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Mosco convergence of strong laws of large numbers for triangular array of row-wise exchangeable random sets and fuzzy random sets

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ABSTRACT

In this paper, we obtain strong laws of large numbers for triangular array of row-wise exchangeable random sets and fuzzy random sets in a separable Banach space in the Mosco sense. Our results are obtained without bounded expectation condition, with or without compactly uniformly integrable and reverse martingale hypotheses. They improve some related results in literature. Moreover, some typical examples illustrating this study are provided.

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1. Introduction

In recent decades, the strong laws of large numbers (SLLN) for unbounded random sets, gave rise to applications in several fields, such as optimization and control, stochastic and integral geometry, mathematical economics, statistics and related fields. The first multivalued SLLN was proved by Artstein and Vitale [1] for independent identically distributed (i.i.d.) random variables whose values are compact subsets of $\mathbf{R}^d$. Puri and Ralescu [16] were the first to obtain the SLLN for i.i.d. Banach space-valued compact convex random sets. Later, Hiai [7] and Hess [6] independently proved similar results for random sets in an infinite dimensional Banach space, with respect to the Mosco convergence. Further variants of the multivalued SLLN have been established under various conditions, for example, see Castaing, Quang and Giap [2,3], Fu and Zhang [4,5], Inoue [9,10], Kim [12], Quang and Giap [18,19], Quang and Thuan [20].

The first result on SLLN with respect to Mosco convergence for triangular array of random sets was established by Quang and Giap [18]. In this paper, the authors established the SLLN for triangular array of row-wise independent random sets in Banach space with bounded expectation condition. According to this direction, in present paper, we study the Mosco convergence of the SLLN for triangular array of row-wise exchangeable random sets and fuzzy random sets. However, in [18], the SLLN was established under the bounded expectation condition, while in the present paper, this condition is not assumed. To give the main results, we provide a new method in building structure of triangular array of selections to prove the '*lim inf*' part of Mosco convergence. We also use a condition of

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the Mosco convergence in the first column of triangular array of random sets and fuzzy random sets, which was introduced by Hiai [7]. Our results improve some related results in literature. Moreover, some typical examples illustrating this study are provided.

The organization of this paper is as follows: In Section 2, we introduce some basic notions: set-valued random variable, fuzzy-valued random variable, Mosco convergence and exchangeability. Section 3 is concerned with some theorems on Mosco convergence of the SLLN for triangular arrays of row-wise exchangeable random sets and fuzzy random sets in a separable Banach space. A new method in building structure of triangular array of selections to prove the '*lim inf*' part of the Mosco convergence is provided. Illustrative examples are also provided in this section.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space, $(\mathfrak{X}, \|.\|)$ be a real separable Banach space and $\mathfrak{X}^*$ be its topological dual. The σ-field of all Borel sets of $\mathfrak{X}$ is denoted by $\mathcal{B}(\mathfrak{X})$. In the present paper, $\mathbf{R}$ (resp. $\mathbf{N}$) will be denoted the set of real numbers (resp. the set of positive integers).

Let $c(\mathfrak{X})$ be the family of all nonempty closed subsets of $\mathfrak{X}$ and $\mathcal{E}(\mathfrak{X})$ (shortly, $\mathcal{E}$) be the Effros σ-field on $c(\mathfrak{X})$. This σ-field is generated by the subsets $U^- = \{F \in c(\mathfrak{X}) : F \cap U \neq \emptyset\}$, where U ranges over the open subsets of $\mathfrak{X}$. On the other hand, for each $A, C \subset \mathfrak{X}$, cl$C$, co$C$ and $\overline{\text{co}}C$ denote the *norm-closure*, the *convex hull* and the *closed convex hull* of C, respectively; the *distance function* $d(\cdot, C)$ of C, the *Hausdorff distance*$d_H(A, C)$ of A and C, the *norm* $\|C\|$ of C and the *support function* $s(C, \cdot)$ of C are defined by

$$d(x, C) = \inf\{\|x - y\| \ : y \in C\}, (x \in \mathfrak{X}),$$

$$d_H(A, C) = \max\{\sup_{x \in A} d(x, C), \sup_{y \in C} d(y, A)\},$$

$$\|C\| = d_H(C, \{0\}) = \sup\{||x|| \ : x \in C\},$$

$$s(C, x^*) = \sup\{\langle x, x^* \rangle \ : x \in C\}, (x^* \in \mathfrak{X}^*).$$

The space $c(\mathfrak{X})$ has a linear structure induced by Minkowski addition and scalar multiplication:

$$A + B = \{a + b : a \in A, b \in B\},$$

$$\lambda A = \{\lambda a : a \in A\},$$

where $A, B \in c(\mathfrak{X}), \lambda \in \mathbf{R}$.

A multivalued (set-valued) function $X\colon \Omega \to c(\mathfrak{X})$ is said to be $\mathcal{F}$*-measurable* (or *measurable*) if X is $(\mathcal{F}, \mathcal{E})-$measurable, i.e. for every open set U of $\mathfrak{X}$, the subset $X^{-1}(U^-) = \{\omega \in \Omega : X(\omega) \cap U \neq \emptyset\}$ belongs to $\mathcal{F}$. A measurable multivalued function is also called a *closed valued random variable* (or *random set*). The sub-σ-field $X^{-1}(\mathcal{E})$ generated by X is denoted by $\mathcal{F}_X$.

The *distribution* $\mathbf{P}_X$ of the random set $X : \Omega \to c(\mathfrak{X})$ on the measurable space $(c(\mathfrak{X}), \mathcal{E})$ is defined by $\mathbf{P}_X(B) = \mathbf{P}(X^{-1}(B)), \quad$ for all $B \in \mathcal{E}$. A collection of random sets $\{X_i, i \in I\}$ is said to be *identically distributed* (i.d.) if the $\mathbf{P}_{X_i}, i \in I$ are identical.

A random element (Banach space valued random variable) $f : \Omega \to \mathfrak{X}$ is called a *selection* of the random set X if $f(\omega) \in X(\omega)$ for all $\omega \in \Omega$.

For every sub-σ-field $\mathcal{A}$ of $\mathcal{F}$ and for $1 \le p < \infty$, $\mathcal{L}^p(\Omega, \mathcal{A}, \mathbf{P}, \mathfrak{X})$ denotes the Banach space of (equivalence classes of) measurable functions $f : \Omega \to \mathfrak{X}$ such that the norm $\|f\|_p = (\mathrm{E}\|f\|^p)^{\frac{1}{p}} = (\int_\Omega \|f(\omega)\|^p d\mathbf{P})^{\frac{1}{p}}$ is finite. In special case, $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbf{P}, \mathfrak{X})$ (resp. $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{R})$) is denoted by $\mathcal{L}^p(\mathfrak{X})$ (resp. $\mathcal{L}^p$). For each random set X, define the following closed subset of $\mathcal{L}^p(\Omega, \mathcal{A}, \mathbf{P}, \mathfrak{X})$

$$S_X^p(\mathcal{A}) = \{f \in \mathcal{L}^p(\Omega, \mathcal{A}, \mathbf{P}, \mathfrak{X}) : f(\omega) \in X(\omega), \text{ for all } \omega \in \Omega\}.$$

A random set $X : \Omega \to c(\mathfrak{X})$ is called *integrable* if the set $S_X^1(\mathcal{F})$ is nonempty (i.e. $d(0, X(\cdot))$ is in $\mathcal{L}^1$), and it is called *integrable bounded* if the random variable $\|X\|$ is in $\mathcal{L}^1$.

For any random set X and any sub-σ-field $\mathcal{A}$ of $\mathcal{F}$, the *multivalued expectation* of X over Ω, with respect to $\mathcal{A}$, is defined by

$$\mathrm{E}(X, \mathcal{A}) = \{\mathrm{E}(f) : f \in S_X^1(\mathcal{A})\},$$

where $\mathrm{E}(f) = \int_\Omega f d\mathbf{P}$ is the usual Bochner integral of f. Shortly, $\mathbf{E}(X, \mathcal{F})$ is denoted by $\mathbf{E}X$. We note that $\mathrm{E}(X, \mathcal{A})$ is not always closed.

The sequence of random elements $\{X_n : n \ge 1\}$ is called a *martingale sequence* if $\mathbf{E}\|X_n\| < \infty$ and $X_n = \mathbf{E}(X_{n+m}|X_1, X_2, \dots, X_n)$ a.s. for all positive integers m and n. Similarly, $\{X_n : n \ge 1\}$ is called a *reverse martingale sequence* if it is a martingale under the reverse ordering of $\mathbf{N}$, that is, $X_{m+n} = \mathbf{E}(X_n|X_{m+n}, X_{m+n+1}, \dots)$ a.s. for all positive integers m and n.

A sequence of random elements $\{X_n : n \ge 1\}$ is said to be *tight* if for each $\epsilon > 0$ there exists a compact subset K_ϵ of $\mathfrak{X}$ such that $\mathbf{P}[X_n \notin K_\epsilon] < \epsilon$ for every positive integer n. Also, a general condition involving tightness of distributions and moments of the random elements $\{X_n : n \ge 1\}$ called *compact uniform integrability* (CUI) can be stated as: Given $\epsilon > 0$, there exists a compact subset K_ϵ of $\mathfrak{X}$ such that $\sup_n(\mathbf{E}\|X_n I_{[X_n \notin K_\epsilon]}\|) < \epsilon$, where I_A is the indicator function of A.

Next, we describe some basic concepts of fuzzy random sets. A *fuzzy set* in $\mathfrak{X}$ is a function $u : \mathfrak{X} \to [0, 1]$. For each fuzzy set u, the α*-level set* is denoted by

$$L_\alpha u = \{x \in \mathfrak{X} : u(x) \ge \alpha\}, \ 0 < \alpha \le 1.$$

It is easy to see that, for every $\alpha \in (0, 1]$, $L_\alpha u = \cap_{\beta < \alpha} L_\beta u$. Let $F(\mathfrak{X})$ denote the space of fuzzy subsets $u : \mathfrak{X} \to [0, 1]$ such that

(1) u is normal, i.e. the 1-level set $L_1 u \neq \emptyset$,

(2) u is upper semicontinuous, that is, for each $\alpha \in (0, 1]$, the α-level set $L_\alpha u$ is a closed subset of $\mathfrak{X}$.

We note that the relation $L_0(u) = \{x \in \mathfrak{X} : u(x) \ge 0\} = \mathfrak{X}$ is automatically satisfied.

A linear structure in $F(\mathfrak{X})$ is defined by the following operations,

$$(u + v)(x) = \sup_{y+z=x} \min\{u(y), v(z)\},$$

$$(\lambda u)(x) = \begin{cases} u(\lambda^{-1}x) & \text{if} \quad \lambda \neq 0, \\ I_{\{0\}}(x) & \text{if} \quad \lambda = 0, \end{cases}$$

where $u, v \in F(\mathfrak{X})$, $\lambda \in \mathbf{R}$. Then it follows that, for $u, v \in F(\mathfrak{X})$, $\lambda \in \mathbf{R}$, we have $L_\alpha(u + v) = L_\alpha(u) + L_\alpha(v)$ and $L_\alpha(\lambda u) = \lambda L_\alpha(u)$ for each $\alpha \in (0, 1]$.

The concept of a fuzzy random set as a generalization for a random set was extensively studied by Puri and Ralescu [17]. A *fuzzy-valued random variable* (or *fuzzy random set*) is a Borel measurable function $\tilde{X} : \Omega \to F(\mathfrak{X})$ such that $L_\alpha \tilde{X}$ is a random set for each $\alpha \in (0, 1]$.

