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ORIGINAL ARTICLE

On the (p, q)-type strong law of large numbers for sequences of independent random variables

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Abstract

Li, Qi, and Rosalsky (Trans. Amer. Math. Soc., 368 (2016), no. 1, 539–561) introduced a refinement of the Marcinkiewicz–Zygmund strong law of large numbers (SLLN), the so-called (p, q)-type SLLN, where $0 < p < 2$ and $q > 0$. They obtained sets of necessary and sufficient conditions for this new type SLLN for two cases: $0 < p < 1, q > p$, and $1 \leq p < 2, q \geq 1$. Results for the case where $0 < q \leq p < 1$ and $0 < q < 1 \leq p < 2$ remain open problems. This paper gives a complete solution to these problems. We consider random variables taking values in a real separable Banach space $\mathbf{B}$, but the results are new even when $\mathbf{B}$ is the real line. Furthermore, the conditions for a sequence of random variables $\{X_n, n \geq 1\}$ satisfying the (p, q)-type SLLN are shown to provide an exact characterization of stable type p Banach spaces.

KEYWORDS

complete convergence in mean, (p, q)-type strong law of large numbers, real separable Banach space, stable type p Banach space, strong law of large numbers

1 | INTRODUCTION AND MAIN RESULTS

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and taking values in a real separable Banach space $\mathbf{B}$ with norm $\|\cdot\|$. Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n),\ n \geq 1$. The sequence $\{X_n, \mathcal{F}_n, n \geq 1\}$ is said to be a quasimartingale (see, e.g., Pisier [21, p. 55]) if $\mathbb{E}(\|X_n\|) < \infty$ for all $n \geq 1$, and

$$\sum_{n=1}^{\infty} \mathbb{E}(\|\mathbb{E}(X_{n+1}|\mathcal{F}_n) - X_n\|) < \infty.$$

If the random variables are independent with mean zero, then it is easy to see that $\{(X_1 + \cdots + X_n)/n^{\alpha}, \mathcal{F}_n, n \geq 1\}, \alpha > 0$, is a quasimartingale if and only if

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}(\|X_1 + \cdots + X_n\|)}{n^{1+\alpha}} < \infty.$$

The study of limit theorems for random variables taking values in a Banach space is usually linked to the notion of "type" of the space. We refer to Giné and Zinn [5], Hoffmann-Jørgensen and Pisier [8], Kuelbs and Zinn [10], Ledoux and Talagrand [12], Marcus and Woyczyński [18], and Pisier [20] for definitions, equivalent characterizations, properties of a Banach space being of Rademacher type p or of stable type p, $1 \leq p \leq 2$.

Assume that $\{X, X_n, n \geq 1\}$ is a sequence of independent identically distributed (i.i.d.) $\mathbf{B}$-valued random variables, and $1 \leq p < 2$. In order to answer the question "when is $\left\{(X_1 + \cdots + X_n)/n^{1/p}, \mathcal{F}_n, n \geq 1\right\}$ a quasimartingale?" Henchner [6] and Hechner and Heinkel [7] proved the following striking result. Here and thereafter, $\ln x$ denotes the natural logarithm of a positive real number x.

Proposition 1.1 (Henchner [6], Hechner and Heinkel [7]). *Let $1 \leq p < 2$ and $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. mean zero $\mathbf{B}$-valued random variables. Suppose that the Banach space $\mathbf{B}$ is of stable type p. Then,*

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\left(\left\|\sum_{k=1}^{n} X_k\right\|\right)}{n^{1+1/p}} < \infty \tag{1.1}$$

if and only if

$$\begin{cases} \mathbb{E}(\|X\| \ln(1+\|X\|)) < \infty & \text{if } p = 1, \\ \int_0^{\infty} \mathbb{P}^{1/p}(\|X\| > t)\mathrm{d}t < \infty & \text{if } 1 < p < 2. \end{cases}$$

Motivated by the above result, Li, Qi, and Rosalsky [14, 15] provided conditions for

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\frac{\left\|\sum_{k=1}^{n} X_k\right\|}{n^{1/p}}\right)^q < \infty \tag{1.2}$$

for $0 < p < 2$ and $q > 0$. Clearly, (1.2) implies that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|\sum_{k=1}^{n} X_k\|}{n^{1/p}}\right)^q < \infty \text{ almost surely (a.s.).} \tag{1.3}$$

Li, Qi, and Rosalsky [15] proved that if (1.3) holds, then

$$\frac{\sum_{k=1}^{n} X_k}{n^{1/p}} \longrightarrow 0 \text{ a.s.,} \tag{1.4}$$

that is, the sequence $\{X, X_n, n \geq 1\}$ obeys the Marcinkiewicz–Zygmund strong law of large numbers (SLLN). It is well known that if $1 \leq p < 2$ and $\mathbf{B}$ is of Rademacher type p, then (1.4) holds if and only if $\mathbb{E}(\|X\|^p) < \infty$ and $\mathbb{E}(X) = 0$ (see, e.g., de Acosta [2]). For the case where $0 < q < p < 2$, Li, Qi, and Rosalsky [15, Theorem 3] proved that (1.3) implies $\int_0^{\infty} \mathbb{P}^{q/p}(\|X\|^q > t)\mathrm{d}t < \infty$, which is stronger than $\mathbb{E}(\|X\|^p) < \infty$. Precisely, Li, Qi, and Rosalsky [15] proved the following result.

Proposition 1.2 (Li, Qi, and Rosalsky [15]). *Let $0 < p < 2$, $q > 0$, and let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables taking values in a real separable Banach space $\mathbf{B}$. Then (1.2) is equivalent to (1.3) and*

$$\begin{cases} \int_0^{\infty} \mathbb{P}^{q/p}(\|X\|^q > t)\mathrm{d}t < \infty & \text{if } q < p, \\ \mathbb{E}(\|X\|^p \ln(1+\|X\|)) < \infty & \text{if } q = p, \\ \mathbb{E}(\|X\|^q) < \infty & \text{if } q > p. \end{cases} \tag{1.5}$$

Furthermore, each of (1.2) and (1.3) implies the Marcinkiewicz–Zygmund SLLN (1.4). For $0 < q < p < 2$, (1.2) and (1.3) are equivalent so that each of them implies that (1.4) and (1.5) hold.

Motivated by the results in [7, 14, 15], Li, Qi, and Rosalsky [16] introduced an interesting type of SLLN as follows:

Definition 1.3 (Li, Qi, and Rosalsky [16]). Let $0 < p < 2$, $q > 0$, and let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. $\mathbf{B}$-random variables. We say that X satisfies the (p, q)-type SLLN (and write $X \in \mathrm{SLLN}(p, q)$) if (1.3) holds.

Li, Qi, and Rosalsky [16] obtained sets of necessary and sufficient conditions for $X \in \mathrm{SLLN}(p, q)$ for two cases: $0 < p < 1, q > p$ and $1 \leq p < 2, q \geq 1$ ([16, Theorems 2.1, 2.2 and 2.3]). For other cases, necessary and sufficient conditions for $X \in \mathrm{SLLN}(p, q)$ remain open problems even when $\mathbf{B} = \mathbb{R}$ as noted by Li, Qi, and Rosalsky [16, p. 541]. In this note, we give a complete solution to these open problems by providing the necessary and sufficient conditions for the (p, q)-type SLLN for the remaining cases: $0 < q \leq p < 1$ and $0 < q < 1 \leq p < 2$. Our main results for the real-valued random variable case can be summarized in the following theorem. In this paper, the indicator function of a set A will be denoted by $\mathbf{1}(A)$.

Theorem 1.4. *Let $0 < p < 2$ and $q > 0$. Let $\{X_n, n \geq 1\}$ be a sequence of independent copies of a real-valued random variable X, and u_n the quantile of order $1 - 1/n$ of $|X|$, $n \geq 1$. The following two statements are equivalent:*

(i) $X \in \mathrm{SLLN}(p, q)$.

(ii)

$$\begin{cases} \displaystyle\int_0^\infty \mathbb{P}^{q/p}(|X|^q > t)\mathrm{d}t < \infty & \text{if } 0 < q < p < 1, \\ \mathbb{E}(|X|^p) < \infty \text{ and} & \\ \displaystyle\sum_{n=1}^\infty \frac{\mathbb{E}\left(|X|^p \mathbf{1}(\min\{u_n^p, n\} < |X|^p \leq n)\right)}{n} < \infty & \text{if } 0 < q = p < 1, \\ \mathbb{E}(X) = 0 \text{ and } \displaystyle\int_0^\infty \mathbb{P}^{q/p}(|X|^q > t)\mathrm{d}t < \infty & \text{if } 0 < q < 1 \leq p < 2. \end{cases}$$

The following two statements are equivalent:

(iii) $\sum_{n=1}^\infty \frac{1}{n}\mathbb{E}\left(\frac{|\sum_{i=1}^n X_i|}{n^{1/p}}\right)^q < \infty$.

