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The Marcinkiewicz–Zygmund-Type Strong Law of Large Numbers with General Normalizing Sequences

Vu T. N. Anh[1] · Nguyen T. T. Hien[2] · Le V. Thanh[2] · Vo T. H. Van[2]

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Abstract

This paper establishes complete convergence for weighted sums and the Marcinkiewicz–Zygmund-type strong law of large numbers for sequences of negatively associated and identically distributed random variables $\{X, X_n, n \geq 1\}$ with general normalizing constants under a moment condition that $ER(X) < \infty$, where $R(\cdot)$ is a regularly varying function. The result is new even when the random variables are independent and identically distributed (i.i.d.), and a special case of this result comes close to a solution to an open question raised by Chen and Sung (Stat Probab Lett 92:45–52, 2014). The proof exploits some properties of slowly varying functions and the de Bruijn conjugates. A counterpart of the main result obtained by Martikainen (J Math Sci 75(5):1944–1946, 1995) on the Marcinkiewicz–Zygmund-type strong law of large numbers for pairwise i.i.d. random variables is also presented. Two illustrative examples are provided, including a strong law of large numbers for pairwise negatively dependent random variables which have the same distribution as the random variable appearing in the St. Petersburg game.

Keywords Weighted sum · Negative association · Negative dependence · Complete convergence · Strong law of large numbers · Normalizing constant · Slowly varying function

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✉ Le V. Thanh
levt@vinhuni.edu.vn

Vu T. N. Anh
vtnanh@hluv.edu.vn

Nguyen T. T. Hien
hienntt.ktoan@vinhuni.edu.vn

Vo T. H. Van
vanvth@vinhuni.edu.vn

1 Department of Mathematics, Hoa Lu University, Ninh Binh, Vietnam

2 Department of Mathematics, Vinh University, Nghe An, Vietnam

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1 Introduction

The motivation of this paper is an open question raised recently by Chen and Sung [8]. Let $1 < \alpha \le 2$, $\gamma > 0$ and let $\{X, X_n, n \ge 1\}$ be a sequence of negatively associated and identically distributed random variables with $E(X) = 0$. Sung [44] proved that if

$$\begin{cases} E|X|^{\gamma} < \infty \text{ for } \gamma > \alpha, \\ E|X|^{\alpha} \log(|X|+2) < \infty \text{ for } \gamma = \alpha, \\ E|X|^{\alpha} < \infty \text{ for } \gamma < \alpha, \end{cases} \tag{1.1}$$

then

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon n^{1/\alpha}\log^{1/\gamma}(n)\right) < \infty \text{ for all } \varepsilon > 0, \tag{1.2}$$

where $\{a_{ni}, n \ge 1, 1 \le i \le n\}$ are constants satisfying

$$\sup_{n\ge1} \frac{\sum_{i=1}^{n} |a_{ni}|^{\alpha}}{n} < \infty. \tag{1.3}$$

Here and thereafter, log denotes the logarithm to the base 2. Chen and Sung [8] proved that for the case where $\gamma > \alpha$, the condition $E|X|^{\gamma} < \infty$ is optimal. They raised an open question about finding the optimal condition for (1.2) when $\gamma \le \alpha$. For the case where $\gamma < \alpha$, Chen and Sung [8, Corollary 2.2] proved that (1.2) holds under an almost optimal condition that

$$E\left(|X|^{\alpha}\log^{1-\alpha/\gamma}(|X|+2)\right) < \infty.$$

In this note, by using some results related to regularly varying functions, we provide the necessary and sufficient conditions for

$$\sum_{n} n^{-1} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon n^{1/\alpha}\tilde{L}(n^{1/\alpha})\right) < \infty \text{ for all } \varepsilon > 0, \tag{1.4}$$

where $\tilde{L}(\cdot)$ is the de Bruijn conjugate of a slowly varying function $L(\cdot)$, defined on $[A, \infty)$ for some $A > 0$. This result is new even when the random variables are i.i.d. By letting $L(x) \equiv \log^{-1/\gamma}(x),\ x \ge 2$, we obtain optimal moment condition for (1.2).

Weak laws of large numbers with the norming constants $n^{1/\alpha}\tilde{L}(n^{1/\alpha})$ were studied by Gut [21], and Matsumoto and Nakata [33]. The Marcinkiewicz–Zygmund strong law of large numbers has been extended and generalized in many directions by a number of authors; see [1,12,22,24,38,39,46] and references therein. To our best knowledge, there is not any result in the literature that considers strong law of large numbers with general normalizing constants $n^{1/\alpha}\tilde{L}(n^{1/\alpha})$ except Gut and Stadmüller [22] who studied the Kolmogorov strong law of large numbers, but for delay sums. The main result of this paper fills this gap. Recently, Miao et al. [35] have studied the Marcinkiewicz–Zygmund-type strong law of large numbers where the norming constants are of the form $n^{1/\alpha}\log^{\beta/\alpha} n$ for some $\beta \geq 0$, which is a special case of our result.

The concept of negative association of random variables was introduced by Joag-Dev and Proschan [28]. A collection $\{X_1, \ldots, X_n\}$ of random variables is said to be negatively associated if for any disjoint subsets A, B of $\{1, \ldots, n\}$ and any real coordinatewise nondecreasing functions f on $\mathbb{R}^{|A|}$ and g on $\mathbb{R}^{|B|}$,

$$\operatorname{Cov}(f(X_k, k \in A), g(X_k, k \in B)) \leq 0 \tag{1.5}$$

whenever the covariance exists, where $|A|$ denotes the cardinality of A. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be negatively associated if every finite subfamily is negatively associated.

