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The Hsu–Robbins–Erdös theorem for the maximum partial sums of quadruplewise independent random variables

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ABSTRACT

Etemadi (1981) [10] and Rio (1995) [27] provided proofs of the Kolmogorov–Marcinkiewicz–Zygmund strong law of large numbers under optimal moment conditions without using the Kolmogorov-type maximal inequalities. While this famous result holds for sequences of pairwise independent identically distributed real-valued random variables, a closely related result, the Hsu–Robbins–Erdös strong law of large numbers may fail if the underlying random variables are only assumed to be pairwise independent identically distributed. This note develops Rio's method and uses an approximation technique to establish the Hsu–Robbins–Erdös strong law of large numbers for the maximum partial sums of quadruplewise independent identically distributed random variables. We consider random variables taking values in a real separable Banach space $\mathcal{X}$, but the main result is new even when $\mathcal{X}$ is the real line. Previous contributions so far considered the complete convergence of the partial sums or restricted to dependence structures satisfying a Kolmogorov-type maximal inequality.

1. Introduction and the main result

The concept of complete convergence was introduced by Hsu and Robbins [14]. A sequence $\{X_n, n \geq 1\}$ of random variables is said to converge completely to a random variable X if

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|X_n - X| > \varepsilon\right) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables. The so-called Hsu–Robbins–Erdös strong law of large numbers (SLLN) provides the necessary and sufficient conditions for complete convergence of the sample means $(X_1 + \cdots + X_n)/n$, $n \geq 1$. The sufficiency was proved by Hsu and Robbins [14] and the necessity was proved by Erdös [9].

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Proposition 1.1 *(Hsu and Robbins [14] and Erdös [9]). Let $\{X_n,\ n \geq 1\}$ be a sequence of i.i.d. random variables. Then*

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left|\sum_{i=1}^{n} X_i\right| > \varepsilon n\right) < \infty \quad \textit{for all} \quad \varepsilon > 0 \tag{1.1}$$

if and only if

$$\mathbb{E}(X_1) = 0,\ \mathbb{E}(X_1^2) < \infty. \tag{1.2}$$

Let $k \geq 2$ be a given integer. A sequence $\{X_n, n \geq 1\}$ is said to be *k-tuplewise independent* if for every choice of k distinct positive integers $i_1, \ldots, i_k$, the random variables $X_{i_1}, \ldots, X_{i_k}$ are independent. For the case where $k = 2$, we say that the sequence $\{X_n, n \geq 1\}$ is pairwise independent. The term "k-tuplewise independence" is also expressed as "triplewise independence" and "quadruplewise independence" for the case where $k = 3$ and $k = 4$, respectively. It is worth noting that "k-tuplewise independence" is an important concept in probability and statistics [11,18] and has many applications in algorithm design and computer science [3,19,24]. For example, in [24], Peled et al. discussed, *inter alia*, the important role of "k-tuplewise independence" in computer science by giving examples in which k-tuplewise independent distributions are used for derandomization.

On the limit theorems in probability, an interesting research direction is to know if the fundamental limit theorems such as the laws of large numbers and the central limit theorems fail or hold under "k-tuplewise independence" (see, e.g., [7,10,27,30] and the references therein). While the classical Kolmogorov–Marcinkiewicz–Zygmund SLLN still holds for sequences of pairwise independent identically distributed (p.i.i.d.) random variables (see Etemadi [10] and Rio [27]), it was pointed out by Szynal [30] that the Hsu–Robbins–Erdös SLLN can fail if the independence assumption is weakened to pairwise independence. Szynal [30] also proved that for a sequence $\{X_n, n \geq 1\}$ of quadruplewise independent identically distributed (q.i.i.d.) random variables, condition (1.2) implies (1.1). Since (1.1) does not involve the maximum partial sums, the usual proofs of the sufficiency would be carried out without using Kolmogorov-type maximal inequalities. On the other hand, under the quadruplewise independence assumption, it is not clear how can one prove the necessity. Szynal's result, therefore, raises a natural question: Under the q.i.i.d. setting, is (1.2) necessary and sufficient for

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right| > \varepsilon n\right) < \infty \quad \text{for all} \quad \varepsilon > 0?$$

In the present paper, by using a totally different approach which is based on Rio's method [27] and an approximation technique, we establish the Hsu–Robbins–Erdös SLLN for the maximum partial sums of q.i.i.d. random variables, thereby giving a positive answer to this question.

The Hsu–Robbins–Erdös SLLN for Banach space-valued random variables was studied by many authors, see, e.g., [15,16,21] and the references therein, but only few of them consider complete convergence for maximal normed partial sums which is of special interest. The aforementioned works are often connected with various geometric conditions on the Banach space [15,16], and often dealt with independent random variables or martingales [15,16,21]. For the case of real-valued random variables, previous contributions so far considered complete convergence of the partial sums or restricted to dependence structures satisfying a maximal inequality, see, e.g., [13,21,25,26,34]. Some of these works considered the problem for random fields (see, e.g., Gut and Stadtmüller [13], Peligrad and Gut [25]). We also refer to Section 11, Chapter 6 of monograph [12] for an excellent survey on results and methods concerning complete convergence for the i.i.d. case. To our best knowledge, there is no work in the literature that establishes the Hsu–Robbins–Erdös SLLN for the maximum partial sums under optimal moment conditions without using maximal inequalities or a general Rosenthal-type inequality. On the other hand, for processes that do not enjoy a maximal inequality, it is a challenge to prove limit theorems for the maximum partial sums, and usually further conditions have to be assumed. For example, when dealing with weighted sums of φ-mixing sequences with arbitrary mixing rate, Chen and Sung [8] recently imposed a strong condition on the weights to obtain a weighted Marcinkiewicz–Zygmund-type SLLN. In this line of research, Wu et al. [34] required a stronger moment condition of the random variables comparing to the one of the cases where a maximal inequality is available (see Theorems 3.1–3.3 in [34]). By applying Lemmas 2.4 and 2.5 of Chen and Sung [8] and techniques in the proof of Theorem 2.2, we can show that the Hsu–Robbins SLLN holds for φ-mixing sequences under an optimal moment condition without any assumptions on the mixing rate.