The *expected value* of any fuzzy random set $\tilde{X}$, denoted by $\mathbf{E}\tilde{X}$, is a fuzzy set such that, for every $\alpha \in (0, 1]$,

$$L_\alpha(\mathbf{E}\tilde{X}) = \mathbf{E}(L_\alpha \tilde{X}).$$

Next, we shall use a notion of convergence for sequences of subsets which has been introduced by Mosco [13,14] and which related to that of Kuratowski. Let t be a topology on $\mathfrak{X}$ and $(C_n)_{n\geq 1}$ be a sequence in $c(\mathfrak{X})$. We put

$$t\text{-}liC_n = \{x \in \mathfrak{X} : x = t\text{-}\lim x_n,\ x_n \in C_n, \forall n \geq 1\},$$
$$t\text{-}lsC_n = \{x \in \mathfrak{X} : x = t\text{-}\lim x_k,\ x_k \in C_{n(k)}, \forall k \geq 1\}$$

where $(C_{n(k)})_{k\geq 1}$ is a subsequence of $(C_n)_{n\geq 1}$. The subsets $t\text{-}liC_n$ and $t\text{-}lsC_n$ are the *lower limit* and the *upper limit* of $(C_n)_{n\geq 1}$, relative to topology t. We obviously have $t\text{-}liC_n \subset t\text{-}lsC_n$.

A sequence $(C_n)_{n\geq 1}$ converges to C_∞, in the *sense of Kuratowski*, relatively to the topology t, if the two following equalities are satisfied: $t\text{-}lsC_n = t\text{-}liC_n = C_\infty$. In this case, we shall write $C_\infty = t\text{-}\mathrm{lim}C_n$; this is true if and only if the next two inclusions hold $t\text{-}lsC_n \subset C_\infty \subset t\text{-}liC_n$.

Let us denote by s (resp. w) the strong (resp. weak) topology of $\mathfrak{X}$. It is easily seen that $s\text{-}liC_n \subset w\text{-}lsC_n$ and $s\text{-}liC_n \in c(\mathfrak{X})$ unless it is empty. A subset C_∞ is said to be the *Mosco limit* of the sequence $(C_n)_{n\geq 1}$ denoted by $M\text{-}\lim C_n$ if $w\text{-}lsC_n = s\text{-}liC_n = C_\infty$ which is true if and only if

$$w\text{-}lsC_n \subset C_\infty \subset s\text{-}liC_n.$$

The corresponding definitions of pointwise convergence and almost sure convergence for a sequence $\{X_n : n \geq 1\}$ of multivalued functions defined on Ω are clear. In fact, in the above definitions, it suffices to replace C_n by $X_n(\omega)$ and C_∞ by $X_\infty(\omega)$ for almost surely $\omega \in \Omega$. Also, a fuzzy random set X_∞ is said to be the *Mosco limit* of the sequence of fuzzy random sets $\{X_n : n \geq 1\}$ denoted by $M\text{-}\lim X_n$ if $L_\alpha X_\infty = M\text{-}\lim L_\alpha X_n$ for every $\alpha \in (0, 1]$ a.s.

At the end of this section, we introduce some concepts of exchangeability. A sequence $\{X_1, X_2, \ldots, X_n\}$ of random sets is said to be *exchangeable* if the joint probability law of random sets, $(X_1, X_2, \ldots, X_n)$, is permutation invariant, that is,

$$\mathbf{P}(X_1 \in B_1, \ldots, X_n \in B_n) = \mathbf{P}(X_{\pi(1)} \in B_1, \ldots, X_{\pi(n)} \in B_n),$$

for all $B_1, \ldots, B_n \in \mathcal{E}$ and each permutation π of $\{1, 2, \ldots, n\}$.

Also, a sequence $\{\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n\}$ of fuzzy random sets is said to be *exchangeable* if for each $\alpha \in (0, 1]$, the sequence of random sets $\{L_\alpha \tilde{X}_1, L_\alpha \tilde{X}_2, \ldots, L_\alpha \tilde{X}_n\}$ is exchangeable.

Exchangeability for an infinite sequence is related to i.i.d. in the following sense. It is obvious that a sequence of $\{X_k : k \geq 1\}$ being i.i.d. random sets implies $\{X_k : k \geq 1\}$ are pairwise independent and exchangeable. However, if $\{X_k : k \geq 1\}$ is a sequence of exchangeable random sets, then $\{X_k : k \geq 1\}$ are i.d. random sets. Moreover, if $\{X_k : k \geq 1\}$ is a sequence of exchangeable random sets and pairwise independent, then these random sets are i.i.d (see Hu [8]). We note that the above results are also true for a finite sequence $\{X_k : 1 \leq k \leq n\}$ if this sequence can be embedded into an infinite sequence of exchangeable random sets. Thus, we can see the concept of exchangeability is an extension of the concept of i.i.d. random sets.

3. SLLN in Mosco convergence for triangular array of rowwise exchangeable random sets

Let X, Y be two random sets and f (resp. g) belongs to $S^1_X(\mathcal{F})$ (resp. $S^1_Y(\mathcal{F})$). If X, Y are independent, then in general case, f and g are not independent. However, if $f \in S^1_X(\mathcal{F}_X)$ and $g \in S^1_Y(\mathcal{F}_Y)$ then the pair of X, Y being independent random sets implies independence of the selections f, g. Similarly, if X, Y are exchangeable random sets, then in general case, f and g are not exchangeable. However, Inoue and Taylor [11] proved the following result.

Lemma 3.1 (Inoue and Taylor [11, Lemma 4.2]): (1) *For each random set X and $S^1_X(\mathcal{F}) \neq \emptyset$, we have*

$$\overline{co}\mathbf{E}(X) = \overline{co}\mathbf{E}(X, \mathcal{F}_X).$$

(2) *Let X, Y be exchangeable random sets. For each $f \in S^1_X(\mathcal{F}_X)$, there exists $g \in S^1_Y(\mathcal{F}_Y)$ such that f and g are exchangeable.*

(3) *For exchangeable random sets X, Y and $S^1_X(\mathcal{F}) \neq \emptyset$,*

$$\mathbf{E}(X, \mathcal{F}_X) = \mathbf{E}(Y, \mathcal{F}_Y).$$

Remark 3.1: Lemma 3.1(2) is also true for a finite or infinite collection of random sets. Especially, we also obtain the stronger conclusion that, let $\{X_n : n \geq 1\}$ (resp. $\{X_k : 1 \leq k \leq n\}$) be a sequence of exchangeable random sets, then for each $f_1 \in S^1_{X_1}(\mathcal{F}_{X_1})$, there exists a sequence $\{f_n : n \geq 2\}$ (resp. $\{f_k : 2 \leq k \leq n\}$) of $f_n \in S^1_{X_n}(\mathcal{F}_{X_n})$ and a measurable function $\varphi : c(\mathfrak{X}) \to \mathfrak{X}$ such that the sequence $\{f_n : n \geq 1\}$ (resp. $\{f_k : 1 \leq k \leq n\}$) is exchangeable and for every $n \geq 1, \omega \in \Omega, f_n(\omega) = \varphi(X_n(\omega))$.

The two following lemmas established the SLLN for triangular array of row-wise exchangeable random variables taking values in a separable Banach space.

Lemma 3.2 (Taylor and Patterson [21, Theorem 1]): *Let $\{X_{nk} : n \geq 1, 1 \leq k \leq n\}$ be an array of random elements in the separable Banach space $\mathfrak{X}$. Let $\{X_{nk}\}$ be row-wise exchangeable. Let $\{X_{nk} : n \geq 1\}$ converge in the second mean to $X_{\infty k}$ for each k and $\|X_{n1} - X_{\infty 1}\| \geq$*

$\|X_{(n+1),1} - X_{\infty 1}\|$ *for each n. If*

$$\rho_n(f) = \mathbf{E}(f(X_{n1})f(X_{n2})) \to 0 \text{ as } n \to \infty \text{ for each } f \in \mathfrak{X}^*$$

then

$$\frac{1}{n}\sum_{k=1}^{n} X_{nk} \to 0 \text{ a.s. as } n \to \infty.$$

The following lemma was obtained with CUI and reverse martingale hypotheses for the case of single-valued random variables

Lemma 3.3 (Patterson and Taylor [15, Theorem 3.4]): *Let* $\{X_{nk} : n \geq 1, 1 \leq k \leq n\}$ *be an array of row-wise exchangeable random elements in the separable Banach space* $\mathfrak{X}$ *such that the sequence* $\{X_{n1} : n \geq 1\}$ *is CUI. If*

$$\text{(i) } \{\mathbf{E}(X_{n1}|G_n) : n \geq 1\} \text{ is a reverse martingale}$$

$$\text{(where } G_n = \sigma\{\sum_{k=1}^{n} X_{nk}, \sum_{k=1}^{n+1} X_{(n+1),k}, \ldots\}),$$

$$\text{(ii) } \mathbf{E}(f(X_{n1})f(X_{n2})) \to 0 \text{ as } n \to \infty, \text{ for each; } f \in \mathfrak{X}^*,$$

$$\text{(iii) } \mathbf{E}(f^2(X_{n1})) = o(n) \text{ for each } f \in \mathfrak{X}^*,$$

then

$$\frac{1}{n}\sum_{k=1}^{n} X_{nk} \to 0 \text{ a.s. as } n \to \infty.$$

In the case of real-valued random variables, we have the following result.

Lemma 3.4 (Patterson and Taylor [15, Theorem 2.1]): *Let* $\{X_{nk} : n \geq 1, 1 \leq k \leq n\}$ *be an array of row-wise exchangeable real-valued random variables. If*

$$\text{(i) } \mathbf{E}(X_{n1}X_{n2}) \to 0 \text{ as } n \to \infty,$$

$$\text{(ii) } \mathbf{E}(X_{n1}^2) = o(n),$$

$$\text{(iii) } \{\mathbf{E}(X_{n1}|G_n) : n \geq 1\} \text{ is a reverse martingale}$$

$$\text{(where } G_n = \sigma\{\sum_{k=1}^{n} X_{nk}, \sum_{k=1}^{n+1} X_{(n+1),k}, \ldots\}),$$

then

$$\frac{1}{n}\sum_{k=1}^{n} X_{nk} \to 0 \text{ a.s. as } n \to \infty.$$

Now, we give a lemma which will be used to prove the main results.

Lemma 3.5 (Quang and Giap [18, Lemma 3.3]): *Let $\{x_{ni} : n \geq 1, 1 \leq i \leq n\}$ be a triangular array of elements in a Banach space satisfying the conditions:*

(i) $\lim_{i\to\infty} x_{ni} = 0$,

(ii) *there exists a positive constant C such that $\|x_{ni}\| \leq C$, for all $n \geq 1, 1 \leq i \leq n$.*

Then, $\frac{1}{n}\sum_{i=1}^{n} x_{ni} \to 0$ as $n \to \infty$.