(iv)

$$\begin{cases} \displaystyle\int_0^\infty \mathbb{P}^{q/p}(|X|^q > t)\mathrm{d}t < \infty & \text{if } 0 < q < p < 1, \\ \mathbb{E}(|X|^p \ln(1 + |X|)) < \infty & \text{if } 0 < q = p < 1, \\ \mathbb{E}(X) = 0 \text{ and } \displaystyle\int_0^\infty \mathbb{P}^{q/p}(|X|^q > t)\mathrm{d}t < \infty & \text{if } 0 < q < 1 \leq p < 2. \end{cases}$$

Versions of the above results in the Banach space setting are also given, and, especially, the conditions for the sequence $\{X_n, n \geq 1\}$ satisfying the (p, q)-type SLLN are shown to provide an exact characterization of stable type p Banach spaces. The latter result was not discovered by Li, Qi, and Rosalsky [16] even for the case $1 \leq p < 2, q \geq 1$. The results are obtained by developing some techniques in Hechner and Heinkel [7], and in Li, Qi, and Rosalsky [14–16], and by using some results regarding the notion of complete convergence in mean of order p developed by Rosalsky, Thanh, and Volodin [22].

In the rest of the paper, we always consider random variables that take values in a real separable Banach space $\mathbf{B}$ if no further clarification is needed. For a random variable X and for each $n \geq 1$, u_n denotes the quantile of order $1 - 1/n$ of $\|X\|$, that is,

$$u_n = \inf\left\{t : \mathbb{P}(\|X\| \leq t) > 1 - \frac{1}{n}\right\} = \inf\left\{t : \mathbb{P}(\|X\| > t) < \frac{1}{n}\right\}.$$

We now present Banach space versions of Theorem 1.4. Theorem 1.5 provides the necessary and sufficient conditions for $X \in \mathrm{SLLN}(p, q)$ for the case where $0 < q \leq p < 1$, while Theorem 1.7 deals with the case where $0 < q < 1 \leq p < 2$.

Theorem 1.5. *Let $0 < q \le p < 1$ and let $\{X, X_n, n \ge 1\}$ be a sequence of i.i.d. random variables. Then,*

$$X \in \mathrm{SLLN}(p, q) \tag{1.6}$$

if and only if

$$\begin{cases} \int_0^\infty \mathbb{P}^{q/p}(\|X\|^q > t)\mathrm{d}t < \infty & \text{if } q < p, \\ \mathbb{E}(\|X\|^p) < \infty \text{ and} & \\ \sum_{n=1}^\infty \frac{\mathbb{E}\left(\|X\|^p \mathbf{1}(\min\{u_n^p, n\} < \|X\|^p \le n)\right)}{n} < \infty & \text{if } q = p. \end{cases} \tag{1.7}$$

Remark 1.6. We make some comments on Theorem 1.5 as follows.

(i) As noted by Li, Qi, and Rosalsky [15], if $X \in \mathrm{SLLN}(p, q)$ for some $q > 0$, then $X \in \mathrm{SLLN}(p, q_1)$ for all $q_1 > q$. By Theorem 1.5 we will show that, for $0 < p < 1$, there exists a random variable X such that $X \in \mathrm{SLLN}(p, p)$ but $X \notin \mathrm{SLLN}(p, q)$ for all $0 < q < p$ (see Example 4.3 in Section 4).
(ii) For the case where $q = p$, each of two conditions $\mathbb{E}(\|X\|^p) < \infty$ and

$$\sum_{n=1}^\infty \frac{\mathbb{E}\left(\|X\|^p \mathbf{1}(\min\{u_n^p, n\} < \|X\|^p \le n)\right)}{n} < \infty$$

do not imply each other (see Examples 4.4 and 4.5 in Section 4).

Theorem 1.7. *Let $0 < q < 1 \le p < 2$ and let $\{X, X_n, n \ge 1\}$ be a sequence of i.i.d. random variables taking values in a real separable Banach space* **B**. *If* **B** *is of stable type p, then*

$$X \in \mathrm{SLLN}(p, q) \tag{1.8}$$

if and only if

$$\mathbb{E}(X) = 0 \text{ and } \int_0^\infty \mathbb{P}^{q/p}(\|X\|^q > t)\mathrm{d}t < \infty. \tag{1.9}$$

Li, Qi, and Rosalsky [16] also provided necessary and sufficient conditions for

$$\sum_{n=1}^\infty \frac{1}{n}\mathbb{E}\left(\frac{\left\|\sum_{k=1}^n X_k\right\|}{n^{1/p}}\right)^q < \infty \tag{1.10}$$

for the case where $0 < p < 1, q > p$ and for the case where $1 \le p < 2, q \ge 1$ (see [16, Theorem 2.1 and Corollaries 2.2 and 2.3]). From Theorems 1.5 and 1.7, we have the following corollary.

Corollary 1.8. *Let $\{X, X_n, n \ge 1\}$ be a sequence of i.i.d. random variables taking values in a real separable Banach space* **B**.

(i) If $0 < q \le p < 1$, then (1.10) is equivalent to

$$\begin{cases} \int_0^\infty \mathbb{P}^{q/p}(\|X\|^q > t)\mathrm{d}t < \infty & \text{if } q < p, \\ \mathbb{E}(\|X\|^p \ln(1 + \|X\|)) < \infty & \text{if } q = p. \end{cases} \tag{1.11}$$

(ii) If $0 < q < 1 \leq p < 2$*, and* $\mathbf{B}$ *is of stable type* p*, then (1.10) is equivalent to*

$$\mathbb{E}(X) = 0 \ \textit{and} \ \int_0^\infty \mathbb{P}^{q/p}(\|X\|^q > t)\mathrm{d}t < \infty. \tag{1.12}$$

By combining Theorems 1.5 and 1.7, and Corollary 1.8, we obtain Theorem 1.4. Characterizations of SLLN in Banach spaces was proved by Hoffmann-Jørgensen and Pisier [8], de Acosta [2], and Mikosch and Norvaiša [19]. Ledoux and Talagrand [11], and more recently Einmahl and Li [3], discovered characterizations of the law of the iterated logarithm for Banach-valued random variables. Our Theorems 1.5 and 1.7, Corollary 1.8, and the findings by Li, Qi, and Rosalsky [16], and Hechner and Heinkel [7] complete a picture of characterizations of the (p, q)-type SLLN in Banach spaces, as well as characterizations of

$$\sum_{n=1}^{\infty} \frac{1}{n}\mathbb{E}\left(\frac{\left\|\sum_{k=1}^{n} X_k\right\|}{n^{1/p}}\right)^q < \infty, \ 0 < p < 2, \ q > 0.$$

The rest of the paper is organized as follows. In Section 2, we prove that the (p, q)-type SLLN implies the Marcinkiewicz–Zygmund SLLN without assuming that the random variables are identically distributed. This result allows us to provide an exact characterization of stable type p Banach spaces through the (p, q)-type SLLN, which we present and prove in Section 3. In Section 4, we will prove Theorems 1.5, 1.7, and Corollary 1.8. Finally, the paper is concluded with further remarks in Section 5.

2 | THE (p, q)-TYPE SLLN IMPLIES THE MARCINKIEWICZ–ZYGMUND SLLN

A sequence of random variables $\{X_n, n \geq 1\}$ is said to be stochastically dominated by a random variable X if

$$\sup_{n\geq 1} \mathbb{P}(\|X_n\| > t) \leq \mathbb{P}(\|X\| > t), \ t \geq 0. \tag{2.1}$$

It is well known that for a sequence of independent mean zero random variables $\{X_n, n \geq 1\}$ taking values in a real separable stable type p Banach space $\mathbf{B}$, $1 \leq p < 2$, the condition that $\{X_n, n \geq 1\}$ are stochastically dominated by a random variable X with $\mathbb{E}(\|X\|^p) < \infty$ implies the Marcinkiewicz–Zygmund SLLN, that is,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n^{1/p}} = 0 \text{ a.s.}$$

However, this is no longer true if $\mathbf{B}$ is of Rademacher type p only. To see this, let $1 \leq p < 2$, and ℓ_p denote the real separable Rademacher type p Banach space of absolute pth power summable real sequences $v = \{v_i, i \geq 1\}$ with norm

$$\|v\| = \left(\sum_{i=1}^{\infty} |v_i|^p\right)^{1/p},$$

and define a sequence $\{V_n, n \geq 1\}$ of independent random variables in ℓ_p by requiring the $\{V_n, n \geq 1\}$ to be independent with

$$\mathbb{P}(V_n = -v^{(n)}) = \mathbb{P}(V_n = v^{(n)}) = \frac{1}{2}, \ n \geq 1,$$

where for $n \geq 1$, $v^{(n)}$ is the element of ℓ_p having 1 in its nth position and 0 elsewhere. Then, the sequence $\{V_n, n \geq 1\}$ is stochastically dominated by V_1 with $\mathbb{E}(\|V_1\|)^p = 1$. However, $\{V_n, n \geq 1\}$ does not obey the Marcinkiewicz–Zygmund