There is a weaker concept of dependence called negative dependence, which was introduced by Lehmann [31] and further investigated by Ebrahimi and Ghosh [13] and Block et al. [4]. A collection of random variables $\{X_1, \ldots, X_n\}$ is said to be negatively dependent if for all $x_1, \ldots, x_n \in \mathbb{R}$,

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) \leq P(X_1 \leq x_1) \ldots P(X_n \leq x_n),$$

and

$$P(X_1 > x_1, \ldots, X_n > x_n) \leq P(X_1 > x_1) \ldots P(X_n > x_n).$$

A sequence of random variables $\{X_i, i \geq 1\}$ is said to be negatively dependent if for any $n \geq 1$, the collection $\{X_1, \ldots, X_n\}$ is negatively dependent. A sequence of random variables $\{X_i, i \geq 1\}$ is said to be pairwise negatively dependent if for all $x, y \in \mathbb{R}$ and for all $i \neq j$,

$$P(X_i \leq x, X_j \leq y) \leq P(X_i \leq x)P(X_j \leq y).$$

It is well known and easy to prove that $\{X_i, i \geq 1\}$ is pairwise negatively dependent if and only if for all $x, y \in \mathbb{R}$ and for all $i \neq j$,

$$P(X_i > x, X_j > y) \leq P(X_i > x)P(X_j > y).$$

By Joag-Dev and Proschan [28, Property P3], negative association implies negative dependence. For examples about negatively dependent random variables which are not negatively associated, see [28, p. 289]. Of course, pairwise independence implies pairwise negative dependence, but pairwise independence and negative dependence do not imply each other. Joag-Dev and Proschan [28] pointed out that many useful distributions enjoy the negative association properties (and therefore, they are negatively dependent) including multinomial distribution, multivariate hypergeometric distribution, Dirichlet distribution, strongly Rayleigh distribution and distribution of random sampling without replacement. Limit theorems for negatively associated and negatively dependent random variables have received extensive attention. We refer to [27,34,42] and references therein. These concepts of dependence can be extended to the Hilbert space-valued random variables; see, e.g., [6,25,29,47], among others.

The rest of the paper is arranged as follows. Section 2 presents some results on slowly varying functions needed in proving the main results. Section 3 focuses on complete convergence for weighted sums of negatively associated and identically distributed random variables. In Sect. 4, we apply a result concerning slowly varying functions developed in Sect. 2 to give a counterpart of Martikainen's strong law of large numbers (see [32]) for sequences of pairwise negatively dependent and identically distributed random variables. As an application, we prove a strong law of large numbers for pairwise negatively dependent random variables which have the same distribution as the random variable appearing in the St. Petersburg game.

2 Some Facts Concerning Slowly Varying Functions

Some technical results concerning slowly varying functions will be presented in this section.

The notion of regularly varying function can be found in [41, Chapter 1]. A real-valued function $R(\cdot)$ is said to be regularly varying with index of regular variation ρ ($\rho \in \mathbb{R}$) if it is a positive and measurable function on $[A, \infty)$ for some $A > 0$, and for each $\lambda > 0$,

$$\lim_{x\to\infty} \frac{R(\lambda x)}{R(x)} = \lambda^{\rho}. \tag{2.1}$$

A regularly varying function with the index of regular variation $\rho = 0$ is called slowly varying. It is well known that a function $R(\cdot)$ is regularly varying with the index of regular variation ρ if and only if it can be written in the form

$$R(x) = x^{\rho} L(x), \tag{2.2}$$

where $L(\cdot)$ is a slowly varying function (see, e.g., [41, p. 2]). On the regularly varying functions and their important role in probability, we refer to Seneta [41], Bingham, Goldie and Teugels [3], and more recent survey paper by Jessen and Mikosch [26]. Regular variation is also one of the key notions for modeling the behavior of large telecommunications networks; see, e.g., Heath et al. [23], Mikosch et al. [36].

The basic result in the theory of slowly varying functions is the representation theorem (see, e.g., [3, Theorem 1.3.1]) which states that for a positive and measurable function $L(\cdot)$ defined on $[A, \infty)$ for some $A > 0$, $L(\cdot)$ is slowly varying if and only if it can be written in the form

$$L(x) = c(x) \exp\left(\int_B^x \frac{\varepsilon(u)\mathrm{d}u}{u}\right)$$

for some $B \geq A$ and for all $x \geq B$, where $c(\cdot)$ is a positive bounded measurable function defined on $[B, \infty)$ satisfying $\lim_{x\to\infty} c(x) = c \in (0, \infty)$ and $\varepsilon(\cdot)$ is a continuous function defined on $[B, \infty)$ satisfying $\lim_{x\to\infty} \varepsilon(x) = 0$. Seneta [40] (see also in [3, Lemma 1.3.2]) proved that if $L(\cdot)$ is a slowly varying function defined on $[A, \infty)$ for some $A > 0$, then there exists $B \geq A$ such that $L(x)$ is bounded on every finite closed interval $[a, b] \subset [B, \infty)$.

