ON CONVERGENCE IN MEAN FOR DOUBLE ARRAYS OF PAIRWISE INDEPENDENT RANDOM VARIABLES

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Abstract: In this paper, we prove a theorem on convergence in mean of order p for double arrays of pairwise independent random variables, where $1 \le p < 2$. We obtain Theorem 2.5 in Hong and Hwang [3] as a special case of the main result. The proof is based on the von Bahr–Essen inequality for pairwise independent random variables which is proved recently by Chen, Bai and Sung [1].

Keywords: Double array; convergence in mean; pairwise independent random variables; uniform integrability.

1 Introduction

Convergence in mean for multidimensional arrays of random variables has been studied by a number of authors, we refer to [2, 3, 9, 10] and references therein. In 1978, Gut [2, Theorem 3.1] established a theorem on convergence in mean of order p, $0 < p < 2$, for the multidimensional array of independent and identically distributed random variables under the condition that the pth moment of the random variables is finite. The independence condition is then replaced by pairwise independent condition by Hong and Hwang [3].

A family of random variables $\{X_i, i \in I\}$ is said to be stochastically dominated by a random variable X if

$$\sup_{i\in I}\mathbb{P}(|X_i| > t) \le \mathbb{P}(|X| > t) \text{ for all } t \ge 0. \tag{1.1}$$

We note that many authors use an apparently weaker definition of $\{X_i, i \in I\}$ being stochastically dominated by a random variable X, namely that

$$\sup_{i\in I}\mathbb{P}(|X_i| > x) \le C\mathbb{P}(|X| > x), \text{ for all } x \in \mathbb{R} \tag{1.2}$$

for some constant $C \in (0,\infty)$, but it is shown by Rosalsky and Thanh [7], *inter alia*, that (1.1) and (1.2) are indeed equivalent.

For a double array $\{X_{mn}, m \ge 1, n \ge 1\}$ of pairwise independent random variables which is stochastically dominated by a random variable X, Hong and Hwang [3, Theorem 2.5] proved that if

$$\mathbb{E}(|X|^p \log(1 + |X|)) < \infty,\ 1 < p < 2, \tag{1.3}$$

then

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij} - \mathbb{E}(X_{ij}))}{(mn)^{1/p}} \xrightarrow{\mathcal{L}_1} 0 \tag{1.4}$$

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as $m \vee n \to \infty$. Here and thereafter, $\max\{m,n\}$ is denoted by $m \vee n$, and the natural logarithm of a positive number x is denoted by $\log(x)$. Thanh [9] proved that

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-\mathbb{E}(X_{ij}))}{(mn)^{1/p}} \xrightarrow{\mathcal{L}_p} 0, \tag{1.5}$$

$1 < p < 2$, when $m \vee n \to \infty$ under the condition that the array $\{X_{mn}, m \geq 1, n \geq 1\}$ is independent and $\{|X_{mn}|^p, m \geq 1, n \geq 1\}$ is uniformly integrable. It is easy to see that the condition $\{|X_{mn}|^p, m \geq 1, n \geq 1\}$ being uniformly integrable is weaker than the condition $\{X_{mn}, m \geq 1, n \geq 1\}$ being stochastically dominated by a random variable X with $\mathbb{E}(|X|^p) < \infty$. However, the result of Thanh [9] does not extend Theorem 2.5 of [3] since Thanh [9] requires independent assumption instead of pairwise independent assumption. The purpose of this article is to extend the results of Thanh [9] to double arrays of pairwise independent random variables. Our result therefore extends Theorem 2.5 of Hong and Hwang [3]. To establish the main result, we use the von Bahr–Essen inequality for pairwise independent random variables which is proved by Chen, Bai and Sung [1, Theorem 2.1]. This result is presented in the following lemma.