SLLN since for all $n \geq 1$,

$$\frac{\| \sum_{i=1}^{n} V_i \|}{n^{1/p}} = 1.$$

In this section, we will prove that for $1 \leq p < 2$, $q > 0$, and for a sequence of independent mean zero random variables $\{X_n, n \geq 1\}$, which is stochastically dominated by a random variable X with $\mathbb{E}(\|X\|^p) < \infty$, the (p, q)-type SLLN implies the Marcinkiewicz–Zygmund SLLN. Li, Qi, and Rosalsky [15, Lemma 3] proved this result for i.i.d. random variables $\{X, X_n, n \geq 1\}$ by using a generalization of Ottaviani's inequality developed by Li and Rosalsky [13] and the strong stationary property of the sequence $\{X_n, n \geq 1\}$ without assuming that $\mathbb{E}(\|X\|^p) < \infty$. In our setting, $\{X_n, n \geq 1\}$ is no longer stationary. The method we present here is completely different from that of Li, Qi, and Rosalsky [15, Lemma 3]. We involve a symmetrization argument and some techniques regarding the notion of complete convergence in mean of order p developed by Rosalsky, Thanh, and Volodin [22]. The result of this section will be used to show that the conditions for the sequence $\{X_n, n \geq 1\}$ satisfying the (p, q)-type SLLN in Theorem 1.7 are shown to provide an exact characterization of stable type p Banach spaces.

First, we will need the following two lemmas. The first lemma is a simple modification of Theorems 1 and 2 of Etemadi [4].

Lemma 2.1. *Let $\alpha > 0$, and let $\{X_n, n \geq 1\}$ be a sequence of independent random variables. Then,*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n^{\alpha}} = 0 \text{ a.s.} \tag{2.2}$$

if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}\left(\left\| \sum_{i=n+1}^{2n} X_i \right\| > n^{\alpha}\varepsilon \right) < \infty \text{ for all } \varepsilon > 0 \tag{2.3}$$

and

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n^{\alpha}} = 0 \text{ in probability.} \tag{2.4}$$

If we assume further that $\{X_n, n \geq 1\}$ are symmetric random variables, then (2.2) and (2.3) are equivalent.

Proof. The proof of the first part is the same as that of Theorem 2 of Etemadi [4]. The proof of the last part is the same as that of Theorem 1 of Etemadi [4]. □

The next lemma shows that for independent (not necessary identically distributed) random variables $\{X, X_n, n \geq 1\}$, (1.10) implies a SLLN. When $\alpha = 1$ and $1 \leq q \leq 2$, Lemma 2.2 is Theorem 3 of Rosalsky, Thanh, and Volodin [22]. The double sum version of Theorem 3 of Rosalsky, Thanh, and Volodin [22] was proved in [23].

Lemma 2.2. *Let $\alpha > 0$, $q \geq 1$, and let $\{X_n.n \geq 1\}$ be a sequence of independent mean zero random variables. If*

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\frac{\left\| \sum_{k=1}^{n} X_k \right\|}{n^{\alpha}} \right)^q < \infty, \tag{2.5}$$

then

$$\frac{\sum_{i=1}^{n} X_i}{n^{\alpha}} \xrightarrow{\mathcal{L}_q} 0, \text{ and } \frac{\sum_{i=1}^{n} X_i}{n^{\alpha}} \xrightarrow{a.s.} 0. \tag{2.6}$$

Proof. For $n \geq 1$, set

$$S_n = \sum_{i=1}^{n} X_i.$$

Then, $\{\mathbb{E}\|S_n\|^q, n \geq 1\}$ is a nondecreasing sequence (see, e.g., [22, Lemma 2]). Therefore, by applying (2.5), we have

$$\begin{aligned}\mathbb{E}\left(\left\|\frac{S_n}{n^\alpha}\right\|^q\right) &\leq \alpha q \sum_{m=n}^{\infty} \frac{1}{m^{1+\alpha q}} \mathbb{E}(\|S_n\|^q) \\ &\leq \alpha q \sum_{m=n}^{\infty} \frac{1}{m^{1+\alpha q}} \mathbb{E}(\|S_m\|^q) \to 0 \text{ as } n \to \infty,\end{aligned} \tag{2.7}$$

thereby proving the first half of (2.6). Moreover, it follows from (2.5) and Markov's inequality that for arbitrary $\varepsilon > 0$,

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}\left(\left\|\sum_{i=n+1}^{2n} X_i\right\| > n^\alpha \varepsilon\right) \\ &\leq \left(\frac{2}{\varepsilon}\right)^q \left(\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\left\|\frac{S_{2n}}{n^\alpha}\right\|^q\right) + \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\left\|\frac{S_n}{n^\alpha}\right\|^q\right)\right) < \infty.\end{aligned} \tag{2.8}$$

The second half of (2.6) then follows from the first part of Lemma 2.1, (2.8), and the first part of (2.6). □

The main result of this section is the following proposition.

Proposition 2.3. *Let $1 \leq p < 2$ and $q > 0$, and let $\{X_n, n \geq 1\}$ be a sequence of independent mean zero random variables, which is stochastically dominated by a random variable X with $\mathbb{E}(\|X\|^p) < \infty$. We assume further that the random variables $X_n, n \geq 1$ are symmetric when $0 < q < 1$. If*

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|S_n\|}{n^{1/p}}\right)^q < \infty \text{ a.s.,} \tag{2.9}$$

then

$$\lim_{n\to\infty} \frac{S_n}{n^{1/p}} = 0 \text{ a.s.} \tag{2.10}$$

Proof. Set

$$Y_n = X_n \mathbf{1}(\|X_n\|^p \leq n),\ S_n^{(1)} = \sum_{i=1}^{n} Y_i,\ n \geq 1.$$

Since $\mathbb{E}(\|X\|^p) < \infty$ and the sequence $\{X_n, n \geq 1\}$ is stochastically dominated by X,

$$\sum_{n=1}^{\infty} \mathbb{P}(\|X_n\|^p > n) \leq \sum_{n=1}^{\infty} \mathbb{P}(\|X\|^p > n) < \infty. \tag{2.11}$$

By the Borel–Cantelli lemma, it follows from (2.11) that

$$\mathbb{P}(\|X_n\|^p > n \text{ i.o. } (n)) = 0. \tag{2.12}$$

Combining (2.9) and (2.12), we have

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\|S_n^{(1)}\|}{n^{1/p}}\right)^q < \infty \text{ a.s.} \tag{2.13}$$

To prove (2.10), recalling (2.12), it suffices to show that

$$\lim_{n\to\infty}\frac{S_n^{(1)}}{n^{1/p}} = 0 \text{ a.s.} \tag{2.14}$$

For $n \geq 1$, set

$$a_n = \frac{1}{n^{1+q/p}},\ b_n = \sum_{k=n}^{\infty} a_k.$$

Then,

$$\mathbb{E}\left(\sup_{n\geq 1} b_n\|Y_n\|^q\right) \leq \mathbb{E}\left(\sup_{n\geq 1}\left(1+\frac{p}{q}\right)\frac{\|Y_n\|^q}{n^{q/p}}\right) \leq 1+\frac{q}{p}. \tag{2.15}$$

First, we consider the case where $0 < q < 1$. Since $\{X_n, n \geq 1\}$ are symmetric random variables, $\{Y_n, n \geq 1\}$ are also symmetric. By applying inequality (11) in Theorem 7 of Li, Qi, and Rosalsky [15], we conclude from (2.13) and (2.15) that

$$\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{E}\left(\frac{\|S_n^{(1)}\|}{n^{1/p}}\right)^q < \infty. \tag{2.16}$$

It follows from (2.16) and Markov's inequality that for arbitrary $\varepsilon > 0$,

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{P}\left(\left\|\sum_{i=n+1}^{2n} Y_i\right\| > n^{1/p}\varepsilon\right)\\ &\leq \left(\frac{2}{\varepsilon}\right)^q\left(\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{E}\left(\left\|\frac{S_{2n}^{(1)}}{n^{1/p}}\right\|^q\right)+\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{E}\left(\left\|\frac{S_n^{(1)}}{n^{1/p}}\right\|^q\right)\right) < \infty.\end{aligned} \tag{2.17}$$

The conclusion (2.14) then follows from the last part of Lemma 2.1.