Let $L(\cdot)$ be a slowly varying function. Then by [3, Theorem 1.5.13], there exists a slowly varying function $\tilde{L}(\cdot)$, unique up to asymptotic equivalence, satisfying

$$\lim_{x\to\infty} L(x)\tilde{L}\left(xL(x)\right) = 1 \text{ and } \lim_{x\to\infty} \tilde{L}(x)L\left(x\tilde{L}(x)\right) = 1. \tag{2.3}$$

The function $\tilde{L}$ is called the de Bruijn conjugate of L, and $\left(L, \tilde{L}\right)$ is called a (slowly varying) conjugate pair (see, e.g., [3, p. 29]). By [3, Proposition 1.5.14], if $\left(L, \tilde{L}\right)$ is a conjugate pair, then for $a, b, \alpha > 0$, each of $\left(L(ax), \tilde{L}(bx)\right)$, $\left(aL(x), a^{-1}\tilde{L}(x)\right)$, $\left((L(x^\alpha))^{1/\alpha}, (\tilde{L}(x^\alpha))^{1/\alpha}\right)$ is a conjugate pair. Bojanić, R. and Seneta [5] (see also Theorem 2.3.3 and Corollary 2.3.4 in [3]) proved that if $L(\cdot)$ is a slowly varying function satisfying

$$\lim_{x\to\infty} \left(\frac{L(\lambda_0 x)}{L(x)} - 1\right) \log(L(x)) = 0, \tag{2.4}$$

for some $\lambda_0 > 1$, then for every $\alpha \in \mathbb{R}$,

$$\lim_{x\to\infty} \frac{L(xL^\alpha(x))}{L(x)} = 1, \tag{2.5}$$

and therefore, we can choose (up to asymptotic equivalence) $\tilde{L}(x) = 1/L(x)$. In particular, if $L(x) = \log(x)$ then $\tilde{L}(x) = 1/\log(x)$.

The following lemma follows from Theorem 1.5.12 and Proposition 1.5.15 in [3]. Here and thereafter, for a slowly varying function $L(\cdot)$ defined on $[A, \infty)$ for some $A > 0$, we denote the de Bruijn conjugate of $L(\cdot)$ by $\tilde{L}(\cdot)$. Without loss of generality, we assume that $\tilde{L}(\cdot)$ is also defined on $[A, \infty)$ and that $L(x)$ and $\tilde{L}(x)$ are both bounded on finite closed intervals.

Lemma 2.1 *Let $\alpha, \beta > 0$ and $L(\cdot)$ be a slowly varying function. Let $f(x) = x^{\alpha\beta}L^\alpha(x^\beta)$ and $g(x) = x^{1/(\alpha\beta)}\tilde{L}^{1/\beta}(x^{1/\alpha})$. Then*

$$\lim_{x\to\infty} \frac{f(g(x))}{x} = \lim_{x\to\infty} \frac{g(f(x))}{x} = 1. \tag{2.6}$$

The second lemma shows that we can approximate a slowly varying function $L(\cdot)$ by a differentiable slowly varying function $L_1(\cdot)$. See Galambos and Seneta [20, p. 111] for a proof.

Lemma 2.2 *For any slowly varying function $L(\cdot)$ defined on $[A, \infty)$ for some $A > 0$, there exists a differentiable slowly varying function $L_1(\cdot)$ defined on $[B, \infty)$ for some $B \geq A$ such that*

$$\lim_{x\to\infty} \frac{L(x)}{L_1(x)} = 1 \quad \textit{and} \quad \lim_{x\to\infty} \frac{xL_1'(x)}{L_1(x)} = 0.$$

Conversely, if $L(\cdot)$ is a positive differentiable function satisfying

$$\lim_{x\to\infty} \frac{xL'(x)}{L(x)} = 0, \tag{2.7}$$

then $L(\cdot)$ is a slowly varying function.

Because of Lemma 2.2, we can work with differentiable slowly varying functions $L(\cdot)$ that satisfy (2.7) in our setting.

The proof of Lemma 2.3 (i) follows from direct calculations (by taking the derivative). Lemma 2.3 (ii) is an easy consequence of the representation theorem stated above.

Lemma 2.3 *Let $p > 0$ and let $L(\cdot)$ be a slowly varying function defined on $[A, \infty)$ for some $A > 0$, satisfying (2.7). Then the following statements hold.*

(i) *There exists $B \geq A$ such that $x^p L(x)$ is increasing on $[B, \infty)$, $x^{-p}L(x)$ is decreasing on $[B, \infty)$, and $\lim_{x\to\infty} x^p L(x) = \infty$, $\lim_{x\to\infty} x^{-p}L(x) = 0$.*

(ii) *For all $\lambda > 0$,*

$$\lim_{x\to\infty} \frac{L(x)}{L(x+\lambda)} = 1.$$

Remark 2.4 If we do not have the assumption that $L(\cdot)$ satisfies (2.7), then we still have $x^p L(x) \to \infty$, $x^{-p}L(x) \to 0$ as $x \to \infty$ (see Seneta [41, p. 18]), but we do not have the monotonicity as in Lemma 2.3 (i).

The following lemma is a direct consequence of Karamata's theorem (see [3, Theorem 1.5.10]) as was so kindly pointed out to us by the referee.

Lemma 2.5 *Let $p > 1$, $q \in \mathbb{R}$ and $L(\cdot)$ be a differentiable slowly varying function defined on $[A, \infty)$ for some $A > 0$. Then*

$$\sum_{k=n}^{\infty} \frac{L^q(k)}{k^p} \sim \frac{L^q(n)}{(p-1)n^{p-1}}. \tag{2.8}$$

The following proposition gives a criterion for $E\left(|X|^\alpha L^\alpha(|X| + A)\right) < \infty$.

