^3 : x^{\mathbf{a}}}$. By Lemma 2.24, there are two cases.

Case 1. $f = \sqrt{x_C/\gcd(x_C, x^{\mathbf{a}})}$, where C is a clique of size 5 of G. Since $f \neq 1$, $\emptyset \neq \operatorname{supp} f \subseteq \operatorname{supp} C$. Since $x_jx_C \in I^3$ for all $j \in \operatorname{supp} C$, we deduce that $fx_C \in I^3$. By Lemma 2.26, $f \in \sqrt{I^3 : x^{\mathbf{a}}}$.

Case 2. $f = \sqrt{x_Cx_D/\gcd(x_Cx_D, x^{\mathbf{a}})}$, where C and D are triangles of G. Since $f \neq 1$, $\emptyset \neq \operatorname{supp} f \subseteq \operatorname{supp} C \cup \operatorname{supp} D$. Since $x_jx_Cx_D \in I^3$ for all $j \in \operatorname{supp} C \cup \operatorname{supp} D$, we deduce that $fx_Cx_D \in I^3$. By Lemma 2.26, $f \in \sqrt{I^3 : x^{\mathbf{a}}}$.

Thus, $\Delta_{\mathbf{a}}(H) = \Delta_{\mathbf{a}}(I^3)$. By Lemma 2.18, $\operatorname{reg} H \leq \operatorname{reg} I^3 \leq \operatorname{reg} I + 4$, where the last inequality follows from Theorem 5.1. This concludes our proof. □

Remark 5.5. Theorem 5.3 implies that bounds for regularity of edge ideals generalize to bounds for regularity of s-th symbolic powers of edge ideals ($s \leq 4$). In particular, upper bounds given by Herzog and Hibi [19], and Fakhari and Yassemi [16] hold for s-th symbolic powers of edge ideals ($s \leq 4$).

By Theorem 1.1 and Theorem 5.3, we obtain a formula of the regularity of small symbolic powers of edge ideals of some new classes of graphs.

Corollary 5.6. *Let $I = I(G)$ be the edge ideal of a simple graph G.*

(1) *If I^s has a linear resolution (for $s = 2$ or 3) then* $\operatorname{reg}(I^{(s)}) = 2s$.
(2) *If* $\operatorname{reg} I = \mu(G)+1$, *where $\mu(G)$ is the induced matching number of G, then* $\operatorname{reg} I^{(s)} = 2s + \mu(G) - 1$ *for $s = 2, 3, 4$.*

Proof. The first statement is a direct consequence of Theorem 1.1. The second statement follows from Theorem 5.3 and Corollary 2.8. □

Remark 5.7.

(1) There is an infinite family of graphs G with the property that although each edge ideal $I(G)$ does not have a linear resolution, a higher power does (see [31]).
(2) There are classes of graphs for which $\operatorname{reg} I = \mu(G) + 1$, while regularity of their symbolic powers was not known. Such examples include the class of very-well covered graphs [24] and weakly chordal graphs [32].

We end the paper with the following remark.

Remark 5.8. From the proof of Theorem 5.3, we see that

$$\operatorname{reg}(I^3 + J_1(G)J_1(G)) \leq \operatorname{reg} I^3.$$

Note that $I^3 + J_1(G)J_1(G) = \overline{I^3}$ is the integral closure of I^3. The regularity of integral closure of powers of edge ideals will be studied in detail in subsequent work.

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