Next, we consider the case where $q \geq 1$. Let $\left\{X', X_n', n \geq 1\right\}$ be an independent copy of $\{X, X_n, n \geq 1\}$. For $n \geq 1$, set

$$V_n = Y_n - X_n'\mathbf{1}(\|X_n'\|^p \leq n),$$

and

$$\hat{S}_n^{(1)} = \sum_{i=1}^{n} V_i.$$

By (2.13), we have

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\|\hat{S}_n^{(1)}\|}{n^{1/p}}\right)^q < \infty \text{ a.s.} \tag{2.18}$$

Similar to the proof of (2.16), (2.18) leads to

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\frac{\|\hat{S}_n^{(1)}\|}{n^{1/p}} \right)^q < \infty. \tag{2.19}$$

By Lemma 4 of Li, Qi, and Rosalsky [15], under (2.18), (2.19) is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\frac{\|S_n^{(1)}\|}{n^{1/p}} \right)^q < \infty. \tag{2.20}$$

This implies

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\frac{\| \sum_{i=1}^{n} (Y_i - \mathbb{E}(Y_i))\|}{n^{1/p}} \right)^q \le 2^{q-1} \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\frac{\|S_n^{(1)}\|}{n^{1/p}} \right)^q < \infty. \tag{2.21}$$

By Lemma 2.2, we have from (2.21) that

$$\frac{\sum_{i=1}^{n} (Y_i - \mathbb{E}(Y_i))}{n^{1/p}} \to 0 \text{ a.s.} \tag{2.22}$$

Since $\mathbb{E}(X_n) = 0$ and $\{X_n, n \ge 1\}$ is stochastically dominated by X with $\mathbb{E}(\|X\|^p) < \infty$, it is routine to prove that

$$\lim_{n\to\infty} \left\| \frac{\sum_{i=1}^{n} \mathbb{E}(Y_i)}{n^{1/p}} \right\| \le \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \mathbb{E}(\|X\| \mathbf{1}(\|X\|^p > i))}{n^{1/p}} = 0. \tag{2.23}$$

Combining (2.22) and (2.23), we obtain (2.14). □

3 | CHARACTERIZATIONS OF STABLE TYPE p BANACH SPACES

This section shows that for the sufficiency part of Theorem 1.7, we can relax the identically distributed condition of the random variables $\{X_n, n \ge 1\}$. Furthermore, the conditions for the sequence $\{X_n, n \ge 1\}$ satisfying the (p, q)-type SLLN are shown to provide an exact characterization of stable type p Banach spaces.

Theorem 3.1. *Let $0 < q < 1 \le p < 2$ and let $\mathbf{B}$ be a separable Banach space. Then, the following statements are equivalent.*

(i) *$\mathbf{B}$ is of stable type p.*
(ii) *For every sequence $\{X_n, n \ge 1\}$ of independent mean zero $\mathbf{B}$-valued random variables, which is stochastically dominated by a random variable X, the condition*

$$\int_0^{\infty} \mathbb{P}^{q/p}(\|X\|^q > t) \mathrm{d}t < \infty \tag{3.1}$$

implies

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\|S_n\|}{n^{1/p}} \right)^q < \infty \text{ a.s.} \tag{3.2}$$

To prove Theorem 3.1, we first present some preliminaries. Let $\{X_k, 1 \le k \le n\}$ be n independent real-valued random variables and $\{X_k^*, 1 \le k \le n\}$ the nonincreasing rearrangement of the sequence $\{|X_k|, 1 \le k \le n\}$. Then, the Marcus–Pisier inequality [17] (see also Pisier [20, Lemma 4.11]) asserts that for all $r \geq 1$,

$$\mathbb{P}\left(\sup_{1\leq k\leq n} k^{1/r}X_k^* > u\right) \leq \frac{2e}{u^r}\sup_{t>0}\left(t^r\sum_{k=1}^n \mathbb{P}(|X_k| > t)\right) \text{ for all } u > 0. \tag{3.3}$$

When $q = 1$, the following lemma is Lemma 1 of Hechner and Heinkel [7]. Li, Qi, and Rosalsky [16] generalized Lemma 1 of Hechner and Heinkel [7] for the case where $1 \leq q < r < 2$ (see Lemma 3.1 Li, Qi, and Rosalsky [16]). Lemma 3.2 shows that their results also hold when $0 < q \leq 1 < r < 2$.

Lemma 3.2. *Let $0 < q \leq 1 < r < 2$ and let $\mathbf{B}$ be a Banach space of stable type r. Then, for every finite sequence $\{X_k, 1 \leq k \leq n\}$ of independent $\mathbf{B}$-valued random variables with $\max_{1\leq k\leq n}\mathbb{E}(\|X_k\|^q) < \infty$, there exists a constant $C(q,r) > 0$ depending only on q and r such that*

$$\mathbb{E}\left(\left\|\sum_{k=1}^n (X_k - EX_k)\right\|^q\right) \leq C(q,r)\left(\sup_{t>0} t^{r/q}\sum_{k=1}^n \mathbb{P}(\|X_k\|^q > t)\right)^{q/r}. \tag{3.4}$$

Proof. Since $0 < q \leq 1 < r < 2$, we have

$$\begin{aligned}
\mathbb{E}\left(\left\|\sum_{k=1}^n (X_k - EX_k)\right\|^q\right) &\leq \left(\mathbb{E}\left\|\sum_{k=1}^n (X_k - EX_k)\right\|\right)^q \\
&\leq \left(C(r)\left(\sup_{t>0} t^r\sum_{k=1}^n \mathbb{P}(\|X_k\| > t)\right)^{1/r}\right)^q \\
&= (C(r))^q\left(\sup_{t>0} t^r\sum_{k=1}^n \mathbb{P}(\|X_k\| > t)\right)^{q/r} \\
&:= C(q,r)\left(\sup_{t>0} t^{r/q}\sum_{k=1}^n \mathbb{P}(\|X_k\|^q > t)\right)^{q/r},
\end{aligned} \tag{3.5}$$

where we have applied Liapunov's inequality in the first inequality and Lemma 1 of Hechner and Heinkel [7] in the second inequality. This completes the proof of Lemma 3.2. □

The following result is a variation of Lemma 3.2 for the case where $0 < q < r < 1$.

Lemma 3.3. *Let $0 < q < r < 1$. Then for every finite sequence $\{X_k, 1 \leq k \leq n\}$ of independent random variables with $\max_{1\leq k\leq n}\mathbb{E}(\|X_k\|^q) < \infty$, we have*

$$\mathbb{E}\left(\left\|\sum_{k=1}^n X_k\right\|^q\right) \leq C_1(q,r)\left(\sup_{t>0} t^{r/q}\sum_{k=1}^n \mathbb{P}(\|X_k\|^q > t)\right)^{q/r}, \tag{3.6}$$

where

$$C_1(q,r) = \left(\frac{1}{1-r}\right)^q\left(1 + \frac{2qe}{r-q}\right).$$

Proof. Let $\{\|X_k\|^*, 1 \le k \le n\}$ be the nonincreasing rearrangement of $\{\|X_k\|, 1 \le k \le n\}$. Since $0 < q < r < 1$,

$$\begin{aligned}
\mathbb{E}\left(\left\|\sum_{k=1}^{n} X_k\right\|^q\right) &\le \mathbb{E}\left(\sum_{k=1}^{n} \|X_k\|\right)^q \\
&= \mathbb{E}\left(\sum_{k=1}^{n} \left(k^{1/r}\|X_k\|^*\right)k^{-1/r}\right)^q \\
&\le \mathbb{E}\left[\sup_{1\le k\le n} \left(k^{q/r}(\|X_k\|^*)^q\right)\left(\sum_{k=1}^{n} k^{-1/r}\right)^q\right] \\
&= \mathbb{E}\left[\sup_{1\le k\le n} \left(k^{q/r}(\|X_k\|^q)^*\right)\left(\sum_{k=1}^{n} k^{-1/r}\right)^q\right] \\
&\le \left(\frac{1}{1-r}\right)^q \mathbb{E}\left(\sup_{1\le k\le n} \left(k^{q/r}(\|X_k\|^q)^*\right)\right) \\
&= \left(\frac{1}{1-r}\right)^q \int_0^\infty \mathbb{P}\left(\sup_{1\le k\le n} \left(k^{q/r}(\|X_k\|^q)^*\right) > u\right)\mathrm{d}u.
\end{aligned} \tag{3.7}$$

Let $\Delta = \sup_{t>0} t^{r/q} \sum_{k=1}^{n} \mathbb{P}(\|X_k\|^q > t)$. Applying (3.3), we have

$$\begin{aligned}
&\int_0^\infty \mathbb{P}\left(\sup_{1\le k\le n} \left(k^{q/r}(\|X_k\|^q)^*\right) > u\right)\mathrm{d}u \\
&= \left(\int_0^{\Delta^{q/r}} + \int_{\Delta^{q/r}}^\infty\right)\mathbb{P}\left(\sup_{1\le k\le n} \left(k^{q/r}(\|X_k\|^q)^*\right) > u\right)\mathrm{d}u \\
&\le \Delta^{q/r} + 2\mathrm{e}\int_{\Delta^{q/r}}^\infty \frac{\Delta}{u^{r/q}}\mathrm{d}u \\
&= \left(1 + \frac{2q\mathrm{e}}{r-q}\right)\Delta^{q/r}.
\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we obtain (3.6). □

Motivated by Lemma 3.4 of Li, Qi, and Rosalsky [16], which considered the case where $1 \le q \le p < 2$ and i.i.d. random variables, we have the following lemma.