Proposition 2.6 *Let $\alpha \geq 1$, and let X be a random variable. Let $L(\cdot)$ be a slowly varying function defined on $[A, \infty)$ for some $A > 0$. Assume that $x^\alpha L^\alpha(x)$ and $x^{1/\alpha}\tilde{L}(x^{1/\alpha})$ are increasing on $[A, \infty)$. Then*

$$E\left(|X|^\alpha L^\alpha(|X| + A)\right) < \infty \ \textit{if and only if} \ \sum_{n\geq A^\alpha} P(|X| > b_n) < \infty, \tag{2.9}$$

where $b_n = n^{1/\alpha}\tilde{L}\left(n^{1/\alpha}\right)$, $n \geq A^{\alpha}$.

Proof Let $f(x) = x^{\alpha}L^{\alpha}(x)$, $g(x) = x^{1/\alpha}\tilde{L}(x^{1/\alpha})$. Since $L(\cdot)$ is positive and bounded on finite closed intervals,

$$E\left(|X|^{\alpha}L^{\alpha}(|X|+A)\right) < \infty \text{ if and only if } E\left(f(|X|+A)\right) < \infty.$$

For a nonnegative random variable Y, $EY < \infty$ if and only if $\sum_{n=1}^{\infty} P(Y > n) < \infty$. Applying this, we have that $E\left(f(|X|+A)\right) < \infty$ if and only if

$$\sum_{n=1}^{\infty} P\left(f(|X|+A) > n\right) < \infty. \tag{2.10}$$

By using Lemma 2.1 with $\beta = 1$, we have $f(g(x)) \sim g(f(x)) \sim x$ as $x \to \infty$. Combining this with the assumption that $f(x)$ and $g(x)$ are increasing on $[A, \infty)$, we see that (2.10) is equivalent to

$$\sum_{n \geq A^{\alpha}} P\left(|X| > n^{1/\alpha}\tilde{L}(n^{1/\alpha})\right) < \infty. \tag{2.11}$$

The proof of the proposition is completed. □

3 Complete Convergence for Weighted Sums of Negatively Associated and Identically Distributed Random Variables

In the following theorem, we establish complete convergence for weighted sums of negatively associated and identically distributed random variables. Theorem 3.1 is new even when the random variables are i.i.d. A special case of this result comes close to a solution of an open question of Chen and Sung [8]. In subsequent derivations, the symbol C denotes a generic positive constant whose value may be different for each appearance.

Theorem 3.1 *Let* $1 \leq \alpha < 2$, $\{X, X_n, n \geq 1\}$ *be a sequence of negatively associated and identically distributed random variables and* $L(\cdot)$ *be a slowly varying function defined on* $[A, \infty)$ *for some* $A > 0$. *When* $\alpha = 1$, *we assume further that* $L(x) \geq 1$ *and is increasing on* $[A, \infty)$. *Let* $b_n = n^{1/\alpha}\tilde{L}(n^{1/\alpha})$, $n \geq A^{\alpha}$. *Then the following four statements are equivalent.*

(i) *The random variable* X *satisfies*

$$E(X) = 0, \ E\left(|X|^{\alpha}L^{\alpha}(|X|+A)\right) < \infty. \tag{3.1}$$

(ii) *For every array of constants* $\{a_{ni}, n \geq 1, 1 \leq i \leq n\}$ *satisfying*

$$\sum_{i=1}^{n} a_{ni}^2 \leq Cn, \quad n \geq 1, \tag{3.2}$$

we have

$$\sum_{n\ge A^\alpha} n^{-1} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon b_n\right) < \infty \quad \textit{for all } \varepsilon > 0. \tag{3.3}$$

(iii)

$$\sum_{n\ge A^\alpha} n^{-1} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_i\right| > \varepsilon b_n\right) < \infty \quad \textit{for all } \varepsilon > 0. \tag{3.4}$$

(iv) *The strong law of large numbers*

$$\lim_{n\to\infty} \frac{\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_i\right|}{b_n} = 0 \ \textit{a.s.} \tag{3.5}$$

holds.

Proof For simplicity, we assume that A^α is an integer number since we can take $[A^\alpha]+1$ otherwise. By Lemmas 2.2 and 2.3, without loss of generality, we can assume that $x^{1/\alpha}\tilde{L}(x^{1/\alpha})$ and $x^{\alpha-1}L^\alpha(x)$ are increasing on $[A,\infty)$ and that $x^{\alpha-2}L^\alpha(x)$ is decreasing on $[A,\infty)$. We may also assume that $a_{ni}\ge 0$ since we can using the identity $a_{ni}=a_{ni}^{+}-a_{ni}^{-}$ in the general case.

Firstly, we prove the implication ((i) ⇒ (ii)). Assume that (3.1) holds and $\{a_{ni}, n\ge 1, i\ge 1\}$ are constants satisfying (3.2), we will prove that (3.3) holds. For $n\ge A^\alpha$, set

$$X_{ni} = -b_n I(X_i < -b_n) + X_i I(|X_i|\le b_n) + b_n I(X_i > b_n),\ 1\le i\le n,$$

and

$$S_{nk} = \sum_{i=1}^{k}\Big(a_{ni}X_{ni} - E(a_{ni}X_{ni})\Big),\ 1\le k\le n.$$

Let $\varepsilon > 0$ be arbitrary. For $n\ge A^\alpha$,

$$\begin{aligned} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon b_n\right) &\le P\left(\max_{1\le k\le n}|X_k| > b_n\right) + P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right| > \varepsilon b_n\right)\\ &\le P\left(\max_{1\le k\le n}|X_k| > b_n\right) + P\left(\max_{1\le k\le n}|S_{nk}| > \varepsilon b_n - \sum_{i=1}^{n}|E(a_{ni}X_{ni})|\right). \end{aligned} \tag{3.6}$$