Lemma 3.6: *Let $\mathfrak{X}, \mathfrak{Y}$ be two Banach space. Let $\{X_i : 1 \leq i \leq n\}$ be a sequence of exchangeable random sets taking values of closed subsets of the Banach space $\mathfrak{X}$ and let $\varphi : c(\mathfrak{X}) \to c(\mathfrak{Y})$ be a $(\mathcal{E}(\mathfrak{X}), \mathcal{E}(\mathfrak{Y}))$-measurable mapping. Then, the sequence $\{\varphi(X_i) : 1 \leq i \leq n\}$ of random sets taking values of closed subsets of the Banach space $\mathfrak{Y}$ is exchangeable.*

Proof: For any permutation π of $\{1, 2, \ldots, n\}$ and the subsets $\{B_1, B_2, \ldots, B_n\}$ of $\mathcal{E}(\mathfrak{Y})$, we have

$$\mathbf{P}\left(\bigcap_{i=1}^{n}[\varphi(X_{\pi(i)}) \in B_i]\right)$$

$$= \mathbf{P}\left(\bigcap_{i=1}^{n}[X_{\pi(i)} \in \varphi^{-1}(B_i)]\right)$$

$$= \mathbf{P}\left(\bigcap_{i=1}^{n}[X_i \in \varphi^{-1}(B_i)]\right)$$

(by the exchangeability of collection $\{X_i, 1 \leq i \leq n\}$ and $\varphi^{-1}(B_i) \in \mathcal{E}(\mathfrak{X})$)

$$= \mathbf{P}\left(\bigcap_{i=1}^{n}[\varphi(X_i) \in B_i]\right).$$

Since then, the lemma is proved. ■

Remark 3.2: Lemma 3.6 is also true if the $(\mathcal{E}(\mathfrak{X}), \mathcal{E}(\mathfrak{Y}))$-measurable function $\varphi : c(\mathfrak{X}) \to c(\mathfrak{Y})$ is replaced by one of the following functions:

(i) the $(\mathcal{E}(\mathfrak{X}), \mathcal{B}(\mathfrak{Y}))$-measurable function $\varphi : c(\mathfrak{X}) \to \mathfrak{Y}$,

(ii) the $(\mathcal{B}(\mathfrak{X}), \mathcal{B}(\mathfrak{Y}))$-measurable function $\varphi : \mathfrak{X} \to \mathfrak{Y}$ (Here, $\{X_i : 1 \leq i \leq n\}$ is a finite sequence of single-valued random variables in the Banach space $\mathfrak{X}$).

It is known that for each random set X, if f is a $\mathcal{F}_X$-measurable selection of X then there exists a measurable function $g : c(\mathfrak{X}) \to \mathfrak{X}$ such that $g(X) = f$. For the collection of random sets $(X_i, i \in I)$, $I_1(X_i, i \in I)$ denotes the family of all the measurable functions $g : c(\mathfrak{X}) \to \mathfrak{X}$ such that $g(X_i) \in S^1_{X_i}(\mathcal{F}_{X_i})$ for every $i \in I$.

The following two theorems will prove the SLLN for triangular array of row-wise exchangeable random sets without CUI and reverse martingale hypotheses. Our first theorem is an extension of a result of Inoue and Taylor [11, Theorem 4.3]. Also, the second theorem extends a result of Taylor and Patterson [21, Theorem 1] to the case of set-valued random variables. To establish these theorems, we provide a new method in building structure of triangular array of selections to prove the '*lim inf*' part of Mosco convergence. Also, as in the proving of [18, Theorem 4.2], to give conclusions, we have to use Lemma 3.5. However, in [18, Theorem 4.2], the SLLN was established under the bounded expectation condition, while in the present paper, this condition is not assumed.

Theorem 3.7: *Let $\{X_{ni} : n \geq 1, 1 \leq i \leq n\}$ be a triangular array of row-wise exchangeable random sets taking values of closed subsets of the separable Banach space $\mathfrak{X}$. Suppose that*

$$\rho_n(f) = Cov\left(f(g_n(\mathrm{co}X_{n1})), f(g_n(\mathrm{co}X_{n2}))\right) \to 0 \text{ as } n \to \infty, \tag{1}$$

for each $f \in \mathfrak{X}^*$ and $g_n \in I_1(\mathrm{co}X_{n1}, \mathrm{co}X_{n2})$, $n \geq 1$. If there exists a nonempty subset X of $\mathfrak{X}$ such that

+) *For each $x \in X$, there exists a sequence $\{f_n : n \geq 1\}$* of *$f_n \in S^1_{X_{n1}}(\mathcal{F}_{X_{n1}})$ such that*

$$\|f_{n1} - \mathbf{E}f_{n1}\| \geq \|f_{(n+1),1} - \mathbf{E}f_{(n+1),1}\| \text{ for each } n \text{ and } f_n \xrightarrow{\mathbf{L}^2} x \text{ as } n \to \infty. \tag{2}$$

+) Foreach $x^* \in \mathfrak{X}^*, s(X_{n1}, x^*) \xrightarrow{\mathbf{L}^2} s(X, x^*)$ *as $n \to \infty$, and*

$$|s(X_{n1}, x^*) - \mathbf{E}s(X_{n1}, x^*)| \geq |s(X_{(n+1),1}, x^*) - \mathbf{E}s(X_{(n+1),1}, x^*)| \text{ for each } n, \tag{3}$$

then

$$M\text{-}\lim \frac{1}{n} cl \sum_{i=1}^{n} X_{ni}(\omega) = \overline{co}X \text{ a.s.}$$

Proof: Let $G_n(\omega) = \frac{1}{n}\mathrm{cl}\sum_{i=1}^{n} X_{ni}(\omega)$. At first, we will show that $\overline{\mathrm{co}}X \subset s\text{-}liG_n(\omega)$ a.s. To do this, we will use Lemma 3.5. For each $x \in \overline{\mathrm{co}}X$ and $\epsilon > 0$, by [2, Lemma 3.6], we can choose $x_1, x_2, \ldots, x_m \in X$ (the elements $x_1, x_2, \ldots, x_m$ only depend on x and ϵ) such that

$$\|\frac{1}{m}\sum_{j=1}^{m} x_j - x\| < \epsilon.$$

Therefore, we only need to show that there exists a triangular array $\{f_{ni} : n \geq 1, 1 \leq i \leq n\}$ of selections of $\{X_{ni}\}$ such that

$$\frac{1}{n}\sum_{i=1}^{n} f_{ni}(\omega) \to \frac{1}{m}\sum_{j=1}^{m} x_j \text{ a.s. as } n \to \infty. \tag{4}$$

Indeed, let $z_m = \frac{1}{m}\sum_{j=1}^{m} x_j$. The statement (4) means that $z_m \in s\text{-}liG_n(\omega)$ a.s. Since the space $\mathfrak{X}$ is separable, there exists a countable dense set $D_{\overline{\mathrm{co}}X}$ of $\overline{\mathrm{co}}X$. For each fixed $x^{(j)} \in D_{\overline{\mathrm{co}}X}$ and for every $\epsilon_k = \frac{1}{k}$ $(k \geq 1)$, by (4), there exists a positive integer m_k, which depends on $x^{(j)}$ and ϵ_k, such that $z_{m_k} \in s\text{-}liG_n(\omega)$ a.s. Therefore, there exists $N_k \in \mathcal{F}$ such that $\mathbf{P}(N_k) = 1$ and $z_{m_k} \in s\text{-}liG_n(\omega)$ for all $\omega \in N_k$. Let $N = \bigcap_{k=1}^{\infty} N_k$, then $\mathbf{P}(N) = 1$. For each $\omega \in N$, it follows from the set $s\text{-}liG_n(\omega)$ is closed, $z_{m_k} \in s\text{-}liG_n(\omega)$ for all k and

$z_{m_k} \to x^{(j)}$ as $k \to \infty$, that $x^{(j)} \in s\text{-}liG_n(\omega)$. This means that $x^{(j)} \in s\text{-}liG_n(\omega)$ a.s., for each $j \geq 1$. Noting that $D_{\overline{\text{co}}X}$ is a countable set, we obtain $D_{\overline{\text{co}}X} \subset s\text{-}liG_n(\omega)$ a.s. Since the set $s\text{-}liG_n(\omega)$ is closed for each ω, by taking the closure of both sides of the above relation, we have $\overline{\text{co}}X \subset s\text{-}liG_n(\omega)$ a.s. Therefore, the statement (4) is proved.

By (2), for each $j \in \{1, 2, \ldots, m\}$, there exists a sequence $\{g_{n1}^{(j)} : n \geq 1\}$ of $g_{n1}^{(j)} \in S^1_{X_{n1}}(\mathcal{F}_{X_{n1}})$ such that $\|g_{n1}^{(j)} - \mathbf{E}g_{n1}^{(j)}\| \geq \|g_{(n+1),1}^{(j)} - \mathbf{E}g_{(n+1),1}^{(j)}\|$ for each n and $g_{n1}^{(j)} \xrightarrow{\mathbf{L}^2} x_j$ as $n \to \infty$.

Since $\{X_{ni} : n \geq 1, 1 \leq i \leq n\}$ is row-wise exchangeable and by virtue of Lemma 3.1(2), it follows that for each $j \in \{1, 2, \ldots, m\}$ and for each $n \geq 1$, there exists a sequence $\{g_{ni}^{(j)} : 1 \leq i \leq n\}$ of $g_{ni}^{(j)} \in S^1_{X_{ni}}(\mathcal{F}_{X_{ni}})$ such that the sequence $\{g_{ni}^{(j)} : 1 \leq i \leq n\}$ is exchangeable. By Lemma 3.6 for the case of single-valued random variables, we get $\mathbf{E}\|g_{ni}^{(j)} - x_j\|^2 = \mathbf{E}\|g_{n1}^{(j)} - x_j\|^2$ for all $i \in \{1, 2, \ldots, n\}$. It follows that $g_{ni}^{(j)} \xrightarrow{\mathbf{L}^2} x_j$ as $n \to \infty$ for each i and j.

Now, we will construct a triangular array $\{f_{ni} : n \geq 1, 1 \leq i \leq n\}$ of $f_{ni} \in S^1_{X_{ni}}(\mathcal{F}_{X_{ni}})$ satisfying (4) as follows:

$$f_{ni}(\omega) := g_{ni}^{(j)}(\omega) \text{ if } i \equiv j \pmod{m}, \text{ where } j \in \{1, 2, \ldots, m\} \text{ and for all } \omega \in \Omega. \tag{5}$$

This means that

$$\Big(f_{ni}\Big)_{n\geq 1, 1\leq i\leq n} = \begin{pmatrix} g_{11}^{(1)} & & & & & \\ g_{21}^{(1)} & g_{22}^{(2)} & & & & \\ \vdots & \vdots & \ddots & & & \\ g_{m,1}^{(1)} & g_{m,2}^{(2)} & \cdots & g_{m,m}^{(m)} & & \\ g_{m+1,1}^{(1)} & g_{m+1,2}^{(2)} & \cdots & g_{m+1,m}^{(m)} & g_{m+1,m+1}^{(1)} & \\ \underbrace{\vdots}_{\substack{1^{\text{st}} \text{ column} \\ \text{of } \{g_{ni}^{(1)}\}}} & \underbrace{\vdots}_{\substack{2^{\text{nd}} \text{ column} \\ \text{of } \{g_{ni}^{(2)}\}}} & \cdots & \underbrace{\vdots}_{\substack{m^{\text{th}} \text{ column} \\ \text{of } \{g_{ni}^{(m)}\}}} & \underbrace{\vdots}_{\substack{(m+1)^{\text{th}} \text{ column} \\ \text{of } \{g_{ni}^{(1)}\}}} & \ddots \end{pmatrix}$$

Then, for each $m \geq 1$ and $j \in \{1, 2, \ldots, m\}$, the array $\{f_{n,(i-1)m+j}\}$ is row-wise exchangeable and

$$f_{n,\,(i-1)m+j} \xrightarrow{\mathbf{L}^2} x_j \text{ as } n \to \infty, \text{ for each } i \geq 1. \tag{6}$$

(Let us note that $\{f_{n,(i-1)m+j}\}$ is not a triangular array of random elements .)