Lemma 1.1. (Chen, Bai and Sung [1]). Let $1 \leq p \leq 2$ and let $\{X_{mn}, m \geq 1, n \geq 1\}$ be a double array of pairwise independent random variables satisfying $\mathbb{E}(X_{ij}) = 0$ and $\mathbb{E}(|X_{ij}|^p) < \infty$ for all $i, j \geq 1$. Then

$$\mathbb{E}\left(\left|\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}\right|^p\right) \leq C_p \sum_{i=1}^{m}\sum_{j=1}^{n} \mathbb{E}\left(|X_{ij}|^p\right),\ m,n \geq 1, \tag{1.6}$$

where C_p is a positive constant depending only on p.

Obviously if $p = 1$ or $p = 2$, then (1.6) holds with $C_p = 1$. We also note that, in Thanh [9], the author considered d-dimensional arrays. The result presented in this paper can also be extended to d-dimensional arrays with the same argument.

2 Main result

The main result of the paper is the following theorem. This result extends the main result of Thanh [9] to the pairwise independent case.

Theorem 2.1. *Let $1 \leq p < 2$ and let $\{X_{mn}, m \geq 1, n \geq 1\}$ be a double array of pairwise independent random variables such that $\{|X_{mn}|^p, m \geq 1, n \geq 1\}$ is uniformly integrable. Then*

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-\mathbb{E}(X_{ij}))}{(mn)^{1/p}} \xrightarrow{\mathcal{L}_p} 0 \text{ as } m \vee n \to \infty. \tag{2.1}$$

Proof. Since the array $\{|X_{mn}|^p, m \geq 1, n \geq 1\}$ is uniformly integrable, we have

$$\lim_{a\to\infty} \sup_{m\geq 1, n\geq 1} \mathbb{E}\left(|X_{mn}|^p \mathbf{1}(|X_{mn}| > a)\right) = 0. \tag{2.2}$$

Let $\varepsilon > 0$ be given. From (2.2), there exists $M > 0$ such that

$$\mathbb{E}(|X_{mn}|^p \mathbf{1}(|X_{mn}| > M)) < \varepsilon \text{ for all } m \geq 1, n \geq 1. \tag{2.3}$$

For $m \geq 1, n \geq 1$, set

$$X'_{mn} = X_{mn}\mathbf{1}(|X_{mn}| \leq M),\ X''_{mn} = X_{mn}\mathbf{1}(|X_{mn}| > M). \tag{2.4}$$

Then, for all $m \geq 1, n \geq 1$, we have

$$\mathbb{E}\left|X''_{mn} - EX''_{mn}\right|^p \leq 4\mathbb{E}\left|X''_{mn}\right|^p < 4\varepsilon. \tag{2.5}$$

To prove (2.1), we need to show that

$$\lim_{m\vee n\to\infty} \frac{\mathbb{E}\left|\sum_{i=1}^m \sum_{j=1}^n (X_{ij} - \mathbb{E}(X_{ij}))\right|^p}{mn} = 0. \tag{2.6}$$

First, we use the inequality $|x+y|^p \leq 2^{p-1}(|x|^p + |y|^p)$ $(x, y \in \mathbb{R},\ p \geq 1)$ to estimate

$$\mathbb{E}\left|\sum_{i=1}^m \sum_{j=1}^n (X_{ij} - \mathbb{E}(X_{ij}))\right|^p$$

as follows:

$$\mathbb{E}\left|\sum_{i=1}^m \sum_{j=1}^n (X_{ij} - \mathbb{E}(X_{ij}))\right|^p \leq 2^{p-1}\left(\mathbb{E}\left|\sum_{i=1}^m \sum_{j=1}^n (X'_{ij} - \mathbb{E}(X'_{ij}))\right|^p + \mathbb{E}\left|\sum_{i=1}^m \sum_{j=1}^n (X''_{ij} - \mathbb{E}(X''_{ij}))\right|^p\right). \tag{2.7}$$