The following theorem is the main result of the paper. It is new even when the underlying Banach space is the real line.

Theorem 1.2. *Let $\{X_n, n \geq 1\}$ be a sequence of q.i.i.d. random variables taking values in a real separable Banach space $\mathcal{X}$. Then*

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k} X_i\right\| > n\varepsilon\right) < \infty \quad \text{for all} \quad \varepsilon > 0 \tag{1.3}$$

if and only if

$$\mathbb{E}(X_1) = 0,\ \mathbb{E}\|X_1\|^2 < \infty. \tag{1.4}$$

We note that in Theorem 1.2, no geometric conditions are imposed on the Banach space. Our proof of Theorem 1.2 is completely different from that of the aforementioned works. Firstly, we develop Rio's method [27] to prove the Hsu–Robbins–Erdös SLLN for general dependent real-valued random variables with regularly varying normalizing sequences without using the maximal inequalities. Then, we apply an approximation technique to deal with the underlying Banach space-valued random variables. The proof of Theorem 1.2 is presented in Section 3.

The Marcinkiewicz–Zygmund SLLN with regularly varying normalizing sequences was studied in [4] by using a Kolmogorov-type maximal inequality. Let $\rho \in \mathbb{R}$. A real-valued function $R(\cdot)$ is said to be *regularly varying* (at infinity) with index of regular variation ρ if it is a positive and measurable function on $[A, \infty)$ for some $A \geq 0$, and for each $\lambda > 0$,

$$\lim_{x\to\infty} \frac{R(\lambda x)}{R(x)} = \lambda^{\rho}.$$

A regularly varying function with the index of regular variation $\rho = 0$ is called *slowly varying* (at infinity). We refer to Bingham et al. [5], Jessen and Mikosch [17] for definition, properties of regularly varying functions and their important role in probability and analysis.

Let $L(\cdot)$ be a slowly varying function. By Theorem 1.5.13 in Bingham et al. [5], there exists a slowly varying function $\tilde{L}(\cdot)$, unique up to asymptotic equivalence, satisfying

$$\lim_{x\to\infty} L(x)\tilde{L}\left(xL(x)\right) = 1 \quad \text{and} \quad \lim_{x\to\infty} \tilde{L}(x)L\left(x\tilde{L}(x)\right) = 1. \tag{1.5}$$

The function $\tilde{L}$ is called the de Bruijn conjugate of L (see p. 29 in Bingham et al. [5]). Bojanić and Seneta [6] showed that for most of "nice" slowly varying functions, we can choose (up to asymptotic equivalence) $\tilde{L}(x) = 1/L(x)$. Especially, if $L(x) = \log^{\gamma} x$ or $L(x) = \log^{\gamma}(\log x)$ for some $\gamma \in \mathbb{R}$, then $\tilde{L}(x) = 1/L(x)$.

Notation. Throughout this paper, C denotes a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance. For a set S, $\mathbf{1}(S)$ denotes the indicator function of S, and $|S|$ denotes the cardinality of S. For $x \geq 0$, $\log x$ denotes the natural logarithm (base e) of $\max\{x, \mathrm{e}\}$. For a slowly varying function $L(\cdot)$ defined on $[0, \infty)$, we denote the Brujin conjugate of $L(\cdot)$ by $\tilde{L}(\cdot)$.

General assumptions on slowly varying functions. By using a suitable asymptotic equivalence version (see Lemma 2.2 and Lemma 2.3 (i) in Anh et al. [4]), we can assume in this paper, without loss of generality, that every slowly varying function $L(\cdot)$ is continuous on $[0, \infty)$, differentiable on $[a, \infty)$ for some large a, and $x^\alpha L(x)$ is strictly increasing on $[0, \infty)$ and $x^{-\alpha}L(x)$ is strictly decreasing on $[0, \infty)$ for all $\alpha > 0$.

2. The Hsu–Robbins–Erdös SLLN for dependent real-valued random variables: regularly varying normalizing sequences

In this section, we establish the Hsu–Robbins–Erdös SLLN for real-valued random variables with regularly varying normalizing sequences. We consider a dependence structure which is much more general than quadruplewise independence, defined as follows:

Condition (H)**.** A family of random variables $\{X_\lambda, \lambda \in \Lambda\}$ is said to satisfy *Condition* (H) if for all finite subset $I \subset \Lambda$ and for all family of nondecreasing functions $\{f_\lambda, \lambda \in I\}$ with $\mathbb{E}(f_\lambda(X_\lambda)) = 0$ for all $\lambda \in I$, there exists a finite constant C_0 such that

$$\mathbb{E}\left(\sum_{\lambda\in I} f_\lambda(X_\lambda)\right)^4 \leq C_0\left(|I|\max_{\lambda\in I}\mathbb{E}\left(f_\lambda^4(X_\lambda)\right) + |I|^2\max_{\lambda\in I}\left(\mathbb{E}\left(f_\lambda^2(X_\lambda)\right)\right)^2\right) \tag{2.1}$$

provided the expectations are finite.

It is easy to see that quadruplewise independent random variables satisfy Condition (H). This condition is also fulfilled for many other weak dependence structures, including m-extended negative dependence (Wu and Wang [33, Lemma 3.3]), and various mixing processes such as φ-mixing (Chen and Sung [8, Lemma 2.4]), ρ^*-mixing (Peligrad and Gut [25, Theorem 1]), ρ-mixing (Shao [29, Theorem 1.1]), and others. Recently, Wu et al. [34] established various strong limit theorems for weighted sums by assuming very general conditions which are much stronger than (2.1) (see Equations (1) and (2) in [34]). Especially, the dependence structures assumed by Wu et al. [34] are not fulfilled for quadruplewise independent random variables. Also, for non-identical distributed ρ-mixing random variables with mixing rate $\sum_{n=1}^\infty \rho^{1/2}(2^n) < \infty$, Condition (H) is satisfied (see Theorem 1.1 of Shao [29]) while, as far as we know, there is no available Rosenthal-type inequality for such processes to guarantee the dependence structures assumed in [34] (see also [34, Remark 1.1]).