By the second half of (3.1) and Proposition 2.6, we have

$$\sum_{n\geq A^\alpha} n^{-1} P\Big(\max_{1\leq k\leq n} |X_k| > b_n\Big) \leq \sum_{n\geq A^\alpha} n^{-1} \sum_{k=1}^{n} P\Big(|X_k| > b_n\Big) = \sum_{n\geq A^\alpha} P(|X| > b_n) < \infty. \tag{3.7}$$

For $n \geq 1$, by the Cauchy–Schwarz inequality and (3.2),

$$\left(\sum_{i=1}^{n} |a_{ni}|\right)^2 \leq n \left(\sum_{i=1}^{n} a_{ni}^2\right) \leq Cn^2. \tag{3.8}$$

For $n \geq A^\alpha$, the first half of (3.1) and (3.8) implies that

$$\begin{aligned} \frac{\sum_{i=1}^{n} |E(a_{ni}X_{ni})|}{b_n} &\leq \frac{\sum_{i=1}^{n} |a_{ni}| \left(|EX_i I(|X_i| \leq b_n)| + b_n P(|X_i| > b_n)\right)}{b_n} \\ &\leq \frac{Cn\left(\left|E(XI(|X| \leq b_n))\right| + b_n P(|X| > b_n)\right)}{b_n} \\ &= \frac{Cn\left(\left|E(XI(|X| > b_n))\right| + b_n P(|X| > b_n)\right)}{b_n} \\ &\leq \frac{CnE|X|I(|X| > b_n)}{b_n}. \end{aligned} \tag{3.9}$$

For n large enough and for $\omega \in (|X| > b_n)$, we have

$$\begin{aligned} \frac{n}{b_n} &= \frac{n^{(\alpha-1)/\alpha} \tilde{L}^{\alpha-1}(n^{1/\alpha})}{\tilde{L}^\alpha(n^{1/\alpha})} \\ &= \frac{\left(n^{1/\alpha}\tilde{L}(n^{1/\alpha})\right)^{\alpha-1} L^\alpha\left(n^{1/\alpha}\tilde{L}(n^{1/\alpha})\right)}{\tilde{L}^\alpha(n^{1/\alpha}) L^\alpha\left(n^{1/\alpha}\tilde{L}(n^{1/\alpha})\right)} \\ &\leq Cb_n^{\alpha-1} L^\alpha(b_n) \leq C|X(\omega)|^{\alpha-1} L^\alpha(|X(\omega)|), \end{aligned} \tag{3.10}$$

where we have applied (2.3) in the first inequality and the monotonicity of $x^{\alpha-1}L^\alpha(x)$ in the second inequality. Combining (3.9), (3.10), the second half of (3.1) and using Lemma 2.3 (ii), we have

$$\begin{aligned} \frac{\sum_{i=1}^{n} |E(a_{ni}X_{ni})|}{b_n} &\leq CE\left(|X|^\alpha L^\alpha(|X|) I(|X| > b_n)\right) \\ &\leq CE\left(|X|^\alpha L^\alpha(|X| + A) I(|X| > b_n)\right) \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.11}$$

From (3.6), (3.7) and (3.11), to obtain (3.3), it remains to show that

$$\sum_{n\geq A^\alpha} n^{-1} P\Big(\max_{1\leq j\leq n} |S_{nj}| > b_n\varepsilon/2\Big) < \infty. \tag{3.12}$$

Set $b_{A^\alpha-1} = 0$. For B large enough, we have

$$\begin{aligned}
\sum_{n\geq A^\alpha} \frac{1}{n} P\Big(\max_{1\leq k\leq n} |S_{nk}| > b_n\varepsilon/2\Big) &\leq \sum_{n\geq A^\alpha} \frac{4}{\varepsilon^2 n b_n^2} E\Big(\max_{1\leq j\leq n} |S_{nj}|\Big)^2 \\
&\leq \sum_{n\geq A^\alpha} \frac{4}{\varepsilon^2 n b_n^2} \sum_{i=1}^{n} E\Big(a_{ni}X_{ni} - E(a_{ni}X_{ni})\Big)^2 \\
&\leq \sum_{n\geq A^\alpha} \frac{4\left(\sum_{i=1}^n a_{ni}^2\right)\left(EX^2 I(|X|\leq b_n) + b_n^2 P(|X|>b_n)\right)}{\varepsilon^2 n b_n^2} \\
&\leq C \sum_{n\geq A^\alpha} \left(\frac{E(X^2 I(|X|\leq b_n))}{b_n^2} + P(|X|>b_n)\right) \\
&\leq C + C \sum_{n\geq A^\alpha} \frac{1}{n^{2/\alpha}\tilde{L}^2(n^{1/\alpha})} \sum_{A^\alpha\leq i\leq n} E\left(X^2 I(b_{i-1} < |X| \leq b_i)\right) \\
&= C + C \sum_{i\geq B} \left(\sum_{n\geq i} \frac{1}{n^{2/\alpha}\tilde{L}^2(n^{1/\alpha})}\right) E\left(X^2 I(b_{i-1} < |X| \leq b_i)\right) \\
&\leq C + C \sum_{i\geq B} i^{(\alpha-2)/\alpha} \tilde{L}^{-2}(i^{1/\alpha}) E\left(X^2 I(b_{i-1} < |X| \leq b_i)\right),
\end{aligned} \tag{3.13}$$

where we have used Chebyshev's inequality in the first inequality, the Kolmogorov maximal inequality (see Shao [42, Theorem 2]) in the second inequality, (3.2) in the fourth inequality, Proposition 2.6 and the second half of (3.1) in the fifth inequality and Lemma 2.5 in the last inequality. For $\omega \in (b_{i-1} < |X| \leq b_i)$, we have