Lemma 3.4. *Let $0 < q \le p < 2$, and let $\{X_n\}$ be a sequence of independent* **B***-valued random variables. Suppose that $\{X_n, n \ge 1\}$ is stochastically dominated by a random variable X satisfying*

$$\int_0^\infty \mathbb{P}^{q/p}(\|X\|^q > t)\mathrm{d}t < \infty. \tag{3.9}$$

For each $n \ge 1$, let the quantile u_n of order $1 - 1/n$ of $\|X\|$ be defined as in Section 1, and set

$$Y_{n,k} = X_k\mathbf{1}(\|X_k\|^p \le u_n),\ Z_{n,k} = X_k\mathbf{1}(\|X_k\|^p \le n),$$

$$U_{n,k} = \sum_{i=1}^{k} Z_{n,i},\ U_{n,k}^{(1)} = \sum_{i=1}^{k} Y_{n,i},\ U_{n,k}^{(2)} = U_{n,k} - U_{n,k}^{(1)}.$$

Then the following statement holds.

(i) If $0 < p < 1$, *then*

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\left(\|U_{n,n}^{(1)}\|^q\right)}{n^{1+q/p}} < \infty. \tag{3.10}$$

(ii) If $1 \le p < 2$ *and* $\mathbf{B}$ *is of stable type* p, *then*

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\left(\|U_{n,n}^{(1)} - \mathbb{E}U_{n,n}^{(1)}\|^q\right)}{n^{1+q/p}} < \infty. \tag{3.11}$$

In particular, if $\mathbb{E}(\|X\|^p) < \infty$, *then the following statement holds.*

(iii) If $0 < p < 1$, *then*

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\left(\|U_{n,n}^{(1)}\|^p\right)}{n^2} < \infty. \tag{3.12}$$

(iv) If $1 \le p < 2$ *and* $\mathbf{B}$ *is of stable type* p, *then*

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\left(\|U_{n,n}^{(1)} - \mathbb{E}U_{n,n}^{(1)}\|^p\right)}{n^2} < \infty. \tag{3.13}$$

Proof. First, we consider the case where $0 < q \le p < 1$. Let $p < r < 1$, and $C_1(q,r)$ be as in Lemma 3.3. By applying Lemma 3.3, we obtain

$$\begin{aligned}
\mathbb{E}\left(\left\|U_{n,n}^{(1)}\right\|^q\right) &\le C_1(q,r)\left(\sup_{t>0} t^{r/q}\sum_{k=1}^{n}\mathbb{P}(\|X_k\|^q I\{\|X_k\| \le u_n\} > t)\right)^{q/r} \\
&= C_1(q,r)\left(\sup_{0\le t\le u_n^q} t^{r/q}\sum_{k=1}^{n}\mathbb{P}(\|X_k\|^q I\{\|X_k\| \le u_n\} > t)\right)^{q/r} \\
&\le C_1(q,r)\left(n \sup_{0\le t\le u_n^q} t^{r/q}\mathbb{P}(\|X\|^q > t)\right)^{q/r} \\
&= C_1(q,r)\left(n \sup_{0\le t\le u_n^q}\left(\int_0^t \mathbb{P}^{q/r}(\|X\|^q > t)\mathrm{d}x\right)^{r/q}\right)^{q/r} \\
&\le C_1(q,r)\left(n \sup_{0\le t\le u_n^q}\left(\int_0^t \mathbb{P}^{q/r}(\|X\|^q > x)\mathrm{d}x\right)^{r/q}\right)^{q/r} \\
&= C_1(q,r)n^{q/r}\int_0^{u_n^q} \mathbb{P}^{q/r}(\|X\|^q > x)\mathrm{d}x \\
&= C_1(q,r)n^{q/r}\sum_{k=1}^{n}\int_{u_{k-1}^q}^{u_k^q} \mathbb{P}^{q/r}(\|X\|^q > x)\mathrm{d}x.
\end{aligned} \tag{3.14}$$

For $k \geq 1$ and $u_{k-1}^q \leq x < u_k^q$, we have $\mathbb{P}(\|X\|^q > x) \geq 1/k$. It thus follows from (3.14) that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\mathbb{E}\left(\|U_{n,n}^{(1)}\|^q\right)}{n^{1+q/p}} &\leq C_1(q,r) \sum_{n=1}^{\infty} \frac{1}{n^{1+q/p-q/r}} \sum_{k=1}^{n} \int_{u_{k-1}^q}^{u_k^q} \mathbb{P}^{q/r}(\|X\|^q > x)\mathrm{d}x \\
&= C_1(q,r) \sum_{k=1}^{\infty} \left(\int_{u_{k-1}^q}^{u_k^q} \mathbb{P}^{q/r}(\|X\|^q > x)\mathrm{d}x \right) \left(\sum_{n=k}^{\infty} \frac{1}{n^{1+q/p-q/r}} \right) \\
&\leq \left(1 + \frac{pr}{q(r-p)}\right) C_1(q,r) \sum_{k=1}^{\infty} \frac{1}{k^{q/p-q/r}} \int_{u_{k-1}^q}^{u_k^q} \mathbb{P}^{q/r}(\|X\|^q > x)\mathrm{d}x \\
&\leq \left(1 + \frac{pr}{q(r-p)}\right) C_1(q,r) \sum_{k=1}^{\infty} \int_{u_{k-1}^q}^{u_k^q} \mathbb{P}^{q/p}(\|X\|^q > x)\mathrm{d}x \\
&= \left(1 + \frac{pr}{q(r-p)}\right) C_1(q,r) \int_0^{\infty} \mathbb{P}^{q/p}(\|X\|^q > x)\mathrm{d}x < \infty,
\end{aligned}$$

thereby proving (3.10) for the case where $0 < q \leq p < 1$.

For the case where $1 \leq q \leq p < 2$, Li, Qi, and Rosalsky [16] proved (3.10) under a stronger assumption that $\{X, X_n, n \geq 1\}$ are identically distributed random variables (see [16, Lemma 3.4]). When the sequence $\{X_n, n \geq 1\}$ is stochastically dominated by X, their proof will be unchanged except for some simple modifications and therefore we conclude that Lemma 3.4 holds for the case where $1 \leq q \leq p < 2$.

Next, we consider the case where $0 < q < 1 \leq p < 2$. Li, Qi, and Rosalsky [16] proved their Lemma 3.4 ([16, p. 548]) by applying (3.4) for $1 \leq q < r < 2$. In our Lemma 3.2, we have showed that (3.4) holds for the case where $0 < q < 1 < r < 2$. Then, by using the same argument as in the proof of Lemma 3.4 of Li, Qi, and Rosalsky [16], we obtain (3.10) for the case where $0 < q < 1 \leq p < 2$.

Finally, by taking $q = p$, (3.9) holds if and only if $\mathbb{E}(\|X\|^p) < \infty$, and (3.10) coincides with (3.12), (3.11) coincides with (3.13). Therefore, the last part of the lemma follows from the first part. This completes the proof. □

Proof of Theorem 3.1. First, we verify the implication (i)⇒(ii). For each $n \geq 1$, let the quantile u_n of order $1 - 1/n$ of $\|X\|$ be defined as in Section 1, and set for $1 \leq k \leq n$,

$$Y_{n,k} = X_k \mathbf{1}(\|X_k\|^p \leq u_n),\ U_{n,k}^{(1)} = \sum_{i=1}^{k} Y_{n,i}.$$

By following the proof of Lemma 3.3 of Li, Qi, and Rosalsky [16] and noting that every real separable Banach space is of Rademacher type q for all $0 < q \leq 1$, we have

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\left(\left\|\left(S_n - U_{n,n}^{(1)}\right) - \mathbb{E}\left(S_n - U_{n,n}^{(1)}\right)\right\|^q\right)}{n^{1+q/p}} < \infty. \tag{3.15}$$

Noting that $1 \leq p < 2$ and therefore applying (3.11), we have

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\left(\left\|U_{n,n}^{(1)} - \mathbb{E}U_{n,n}^{(1)}\right\|^q\right)}{n^{1+q/p}} < \infty. \tag{3.16}$$

Combining (3.15) and (3.16) and noting that $\mathbb{E}(S_n) = 0$, we obtain

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}(\|S_n\|^q)}{n^{1+q/p}} < \infty,$$

which yields (3.2).