Remark 2.1. The Referee so kindly brought to our attention an interesting dependence structure so-called multiplicative orthogonal system which is much weaker than independence. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be a *multiplicative orthogonal system* (see, e.g., Alexits and Sharma [2]) if

$$\mathbb{E}\left(X_{i_1}X_{i_2}\cdots X_{i_m}\right) = 0$$

for any $m \geq 1$ and for every choice of distinct positive integers $i_1, \ldots, i_m$. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be a *strongly multiplicative orthogonal system* (Alexits [1]) if

$$\mathbb{E}\left(X_{i_1}^{r_1}X_{i_2}^{r_2}\cdots X_{i_m}^{r_m}\right) = 0$$

for any $m \geq 1$ and for every choice of distinct positive integers $i_1, \ldots, i_m$, $r_k = 1$ or 2 and at least one $r_k = 1$. We refer to [1,2,22] for sequences of multiplicative orthogonal and strongly multiplicative orthogonal random variables which are not independent. Some fundamental limit theorems hold for strongly multiplicative orthogonal systems such as the Khintchine–Kolmogorov convergence theorem (Alexits [1, Theorem 1]), the central limit theorem and the (weak form of) the law of the iterated logarithm (Móricz [22, Theorems 5 and 6]). Following Szynal [30, Definition 2], we say that a sequence $\{X_n, n \geq 1\}$ of random variables is a *quadruple-wise strongly multiplicative orthogonal system* if

$$\mathbb{E}\left(X_{i_1}^{r_1} X_{i_2}^{r_2} X_{i_3}^{r_3} X_{i_4}^{r_4}\right) = 0$$

for every $i_1 < i_2 < i_3 < i_4$, $r_k = 0$, 1, or 2 and at least one element of r_k is 1. It is clear that if $\{X_n, n \geq 1\}$ is a sequence of pairwise independent and quadruple-wise strongly multiplicative orthogonal mean random variables, then we have (see the proof of Theorem 3 in Szynal [30])

$$\mathbb{E}\left(\sum_{\lambda \in I} X_\lambda\right)^4 \leq C\left(|I| \max_{\lambda \in I} \mathbb{E}X_\lambda^4 + |I|^2 \max_{\lambda \in I}\left(\mathbb{E}X_\lambda^2\right)^2\right)$$

for any finite set $I \subset \{1, 2, \ldots\}$ provided the expectations are finite. In order to ensure that Condition (H) is fulfilled, we would have to require that the sequence $\{f_n(X_n), n \geq 1\}$ is also a quadruple-wise strongly multiplicative orthogonal system for all non-decreasing function $f_n, n \geq 1$. It would be interesting to find non-trivial sequences of random variables satisfying this requirement. □

The main result of this section is the following theorem. This is new even when $L(x) \equiv \tilde{L}(x) \equiv 1$.

Theorem 2.2. *Let $\{X_n,\ n \geq 1\}$ be a sequence of identically distributed random variables satisfying Condition (H). Let $L(\cdot)$ be a slowly varying function satisfying $L(x) \geq 1$ for all $x \geq 0$. Then*

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1 \leq k \leq n}\left|\sum_{i=1}^{k} X_i\right| > \varepsilon n \tilde{L}(n)\right) < \infty \text{ for all } \varepsilon > 0 \tag{2.2}$$

if and only if

$$\mathbb{E}(X_1) = 0 \quad \text{and} \quad \mathbb{E}\left(X_1^2 L^2(|X_1|)\right) < \infty. \tag{2.3}$$

A family of random variables $\{X_\lambda, \lambda \in \Lambda\}$ is said to be *stochastically dominated* by a random variable X if

$$\sup_{\lambda \in \Lambda} \mathbb{P}(|X_\lambda| > t) \leq \mathbb{P}(|X| > t), \quad \text{for all } t \in \mathbb{R}. \tag{2.4}$$

For all $r, b > 0$ and $\lambda \in \Lambda$, it follows from integration by parts and (2.4) that

$$\mathbb{E}(|X_\lambda|^r \mathbf{1}(|X_\lambda| > b)) \leq \mathbb{E}(|X|^r \mathbf{1}(|X| > b)), \tag{2.5}$$

and

$$\mathbb{E}\left(|X_\lambda|^r \mathbf{1}(|X_\lambda| \leq b)\right) \leq \mathbb{E}\left(|X|^r \mathbf{1}(|X| \leq b)\right) + b^r \mathbb{P}(|X| > b). \tag{2.6}$$

We will use (2.5) and (2.6) in the proofs without further mention. The sufficiency part of Theorem 2.2 follows from the following general proposition.

Proposition 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables satisfying Condition (H). Let $L(\cdot)$ be a slowly varying function satisfying $L(x) \geq 1$ for all $x \geq 0$. Assume that $\{X_n, n \geq 1\}$ is stochastically dominated by a random variable X. If*

$$\mathbb{E}\left(X^2 L^2(|X|)\right) < \infty, \tag{2.7}$$

then

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}(X_i - \mathbb{E}(X_i))\right| > \varepsilon n\tilde{L}(n)\right) < \infty \text{ for all } \varepsilon > 0. \tag{2.8}$$

Proof. Since $\{X_n^+, n \geq 1\}$ and $\{X_n^-, n \geq 1\}$ satisfy the assumptions of the theorem, we can assume, without loss of generality, that $X_n \geq 0$ for all $n \geq 1$. For $n \geq 1$, set $b_n = n\tilde{L}(n)$,

$$X_{i,n} = X_i \mathbf{1}(X_i \leq b_n) + b_n \mathbf{1}(X_i > b_n),\ 1 \leq i \leq n, \tag{2.9}$$

and

$$Y_{i,m} = \left(X_{i,2^m} - X_{i,2^{m-1}}\right) - \mathbb{E}\left(X_{i,2^m} - X_{i,2^{m-1}}\right),\ m \geq 1,\ i \geq 1. \tag{2.10}$$