$$\begin{aligned}
i^{(\alpha-2)/\alpha} \tilde{L}^{-2}(i^{1/\alpha}) &= \frac{i^{(\alpha-2)/\alpha} \tilde{L}^{\alpha-2}(i^{1/\alpha}) L^\alpha\left(i^{1/\alpha}\tilde{L}(i^{1/\alpha})\right)}{\tilde{L}^\alpha(i^{1/\alpha}) L^\alpha\left(i^{1/\alpha}\tilde{L}(i^{1/\alpha})\right)} \\
&\leq C\left(i^{1/\alpha}\tilde{L}(i^{1/\alpha})\right)^{\alpha-2} L^\alpha\left(i^{1/\alpha}\tilde{L}(i^{1/\alpha})\right) \\
&= C b_i^{\alpha-2} L^\alpha(b_i) \leq C|X(\omega)|^{\alpha-2} L^\alpha(|X(\omega)|),
\end{aligned} \tag{3.14}$$

where we have applied (2.3) in the first inequality and the monotonicity of $x^{2-\alpha}L^\alpha(x)$ in the second inequality. Combining (3.13), (3.14), the second half of (3.1) and using Lemma 2.3 (ii), we have

$$\sum_{n\ge A^\alpha}\frac{1}{n}P\Big(\max_{1\le k\le n}|S_{nk}|>b_n\varepsilon/2\Big)\le C+CE\left(|X|^\alpha L^\alpha(|X|+A)\right)<\infty, \tag{3.15}$$

thereby proving (3.12).

The implication [(ii) ⇒ (iii)] is immediate by letting $a_{ni}\equiv 1$. Now, we assume that (iii) holds. Since

$$b_n=n^{1/\alpha}\tilde{L}(n^{1/\alpha})\uparrow\infty \text{ and } \frac{b_{2n}}{b_n}=\frac{2^{1/\alpha}\tilde{L}((2n)^{1/\alpha})}{\tilde{L}(n^{1/\alpha})}\le C,$$

it follows from the proof of [45, Lemma 2.4] that (see (2.1) in [45])

$$\lim_{k\to\infty}\frac{\max_{1\le i\le 2^{k+1}}|\sum_{j=1}^i X_j|}{b_{2^k}}=0 \text{ a.s.} \tag{3.16}$$

For $2^k\le n<2^{k+1}$,

$$\frac{\max_{1\le i\le n}|\sum_{j=1}^i X_j|}{b_n}\le\frac{\max_{1\le i\le 2^{k+1}}|\sum_{j=1}^i X_j|}{b_{2^k}}. \tag{3.17}$$

Combining (3.16) and (3.17), we obtain (3.5).

Finally, we prove the implication [(iv)⇒(i)]. It follows from (3.5) that

$$\lim_{n\to\infty}\frac{\max_{1\le k\le n}|X_k|}{b_n}=0 \text{ a.s.} \tag{3.18}$$

Since $\{X, X_n, n\ge 1\}$ is a sequence of negatively associated random variables, $\{(X_n>b_n), n\ge 1\}$ are pairwise negatively correlated events, and so are $\{(X_n<-b_n), n\ge 1\}$. By the generalized Borel–Cantelli lemma (see, e.g., [37]), it follows from (3.18) that

$$\sum_{n\ge A^\alpha}P(|X|>b_n)=\sum_{n\ge A^\alpha}P(|X_n|>b_n)<\infty \tag{3.19}$$

which, by Proposition 2.6, is equivalent to

$$E\left(|X|^\alpha L^\alpha(|X|+A)\right)<\infty. \tag{3.20}$$

From (3.20), we have $E|X|<\infty$. Since $|X-EX|\le|X|+E|X|$ and $L(\cdot)$ is differentiable slowly varying, (3.20) further implies

$$E\left(|X-EX|^\alpha L^\alpha(|X-EX|+A)\right)<\infty. \tag{3.21}$$

From (3.21) and the proof of ((i)⇒(iv)), we have

$$\lim_{n\to\infty}\left(\frac{\sum_{i=1}^{n} X_i}{b_n} - \frac{n^{(\alpha-1)/\alpha} EX}{\tilde{L}(n^{1/\alpha})}\right) = \lim_{n\to\infty}\frac{\sum_{i=1}^{n}(X_i - EX_i)}{b_n} = 0 \text{ a.s.} \tag{3.22}$$

For the case where $1 < \alpha < 2$, we have from Remark 2.4 that $n^{(\alpha-1)/\alpha}/\tilde{L}(n^{1/\alpha}) \to \infty$ as $n \to \infty$. For the case where $\alpha = 1$, we have from (2.3) that $n^{(\alpha-1)/\alpha}/\tilde{L}(n^{1/\alpha}) = 1/\tilde{L}(n) \sim L(n\tilde{L}(n)) \geq 1$. It thus follows from (3.5) and (3.22) that $E(X) = 0$, i.e., the first half of (3.1) holds. The proof is completed. □

By letting $L(x) \equiv 1$, Theorem 3.1 generalizes a seminal result of Baum and Katz [1] on complete convergence for sums of independent random variables to weighted sums of negatively associated random variables. Recently, Miao et al. [35] have proved the following proposition.