Let $y_{ni} = \mathbf{E}f_{ni}$, $n \geq 1, 1 \leq i \leq n$. If $n = (k-1)m + l$, where $1 \leq l \leq m$, then the following estimations hold:

$$\begin{aligned} &\Big\|\frac{1}{n}\sum_{i=1}^{n} f_{ni}(\omega) - \frac{1}{m}\sum_{j=1}^{m} x_j\Big\| \\ &\quad = \Big\|\frac{1}{n}\sum_{j=1}^{m}\sum_{i=1}^{k} f_{n,(i-1)m+j}(\omega) - \frac{1}{n}\sum_{j=l+1}^{m} f_{n,(k-1)m+j}(\omega) - \frac{1}{m}\sum_{j=1}^{m} x_j\Big\| \end{aligned}$$

$$\leq \frac{k}{n}\sum_{j=1}^{m}\left\|\frac{1}{k}\sum_{i=1}^{k}(f_{n,(i-1)m+j}(\omega)-x_j)\right\|+\frac{1}{n}\sum_{j=l+1}^{m}\|f_{n,(k-1)m+j}(\omega)\|$$

$$+\left(\frac{k}{n}-\frac{1}{m}\right)\|\sum_{j=1}^{m}x_j\|$$

$$\leq \frac{k}{n}\sum_{j=1}^{m}\left\|\frac{1}{k}\sum_{i=1}^{k}(f_{n,(i-1)m+j}(\omega)-y_{n,(i-1)m+j})\right\|+\frac{k}{n}\sum_{j=1}^{m}\frac{1}{k}\sum_{i=1}^{k}\|y_{n,(i-1)m+j}-x_j\|$$

$$+\frac{1}{n}\sum_{j=l+1}^{m}\|f_{n,(k-1)m+j}(\omega)-y_{n,(k-1)m+j}\|+\frac{1}{n}\sum_{j=l+1}^{m}\|y_{n,(k-1)m+j}\|$$

$$+\left(\frac{k}{n}-\frac{1}{m}\right)\|\sum_{j=1}^{m}x_j\|. \tag{7}$$

Let $g_{ni}(\omega)=f_{ni}(\omega)-y_{ni}$, for all $\omega\in\Omega$, $n\geq 1$ and $1\leq i\leq n$. By Lemma 3.6 for the case of single-valued random variables, we get that if a sequence $\{f_k : k\geq 1\}$ of random elements is exchangeable then the sequence $\{f_k+c : k\geq 1\}$ is exchangeable, too (where c is a constant in $\mathfrak{X}$). Therefore, since the array $\{f_{n,(i-1)m+j} : n\geq 1, 1\leq i\leq k\}$ of random elements is row-wise exchangeable, we obtain $\mathbf{E}f_{n,(i-1)m+j}=c$ for all i (here, n and j are fixed). From the above statements, we deduce that the array $\{g_{n,(i-1)m+j} : n\geq 1, 1\leq i\leq k\}$ is row-wise exchangeable, too.

By (6), for each $s=(i-1)m+j$ $(1\leq j\leq m)$, we have $f_{ns}\overset{\mathbf{L}^2}{\to} x_j$ as $n\to\infty$; namely, $\mathbf{E}\|f_{ns}-x_j\|^2\to 0$ as $n\to\infty$, and so

$$\begin{aligned}0\leq \|\mathbf{E}f_{ns}-x_j\|^2 &= \|\mathbf{E}(f_{ns}-x_j)\|^2\\ &\leq \big(\mathbf{E}\|f_{ns}-x_j\|\big)^2 \quad (\text{by } \|\mathbf{E}X\|\leq\mathbf{E}\|X\|)\\ &\leq \mathbf{E}\|f_{ns}-x_j\|^2\to 0 \text{ as } n\to\infty.\\ &\quad(\text{by the inequality for convex function})\end{aligned}$$

Since then, we get

$$0\leq\|g_{ns}\|_2=\|(f_{ns}-x_j)-(\mathbf{E}f_{ns}-x_j)\|_2\leq\|f_{ns}-x_j\|_2+\|\mathbf{E}f_{ns}-x_j\|_2\overset{n\to\infty}{\to}0.$$

This means that

$$g_{ns}\overset{\mathbf{L}^2}{\to}0 \text{ as } n\to\infty. \tag{8}$$

Further, for each $f\in\mathfrak{X}^*$, we have

$$\begin{aligned}\rho_n(f)&=\mathbf{E}\left(f(g_{ni})f(g_{nj})\right)\\ &=\mathbf{E}\left(f(f_{ni}-\mathbf{E}f_{ni}).f(f_{nj}-\mathbf{E}f_{nj})\right)\\ &=\mathbf{E}\big((f(f_{ni})-f(\mathbf{E}f_{ni})).(f(f_{nj})-f(\mathbf{E}f_{nj}))\big)\quad(\text{by } f \text{ is a linear mapping})\\ &=\mathbf{E}\big((f(f_{ni})-\mathbf{E}(f(f_{ni}))).(f(f_{nj})-\mathbf{E}(f(f_{nj})))\big)\end{aligned}$$

(by the definition of expectation of random elements)

$= Cov\left(f(f_{ni}), f(f_{nj})\right)$

$= Cov\left(f(g_n(\text{co}X_{n1})), f(g_n(\text{co}X_{n2}))\right) \to 0$ as $n \to \infty$ for all $i \neq j$ and

$i \equiv j \pmod{m}$. (by (1)) (9)

For each $n = (k-1)m + l$, set $S_n^{(j)}(\omega) = \frac{1}{k}\sum_{i=1}^{k} g_{n,(i-1)m+j}(\omega)$ for all $\omega \in \Omega$. For each $j \in \{1, 2, \ldots, m\}$, the sequence $\{S_n^{(j)} : n \geq 1\}$ of random elements is divided into m subsequences $\{S_{(k-1)m+l}^{(j)} : k \geq 1\}, l \in \{1, 2, \ldots, m\}$.

From the above statements, the triangular array $\{g_{(k-1)m+l,(i-1)m+j} : k \geq 1, 1 \leq i \leq k\}$ satisfies all the conditions of Lemma 3.2 for each $l, j \in \{1, 2, \ldots, m\}$. Applying this lemma, we obtain

$$S_{(k-1)m+l}^{(j)}(\omega) = \frac{1}{k}\sum_{i=1}^{k} g_{(k-1)m+l,(i-1)m+j}(\omega) \to 0 \text{ a.s. as } k \to \infty, \tag{10}$$

for each $l, j \in \{1, 2, \ldots, m\}$.

It is equivalent to

$$S_n^{(j)}(\omega) = \frac{1}{k}\sum_{i=1}^{k} g_{n,(i-1)m+j}(\omega) \to 0 \text{ a.s. as } n \to \infty, \text{ for each } j \in \{1, 2, \ldots, m\}. \tag{11}$$

For each $n \geq 1$ and $j \in \{1, 2, \ldots, m\}$, we set $V_n^{(j)} = \frac{1}{k}\sum_{i=1}^{k} \|y_{n,(i-1)m+j} - x_j\|$. The sequence $\{V_n^{(j)} : n \geq 1\}$ of real numbers is divided into m subsequences $\{V_{(k-1)m+l}^{(j)} : k \geq 1\}, l \in \{1, 2, \ldots, m\}$.

For each $l, j \in \{1, 2, \ldots, m\}$, we put

$$z_{ki}^{(l,j)} = \|y_{(k-1)m+l,(i-1)m+j} - x_j\|.$$

For each $j \in \{1, 2, \ldots, m\}$, by the assumption that the array $\{f_{n,(i-1)m+j} : n \geq 1, 1 \leq i \leq k\}$ is row-wise exchangeable and converges in the second mean to x_j as $n \to \infty$ for each column, we get that the elements of this array have bounded expectations. Therefore,

$$|z_{ki}^{(l,j)}| \leq \|\mathbf{E}f_{(k-1)m+l,(i-1)m+j}\| + \|x_j\| \leq C + \|x_j\|, \tag{12}$$

for all $k \geq 1, 1 \leq i \leq k$.

Since the convergence in $\mathbf{L}^2$ implies the convergence in $\mathbf{L}^1$ and by (6), we have that $z_{ki}^{(l,j)} \to 0$ as $k \to \infty$, for each $i \geq 1$. Since then, $z_{ki}^{(l,j)} \to 0$ as $i \to \infty$.

Combining this with (12), we have that for each $l, j \in \{1, 2, \ldots, m\}$, the triangular array $\{z_{ki}^{(l,j)} : k \geq 1, 1 \leq i \leq k\}$ of real numbers satisfies all the conditions of Lemma 3.5. Applying this lemma, we obtain

$$V^{(j)}_{(k-1)m+l} = \frac{1}{k}\sum_{i=1}^{k} \|y_{(k-1)m+l,(i-1)m+j} - x_j\| \to 0 \text{ as } k \to \infty \text{ for each } l, j \in \{1, 2, \dots, m\}.$$

Hence,

$$V^{(j)}_n = \frac{1}{k}\sum_{i=1}^{k} \|y_{n,(i-1)m+j} - x_j\| \to 0 \text{ as } n \to \infty. \tag{13}$$

By (11), we have

$$\begin{aligned}
&\frac{1}{n}\sum_{j=l+1}^{m} \|f_{n,(k-1)m+j}(\omega) - y_{n,(k-1)m+j}\| \\
&= \frac{1}{n}\sum_{j=l+1}^{m} \|g_{n,(k-1)m+j}(\omega)\| \\
&= \frac{k}{n}\sum_{j=l+1}^{m} \left\| \frac{1}{k}\sum_{i=1}^{k} g_{n,(i-1)m+j}(\omega) - \left(\frac{k-1}{k}\right)\frac{1}{k-1}\sum_{i=1}^{k-1} g_{n,(i-1)m+j}(\omega) \right\| \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{14}$$

Similarly, by (13), we obtain

$$\begin{aligned}
&\frac{1}{n}\sum_{j=l+1}^{m} \|y_{n,(k-1)m+j}\| \\
&\le \frac{1}{n}\sum_{j=l+1}^{m} \|y_{n,(k-1)m+j} - x_j\| + \frac{1}{n}\sum_{j=l+1}^{m} \|x_j\| \\
&= \frac{k}{n}\sum_{j=l+1}^{m} \left(\frac{1}{k}\sum_{i=1}^{k} \|y_{n,(i-1)m+j} - x_j\| - \left(\frac{k-1}{k}\right)\frac{1}{k-1}\sum_{i=1}^{k-1} \|y_{n,(i-1)m+j} - x_j\| \right) \\
&\quad + \frac{1}{n}\sum_{j=l+1}^{m} \|x_j\| \to 0 \ \text{ as } n \to \infty.
\end{aligned} \tag{15}$$

We also have $(\frac{k}{n} - \frac{1}{m}) \to 0$ as $n \to \infty$. Therefore, combining (7), (11), (13), (14) and (15), we get

$$\frac{1}{n}\sum_{i=1}^{n} f_{ni}(\omega) - \frac{1}{m}\sum_{j=1}^{m} x_j \to 0 \text{ a.s. as } n \to \infty.$$

This yields $\frac{1}{m}\sum_{j=1}^{m} x_j \in s\text{-}liG_n(\omega)$ a.s. Hence $\overline{\text{co}}X \subset s\text{-}liG_n(\omega)$ a.s.