It is easy to see that (2.8) is equivalent to

$$\sum_{n=1}^{\infty} 2^n \mathbb{P}\left(\max_{1\leq j<2^n}\left|\sum_{i=1}^{j}(X_i - \mathbb{E}(X_i))\right| > \varepsilon b_{2^n}\right) < \infty \text{ for all } \varepsilon > 0. \tag{2.11}$$

Using argument as in Rio [27, Proposition 1] (see also the proof of Theorem 1 in Thành [31]), the proof of (2.11) will be completed if we can show that

$$\sum_{n=1}^{\infty} 2^n \mathbb{P}\left(\max_{1\leq j<2^n}\left|\sum_{i=1}^{j}(X_{i,2^n} - \mathbb{E}X_{i,2^n})\right| > \varepsilon b_{2^n}\right) < \infty \text{ for all } \varepsilon > 0. \tag{2.12}$$

For $m \geq 0$, set $S_{0,m} = 0$ and

$$S_{j,m} = \sum_{i=1}^{j}(X_{i,2^m} - \mathbb{E}\left(X_{i,2^m}\right)),\ j \geq 1.$$

Then (see [31, Equation (28)])

$$\begin{aligned}\max_{1\leq j<2^n}|S_{j,n}| \leq \sum_{m=1}^{n}\max_{0\leq k<2^{n-m}}&\left|\sum_{i=k2^m+1}^{k2^m+2^{m-1}}\left(X_{i,2^{m-1}} - \mathbb{E}(X_{i,2^{m-1}})\right)\right| \\ &+ \sum_{m=1}^{n}\max_{0\leq k<2^{n-m}}\left|\sum_{i=k2^m+1}^{(k+1)2^m} Y_{i,m}\right| + \sum_{m=1}^{n} 2^{m+1}\mathbb{E}\left(|X|\mathbf{1}(|X| > b_{2^{m-1}})\right).\end{aligned} \tag{2.13}$$

It follows from Toeplitz's lemma, (2.7), and the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty}\frac{\sum_{m=1}^{n} 2^{m+1}\mathbb{E}\left(|X|\mathbf{1}(|X| > b_{2^{m-1}})\right)}{b_{2^n}} = 0. \tag{2.14}$$

By using (2.13) and (2.14), to prove (2.12), it remains to show that

$$\sum_{n=1}^{\infty} 2^n \mathbb{P}\left(\sum_{m=1}^{n} \max_{0\le k<2^{n-m}} \left|\sum_{i=k2^m+1}^{k2^m+2^{m-1}} \left(X_{i,2^{m-1}} - \mathbb{E}(X_{i,2^{m-1}})\right)\right| > \varepsilon b_{2^n}\right) < \infty \quad \text{for all } \varepsilon > 0, \tag{2.15}$$

and

$$\sum_{n=1}^{\infty} 2^n \mathbb{P}\left(\sum_{m=1}^{n} \max_{0\le k<2^{n-m}} \left|\sum_{i=k2^m+1}^{(k+1)2^m} Y_{i,m}\right| > \varepsilon b_{2^n}\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{2.16}$$

Let $\varepsilon_1 > 0$ be arbitrary, let a and b be constants satisfying

$$a + b = 1, 0 < b < 1/4, \tag{2.17}$$

and let

$$\lambda_{m,n} = \varepsilon_1 2^{bm} 2^{an} \tilde{L}(2^n),\ n \ge 1,\ 1 \le m \le n. \tag{2.18}$$

Then

$$\sum_{m=1}^{n} \lambda_{m,n} \le \frac{2^b \varepsilon_1 b_{2^n}}{2^b - 1} := C_1(b)\varepsilon_1 b_{2^n}. \tag{2.19}$$

Therefore

$$\begin{aligned}
&\mathbb{P}\left(\sum_{m=1}^{n} \max_{0\le k<2^{n-m}} \left|\sum_{i=k2^m+1}^{(k+1)2^m} Y_{i,m}\right| \ge C_1(b)\varepsilon_1 b_{2^n}\right) \le \sum_{m=1}^{n} \mathbb{P}\left(\max_{0\le k<2^{n-m}} \left|\sum_{i=k2^m+1}^{(k+1)2^m} Y_{i,m}\right| \ge \lambda_{m,n}\right) \\
&\le \sum_{m=1}^{n} \lambda_{m,n}^{-4} \mathbb{E}\left(\max_{0\le k<2^{n-m}} \left|\sum_{i=k2^m+1}^{(k+1)2^m} Y_{i,m}\right|\right)^4 \\
&\le \sum_{m=1}^{n} \lambda_{m,n}^{-4} \sum_{k=0}^{2^{n-m}-1} \mathbb{E}\left(\sum_{i=k2^m+1}^{(k+1)2^m} Y_{i,m}\right)^4 \\
&\le C_0 \sum_{m=1}^{n} \lambda_{m,n}^{-4} \sum_{k=0}^{2^{n-m}-1} \left(2^m \max_{k2^m+1\le i\le (k+1)2^m} \mathbb{E}Y_{i,m}^4 + 2^{2m} \max_{k2^m+1\le i\le (k+1)2^m} \left(\mathbb{E}Y_{i,m}^2\right)^2\right),
\end{aligned} \tag{2.20}$$

where we have applied (2.19) in the first inequality, Markov's inequality in the second inequality, (2.1) in the last inequality. By (2.9) and (2.10), we have

$$\begin{aligned}
Y_{i,m} &\le (X_i - b_{2^{m-1}})\,\mathbf{1}(b_{2^{m-1}} < X_i \le b_{2^m}) + (b_{2^m} - b_{2^{m-1}})\,\mathbf{1}(X_i > b_{2^{m-1}}) \\
&\le b_{2^m}\mathbf{1}(X_i > b_{2^{m-1}}),\ m \ge 1, i \ge 1.
\end{aligned}$$