We will now prove the implication (ii)⇒(i). Let $\{\varepsilon_k, k \geq 1\}$ be a Rademacher sequence and let $\{x_k, k \geq 1\}$ be a sequence of elements in $\mathbf{B}$ such that

$$X := \sup_{k\geq 1} \|x_k\| < \infty. \tag{3.17}$$

By Theorem V.9.3 in [24], (i) will holds if

$$\lim_{n\to\infty} \frac{1}{n^{1/p}} \sum_{k=1}^{n} x_k\varepsilon_k = 0 \text{ a.s.} \tag{3.18}$$

Set

$$X_k = x_k\varepsilon_k, k \geq 1.$$

Then, $\{X_k, k \geq 1\}$ is a sequence of independent symmetric $\mathbf{B}$-valued random variables, stochastically dominated by X. Since X is bounded, (3.1) holds. Therefore, by (ii), we have

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\|\sum_{k=1}^{n} X_k\|}{n^{1/p}}\right)^q < \infty \text{ a.s.} \tag{3.19}$$

By applying Proposition 2.3, we obtain (3.18). □

Now, we consider the case where $q \geq 1$ and $1 \leq p < 2$. Li, Qi, and Rosalsky [16] provided a set of necessary and sufficient conditions for the (p, q)-SLLN. Theorem 2.2 of Li, Qi, and Rosalsky [16] is as follows.

Proposition 3.5 (Theorem 2.2 of [16]). *Let $1 < p < 2$, $q \geq 1$, and let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables taking values in a real separable stable type p Banach space* $\mathbf{B}$. *Then, $X \in \mathrm{SLLN}(p, q)$ if and only if $\mathbb{E}(X) = 0$ and*

$$\begin{cases} \int_0^\infty \mathbb{P}^{q/p}(\|X\|^q > t)\mathrm{d}t < \infty & \text{if } q < p, \\ \mathbb{E}(\|X\|^p) < \infty, \sum_{n=1}^{\infty} \frac{\int_{\min\{u_n^p, n\}}^{n} \mathbb{P}(\|X\|^p > t)\mathrm{d}t}{n} < \infty & \text{if } q = p, \\ \mathbb{E}(\|X\|^p) < \infty & \text{if } q > p. \end{cases} \tag{3.20}$$

Similar to Theorem 3.1, the following theorem is a complement of Proposition 3.5 (i.e., Theorem 2.2 of Li, Qi, and Rosalsky [16]).

Theorem 3.6. *Let $1 < p < 2$, $q \geq 1$, and let* $\mathbf{B}$ *be a real separable Banach space. Then the following statements are equivalent.*

(i) $\mathbf{B}$ *is of stable type p.*
(ii) For every sequence $\{X_n, n \geq 1\}$ of independent mean zero $\mathbf{B}$*-valued random variables, which is stochastically dominated by a random variable X, condition (3.20) implies*

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\|S_n\|}{n^{1/p}}\right)^q < \infty \text{ a.s.}$$

Proof. The proof of the implication (ii)⇒(i) is exactly the same as that of Theorem 3.1. The proof of the implication (i)⇒(ii) is similar to that of the sufficient part of Theorem 2.2 of Li, Qi and Rosalsky [16] with some simple changes. We omit the details. □

Similarly, we have the following theorem for the case where $p = 1$. It is a complement of Theorem 2.3 of Li, Qi, and Rosalsky [16].

Theorem 3.7. *Let $q \geq 1$, and let $\mathbf{B}$ be a real separable Banach space. Then, the following statements are equivalent.*

(i) *$\mathbf{B}$ is of stable type 1.*

(ii) *For every sequence $\{X_n, n \geq 1\}$ of independent mean zero $\mathbf{B}$-valued random variables, which is stochastically dominated by a random variable X, the conditions*

$$\mathbb{E}(\|X\|) < \infty,\ \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\sum_{i=1}^{n} \|\mathbb{E}(X_i \mathbf{1}(\|X_i\| \leq n))\|^q\right) < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{\mathbf{1}(\{q=1\}) \int_{\min\{u_n,n\}}^{n} \mathbb{P}(\|X\| > t)\mathrm{d}t}{n} < \infty$$

imply

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\|S_n\|}{n}\right)^q < \infty \text{ a.s.} \tag{3.21}$$

(iii) *For every sequence $\{X_n, n \geq 1\}$ of independent symmetric $\mathbf{B}$-valued random variables, which is stochastically dominated by a random variable X, the conditions $\mathbb{E}(\|X\|) < \infty$ and*

$$\sum_{n=1}^{\infty} \frac{\mathbf{1}(\{q=1\}) \int_{\min\{u_n,n\}}^{n} \mathbb{P}(\|X\| > t)\mathrm{d}t}{n} < \infty$$

imply (3.21).

4 | PROOF OF THE MAIN RESULTS

In this section, we will prove Theorems 1.5 and 1.7. The following lemma generalizes Lemma 5.4 of [14].

Lemma 4.1. *Let $Y_1, \ldots, Y_n$ be i.i.d. nonnegative real-valued random variables such that*

$$\mathbb{P}(Y_1 > 0) \leq \frac{K}{n} \text{ for some constant } K \geq 1. \tag{4.1}$$

Then,

$$\mathbb{E}\left(\max_{1\leq k\leq n} Y_k\right) \geq \frac{n}{2K}\mathbb{E}(Y_1). \tag{4.2}$$

Proof. For all $t \geq 0$, we have

$$\mathbb{P}\left(\max_{1\leq k\leq n} Y_k > t\right) = 1 - (1 - \mathbb{P}(Y_1 \leq t))^n \geq 1 - \mathrm{e}^{-n\mathbb{P}(Y_1 > t)}. \tag{4.3}$$

Elementary calculus shows that

$$1 - \mathrm{e}^{-x} \geq \frac{x}{2K} \text{ for all } 0 \leq x \leq K.$$

It thus follows from (4.1) and (4.3) that

$$\mathbb{P}\left(\max_{1\leq k\leq n} Y_k > t\right) \geq \frac{n}{2K}\mathbb{P}(Y_1 > t) \text{ for all } t \geq 0,$$

which implies (4.2). □

The next lemma is a special case of Lemma 3.2 of Li and Rosalsky [13]. This useful result will be used in our symmetrization procedure.

Lemma 4.2. *Let $g : \mathbf{B} \to [0, \infty]$ be a measurable even function such that for all $x, y \in \mathbf{B}$,*

$$g(x + y) \le \beta(g(x) + g(y)),$$

where $\beta \ge 1$ is a constant, depending only on the function g. If V is a $\mathbf{B}$-valued random variable and $\hat{V}$ is a symmetrized version of V, then for all $t \ge 0$, we have that

$$\mathbb{P}(g(V) \le t)\mathbb{E}(g(V)) \le \mathbb{E}(g(\hat{V})) + \beta t.$$

Proof of Theorem 1.5 (Sufficiency). First, we consider the case where $0 < q < p < 1$. We will prove that

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\frac{\|S_n\|}{n^{1/p}}\right)^q < \infty. \tag{4.4}$$

For each $n \ge 1$, let the quantile u_n of order $1 - 1/n$ of $\|X\|$ be defined as in Section 1. For $n \ge 1$, $1 \le k \le n$, set

$$Y_{n,k} = X_k \mathbf{1}(\|X_k\|^p \le u_n),\ Z_{n,k} = X_k \mathbf{1}(\|X_k\|^p \le n),$$

and

$$U_{n,k} = \sum_{i=1}^{k} Z_{n,i},\ U_{n,k}^{(1)} = \sum_{i=1}^{k} Y_{n,i},\ U_{n,k}^{(2)} = U_{n,k} - U_{n,k}^{(1)}.$$

By (3.10) in Lemma 3.4, (4.4) holds if we can show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\frac{\|S_n - U_{n,n}^{(1)}\|}{n^{1/p}}\right)^q < \infty. \tag{4.5}$$

Since $0 < q < 1$, we have

$$\begin{aligned}
\mathbb{E}\left(\|S_n - U_{n,n}^{(1)}\|^q\right) &\le n\mathbb{E}\left(\|X\|^q \mathbf{1}(\|X\|^q > u_n^q)\right) \\
&= n u_n^q \mathbb{P}(\|X\|^q > u_n^q) + n \int_{u_n^q}^{\infty} \mathbb{P}(\|X\|^q > t)\mathrm{d}t \\
&\le u_n^q + n \int_{u_n^q}^{\infty} \mathbb{P}(\|X\|^q > t)\mathrm{d}t.
\end{aligned} \tag{4.6}$$