Thus, in the above proving, the triangular array $\{g_{ni} : n \ge 1, 1 \le i \le n\}$ of random elements has been divided into m^2 triangular sub-arrays $\{g_{(k-1)m+l,(i-1)m+j} : k \ge 1, 1 \le i \le k\}$. Also, for each $j \in \{1, 2, \dots, m\}$, the array $\{\|y_{n,(i-1)m+j} - x_j\| : n \ge 1, 1 \le i \le k\}$

of real numbers has been divided into m triangular sub-arrays $\{z_{ki}^{(l,j)} : k \geq 1, 1 \leq i \leq k\}$, $l \in \{1, 2, \ldots, m\}$. By using Lemma 3.2 (resp. Lemma 3.5) for each above triangular sub-array of random elements (resp. of real numbers), we obtain the '*lim inf*' path of the Mosco convergence.

Next, let $\{x_j : j \geq 1\}$ be a dense sequence of $\mathfrak{X} \setminus \overline{\text{co}}X$. By the separation theorem, there exists a sequence $\{x_j^* : j \geq 1\}$ in $\mathfrak{X}^*$ with $\|x_j^*\| = 1$ such that

$$\langle x_j, x_j^* \rangle - d(x_j, \overline{\text{co}}X) \geq s(\overline{\text{co}}X, x_j^*), \text{ for every } j \geq 1. \tag{16}$$

Then $x \in \overline{\text{co}}X$ if and only if $\langle x, x_j^* \rangle \leq s(\overline{\text{co}}X, x_j^*)$ for every $j \geq 1$.

Note that the function $X \mapsto s(X, x_j^*)$ of $c(\mathfrak{X})$ into $(-\infty, \infty]$ is $(\mathcal{E}, \mathcal{B}(\mathbf{R}))$-measurable.

Using the above statement, the inequality (16), the hypotheses of this theorem and Lemma 3.6, we have that $\{s(X_{ni}, x_j^*) : n \geq 1, 1 \leq i \leq n\}$ is a triangular array of row-wise exchangeable random variables in $\mathcal{L}^1$, for each $j \geq 1$. Set $h_{ni}^{(j)} = s(X_{ni}, x_j^*) - \mathbf{E}(s(X_{ni}, x_j^*))$. Then, $\{h_{ni}^{(j)} : n \geq 1, 1 \leq i \leq n\}$ is the triangular array of row-wise exchangeable random variables.

By the condition (3), using the arguments as in the proof of (8), we get $h_{n1}^{(j)} \xrightarrow{\mathbf{L}^2} 0$ as $n \to \infty$. It implies that $h_{ni}^{(j)} \xrightarrow{\mathbf{L}^2} 0$ as $n \to \infty$, for each $i \geq 1$.

By the condition (1), we have that $\rho_n(f) = \mathbf{E}(h_{ni}^{(j)} h_{nk}^{(j)}) \to 0$ as $n \to \infty$ for all $i \neq k$.

From the above statements, we get that the triangular array $\{h_{ni}^{(j)} : n \geq 1, 1 \leq i \leq n\}$ satisfies all the conditions of Lemma 3.2 for real-valued random variables, for each $j \geq 1$. Then, applying this lemma, we have

$$\frac{1}{n} \sum_{i=1}^{n} h_{ni}^{(j)}(\omega) \to 0 \text{ a.s. as } n \to \infty, \text{ for every } j \geq 1.$$

This means that

$$\frac{1}{n} \sum_{i=1}^{n} s(X_{ni}, x_j^*) - \frac{1}{n} \sum_{i=1}^{n} \mathbf{E}(s(X_{ni}, x_j^*)) \to 0 \text{ a.s. as } n \to \infty, \text{ for every } j \geq 1.$$

Moreover, by (3) and (16), we get

$$\mathbf{E}s(X_{ni}, x_j^*) = s(\text{cl}\mathbf{E}(X_{ni}), x_j^*) \to s(X, x_j^*) < \infty \text{ as } n \to \infty \text{ for every } i, j \geq 1.$$

Therefore, for each i and j, the sequence $\{s(X_{ni}, x_j^*) : n \geq 1\}$ has bounded expectation.

Since $\mathbf{E}s(X_{ni}, x_j^*) = \mathbf{E}s(X_{n1}, x_j^*)$ for all $i \in \{1, 2, \ldots, n\}$, the triangular array $\{s(X_{ni}, x_j^*) : n \geq 1, 1 \leq i \leq n\}$ has bounded expectation.

Since then, by applying Lemma 3.5, we have

$$\frac{1}{n} \sum_{i=1}^{n} \mathbf{E}(s(X_{ni}, x_j^*)) \to s(X, x_j^*) \text{ as } n \to \infty, \text{ for every } j \geq 1.$$

Consequently, for each $j \geq 1$, $s(G_n(\omega), x_j^*) \to s(X, x_j^*)$ a.s. as $n \to \infty$. Namely, there exists $N \in \mathcal{F}$, $\mathbf{P}(N) = 0$ such that for each $\omega \in \Omega \backslash N, j \geq 1$, $s(G_n(\omega), x_j^*) \to s(X, x_j^*)$ as $n \to \infty$.

For each $\omega \in \Omega \backslash N$, if $x \in w\text{-}lsG_n(\omega)$ then $x_k \xrightarrow{w} x$ as $k \to \infty$, where $x_k \in G_{n(k)}(\omega)$. Hence,

$$\langle x, x_j^* \rangle = \lim_{k\to\infty} \langle x_k, x_j^* \rangle \le \lim_{k\to\infty} s(G_{n(k)}(\omega), x_j^*) = s(X, x_j^*) = s(\overline{\text{co}}X, x_j^*), \text{ for every } j \ge 1.$$

This implies that $x \in \overline{\text{co}}X$. Thus, $w\text{-}ls\frac{1}{n}\text{cl}\sum_{i=1}^n X_{ni}(\omega) \subset \overline{\text{co}}X$ a.s. ■

By putting $X_{ni}(\omega) = X_i(\omega)$ for every $n \ge 1$, $1 \le i \le n$ and $\omega \in \Omega$, and applying Theorem 3.7 for the triangular array $\{X_{ni} : n \ge 1, 1 \le i \le n\}$, we get

Corollary 3.8 (Inoue and Taylor [11, Theorem 4.3]): *Let $\{X_n : n \ge 1\}$ be an infinite sequence of exchangeable random sets in $c(\mathfrak{X})$. If $\mathbf{E}\|X_1\| < \infty$ and $Cov\{f(g(\text{co}X_1)), f(g(\text{co}X_2))\} = 0$ for each $f \in \mathfrak{X}^*$, then*

$$\frac{1}{n}\sum_{k=1}^{n} X_k \to \overline{\text{co}}\mathbf{E}X_1 \quad \textit{in Mosco sense,}$$

where $g \in I_1(\text{co}X_1, \text{co}X_2)$.

In Lemma 3.2, for each $k \ge 1$, the sequence $\{X_{nk} : n \ge 1\}$ converges in the second mean to $X_{\infty k}$, where $X_{\infty k}$ is a random element. However, in Theorem 3.7, the condition

$$f_n \xrightarrow{\mathbf{L}^2} x \text{ as } n \to \infty \tag{I}$$

cannot be replaced by the weaker one

$$f_n \xrightarrow{\mathbf{L}^2} f \text{ as } n \to \infty, \text{ where } f \text{ is a random element satisfying } \mathbf{E}f = x. \tag{II}$$

Because, in the proof of Theorem 3.7, if the condition (I) is replaced by the condition (II), then the statement $g_{n1}^{(j)} \xrightarrow{\mathbf{L}^2} f_1^{(j)}$ as $n \to \infty$ can not imply $g_{ni}^{(j)} \xrightarrow{\mathbf{L}^2} f_i^{(j)}$ as $n \to \infty$ with $\mathbf{E}f_i^{(j)} = \mathbf{E}f_1^{(j)} = x_j$ for every $i \in \{1, 2, \ldots, n\}$.

The example below shows sequences $\{f_n : n \ge 1\}$ and $\{g_n : n \ge 1\}$ such that $\{f_n : n \ge 1\}$ converges in the second mean to a random variable f and $\{g_n, f_n\}$ is exchangeable for every $n \ge 1$, but $\{g_n : n \ge 1\}$ does not converge in the second mean.

Example 1: Let f be a random variable in $\mathbf{R}$, which is defined as follows:

$$\mathbf{P}(f = 1) = \mathbf{P}(f = -1) = \frac{1}{2}.$$

Putting $f_n := f$ for each $n \ge 1$, then $f_n \xrightarrow{\mathbf{L}^2} f$ as $n \to \infty$.

Define the random variables

$$g_n = \begin{cases} f, & \text{if } n \text{ is an even positive integer,} \\ -f, & \text{otherwise.} \end{cases}$$

For every $x_1, x_2 \in \mathbf{R}$, we have

$$\mathbf{P}(f < x_1, -f < x_2) = \mathbf{P}(1 < x_1, -1 < x_2) + \mathbf{P}(-1 < x_1, 1 < x_2),$$

and

$$\mathbf{P}(-f < x_1, f < x_2) = \mathbf{P}(-1 < x_1, 1 < x_2) + \mathbf{P}(1 < x_1, -1 < x_2).$$

Combining the above two equalities, we obtain

$$\mathbf{P}(f < x_1, -f < x_2) = \mathbf{P}(-f < x_1, f < x_2), \text{ for all } x_1, x_2 \in \mathbf{R}.$$

Hence, f and $-f$ are exchangeable.

Since then, $\{g_n, f_n\}$ is exchangeable for all positive integers n.

For every $k \in \mathbf{N}$, we get

$$\mathbf{P}(|g_{2k+1} - g_{2k}| > 1) = \mathbf{P}(|2f| > 1) = \mathbf{P}(|f| > \frac{1}{2}) = 1.$$

Hence, this sequence does not converge in probability. This implies that $\{g_n : n \geq 1\}$ does not converge in the second mean.

Theorem 3.9: *Let $\{X_{ni} : n \geq 1, 1 \leq i \leq n\}$ be a triangular array of row-wise exchangeable random sets in separable Banach space $\mathfrak{X}$.*

Assume that there exists a triangular array $\{f_{ni} : n \geq 1, 1 \leq i \leq n\}$ of $f_{ni} \in S^1_{X_{ni}}(\mathcal{F}_{X_{ni}})$ such that $\{f_{ni}\}$ is row-wise exchangeable, $f_{nk} \xrightarrow{\mathbf{L}^2} f_{\infty k}$ as $n \to \infty$ for each

$$k \geq 1 \text{ and } \|f_{n1} - f_{\infty 1}\| \geq \|f_{(n+1),1} - f_{\infty 1}\| \text{ for all } n. \tag{17}$$

Suppose that there exists a nonempty subset X of $\mathfrak{X}$ such that:

$$\begin{aligned} &+) \text{ For each } x \in X, \mathbf{E}\big(f(g_{n1}(\mathrm{co}X_{n1}) - x).f(g_{n2}(\mathrm{co}X_{n2}) - x)\big) \to 0 \text{ as } n \to \infty, \\ &\quad \text{for all } f \in \mathfrak{X}^* \text{ and } g_{ni} \in I_1(\mathrm{co}X_{ni}),\ n \geq 1,\ i \in \{1,2\}. \end{aligned} \tag{18}$$

$$\begin{aligned} &+) \text{ For each } x^* \in \mathfrak{X}^* \text{ and } k \geq 1, s(X_{nk}, x^*) \xrightarrow{\mathbf{L}^2} S^{(x^*)}_{\infty k} \text{ as } n \to \infty, \text{ and} \\ &\quad |s(X_{n1}, x^*) - S^{(x^*)}_{\infty 1}| \geq |s(X_{(n+1),1}, x^*) - S^{(x^*)}_{\infty 1}| \text{ for all } n, \end{aligned} \tag{19}$$

then

$$M\text{-}\lim \frac{1}{n}\, cl \sum_{i=1}^{n} X_{ni}(\omega) = \overline{\mathrm{co}}X \text{ a.s.}$$

Proof: By the arguments as in the proof of Theorem 3.7, for each $x \in \overline{\mathrm{co}}X$ and $\epsilon > 0$, there exist $x_1, x_2, \ldots, x_m \in X$ such that

$$\|\frac{1}{m}\sum_{j=1}^{m} x_j - x\| < \epsilon.$$

To prove the '*lim inf*' part $\overline{\mathrm{co}}X \subset s\text{-}liG_n(\omega)$ a.s. in the Mosco convergence, we need to show that $\frac{1}{m}\sum_{j=1}^{m} x_j \in s\text{-}liG_n(\omega)$ a.s.