Since $\{X_n, n \ge 1\}$ is stochastically dominated by X with $\mathbb{E}(X^2 L^2(|X|)) < \infty$ and $L(x) \ge 1$ for all $x \ge 0$, it thus follows that

$$\sup_{i\ge 1, m\ge 1} \mathbb{E}Y_{i,m}^2 \le \mathbb{E}(X^2) < \infty, \tag{2.21}$$

and

$$\sup_{i\geq 1}\mathbb{E}Y_{i,m}^4 \leq b_{2^m}^4\mathbb{P}(|X|>b_{2^{m-1}}),\ m\geq 1. \tag{2.22}$$

Combining (2.20)–(2.22) yields

$$\mathbb{P}\left(\sum_{m=1}^{n}\max_{0\leq k<2^{n-m}}\left|\sum_{i=k2^m+1}^{(k+1)2^m}Y_{i,m}\right|\geq C_1(b)\varepsilon_1 b_{2^n}\right)\leq C\sum_{m=1}^{n}\lambda_{m,n}^{-4}2^n\left(b_{2^m}^4\mathbb{P}(|X|>b_{2^{m-1}})+2^m\right). \tag{2.23}$$

By applying Lemmas A.1 and A.2, we have from (2.23), (2.17)–(2.18), and (2.7) that

$$\begin{aligned}
&\sum_{n=1}^{\infty}2^n\mathbb{P}\left(\sum_{m=1}^{n}\max_{0\leq k<2^{n-m}}\left|\sum_{i=k2^m+1}^{(k+1)2^m}Y_{i,m}\right|\geq C_1(b)\varepsilon_1 b_{2^n}\right)\\
&\leq C\sum_{n=1}^{\infty}2^{2n}\left(2^{-4an}\tilde{L}^{-4}(2^n)\sum_{m=1}^{n}2^{4m(1-b)}\tilde{L}^4(2^m)\mathbb{P}(|X|>b_{2^{m-1}})+2^{-4an}\tilde{L}^{-4}(2^n)\sum_{m=1}^{n}2^{m(1-4b)}\right)\\
&\leq C\sum_{m=1}^{\infty}\left(\sum_{n=m}^{\infty}2^{n(2-4a)}\tilde{L}^{-4}(2^n)\right)2^{4m(1-b)}\tilde{L}^4(2^m)\mathbb{P}(|X|>b_{2^{m-1}})+C\sum_{n=1}^{\infty}2^{-n}\tilde{L}^{-4}(2^n)\\
&\leq C\left(\sum_{m=1}^{\infty}2^{2m}\mathbb{P}(|X|>b_{2^{m-1}})+1\right)\\
&\leq C\left(\mathbb{E}(X^2L^2(|X|))+1\right)<\infty
\end{aligned}$$

thereby proving (2.16) since $\varepsilon_1>0$ is arbitrary. By using a similar argument, we obtain (2.15). The proof of the proposition is completed. □

Remark 2.4.

(i) By using Theorem 1 of Shao [29], Proposition 2.3 holds for ρ-mixing sequences with mixing rate $\sum_{n=1}^{\infty}\rho^{1/2}(2^n)<\infty$.
(ii) Along the same lines as the proof of Proposition 2.3, and by using Lemmas 2.4 and 2.5 of Chen and Sung [8], we see that Proposition 2.3 holds for φ-mixing sequences without any further requirements on the mixing rate. □

The following technical result is used in the proof of the necessity part of Theorem 2.2 and may be of independent interest. The proof is presented in the Appendix. When the random variables are m-extended negatively dependent, a related result was recently established by Wu and Wang [33, Lemma 3.5].

Proposition 2.5. *Let $\{A_i, 1\leq i\leq n\}$ be events and let $\xi_i=\mathbf{1}(A_i)-\mathbb{P}(A_i)$, $1\leq i\leq n$. If there exist a positive integer r and a positive constant C_1 such that*

$$\mathbb{E}\left(\sum_{i=1}^{n}\xi_i\right)^{2r}\leq C_1\max\left\{\sum_{i=1}^{n}\mathbb{P}(A_i),\left(\sum_{i=1}^{n}\mathbb{P}(A_i)\right)^r\right\}, \tag{2.24}$$

then

$$\left(1-\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right)\right)^2 \sum_{i=1}^{n} \mathbb{P}(A_i) \leq C_2 \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right), \tag{2.25}$$

where C_2 is a positive constant depending only on r and C_1.

Proof of Theorem 2.2. The sufficiency part follows immediately from Proposition 2.3. We now prove the necessity part. Assume that (2.2) holds. Then

$$\lim_{n\to\infty} \frac{\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|}{n\tilde{L}(n)} = 0 \text{ almost surely (a.s.)}, \tag{2.26}$$

and so

$$\lim_{n\to\infty} \frac{\max_{1\leq i\leq n}|X_i|}{n\tilde{L}(n)} = \lim_{n\to\infty} \frac{\max_{1\leq i\leq n}\left(X_i^+ + X_i^-\right)}{n\tilde{L}(n)} = 0 \text{ a.s.} \tag{2.27}$$

This implies

$$\lim_{n\to\infty} \mathbb{P}\left(\bigcup_{i=1}^{n}(X_i^+ > n\tilde{L}(n))\right) = \lim_{n\to\infty} \mathbb{P}\left(\max_{1\leq i\leq n} X_i^+ > n\tilde{L}(n)\right) = 0. \tag{2.28}$$

For fixed n, let

$$A_i = (X_i^+ > n\tilde{L}(n)), \text{ and } \xi_i = \mathbf{1}(X_i^+ > n\tilde{L}(n)) - \mathbb{P}(X_i^+ > n\tilde{L}(n)),\ 1\leq i\leq n.$$

Since the sequence $\{X_n,\ n\geq 1\}$ satisfies Condition (H) and $\mathbb{P}(A_1) = \cdots = \mathbb{P}(A_n)$, (2.24) holds for $r=2$. Applying Proposition 2.5, we have

$$\left(1-\mathbb{P}\left(\max_{k\leq n} X_k^+ > n\tilde{L}(n)\right)\right)^2 \sum_{k=1}^{n} \mathbb{P}(X_k^+ > n\tilde{L}(n)) \leq C\mathbb{P}\left(\max_{k\leq n} X_k^+ > n\tilde{L}(n)\right). \tag{2.29}$$