Proposition 3.2 ([35], Theorem 2.1) *Let $0 < \alpha < 2$ and let $\{X, X_n, n \geq 1\}$ be a strictly stationary negatively associated sequence with $E|X|^\alpha \log^{-\beta}(|X| + 2) < \infty$ for some $\beta \geq 0$. In the case where $1 < \alpha < 2$, assume further that $EX = 0$. Then for any $\delta \geq (1 + \alpha^2 - \alpha)/\alpha$, we have*

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n} X_i}{n^{1/\alpha}(\log n)^{\beta(1-\alpha+\delta)}} = 0 \ \ a.s. \tag{3.23}$$

We observe that one only needs to verify (3.23) for the case where $\delta = (1 + \alpha^2 - \alpha)/\alpha$. In this case, (3.23) becomes

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n} X_i}{n^{1/\alpha}(\log n)^{\beta/\alpha}} = 0 \ \text{ a.s.} \tag{3.24}$$

For the case where (i) $1 < \alpha < 2, \beta \geq 0$ or (ii) $\alpha = 1, \beta = 0$, by letting $L(x) \equiv \log^{-\beta/\alpha}(x + 2)$, we see that (3.5) reduces to (3.24). Therefore, Proposition 3.2 is a special case of Theorem 3.1. For the case where $\alpha = 1$ and $\beta > 0$, we will show in the next section (Sect. 4) that Proposition 3.2 holds under a weaker condition that $\{X, X_n, n \geq 1\}$ are pairwise negatively dependent.

Now, we consider another special case where $1 < \alpha < 2$, $\gamma > 0$ and $L(x) = \log^{-1/\gamma}(x)$, $x \geq 2$. Then

$$b_n = n^{1/\alpha}L^{-1}(n^{1/\alpha}) = \left(\frac{1}{\alpha}\right)^{1/\gamma} n^{1/\alpha}\log^{1/\gamma}(n), \ n \geq 2,$$

and we have the following corollary. This result comes close to a solution to the open question raised by Chen and Sung [8] which we have mentioned in Introduction.

Corollary 3.3 *Let $1 < \alpha < 2$, $\gamma > 0$ and $\{X, X_n, n \geq 1\}$ be a sequence of negatively associated and identically distributed random variables. Then the following statements are equivalent.*

(i) *The random variable X satisfies*

$$E(X) = 0 \ \ and \ \ E\left(|X|^{\alpha}/\log^{\alpha/\gamma}(|X|+2)\right) < \infty.$$

(ii) *For every array of constants* $\{a_{ni}, n \geq 1, 1 \leq i \leq n\}$ *satisfying* (3.2)*, we have* (1.2).

(iii) *The strong law of large numbers*

$$\lim_{n\to\infty} \frac{\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|}{n^{1/\alpha}\log^{1/\gamma} n} = 0 \ \ a.s.$$

holds.

Remark 3.4 When $\alpha \leq 2$, by Hölder's inequality, our condition (3.2) implies (1.3). From Corollary 3.3, we see that by slightly extending (1.3), we obtain optimal moment condition for (1.2).

In the following example, we show that the moment condition provided by Chen and Sung [8, Corollary 2.2] is violated, but Corollary 3.3 can still be applied.

Example 3.5 Let $1 < \alpha < 2$, $\gamma > 0$ and $\{X, X_n, n \geq 1\}$ be a sequence of negatively associated and identically distributed random variables with the common density function

$$f(x) = \frac{b}{|x|^{\alpha+1}\log^{1-\alpha/\gamma}(|x|+2)\log^{2}(\log(|x|+2))} I(|x| > 1),$$

where b is the normalization constant. Then

$$EX = 0, \ \ E\left(|X|^{\alpha}/\log^{\alpha/\gamma}(|X|+2)\right) < \infty.$$

Therefore, by applying Corollary 3.3 with $a_{ni} \equiv 1$, we obtain

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right| > \varepsilon n^{1/\alpha}\log^{1/\gamma}(n)\right) < \infty \ \ \text{for all} \ \ \varepsilon > 0,$$

and

$$\lim_{n\to\infty} \frac{\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|}{n^{1/\alpha}\log^{1/\gamma} n} = 0 \ \ \text{a.s.}$$

In this example, we cannot apply Corollary 2.2 in Chen and Sung [8] since

$$E\left(|X|^{\alpha}\log^{1-\alpha/\gamma}(|X|+2)\right) = \infty. \tag{3.25}$$

4 Strong Law of Large Numbers for Sequences of Pairwise Negatively Dependent and Identically Distributed Random Variables

For a sequence of i.i.d. random variables $\{X, X_n, n \geq 1\}$, the classical Hartman–Wintner law of the iterated logarithm states that $E(X) = 0$ and $E(X^2) < \infty$ are necessary and sufficient conditions for the law of the iterated logarithm to hold.

By letting $L(x) \equiv 1$ in Theorem 3.1, we see that the Marcinkiewicz–Zygmund strong law of large numbers holds for sequences of negatively associated and identically distributed random variables under optimal condition $E|X|^\alpha < \infty$. However, in Theorem 3.1, for the case where $\alpha = 1$, we require $L(x) \geq 1$ for $x \geq A$. The reason behind this is because we need $E|X| < \infty$ in the proof. The aim of this section is to establish the strong law of large numbers for the case where $E|X| = \infty$. It turns out that a similar strong law of large numbers still holds even for pairwise negatively dependent random variables. On the law of the iterated logarithm, this line of research was initiated by Feller [18] and completely developed by Kuelbs and Zinn [30], Einmahl [14], Einmahl and Li [15,16] where the authors proved general laws of the iterated logarithm for sequences of i.i.d. random variables with $E(X^2) = \infty$. The normalizing sequences in laws of the iterated logarithm in Einmahl and Li [15,16] are also of the form $\sqrt{nL(n)}$, where $L(n)$ is a slowly varying increasing function.