Noting that for $n \ge 1$, u_n^q is the quantile of order $1 - 1/n$ of $\|X\|^q$. Letting $u_0 = 0$, it thus follows from (4.6) that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\frac{\|S_n - U_{n,n}^{(1)}\|}{n^{1/p}}\right)^q &\le \sum_{n=1}^{\infty} \frac{u_n^q}{n^{1+q/p}} + \sum_{n=1}^{\infty} \frac{1}{n^{q/p}} \int_{u_n^q}^{\infty} \mathbb{P}(\|X\|^q > t)\mathrm{d}t \\
&\le \sum_{n=1}^{\infty} \frac{1}{n^{1+q/p}} \sum_{k=1}^{n} (u_k^q - u_{k-1}^q) + \sum_{n=1}^{\infty} \frac{1}{n^{q/p}} \sum_{k=n}^{\infty} \int_{u_k^q}^{u_{k+1}^q} \mathbb{P}(\|X\|^q > t)\mathrm{d}t \\
&= \sum_{k=1}^{\infty} (u_k^q - u_{k-1}^q) \left(\sum_{n=k}^{\infty} \frac{1}{n^{1+q/p}}\right) + \sum_{k=1}^{\infty} \int_{u_k^q}^{u_{k+1}^q} \mathbb{P}(\|X\|^q > t)\mathrm{d}t \left(\sum_{n=1}^{k} \frac{1}{n^{q/p}}\right) \\
&\le \left(1 + \frac{p}{q}\right) \sum_{k=1}^{\infty} \frac{1}{k^{q/p}} (u_k^q - u_{k-1}^q) + \frac{p}{p-q} \sum_{k=1}^{\infty} k^{1-q/p} \int_{u_k^q}^{u_{k+1}^q} \mathbb{P}(\|X\|^q > t)\mathrm{d}t
\end{aligned}$$

$$\le \left(1+\frac{p}{q}\right)\sum_{k=1}^{\infty}\int_{u_{k-1}^q}^{u_k^q}\mathbb{P}^{q/p}(\|X\|^q>t)\mathrm{d}t+\frac{p}{p-q}\sum_{k=1}^{\infty}\int_{u_k^q}^{u_{k+1}^q}\mathbb{P}^{q/p}(\|X\|^q>t)\mathrm{d}t$$
$$\le \left(1+\frac{p}{q}+\frac{p}{p-q}\right)\int_0^{\infty}\mathbb{P}^{q/p}(\|X\|^q>t)\mathrm{d}t<\infty,$$

thereby proving (4.5).

Now we consider the case where $0<q=p<1$. Since $\mathbb{E}(\|X\|^p)<\infty$, it is easy to see that

$$\lim_{n\to\infty}\frac{u_n}{n^{1/p}}=0 \tag{4.7}$$

and

$$\sum_{n=1}^{\infty}\mathbb{P}(\|X_n\|^p>n)=\sum_{n=1}^{\infty}\mathbb{P}(\|X\|^p>n)<\infty. \tag{4.8}$$

From (4.7), we can assume that $u_n<n^{1/p}$ for all $n\ge 1$. We then write

$$S_n=U_{n,n}^{(1)}+U_{n,n}^{(2)}+\sum_{k=1}^{n}X_k\mathbf{1}(\|X_k\|^p>n),\ n\ge 1.$$

By the Borel–Cantelli lemma, it follows from (4.8) that

$$\mathbb{P}(\|X_n\|^p>n \text{ i.o.}(n))=0$$

and hence,

$$\mathbb{P}\left(\max_{1\le k\le n}\|X_k\|^p>n \text{ i.o.}(n)\right)=0.$$

We thus have

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\left\|\sum_{k=1}^{n}X_k\mathbf{1}(\|X_k\|^p>n)\right\|}{n^{1/p}}\right)^p<\infty \text{ a.s.} \tag{4.9}$$

By using (4.9), (1.6) (with $0<q=p<1$) holds if we can show that

$$\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{E}\left(\frac{\|U_{n,n}^{(1)}\|}{n^{1/p}}\right)^p<\infty \tag{4.10}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{E}\left(\frac{\|U_{n,n}^{(2)}\|}{n^{1/p}}\right)^p<\infty. \tag{4.11}$$

By (3.12) in Lemma 3.4, we have

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\mathbb{E}(\|U_{n,n}^{(1)}\|^p)<\infty,$$

which proves (4.10). Since $0 < p < 1$, it follows from (1.7) (with $p = q$) that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \mathbb{E}\left(\|U_{n,n}^{(2)}\|^p \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} \mathbb{E}\left(\left\| \sum_{k=1}^{n} X_k \mathbf{1}\left(u_n < \|X_k\| \le n^{1/p}\right) \right\|^p \right)$$
$$\le \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\|X\|^p \mathbf{1}\left(u_n < \|X\| \le n^{1/p}\right)\right) < \infty,$$

which proves (4.11). □

Proof of Theorem 1.5 (Necessity). By Proposition 1.2, we only need to prove for the case where $0 < q = p < 1$. Also by applying Proposition 1.2, we have from (1.6) that

$$\lim_{n\to\infty} \frac{S_n}{n^{1/p}} = 0 \text{ a.s.},$$

which ensures $\mathbb{E}(\|X\|^p) < \infty$. Therefore, we only need to show that (1.6) implies

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}(\|X\|^p \mathbf{1}(u_n^p < \|X\|^p \le n))}{n} < \infty. \tag{4.12}$$

Since $0 < p < 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2} \mathbb{E}\left(\left\| U_{n,n} - \sum_{k=1}^{n} X_k \mathbf{1}(\|X_k\|^p \le k) \right\|^p \right) &\le \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \mathbb{E}\left(\|X_k\|^p \mathbf{1}(k < \|X_k\|^p \le n)\right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \sum_{j=k+1}^{n} \mathbb{E}\left(\|X\|^p \mathbf{1}(j-1 < \|X\|^p \le j)\right) \\
&\le \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{j=1}^{n} j\mathbb{E}\left(\|X\|^p \mathbf{1}(j-1 < \|X\|^p \le j)\right) \\
&= \sum_{j=1}^{\infty} j\mathbb{E}\left(\|X\|^p \mathbf{1}(j-1 < \|X\|^p \le j)\right) \left(\sum_{n=j}^{\infty} \frac{1}{n^2} \right) \\
&\le 2 \sum_{j=1}^{\infty} \mathbb{E}\left(\|X\|^p \mathbf{1}(j-1 < \|X\|^p \le j)\right) \\
&= 2\mathbb{E}(\|X\|^p) < \infty.
\end{aligned} \tag{4.13}$$

Using the same argument of proof of (3.3) in [16, p. 556], we have

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\left(\| \sum_{k=1}^{n} X_k \mathbf{1}(\|X_k\|^p \le k) \|^p \right)}{n^2} < \infty. \tag{4.14}$$

Combining (4.13) and (4.14), we have

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\left(\|U_{n,n}\|^p \right)}{n^2} < \infty. \tag{4.15}$$

It follows from (4.15) and (3.12) of Lemma 3.4 that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\mathbb{E}\left(\|U^{(2)}_{n,n}\|^p\right)}{n^2}&=\sum_{n=1}^{\infty}\frac{\mathbb{E}\left(\|U_{n,n}-U^{(1)}_{n,n}\|^p\right)}{n^2}\\&\le\sum_{n=1}^{\infty}\frac{\mathbb{E}\left(\|U_{n,n}\|^p\right)}{n^2}+\sum_{n=1}^{\infty}\frac{\mathbb{E}\left(\|U^{(1)}_{n,n}\|^p\right)}{n^2}<\infty.\end{aligned}\tag{4.16}$$

Let $\left\{X',X'_n,n\ge1\right\}$ be an independent copy of $\{X,X_n,n\ge1\}$. For $n\ge1, 1\le k\le n$, set

$$V_{n,k}=X_k\mathbf{1}(u_n^p<\|X_k\|^p\le n)-X'_k\mathbf{1}(u_n^p<\|X'_k\|^p\le n),\ \hat{U}^{(2)}_{n,k}=\sum_{j=1}^{k}V_{n,j},\ \hat{U}^{(2)}_{n,0}=0.$$

It follows from (4.16) that

$$\sum_{n=1}^{\infty}\frac{\mathbb{E}\left(\|\hat{U}^{(2)}_{nn}\|^p\right)}{n^2}<\infty.\tag{4.17}$$

For $n\ge1$, applying Lévy's inequality (see, e.g., [12, pp. 47–48]) for independent symmetric random variables $\left\{V_{n,k},1\le k\le n\right\}$, we have

$$\begin{aligned}\mathbb{E}\left(\max_{1\le k\le n}\|V_{n,k}\|^p\right)&=\mathbb{E}\left(\max_{1\le k\le n}\left\|\hat{U}^{(2)}_{n,k}-\hat{U}^{(2)}_{n,(k-1)}\right\|^p\right)\\&\le2\mathbb{E}\left(\max_{1\le k\le n}\left\|\hat{U}^{(2)}_{n,k}\right\|^p\right)\\&\le4\mathbb{E}\left(\left\|\hat{U}^{(2)}_{n,n}\right\|^p\right).\end{aligned}\tag{4.18}$$