It follows from (2.28) and (2.29) that there exists n_0 such that

$$\begin{aligned} n\mathbb{P}(X_1^+ > n\tilde{L}(n)) &= \sum_{k=1}^{n} \mathbb{P}(X_k^+ > n\tilde{L}(n)) \\ &\leq C\mathbb{P}\left(\max_{k\leq n} X_k^+ > n\tilde{L}(n)\right) \end{aligned} \tag{2.30}$$

whenever $n\geq n_0$. Combining (2.2) and (2.30) yields

$$\begin{aligned} \sum_{n\geq 1} n\mathbb{P}(X_1^+ > n\tilde{L}(n)) &\leq C + C\sum_{n\geq n_0} \mathbb{P}\left(\max_{k\leq n} X_k^+ > n\tilde{L}(n)\right) \\ &\leq C + C\sum_{n\geq n_0} \mathbb{P}\left(\max_{k\leq n}|X_k| > n\tilde{L}(n)\right) \\ &\leq C + C\sum_{n\geq n_0} \mathbb{P}\left(2\max_{k\leq n}\left|\sum_{i=1}^{k} X_i\right| > n\tilde{L}(n)\right) \\ &< \infty. \end{aligned} \tag{2.31}$$

By using the same arguments, we also have

$$\sum_{n\ge 1} n\mathbb{P}(X_1^- > n\tilde{L}(n)) < \infty. \tag{2.32}$$

Combining (2.31) and (2.32) yields

$$\sum_{n\ge 1} n\mathbb{P}(|X_1|/2 > n\tilde{L}(n)) \le \sum_{n\ge 1} n\mathbb{P}(X_1^+ > n\tilde{L}(n)) + \sum_{n\ge 1} n\mathbb{P}(X_1^- > n\tilde{L}(n)) < \infty. \tag{2.33}$$

Applying Lemma A.2 with $p = 2$ and $\alpha = 1$, we have from (2.33) that $\mathbb{E}\left(|X_1|^2 L^2\left(|X_1|\right)\right) < \infty$, i.e., the second half of (2.3) is satisfied.

From the second half of (2.3), we can apply Proposition 2.3 to obtain

$$\lim_{n\to\infty}\left(\frac{\sum_{i=1}^n X_i}{n\tilde{L}(n)} - \frac{\mathbb{E}(X_1)}{\tilde{L}(n)}\right) = 0 \ \text{ a.s.} \tag{2.34}$$

On the other hand, we have from (2.2) that

$$\lim_{n\to\infty}\frac{\sum_{i=1}^n X_i}{n\tilde{L}(n)} = 0 \ \text{ a.s.} \tag{2.35}$$

From (1.5) and the assumption that $L(x) \ge 1$ for all $x \ge 0$, we have $\tilde{L}^{-1}(n) \sim L(n\tilde{L}(n)) \ge 1$. It thus follows from (2.34) and (2.35) that $\mathbb{E}(X_1) = 0$, i.e., the first half of (2.3) is satisfied. □

3. Proof of Theorem 1.2

In this section, we will present the proof of Theorem 1.2. An open problem is also discussed.

Proof of Theorem 1.2. Firstly, we prove the sufficiency. Assume that (1.4) holds. Let $\varepsilon > 0$ be arbitrary. By [20, p. 42], there exists a compact subset K of $\mathcal{X}$ such that

$$\sup_{n\ge 1}\mathbb{E}\|X_n\|\mathbf{1}(X_n \notin K) = \mathbb{E}\|X_1\|\mathbf{1}(X_1 \notin K) < \varepsilon/6. \tag{3.1}$$

For $n \ge 1$, set

$$V_n = X_n\mathbf{1}(X_n \in K),\ W_n = X_n\mathbf{1}(X_n \notin K).$$

By (3.1), we have

$$\sum_{i=1}^n \mathbb{E}\left\|W_i\right\| \le \varepsilon n/6,\ n \ge 1. \tag{3.2}$$

Since V_i takes values in $K \cup \{0\}$ for all $i \ge 1$, there exist a finite set $\{x_1, \ldots, x_t\} \subset K$ and Borel subsets $\{A_1, \ldots, A_t\}$ of $\mathcal{X}$ such that (see, e.g., [23, Lemma 1])

$$\left\|V_i - \sum_{r=1}^t x_r\mathbf{1}(V_i \in A_r)\right\| < \varepsilon/6 \text{ for all } i \ge 1. \tag{3.3}$$

Set

$$Y_i = \sum_{r=1}^{t} x_r \mathbf{1}(V_i \in A_r),\ i \geq 1.$$

It follows from (3.3) that for all $n \geq 1$,

$$\sum_{i=1}^{n} \|V_i - Y_i\| \leq \varepsilon n/6 \tag{3.4}$$

and so

$$\sum_{i=1}^{n} \mathbb{E}\,\|V_i - Y_i\| \leq \varepsilon n/6. \tag{3.5}$$

Combining (3.2), (3.4), (3.5), and noting that $\mathbb{E}(X_i) \equiv 0$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k} X_i\right\| > \varepsilon n\right) &\leq \sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k}(W_i - \mathbb{E}W_i)\right\| > \varepsilon n/2\right) \\
&\quad + \sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k}(V_i - \mathbb{E}V_i)\right\| > \varepsilon n/2\right) \\
&\leq \sum_{n=1}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n}(\|W_i\| + \mathbb{E}\|W_i\|) > \varepsilon n/2\right) \\
&\quad + \sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k}(Y_i - \mathbb{E}Y_i)\right\| > \varepsilon n/6\right) \\
&\leq \sum_{n=1}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n}(\|W_i\| - \mathbb{E}\|W_i\|) > \varepsilon n/6\right) \\
&\quad + \sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k}(Y_i - \mathbb{E}Y_i)\right\| > \varepsilon n/6\right).
\end{aligned} \tag{3.6}$$