It is worth noting that for pairwise i.i.d. random variables, the Marcinkiewicz–Zygmund strong law of large numbers holds under optimal moment condition $E|X|^\alpha < \infty, 1 \leq \alpha < 2$ (see Etemadi [17] for the case where $\alpha = 1$ and Rio [39] for the case where $1 < \alpha < 2$). On the case where the random variables are pairwise independent, but not identically distributed, Csörgő et al. [11] proved that the Kolmogorov condition alone does not ensure the strong law of large numbers. Bose and Chandra [2], and Chandra and Goswami [7] generalized the Marcinkiewicz–Zygmund-type law of large numbers for pairwise independent case under the so-called Cesàro uniform integrability condition.

For pairwise negatively dependent random variables, Shen et al. [43] established a strong law of large numbers for pairwise negatively dependent and identically distributed random variables under a very general condition. Precisely, Shen et al. [43, Theorems 3 and 5] proved that if $\{b_n, n \geq 1\}$ is a sequence of positive constants with $b_n/n \uparrow \infty$ and if $\{X, X_n, n \geq 1\}$ is a sequence of pairwise negatively dependent and identically distributed random variables, then $\sum_{n=1}^{\infty} P(|X| > b_n) < \infty$ if and only if $\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i| > b_n\varepsilon\right) < \infty$ for all $\varepsilon > 0$. By combining this result of Shen et al. [43] with Proposition 2.6, we have the following theorem.

Theorem 4.1 *Let $\{X, X_n, n \geq 1\}$ be a sequence of pairwise negatively dependent and identically distributed random variables, and let $L(\cdot)$ be a slowly varying function defined on $[A, \infty)$ for some $A > 0$ with $\tilde{L}(x) \uparrow \infty$ as $x \to \infty$. Then the following statements are equivalent.*

(i) *The random variable X satisfies*

$$E\left(|X|L(|X| + A)\right) < \infty. \tag{4.1}$$

(ii)

$$\sum_{n\ge A} n^{-1}P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}X_i\right| > \varepsilon n\tilde{L}(n)\right) < \infty \textit{ for all } \varepsilon > 0. \tag{4.2}$$

(iii) *The following strong law of large numbers holds:*

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}|X_i|}{n\tilde{L}(n)} = 0 \ \ a.s. \tag{4.3}$$

Martikainen [32] proved that if $\{X, X_n, n \ge 1\}$ is a sequence of pairwise i.i.d. mean 0 random variables, then $E|X|\log^{\gamma}(|X|+2) < \infty$ for some $\gamma > 0$ if and only if $\lim_{n\to\infty}\dfrac{\sum_{i=1}^{n}X_i}{n\log^{-\gamma}(n)} = 0$ a.s. In Theorem 4.1, by letting $L(x) \equiv \log^{-\gamma}(x)$ for some $\gamma > 0$, then for sequences of pairwise negatively dependent and identically distributed random variables, we have $E|X|\log^{-\gamma}(|X|+2) < \infty$ if and only if $\lim_{n\to\infty}\dfrac{\sum_{i=1}^{n}X_i}{n\log^{\gamma}(n)} = 0$ a.s. Therefore, a very special case of Theorem 4.1 can be considered as a counterpart of the main result in Martikainen [32]. This special case also extends Proposition 3.2 (for the case where $\alpha = 1, \beta > 0$) to pairwise negatively dependent random variables.

Finally, we present the following example to illustrate Theorem 4.1. This example concerns a random variable appearing in the St. Petersburg game.

Example 4.2 The St. Petersburg game which is defined as follows: Tossing a fair coin repeatedly until the head appears. If this happens at trial number n, you receive 2^n Euro. The random variable X behind the game has probability mass function:

$$P(X = 2^n) = \frac{1}{2^n}, n \ge 1. \tag{4.4}$$

Since $E(X) = \infty$, a fair price for you to participate in the game would be impossible. To set the fee as a function of the number of games, Feller [19, Chapter X] (see also in Gut [21]) proved that

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}X_i}{n\log n} = 1 \text{ in probability,} \tag{4.5}$$

where $\{X_n, n \ge 1\}$ are independent random variables which have the same distribution as X.

By Theorem 2 of Chow and Robbins [9], it is impossible to have almost sure convergence in (4.5). The natural question that comes to mind is what would be an "optimal" (or "smallest") choice of $\{b_n, n \ge 1\}$ in order for

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}X_i}{b_n} = 0 \text{ a.s.}$$

to hold? It turns out that we can have such a strong law of large numbers even by requiring only the random variables $\{X_n, n \geq 1\}$ are pairwise negatively dependent and have the same distribution as X. To see this, let

$$L(x) = \left((\log |x|)(\log(\log(4 + |x|)))^{1+\gamma}\right)^{-1},$$

where γ is positive, arbitrary small, but fixed, then $E\left(|X|L(|X|)\right) < \infty$. By Theorem 4.1, the Borel–Cantelli lemma and some easy computations, we can show that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n(\log n)(\log(\log(4+n)))^{1+\gamma}} = 0 \text{ a.s.}, \tag{4.6}$$

and

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n(\log n)(\log(\log(4+n)))\log(\log(\log(4+n)))} = \infty \text{ a.s.} \tag{4.7}$$

Remark 4.3 For the i.i.d. case, Csörgő and Simons [10] obtained (4.6) and (4.7) by applying their strong law of large numbers for trimmed sums.

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