If $n=(k-1)m+l$, where $1\leq l\leq m$, then the following estimations hold:

$$
\begin{aligned}
&\|\frac{1}{n}\sum_{i=1}^{n}f_{ni}(\omega)-\frac{1}{m}\sum_{j=1}^{m}x_j\|\\
&\qquad=\|\frac{1}{n}\sum_{j=1}^{m}\sum_{i=1}^{k}f_{n,(i-1)m+j}(\omega)-\frac{1}{n}\sum_{j=l+1}^{m}f_{n,(k-1)m+j}(\omega)-\frac{1}{m}\sum_{j=1}^{m}x_j\|\\
&\qquad\leq\frac{k}{n}\sum_{j=1}^{m}\left\|\frac{1}{k}\sum_{i=1}^{k}(f_{n,(i-1)m+j}(\omega)-x_j)\right\|+\frac{1}{n}\sum_{j=l+1}^{m}\|f_{n,(k-1)m+j}(\omega)\|\\
&\qquad+\left(\frac{k}{n}-\frac{1}{m}\right)\|\sum_{j=1}^{m}x_j\|.
\end{aligned}
\tag{20}
$$

By (18), we have

$$
\rho_n(f,x_j)=\mathbf{E}\big(f(f_{ns}-x_j).f(f_{nk}-x_j)\big)\to 0\text{ as }n\to\infty,\tag{21}
$$

for all $s\neq k, f\in\mathfrak{X}^*$, for each $j\in\{1,2,\dots,m\}$.

By the arguments as in the proof of Theorem 3.7, we have that for each $j\in\{1,2,\dots,m\}$, the row-wise exchangeability of array $\{f_{n,(i-1)m+j}\}$ of random elements implies the row-wise exchangeability of array $\{f_{n,(i-1)m+j}-x_j\}$.

For each $n=(k-1)m+l$, we put $S_n^{(j)}(\omega)=\frac{1}{k}\sum_{i=1}^{k}(f_{n,(i-1)m+j}(\omega)-x_j)$ for all $\omega\in\Omega$. For each $j\in\{1,2,\dots,m\}$, the sequence $\{S_n^{(j)}:n\geq 1\}$ of random elements is divided into m subsequences $\{S_{(k-1)m+l}^{(j)}:k\geq 1\}, l\in\{1,2,\dots,m\}$.

By (17), we get that $f_{n,(i-1)m+j}-x_j\xrightarrow{\mathbf{L}^2}f_{\infty,(i-1)m+j}-x_j$ as $n\to\infty$ for each $i\geq 1, j\in\{1,2,\dots,m\}$ and $\|(f_{n,(i-1)m+j}-x_j)-(f_{\infty,(i-1)m+j}-x_j)\|=\|f_{n,(i-1)m+j}-f_{\infty,(i-1)m+j}\|$.

Therefore, the triangular array $\{f_{(k-1)m+l,(i-1)m+j}-x_j:k\geq 1, 1\leq i\leq k\}$ of random elements satisfies all the conditions of Lemma 3.2 for each $l,j\in\{1,2,\dots,m\}$ and we have

$$
S_{(k-1)m+l}^{(j)}(\omega)=\frac{1}{k}\sum_{i=1}^{k}(f_{(k-1)m+l,(i-1)m+j}(\omega)-x_j)\to 0\text{ a.s. as }k\to\infty,\tag{22}
$$

for each $l,j\in\{1,2,\dots,m\}$.

It implies that

$$
S_n^{(j)}(\omega)\to 0\text{ a.s. as }n\to\infty,\ \text{ for each }j\in\{1,2,\dots,m\}.\tag{23}
$$

Since $n\to\infty$ implies $k\to\infty$, by (23), we obtain

$$
\begin{aligned}
\frac{1}{n}\sum_{j=l+1}^{m}\|f_{n,(k-1)m+j}(\omega)\|&=\frac{k}{n}\sum_{j=l+1}^{m}\left\|\frac{1}{k}\sum_{i=1}^{k}(f_{n,(i-1)m+j}(\omega)-x_j)\right.\\
&\qquad\left.-(\frac{k-1}{k})\frac{1}{k-1}\sum_{i=1}^{k-1}(f_{n,(i-1)m+j}(\omega)-x_j)+\frac{1}{k}x_j\right\|
\end{aligned}
$$

$$\to 0 \text{ as } n \to \infty. \tag{24}$$

Since then, combining (20), (23) and (24), we have

$$\frac{1}{n}\sum_{i=1}^{n} f_{ni}(\omega) - \frac{1}{m}\sum_{j=1}^{m} x_j \to 0 \text{ a.s. as } n \to \infty.$$

Hence, $\frac{1}{m}\sum_{j=1}^{m} x_j \in s\text{-}liG_n(\omega)$ a.s. Let $\{x_j^* : j \geq 1\}$ be as in the proof of Theorem 3.7 taken for $\overline{\text{co}}X$. To prove the '*lim sup*' path $w\text{-}ls\frac{1}{n}\text{cl}\sum_{i=1}^{n} X_{ni}(\omega) \subset \overline{\text{co}}X$ a.s. in the Mosco convergence, we argue as in the proof of Theorem 3.7. Detail, for each $j \geq 1$, set $h_{ni}^{(j)}(\omega) = s(X_{ni}(\omega), x_j^*) - s(X, x_j^*)$. By using Lemma 3.2, we obtain

$$\frac{1}{n}\sum_{i=1}^{n} h_{ni}^{(j)}(\omega) \to 0 \text{ a.s. as } n \to \infty, \text{ for each } j \geq 1.$$

This means that

$$\frac{1}{n}\sum_{i=1}^{n} s(X_{ni}, x_j^*) - s(X, x_j^*) \to 0 \text{ a.s. as } n \to \infty, \text{ for each } j \geq 1.$$

It is equivalent to $s(G_n(\omega), x_j^*) \to s(X, x_j^*)$ a.s., for each $j \geq 1$. Thus, we obtain the desired conclusion. ■

Remark 3.3: Let us note that the conclusion of Theorem 3.9 will be only $\overline{\text{co}}X \subset s\text{-}liG_n(\omega)$ a.s., if the condition (18) is not assumed. At this point, Theorem 3.9 extends the result of Taylor and Patterson (21, Theorem 1) for multivalued random variables. Indeed, suppose that the triangular array $\{X_{ni} : n \geq 1, 1 \leq i \leq n\}$ of random elements in a separable Banach space satisfies all the conditions of Lemma 3.2. We can check that the triangular array $\{X_{ni} : n \geq 1, 1 \leq i \leq n\}$ satisfies all the conditions of Theorem 3.9 without the condition (19) for single-valued random variables case with $X = \{0\}$. By using Theorem 3.9, we obtain the SLLN as in Lemma 3.2.

Next, we will establish a multivalued SLLN for triangular array of row-wise exchangeable random sets with CUI and reverse martingale conditions. To do this, we need the following lemma.

Lemma 3.10: *Let $\{f_n : n \geq 1\}$ be a sequence of random elements in $\mathcal{L}^1(\mathfrak{X})$. Suppose that the sequence $\{f_n : n \geq 1\}$ is CUI and $\mathbf{E}f_n \to x$ as $n \to \infty$, where x is an element of $\mathfrak{X}$. Then, the sequence $\{f_n - \mathbf{E}f_n : n \geq 1\}$ is CUI.*

Proof: Given $\epsilon > 0$, there exists a compact subset K_1 of $\mathfrak{X}$ such that

$$\sup_n \mathbf{E}\|f_n I_{[f_n \notin K_1]}\| < \frac{\epsilon}{2}. \tag{25}$$

We put $K_2 = \text{cl}\{\mathbf{E}f_n \mid n \geq 1\}$. By the convergence of the sequence $\{\mathbf{E}f_n : n \geq 1\}$, we have that K_2 is a compact subset of $\mathfrak{X}$. We set $K = K_1 - K_2$, then K is a compact subset.

Now, we will show that

$$K_1 \subset \mathbf{E}f_n + K, \text{ for every } n \geq 1. \tag{26}$$

Indeed, for each $k_1 \in K_1$, it follows from $K = K_1 - K_2$ and $\mathbf{E}f_n \in K_2$ that $k_n = k_1 - \mathbf{E}f_n \in K$. This yields $k_1 = \mathbf{E}f_n + k_n \in \mathbf{E}f_n + K$. Thus, (26) is proved.

By (26), we get

$$[f_n \notin \mathbf{E}f_n + K] \subset [f_n \notin K_1], \text{ for every } n \geq 1. \tag{27}$$

Next, we have that

$$\begin{aligned}\mathbf{E}\|(f_n - \mathbf{E}f_n)I_{[f_n-\mathbf{E}f_n\notin K]}\| &\leq 2\mathbf{E}\|f_n I_{[f_n-\mathbf{E}f_n\notin K]}\| \\ &= 2\mathbf{E}\|f_n I_{[f_n\notin \mathbf{E}f_n+K]}\| \\ &\leq 2\mathbf{E}\|f_n I_{[f_n\notin K_1]}\|, \text{ for every } n \geq 1 \text{ (by (27))}.\end{aligned}$$

By (25), we obtain

$$\sup_n \mathbf{E}\|(f_n - \mathbf{E}f_n)I_{[f_n-\mathbf{E}f_n\notin K]}\| < \epsilon.$$

The lemma is proved completely. ■

Theorem 3.11: *Let $\{X_{ni} : n \geq 1, 1 \leq i \leq n\}$ be a triangular array of row-wise exchangeable random sets in the separable Banach space $\mathfrak{X}$. If for every $f \in \mathfrak{X}^*$,*

+) *the sequence $\{g_n(X_{n1}) : n \geq 1\}$ is CUI, with $g_n \in I_1(\mathrm{co}X_{n1})$,* (28)

+) $\{\mathbf{E}(g_n(\mathrm{co}X_{n1})|G(n,m,j)) : n \geq 1\}$ *is a reverse martingale, for each $m \geq 1$, $j \in \{1, 2, \ldots, m\}$, where $I(n,m,j) = \{(k-1)m + j | 1 \leq (k-1)m + j \leq n, k \in \mathbf{N}\}$, $g_n \in I_1(\mathrm{co}X_{n1})$ and* $G(n,m,j) = \sigma\{\sum_{k\in I(n,m,j)} g_n(\mathrm{co}X_{nk}),$

$$\sum_{k\in I(n+1,m,j)} g_{n+1}(\mathrm{co}X_{n+1,k}), \ldots\}, \tag{29}$$

+) *Cov* $\big(f(g_n(\mathrm{co}X_{n1})), f(g_n(\mathrm{co}X_{n2}))\big) \to 0$ *as* $n \to \infty$, *with* $g_n \in I_1(\mathrm{co}X_{n1}, \mathrm{co}X_{n2})$, (30)