Noting that $\mathbb{E}\left(\|W_1\| - \mathbb{E}\|W_1\|\right)^2 \leq 4\mathbb{E}\left(\|W_1\|\right)^2 \leq 4\mathbb{E}\left(\|X_1\|\right)^2 < \infty$. Applying Proposition 2.3 for $L(x) \equiv 1$ and for the sequence of q.i.i.d. real-valued random variables $\{\|W_n\| - \mathbb{E}\|W_n\|,\ n \geq 1\}$, we obtain

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n}(\|W_i\| - \mathbb{E}\|W_i\|) > \varepsilon n/6\right) < \infty. \tag{3.7}$$

Set

$$A_r^{(n)} = (V_n \in A_r),\ 1 \leq r \leq t,\ n \geq 1.$$

For all $n \geq 1$, we have

$$\begin{aligned}
&\mathbb{P}\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k}(Y_i - \mathbb{E}Y_i)\right\| > \varepsilon n/6\right) \\
&= \mathbb{P}\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k}\sum_{r=1}^{t} x_r(\mathbf{1}(A_r^{(i)}) - \mathbb{P}(A_r^{(i)}))\right\| > \varepsilon n/6\right)
\end{aligned} \tag{3.8}$$

$$\le \mathbb{P}\left(\sum_{r=1}^{t} \max_{1\le k\le n}\left\|\sum_{i=1}^{k} x_r(\mathbf{1}(A_r^{(i)}) - \mathbb{P}(A_r^{(i)}))\right\| > \varepsilon n/6\right)$$
$$\le \sum_{\|x_r\|\ne 0} \mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}(\mathbf{1}(A_r^{(i)}) - \mathbb{P}(A_r^{(i)}))\right| > \frac{\varepsilon n}{6t\|x_r\|}\right).$$

For each $1 \le r \le t$ with $\|x_r\| \ne 0$, applying Proposition 2.3 for the sequence of q.i.i.d. random variables $\{\mathbf{1}(A_r^{(n)}) - \mathbb{E}\mathbf{1}(A_r^{(n)}),\ n \ge 1\}$, we obtain

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}(\mathbf{1}(A_r^{(i)}) - \mathbb{P}(A_r^{(i)}))\right| > \frac{\varepsilon n}{6t\|x_r\|}\right) < \infty. \tag{3.9}$$

Combining (3.8) and (3.9) yields

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k}(Y_i - \mathbb{E}Y_i)\right\| > \varepsilon n/6\right) < \infty. \tag{3.10}$$

The conclusion (1.3) follows from (3.6), (3.7) and (3.10).

By proceeding in a similar manner as the proof of the necessity part of Theorem 2.2 with noting that the events $(\|X_i\| > n), 1 \le i \le n$ are quadruplewise independent for each $n \ge 1$, we obtain the necessity part of Theorem 1.2. $\square$

Remark 3.1. While no geometric conditions are imposed on the Banach space in Theorem 1.2, it is well known that the Marcinkiewicz–Zygmund SLLN for i.i.d. Banach space-valued random variables may fail if the Banach space is not of Rademacher type p, $1 < p < 2$. It is an open problem as to whether or not the Marcinkiewicz–Zygmund SLLN holds for pairwise i.i.d. random variables taking values in a Rademacher type p Banach space. We expect that the techniques developed in this paper may help to shed some light on solving this problem.

4. On the stochastic domination condition

The stochastic domination condition is an extension of the identical distribution condition. In [32, Theorem 2.6] and [28, Theorem 2.5], it was shown that bounded moment type conditions on a family of random variables can accomplish stochastic domination. Based on Theorem 2.6 in [32] and Proposition 2.3, we have the following result. Proposition 4.1 provides an almost optimal moment condition for the Hsu–Robbins–Erdös SLLN. Hereafter, for $x \ge 0$, and for a fixed positive integer ν, we let

$$\log_\nu(x) := (\log x)(\log\log x)\ldots(\log\cdots\log x), \tag{4.1}$$

and

$$\log_\nu^{(2)}(x) := (\log x)(\log\log x)\ldots(\log\cdots\log x)^2, \tag{4.2}$$

where in both (4.1) and (4.2), there are ν factors. For example, $\log_2(x) = (\log x)(\log\log x)$, $\log_3^{(2)}(x) = (\log x)(\log\log x)(\log\log\log x)^2$, and so on.

Proposition 4.1. *Let $\{X_n, n \ge 1\}$ be a sequence of random variables satisfying Condition (H). Let $L(\cdot)$ be a slowly varying function satisfying $L(x) \ge 1$ for all $x \ge 0$, and let ν be a fixed positive integer. If*

$$\sup_{n\geq 1}\mathbb{E}\left(X_n^2 L^2(|X_n|)\log_\nu^{(2)}(|X_n|)\right) < \infty, \tag{4.3}$$

then

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}(X_i - \mathbb{E}(X_i))\right| > \varepsilon n\tilde{L}(n)\right) < \infty. \tag{4.4}$$

Proof. By using a similar argument to that of Theorem 2.6 in [32] (see Theorem 2.5 (iii) in [28] for a slightly weaker version), it follows from (4.3) that there exists a nonnegative random variable X such that $\{X_n, n \geq 1\}$ is stochastically dominated by X and

$$\mathbb{E}\left(X^2 L^2(X)\right) < \infty. \tag{4.5}$$

By applying Proposition 2.3, (4.4) follows from (4.5). □

The following simple example, however, shows that in the Banach space-valued case, even for independent and bounded random variables, we cannot weaken the identical distribution condition to the stochastic domination condition in Theorem 1.2. To see this, consider the real separable Banach space ℓ_1 consisting of absolutely summable real sequences $v = \{v_k, k \geq 1\}$ with norm $\|v\| = \sum_{k=1}^{\infty}|v_k|$. Let $v^{(k)}$ denote the element of ℓ_1 having 1 in its k^{th} position and 0 elsewhere, $k \geq 1$. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables in ℓ_1 by requiring the $\{X_n, n \geq 1\}$ to be independent with

$$\mathbb{P}\left(X_n = v^{(n)}\right) = \mathbb{P}\left(X_n = -v^{(n)}\right) = \frac{1}{2}, n \geq 1.$$