Since

$$\mathbb{P}(\|V_{n,1}\|^p>0)\le\mathbb{P}(\|X_1\|>u_n)+\mathbb{P}(\|X'_1\|>u_n)<\frac{2}{n},$$

by applying Lemma 4.1 with the constant $K=2$, we obtain

$$\mathbb{E}(\|V_{n,1}\|^p)\le\frac{4}{n}\mathbb{E}\left(\max_{1\le k\le n}\|V_{n,k}\|^p\right)\tag{4.19}$$

Combining (4.17)–(4.19), we have

$$\sum_{n=1}^{\infty}\frac{\mathbb{E}(\|V_{n,1}\|^p)}{n}\le\sum_{n=1}^{\infty}\frac{16\mathbb{E}\left(\|\hat{U}^{(2)}_{n,n}\|^p\right)}{n^2}<\infty.\tag{4.20}$$

We see that V_{n1} is a symmetrized version of $X_1\mathbf{1}(u_n^p<\|X_1\|^p\le n), n\ge1$. Applying Lemma 4.2 with $t=1/n$ and $g(x)=\|x\|^p, x\in\mathbf{B}$, we have

$$\mathbb{P}\left(\|X_1\|^p\mathbf{1}(u_n^p<\|X_1\|^p\le n)\le\frac{1}{n}\right)\mathbb{E}(\|X_1\|^p\mathbf{1}(u_n^p<\|X_1\|^p\le n))\le\mathbb{E}(\|V_{n,1}\|^p)+\frac{1}{n}.\tag{4.21}$$

Since

$$\begin{aligned}1-\frac{1}{n}\le\mathbb{P}(\|X\|\le u_n)&\le\mathbb{P}\left(\|X\|^p\mathbf{1}(u_n^p<\|X\|^p\le n)\le\frac{1}{n}\right)\\&=\mathbb{P}\left(\|X_1\|^p\mathbf{1}(u_n^p<\|X_1\|^p\le n)\le\frac{1}{n}\right)\ \text{ for all }\ n\ge1,\end{aligned}$$

it follows from (4.21) that

$$\left(1-\frac{1}{n}\right)\mathbb{E}(\|X\|^p\mathbf{1}(u_n^p < \|X\|^p \le n)) \le \mathbb{E}(\|V_{n1}\|^p) + \frac{1}{n}, n \ge 1. \tag{4.22}$$

Combining (4.20) and (4.22), we have

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{1}{2n}\mathbb{E}(\|X\|^p\mathbf{1}(u_n^p < \|X\|^p \le n)) &\le \sum_{n=2}^{\infty}\frac{1}{n}\left(1-\frac{1}{n}\right)\mathbb{E}(\|X\|^p\mathbf{1}(u_n^p < \|X\|^p \le n))\\ &\le \sum_{n=2}^{\infty}\frac{1}{n}\left(\mathbb{E}\big(\|V_{n,1}\|^p\big)+\frac{1}{n}\right) < \infty,\end{aligned}$$

thereby completing the proof of (4.12). □

Proof of Theorem 1.7. Since $0 < q < 1 \le p < 2$, the necessity follows immediately from Proposition 1.2 and the fact that (1.4) implies $\mathbb{E}(X) = 0$. The sufficiency follows from the implication (i)⇒(ii) of Theorem 3.1 in Section 3. □

Proof of Corollary 1.8. Recalling Proposition 1.2, if $0 < q < p < 2$, then (1.10) is equivalent to $X \in \mathrm{SLLN}(p,q)$. Therefore, the case where $0 < q < p < 1$ follows from Theorem 1.5, and the case where $0 < q < 1 \le p < 2$ follows from Theorem 1.7.

We now consider the case where $0 < q = p < 1$. If (1.10) holds, then by applying Proposition 1.2 again, we obtain (1.11) (with $q = p$). Conversely, if (1.11) (with $q = p$) holds, then by following the proof of Lemma 5.6 of Li, Qi, and Rosalsky [14] with $\|X\|^p$ in the place of $\|X\|$, we obtain

$$\sum_{n=1}^{\infty}\frac{\mathbb{E}\big(\|X\|^p\mathbf{1}(\min\{u_n^p,n\} < \|X\|^p \le n)\big)}{n} < \infty.$$

Therefore, (1.7) (with $q = p$) holds, and by applying Theorem 1.5, we have $X \in \mathrm{SLLN}(p,p)$, that is, (1.6) holds with $q = p$. The conclusion (1.10) then follows from Proposition 1.2. □

We close this section by presenting three simple examples to illustrate Theorem 1.5, as mentioned in Remark 1.6. The first example shows that, for $0 < p < 1$, there exists a random variable X such that $X \in \mathrm{SLLN}(p,p)$ but $X \notin \mathrm{SLLN}(p,q)$ for all $0 < q < p$.

Example 4.3. Let $0 < p < 1$. For $q > 0$, let X be a real-valued random variable such that its tail probability function is

$$\mathbb{P}(X > t) = \mathbf{1}\{t \le \mathrm{e}\} + \frac{\mathrm{e}^q}{t^q(\ln t)^{2p/q}}\mathbf{1}\{t > \mathrm{e}\},\ t \in \mathbb{R}.$$

Then, for all $t > \mathrm{e}^q$, we have

$$\mathbb{P}(|X|^q > t) = \mathbb{P}(X > t^{1/q}) = \frac{\mathrm{e}^q}{t(\ln t^{1/q})^{2p/q}}.$$

Therefore,

$$\int_0^{\infty}\mathbb{P}^{q/p}(|X|^q > t)\mathrm{d}t = \infty\ \text{ if }\ q < p.$$

For $q = p$, elementary calculus also shows that

$$\mathbb{E}\Big(|X|^p\ln^{1/2}(1+|X|^p)\Big) < \infty. \tag{4.23}$$

The proof of Lemma 5.6 of Li, Qi, and Rosalsky [14] shows that, for any random variable X, if

$$\mathbb{E}\Big(\|X\|\ln^{\delta}(1+\|X\|)\Big) < \infty\ \text{ for some }\ \delta > 0,$$

then

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}(\|X\|\mathbf{1}(\min\{u_n, n\} < \|X\| \le n))}{n} < \infty.$$

It thus follows from (4.23) that (1.7) holds for $q = p$. By Theorem 1.5, we see that, for this example, $X \in \mathrm{SLLN}(p, p)$ but $X \notin \mathrm{SLLN}(p, q)$ for all $0 < q < p$.

The next two examples show that each of the two conditions that appeared in (1.7) (for the case where $p = q$) does not imply each other. Examples 4.4 and 4.5 are inspired by Examples 5.2 and 5.3 of Li, Qi, and Rosalsky [14], respectively.

Example 4.4. Let $0 < p < 1$ and let X be a real-valued random variable such that its tail probability function is

$$\mathbb{P}(X > t) = \mathbf{1}\{t \le \mathrm{e}^{\mathrm{e}}\} + \frac{\mathrm{e}^{\mathrm{e}p+1}}{t^p(\ln t)(\ln \ln t)^2}\mathbf{1}\{t > \mathrm{e}^{\mathrm{e}}\},\ t \in \mathbb{R}.$$

Then, $\mathbb{E}(|X|^p) < \infty$ and by the same calculation as in Lemma 5.2 of Li, Qi, and Rosalsky [14], we have

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\left(|X|^p\mathbf{1}(\min\{u_n^p, n\} < |X|^p \le n)\right)}{n} = \infty.$$

Example 4.5. Let $0 < p < 1$ and let X be a real-valued random variable such that its tail probability function is

$$\mathbb{P}(X > t) = \mathbf{1}\{t \le 1\} + \frac{1}{t^p}\mathbf{1}\{t > 1\},\ t \in \mathbb{R}.$$

Then, $\mathbb{E}(|X|^p) = \infty$ and

$$\sum_{n=1}^{\infty} \frac{\mathbb{E}\left(|X|^p\mathbf{1}(\min\{u_n^p, n\} < |X|^p \le n)\right)}{n} = 0$$

since $u_n^p = n$.

5 | FURTHER REMARKS

This work has been devoted to (p, q)-type SLLN and related results for one-parameter processes. As noted by Khoshnevisan [9], "there are a number of compelling reasons for studying random fields, one of which is that, if and when possible, multiparameter processes are a natural extension of existing one-parameter processes." Some of the tools used in this paper such as the generalization of Ottaviani's inequality developed by Li and Rosalsky [13] or Lemma 2.2 are available for multiparameter processes (see [1, 23]), but it is unclear whether the methods of this paper can be pushed through.

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