+) *Var*$(f(g_n(\mathrm{co}X_{n1}))) = \mathrm{o}(n)$, *with* $g_n \in I_1(\mathrm{co}X_{n1})$, (31)

+) *there exists a set $X \in c(\mathfrak{X})$ such that*

$$X \subset s\text{-}li\,\mathrm{cl}\mathbf{E}(X_{n1}, \mathcal{F}_{X_{n1}}), \tag{32}$$

$$\limsup s(\mathrm{cl}\mathbf{E}X_{n1}, x^*) \leq s(X, x^*) \text{ for all } x^* \in \mathfrak{X}^*, \tag{33}$$

then

$$M\text{-}\lim \frac{1}{n}\, cl \sum_{i=1}^{n} X_{ni}(\omega) = \overline{co}X \text{ a.s.}$$

Proof: As in the proof of Theorem 3.7, to prove the '*lim inf*' part in the Mosco convergence, we need to show that there exists a triangular array $\{f_{ni} : n \geq 1, 1 \leq i \leq n\}$ of $f_{ni} \in S^1_{X_{ni}}(\mathcal{F})$ such that

$$\frac{1}{n}\sum_{i=1}^{n} f_{ni}(\omega) \to \frac{1}{m}\sum_{j=1}^{m} x_j \text{ a.s. as } n \to \infty.$$

By (32), for each $j \in \{1, 2, \ldots, m\}$, there exists a sequence $\{g^{(j)}_{n1} : n \geq 1\}$ of $g^{(j)}_{n1} \in S^1_{X_{n1}}(\mathcal{F}_{X_{n1}})$ such that $\mathbf{E}g^{(j)}_{n1} \to x_j$ as $n \to \infty$.

Since the triangular array $\{X_{ni} : n \geq 1, 1 \leq i \leq n\}$ of random sets is row-wise exchangeable and by Lemma 3.1(2), it follows that for each $j \in \{1, 2, \ldots, m\}$ and for each $n \geq 1$, there exists a sequence $\{g^{(j)}_{ni} : 1 \leq i \leq n\}$ of $g^{(j)}_{ni} \in S^1_{X_{ni}}(\mathcal{F}_{X_{ni}})$ such that $\{g^{(j)}_{ni} : 1 \leq i \leq n\}$ is exchangeable. Since then, we get $\mathbf{E}g^{(j)}_{ni} = \mathbf{E}g^{(j)}_{n1}$ for all $i \in \{1, 2, \ldots, n\}$. It follows that $\mathbf{E}g^{(j)}_{ni} \to x_j$ as $n \to \infty$ for each i and j.

Next, we define the triangular array $\{f_{ni} : n \geq 1, 1 \leq i \leq n\}$ of $f_{ni} \in S^1_{X_{ni}}(\mathcal{F}_{X_{ni}})$ as follows:

$$f_{ni}(\omega) := g^{(j)}_{ni}(\omega) \text{ if } i \equiv j \pmod{m}, \text{ where } j \in \{1, 2, \ldots, m\} \text{ and for all } \omega \in \Omega.$$

Let $y_{ni} = \mathbf{E}f_{ni}$ and $g_{ni}(\omega) = f_{ni}(\omega) - y_{ni}$, where $n \geq 1, 1 \leq i \leq n, \omega \in \Omega$.

Let $n = (k-1)m + l$, $1 \leq l \leq m$. By the arguments as in the proof of Theorem 3.7, we get that the array $\{g_{n,(i-1)m+j} : n \geq 1, 1 \leq i \leq k\}$ of random elements is row-wise exchangeable, for each $j \in \{1, 2, \ldots, m\}$.

By the arguments as in (9), for every $f \in \mathfrak{X}^*$,

$$\begin{aligned}
&\mathbf{E}\left(f(g_{ni})f(g_{nj})\right)\\
&\quad = Cov\left(f(f_{ni}), f(f_{nj})\right)\\
&\quad = Cov\left(f(g(\mathrm{co}X_{ni})), f(g(\mathrm{co}X_{nj}))\right)\\
&\quad = Cov\left(f(g_n(\mathrm{co}X_{n1})), f(g_n(\mathrm{co}X_{n2}))\right) \to 0 \text{ as } n \to \infty \text{ for all } i \neq j \text{ and } i \equiv j (\mathrm{mod}\ m).\\
&\qquad \text{(by the condition (30))} \qquad (34)
\end{aligned}$$

Similarly, by the condition (31), we have that

$$\mathbf{E}\left(f^2(g_{n1})\right) = Var(f(g_n(\mathrm{co}X_{n1}))) = \mathrm{o}(n), \text{ for all } f \in \mathfrak{X}^*.$$

As in the proof of Theorem 3.7, for each $n = (k-1)m + l$ and for each $j \in \{1, 2, \ldots, m\}$, set $S^{(j)}_n(\omega) = \frac{1}{k}\sum_{i=1}^{k} g_{n,(i-1)m+j}(\omega)$ for all $\omega \in \Omega$. For each $j \in \{1, 2, \ldots, m\}$, the sequence $\{S^{(j)}_n : n \geq 1\}$ of random elements is divided into m subsequences $\{S^{(j)}_{(k-1)m+l} : k \geq 1\}$, $l \in \{1, 2, \ldots, m\}$.

Since the triangular array $\{g^{(j)}_{ni} : n \geq 1, 1 \leq i \leq n\}$ of random elements is row-wise exchangeable and by the condition (29), the sequence $\{\mathbf{E}(g^{(j)}_{nj} | G(n, m, j)) : n \geq 1\}$ of random elements is a reverse martingale, for each $m \geq 1$ and $j \in \{1, 2, \ldots, m\}$.

For each $m \geq 1$ and $l, j \in \{1, 2, \ldots, m\}$, set

$$G_k^{(l,j)}(f) = \sigma\{\sum_{i=1}^{k} f_{(k-1)m+l,(i-1)m+j}, \sum_{i=1}^{k+1} f_{km+l,(i-1)m+j}, \ldots\}.$$

For each $m \geq 1$ and $l, j \in \{1, 2, \ldots, m\}$, the sequence $\{\mathbf{E}(g^{(j)}_{(k-1)m+l,j} | G((k-1)m + l, m, j)) : k \geq 1\}$ of random elements is a reverse martingale, since every subsequence of a reverse martingale sequence is also a reverse martingale.

Next, we will show that the sequence $\{\mathbf{E}(g^{(j)}_{(k-1)m+l,j} | G_k^{(l,j)}(f)) : k \geq 1\}$ of random elements is a reverse martingale.

Indeed, it suffices to show that

$$\mathbf{E}(\mathbf{E}(g^{(j)}_{(k-1)m+l,j} | G_k^{(l,j)}(f)) | G_{k+1}^{(l,j)}(f)) = \mathbf{E}(g^{(j)}_{km+l,j} | G_{k+1}^{(l,j)}(f)) \text{ a.s.,}$$

which is equivalent to

$$\mathbf{E}(g^{(j)}_{(k-1)m+l,j} | G_{k+1}^{(l,j)}(f)) = \mathbf{E}(g^{(j)}_{km+l,j} | G_{k+1}^{(l,j)}(f)) \text{ a.s.}$$
$$\text{(by the smoothing lemma with } G_{k+1}^{(l,j)}(f) \subset G_k^{(l,j)}(f)). \tag{35}$$

Since the sequence $\{\mathbf{E}(g^{(j)}_{(k-1)m+l,j} | G((k-1)m + l, m, j)) : k \geq 1\}$ of random elements is a reverse martingale and by the similar argument, we obtain

$$\mathbf{E}(g^{(j)}_{(k-1)m+l,j} | G(km + l, m, j)) = \mathbf{E}(g^{(j)}_{km+l,j} | G(km + l, m, j)) \text{ a.s.} \tag{36}$$

We have that

$$\begin{aligned}
\mathbf{E}(g^{(j)}_{km+l,j} | G_{k+1}^{(l,j)}(f)) &= \mathbf{E}(\mathbf{E}(g^{(j)}_{km+l,j} | G(km + l, m, j)) | G_{k+1}^{(l,j)}(f)) \text{ a.s.} \\
&\quad \text{(by the smoothing lemma with } G_{k+1}^{(l,j)}(f) \subset G(km + l, m, j)) \\
&= \mathbf{E}(\mathbf{E}(g^{(j)}_{(k-1)m+l,j} | G(km + l, m, j)) | G_{k+1}^{(l,j)}(f)) \text{ a.s. (by (36))} \\
&= \mathbf{E}(g^{(j)}_{(k-1)m+l,j} | G_{k+1}^{(l,j)}(f)) \\
&\quad \text{(by the smoothing lemma with } G_{k+1}^{(l,j)}(f) \subset G(km + l, m, j)).
\end{aligned}$$

Thus, (35) is proved.

Exchangeability of the sequence $\{f_{(k-1)m+l,(i-1)m+j} : 1 \leq i \leq k\}$ implies that the sequence $\{\mathbf{E}(g_{(k-1)m+l,j} | G_k^{(l,j)}(g)) : k \geq 1\}$ of random elements is a reverse martingale (where

$$G_k^{(l,j)}(g) = \sigma\{\sum_{i=1}^{k} g_{(k-1)m+l,(i-1)m+j}, \sum_{i=1}^{k+1} g_{km+l,(i-1)m+j}, \ldots\}).$$

By (28) and by the exchangeability of $\{g^{(j)}_{ni} : 1 \leq i \leq n\}$, we deduce that the sequence $\{g^{(j)}_{nj} : n \geq 1\}$ is CUI, for each $j \in \{1, 2, \ldots, m\}$. This yields that $\{f_{(k-1)m+l,j} : k \geq 1\}$ is

CUI, for each $l,j \in \{1,2,\ldots,m\}$ (Because $g^{(j)}_{(k-1)m+l,j}(\omega) = f_{(k-1)m+l,j}(\omega)$). Moreover, the sequence $\{\mathbf{E}f_{(k-1)m+l,j} : k \geq 1\}$ converges to x_j as $k \to \infty$. Applying Lemma 3.10, we obtain that the sequence $\{g_{(k-1)m+l,j} : k \geq 1\}$ is CUI, for each $l,j \in \{1,2,\ldots,m\}$.

Hence, for each $l,j \in \{1,2,\ldots,m\}$, the triangular array $\{g_{(k-1)m+l,(i-1)m+j} : k \geq 1, 1 \leq i \leq k\}$ satisfies all the conditions of Lemma 3.3, and so

$$S^{(j)}_{(k-1)m+l}(\omega) = \frac{1}{k}\sum_{i=1}^{k} g_{(k-1)m+l,(i-1)m+j}(\omega) \to 0 \text{ a.s. as } k \to \infty,$$

for each $l,j \in \{1,2,\ldots,m\}$.

It is equivalent to

$$S^{(j)}_n(\omega) = \frac{1}{k}\sum_{i=1}^{k} g_{n,(i-1)m+j}(\omega) \to 0 \text{ a.s. as } n \to \infty, \text{ for each } j \in \{1,2,\ldots,m\}.$$

Suppose that $V^{(j)}_n, z^{(l,j)}_{ki}$ are defined as in the proof of Theorem 3.7.