Then $X_n, n \geq 1$ are not identically distributed but are stochastically dominated by $\|X_1\|$ and

$$\sup_{n\geq 1}\|X_n\| \leq 1 \text{ a.s.}$$

Note that (1.4) holds but (1.3) fails since for all $n \geq 1$,

$$\frac{\max_{1\leq k\leq n}\|\sum_{i=1}^{k}X_i\|}{n} \geq \frac{\|\sum_{i=1}^{n}X_i\|}{n} = \frac{n}{n} = 1 \text{ a.s.}$$

Comparing with the necessary and sufficient condition (2.3) in the Theorem 2.2, we see that (4.3) is nearly optimal for (4.4) to hold. Moreover, in view of Example 4.3 in [28], we conjecture that, even for the independence case, (4.3) is almost impossible to improve in the sense that Corollary 4.1 may fail if (4.3) is weakened to

$$\sup_{n\geq 1}\mathbb{E}\left(X_n^2 L^2(|X_n|)\log_\nu(|X_n|)\right) < \infty.$$

We formulate the case $L(x) \equiv 1$ as follows.

Conjecture 4.2. *Let ν be a fixed positive integer. Then there exists a sequence $\{X_n, n \geq 1\}$ of independent real-valued random variables satisfying*

$$\sup_{n\geq 1}\mathbb{E}\left(X_n^2\log_\nu(|X_n|)\right) < \infty,$$

and

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}(X_i - \mathbb{E}(X_i))\right| > \varepsilon n\right) = \infty$$

for some $\varepsilon > 0$.

Declaration of competing interest

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Appendix A

In this section, we will present two technical lemmas and the proof of Proposition 2.5.

The first lemma is a direct consequence of Karamata's theorem (see Proposition 1.5.10 in Bingham et al. [5]).

Lemma A.1. *Let* $0 < a < 1, b \ge 1$, *and let* $L(\cdot)$ *be a slowly varying function. Then*

$$\sum_{k=n}^{\infty} a^k L(b^k) \le C a^n L(b^n).$$

The following lemma gives simple criteria for $\mathbb{E}\left(|X|^p L^p(|X|)\right) < \infty$, and its proof is standard (see, e.g., Lemma 4 in Thành [31]).

Lemma A.2. *Let* $p \ge 1$, $\alpha p \ge 1$, *and* X *be a random variable. Let* $L(x)$ *be a slowly varying function defined on* $[0,\infty)$, *and* $b_n = n^{\alpha}\tilde{L}(n^{\alpha})$, $n \ge 1$. *Assume that* $x^{1/\alpha}L^{1/\alpha}(x)$ *and* $x^{\alpha}\tilde{L}(x^{\alpha})$ *are strictly increasing on* $[A,\infty)$ *for some* $A > 0$. *Then* $\mathbb{E}\left(|X|^p L^p(|X|)\right) < \infty$ *if and only if either*

$$\sum_{n=1}^{\infty} n^{\alpha p-1}\mathbb{P}(|X| > b_n) < \infty,$$

or

$$\sum_{n=1}^{\infty} 2^{n\alpha p}\mathbb{P}(|X| > b_{2^{n-1}}) < \infty.$$

Finally, we present the proof of Proposition 2.5.

Proof of Proposition 2.5. Let $A = \bigcup_{i=1}^n A_i$. We only need to prove the proposition for the case $p := \mathbb{P}(A) < 1$ since (2.25) is trivial otherwise. Applying Hölder's inequality and (2.24), we have

$$\begin{aligned}(1-p)\sum_{i=1}^n \mathbb{P}(A_i) &= \mathbb{E}\left(\mathbf{1}(A)\sum_{i=1}^n \xi_i\right)\\ &\le (\mathbb{E}(\mathbf{1}(A)))^{(2r-1)/(2r)}\left(\mathbb{E}\left(\sum_{i=1}^n \xi_i\right)^{2r}\right)^{1/(2r)} \\ &\le C_1^{1/(2r)} p^{(2r-1)/(2r)} \max\left\{\left(\sum_{i=1}^n \mathbb{P}(A_i)\right)^{1/(2r)}, \left(\sum_{i=1}^n \mathbb{P}(A_i)\right)^{1/2}\right\}.\end{aligned} \tag{A.1}$$

Since $0 \le p < 1$, (A.1) implies

$$(1-p)\sum_{i=1}^n \mathbb{P}(A_i) \le \max\left\{\left((C_2p)^{2r-1}\sum_{i=1}^n \mathbb{P}(A_i)\right)^{1/(2r)}, \left(C_2p\sum_{i=1}^n \mathbb{P}(A_i)\right)^{1/2}\right\}, \tag{A.2}$$

where $C_2 = \max\{1, C_1^{1/r}\}$. By applying the Cauchy–Schwarz inequality, we have

$$\left((C_2p)^{2r-1}\sum_{i=1}^n \mathbb{P}(A_i)\right)^{1/(2r)} \le \frac{1}{2r}\left(\frac{(2r-1)C_2p}{(1-p)^{1/(2r-1)}} + (1-p)\sum_{i=1}^n \mathbb{P}(A_i)\right), \tag{A.3}$$

and

$$\left(C_2p\sum_{i=1}^n \mathbb{P}(A_i)\right)^{1/2} \le \frac{1}{2}\left(\frac{C_2p}{(1-p)} + (1-p)\sum_{i=1}^n \mathbb{P}(A_i)\right). \tag{A.4}$$

By using (A.2)–(A.4), and elementary computations, we have

$$(1-p)\sum_{i=1}^n \mathbb{P}(A_i) \le \max\left\{\frac{C_2p}{(1-p)^{1/(2r-1)}}, \frac{C_2p}{1-p}\right\} = \frac{C_2p}{1-p} \tag{A.5}$$

thereby proving (2.25). □

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