For each $j \in \{1,2,\ldots,m\}$, since the array $\{f_{n,(i-1)m+j} : n \geq 1, 1 \leq i \leq k\}$ is row-wise exchangeable and $\{\mathbf{E}f_{n,(i-1)m+j}\}$ converges to x_j as $n \to \infty$ for each column, it implies that this array has bounded expectation. Therefore, we have

$$|z^{(l,j)}_{ki}| \leq \|\mathbf{E}f_{n,(i-1)m+j}\| + \|x_j\| \leq C + \|x_j\|,$$

for all $k \geq 1, 1 \leq i \leq k, n = (k-1)m + l$.

By $z^{(l,j)}_{ki} \to 0$ as $k \to \infty$, for each $i \geq 1$, we get $z^{(l,j)}_{ki} \to 0$ as $i \to \infty$.

Combining the above statements, we have that for each $m \geq 1$ and $l,j \in \{1,2,\ldots,m\}$, the triangular array $\{z^{(l,j)}_{ki} : k \geq 1, 1 \leq i \leq k\}$ satisfies all the conditions of Lemma 3.5.

To complete the '*lim inf*' part of the proof in the Mosco convergence, we proceed as in the proof of Theorem 3.7.

Finally, by the arguments as in the proof of Theorem 3.7 and by Lemma 3.4, we obtain the '*lim sup*' path of the Mosco convergence. ■

Remark 3.4: 1. In Theorem 3.11, if the condition (29) is replaced by the following two conditions:

+) $\{\mathbf{E}(f(g(\text{co}X_{n1}))|G^g(n,m,j)) : n \geq 1\}$ is a reverse martingale, for each $m \geq 1$, $1 \leq j \leq m, f \in \mathfrak{X}^*$,

+) $\{\mathbf{E}(g(\text{co}X_{n1})I_{[g(\text{co}X_{n1})\notin K]}|G^g(n,m,j)) : n \geq 1\}$ is a reverse martingale, for each $m \geq 1, 1 \leq j \leq m$, and for each compact subset K of $\mathfrak{X}$,

then by the same arguments as in the proof of Theorem 3.11 and using [15, Theorem 3.3], the SLLN also holds.

2. In the past results, one built the family of selections of random sets to prove the '*lim inf*' path by being the union of the families with respect to $x_j, j \in \{1,2,\ldots,m\}$. However, in present paper, the triangular array $\{f_{ni} : n \geq 1, 1 \leq i \leq n\}$ of selections of random sets is the union of sets which each set is a sub family of triangular array with respect

to $x_j, j \in \{1, 2, \ldots, m\}$. Then, we use the single-valued SLLN for each triangular sub-array to obtain the multivalued SLLN. This is one of the key tools to prove the most difficult 'half' of the multivalued SLLN in the Mosco topology.
The example below shows that Theorem 3.11 is really different from Lemma 3.3, even in the case of single-valued random variables.

Example 2: Consider the Banach space $\mathfrak{X} = \mathbf{R}$. The triangular array $\{X_{ni} : n \geq 1, 1 \leq i \leq n\}$ is defined by $X_{ni}(\omega) = \{1\}$ for every $n \geq 1, 1 \leq i \leq n$ and $\omega \in \Omega$. Then, it is easy to check that the triangular array $\{X_{ni} : n \geq 1, 1 \leq i \leq n\}$ satisfies all the conditions of Theorem 3.11. But, it follows from $\mathbf{E}X_{n1}^2 = 1$ for all $n \geq 1$ that the triangular array $\{X_{ni} : n \geq 1, 1 \leq i \leq n\}$ does not satisfy the condition (iii) of Lemma 3.3.

However, the conditions (29) and (32) in Theorem 3.11 are also necessary. The next example shows that Theorem 3.11 is not true without the conditions (29) and (32).

Example 3: Let $\mathfrak{X} = \ell^2$ be the space of square-summable sequences. Namely, $x \in \ell^2$ if $x = (x_1, x_2, \ldots, x_n, \ldots), x_i \in \mathbf{R}$ and $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. The norm $\| \cdot \|_{l^2}$ is defined by $\|x\|_{l^2} = \sqrt{\sum_{n=1}^{\infty} |x_n|^2}$. Then, ℓ^2 is a Hilbert space with scalar multiplication $(\cdot|\cdot)$ which is given by $(x|y) = \sum_{n=1}^{\infty} x_n y_n$ for each $x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell^2, y = (y_1, y_2, \ldots, y_n, \ldots) \in \ell^2$.

For each $i \geq 1$, let $e_i = \{0, \ldots, 0, 1, 0, \ldots\}$, with number 1 in the i^{th} position. Then, $\{e_1, e_2, \ldots, e_n, \ldots\}$ is a standard basis of $\mathfrak{X}$.

For each $n \geq 1, 1 \leq i \leq n$ and $\omega \in \Omega$, we set $X_{ni}(\omega) = \{e_n\}$. Then, the triangular array $\{X_{ni} : n \geq 1, 1 \leq i \leq n\}$ satisfies all the conditions of Theorem 3.11 without the conditions (29) and (32). We have $G_n(\omega) = \frac{1}{n}\mathrm{cl}\sum_{i=1}^{n} X_{ni}(\omega) = \{e_n\}$ for every $n \geq 1$ and $\omega \in \Omega$. Since $\|e_m - e_n\|_{l^2} = \sqrt{2}$ for all $m \neq n$, the sequence $\{e_n : n \geq 1\}$ is not Cauchy's sequence. Consequently, the sequence $\{e_n : n \geq 1\}$ does not converge in norm. Therefore, we have $0 \notin s\text{-}liG_n(\omega)$ for all $\omega \in \Omega$.

By Riezs's theorem, we have that for each $f \in \mathfrak{X}^*$, there exists $a \in \ell^2$ such that $f(x) = (a|x)$ for all $x \in \mathfrak{X}$. On the other hand, $a = \sum_{n=1}^{\infty}(a|e_n).e_n \in \ell^2$. This series converges to a. It implies that the general term $(a|e_n)$ converges to 0 as $n \to \infty$, which is equivalent to $\lim f(e_n) = f(0)$. It follows that $e_n \overset{w}{\to} 0$ as $n \to \infty$. Since then, $0 \in w\text{-}lsG_n(\omega)$ for all $\omega \in \Omega$.

Since the above statements, we do not obtain the SLLN for the triangular array $\{X_{ni} : n \geq 1, 1 \leq i \leq n\}$ with respect to Mosco convergence.

In Theorems 3.7 and 3.11, we use a condition which is general stronger than the condition (i) of Lemma 3.5, that is,

$$z_{ki}^{(l,j)} \to 0 \text{ as } k \to \infty \text{ for each } i. \qquad (iii)$$

However, the condition (ii) is also necessary in this case. Indeed, the following example shows that if the condition (i) is replaced by the condition (iii) then Lemma 3.5 without condition (ii) is also not true.

Example 4: Let $\{x_{ni} : n \geq 1, 1 \leq i \leq n\}$ be a triangular array of elements in $\mathbf{R}$ and it is defined as follows:

$$x_{ni} = \begin{cases} n^2 & \text{if } i = n, \\ \dfrac{1}{n} & \text{otherwise.} \end{cases}$$

This means that

$$\left(x_{ni}\right)_{n\geq 1,1\leq i\leq n}=\begin{pmatrix} 1^2 & & & & & \\ \frac{1}{2} & 2^2 & & & & \\ \frac{1}{3} & \frac{1}{3} & 3^2 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \dots & n^2 & \\ \vdots & \vdots & \vdots & \dots & \vdots & \ddots \end{pmatrix}$$

It is clear that $\lim_{n\to\infty} x_{ni} = 0$ for each i; namely, the condition (iii) is satisfied.

Moreover, $x_{nn} = n^2 \to \infty$ as $n \to \infty$. This means that the condition (ii) of Lemma 3.5 is not satisfied.

However,

$$\frac{1}{n}\sum_{i=1}^{n} x_{ni} = \frac{1}{n}(\frac{n-1}{n} + n^2) = \frac{n-1}{n^2} + n \to \infty \text{ as } n \to \infty.$$

Now, we extend the previous theorems to fuzzy-valued random sets.

Theorem 3.12: *Let $\{\tilde{X}_{ni} : n \geq 1, 1 \leq i \leq n\}$ be a triangular array of row-wise exchangeable fuzzy random sets such that for each $\alpha \in (0, 1]$, the triangular array of random sets $\{L_\alpha \tilde{X}_{ni} : n \geq 1, 1 \leq i \leq n\}$ satisfies all the conditions of one of three Theorems 3.7, 3.9 and 3.11. Then,*

$$M\text{-}\lim \frac{1}{n}\text{cl}\sum_{i=1}^{n} \tilde{X}_{ni}(\omega) = I_{\overline{\text{co}}X} \text{ a.s.},$$

where $I_{\overline{\text{co}}X}$ is the indicator function of $\overline{\text{co}}X$.

Proof: Let $\tilde{G}_n(\omega) = \frac{1}{n}\text{cl}\sum_{i=1}^{n} \tilde{X}_{ni}(\omega)$. By virtue of the suitable theorem (one of three Theorems 3.7, 3.9 and 3.11), we have that $M\text{-}\lim L_\alpha \tilde{G}_n(\omega) = \overline{\text{co}}X$ a.s. for every fixed $\alpha \in (0, 1]$, in particular, for every $\alpha = r \in \mathbf{Q}$, where $\mathbf{Q}$ is the set of all rational numbers. Since countable set $\mathbf{Q}$ is dense in $[0, 1]$ and $L_\alpha \tilde{G}_n(\omega) = \lim_{r\uparrow\alpha, r\in\mathbf{Q}} L_r \tilde{G}_n(\omega)$, we have $M\text{-}\lim L_\alpha \tilde{G}_n(\omega) = \overline{\text{co}}X$, for every $\alpha \in (0, 1]$, a.s.

Next, for each $C \in c(\mathfrak{X})$, there exists a unit (with probability one) fuzzy-valued random set $\tilde{Y}$ satisfying $L_\alpha \tilde{Y}(\omega) = C$, for all $\alpha \in (0, 1]$, a.s. Indeed, it is easy to check that $L_\alpha I_C = C$, for all $\alpha \in (0, 1]$. Suppose that the fuzzy random set $\tilde{Y}$ satisfying $L_\alpha \tilde{Y}(\omega) = C$ for all $\alpha \in (0, 1]$ a.s. For each $\omega \in N$ with $\mathbf{P}(N) = 1$, put $u = \tilde{Y}(\omega)$. It follows from the sets $L_\alpha u$, $\alpha \in (0, 1]$ are non-increasing monotonic ordered by inclusion as $\alpha \uparrow$ that $L_\alpha u = C$ for all $\alpha \in (0, 1]$ is equivalent to

$$L_{0^+} u \subset C \subset L_1 u,$$

where $L_{0^+} u = \{x \in \mathfrak{X} \mid u(x) > 0\}$.

Since then, it is not hard to prove that $u = I_C$, which implies $\tilde{Y}(\omega) = I_C$ a.s.

Hence, $M\text{-}\lim L_\alpha \tilde{G}_n(\omega) = L_\alpha I_{\overline{\text{co}}X}$ for every $\alpha \in (0, 1]$, a.s., that is, $M\text{-}\lim \tilde{G}_n(\omega) = I_{\overline{\text{co}}X}$ a.s. ■

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