Next, we estimate the terms on the right hand side of (2.7). Since $1 \leq p < 2$, applying Lyapunov's inequality and the first half of (2.4), we have

$$\begin{aligned} \mathbb{E}\left|\sum_{i=1}^m \sum_{j=1}^n (X'_{ij} - \mathbb{E}(X'_{ij}))\right|^p &\leq \left(\mathbb{E}\left(\sum_{i=1}^m \sum_{j=1}^n (X'_{ij} - \mathbb{E}(X'_{ij}))\right)^2\right)^{p/2} \\ &= \left(\sum_{i=1}^m \sum_{j=1}^n \mathbb{E}\left(X'_{ij} - \mathbb{E}(X'_{ij})\right)^2\right)^{p/2} \\ &\leq \left(\sum_{i=1}^m \sum_{j=1}^n \mathbb{E}\left(X'_{ij}\right)^2\right)^{p/2} \leq (mnM^2)^{p/2}. \end{aligned} \tag{2.8}$$

Applying Lemma 1.1 and (2.5), we have

$$\begin{aligned} \mathbb{E}\left|\sum_{i=1}^m \sum_{j=1}^n (X''_{ij} - \mathbb{E}(X''_{ij}))\right|^p &\leq C_p \sum_{i=1}^m \sum_{j=1}^n \mathbb{E}\left|X''_{ij} - EX''_{ij}\right|^p \\ &\leq 4C_p mn\varepsilon. \end{aligned} \tag{2.9}$$

It follows from (2.7)–(2.9) that

$$\mathbb{E}\left|\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-\mathbb{E}(X_{ij}))\right|^p \leq 2^{p-1}\left((mnM^2)^{p/2}+4C_p mn\varepsilon\right). \tag{2.10}$$

This implies

$$\frac{\mathbb{E}\left|\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-\mathbb{E}(X_{ij}))\right|^p}{mn} \leq 2^{p-1}\left(\frac{M^p}{(mn)^{(2-p)/2}}+4C_p\varepsilon\right). \tag{2.11}$$

Since $p<2$, $\varepsilon>0$ is arbitrary, C_p depends only on p, the conclusion (2.6) follows from (2.11). □

Remark 2.2. (i) When $p=1$, this result is proved in Thanh [9].

(ii) Since convergence in mean of order p implies convergence in probability, we obtain from Theorem 2.1 the Marcinkiewicz–Zygmund type weak law of large numbers. The Marcinkiewicz–Zygmund-type strong and weak laws of large numbers for d-dimensional arrays of random variables are also studied by numbers of authors. We refer the reader to [2, 4, 5, 6, 8, 10, 11] and the references therein.

Corollary 2.3. *Let* $1\leq p<2$ *and let* $\{X_{mn}, m\geq 1, n\geq 1\}$ *be a double array of pairwise independent random variables, and* $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ *be a convex monotone function* g *defined on* $[0,\infty)$ *with* $g(0)=0$ *such that*

$$\lim_{x\to\infty}\frac{\varphi(x)}{x}=\infty. \tag{2.12}$$

If $\sup_{m\geq 1,n\geq 1}\mathbb{E}(|X_{mn}|^p\varphi(|X_{mn}|^p))<\infty$*, then*

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-\mathbb{E}(X_{ij}))}{(mn)^{1/p}} \stackrel{\mathcal{L}_p}{\to} 0 \text{ as } m\vee n\to\infty. \tag{2.13}$$

Proof. Applying the de La Vallée Poussin theorem, we conclude that the array $\{|X_{mn}|^p, m\geq 1, n\geq 1\}$ is uniformly integrable. Therefore, the conclusion (2.13) follows immediately from Theorem 2.1. □

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TÓM TẮT

VỀ SỰ HỘI TỤ THEO TRUNG BÌNH CỦA MẢNG KÉP CÁC BIẾN NGẪU NHIÊN ĐỘC LẬP ĐÔI MỘT

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Trong bài báo này, chúng tôi chứng minh một định lý về sự hội tụ theo trung bình cấp p đối với mảng kép các biến ngẫu nhiên độc lập đôi một, với $1 \leq p < 2$. Từ kết quả chính, chúng tôi nhận được Định lý 2.5 của Hong và Hwang [3] như là một trường hợp đặc biệt. Phép chứng minh dựa vào bất đẳng thức von Bahr–Essen inequality cho các biến ngẫu nhiên độc lập đôi một, một kết quả được chứng minh gần đây bởi Chen, Bai và Sung [1].

Từ khóa: Mảng kép; sự hội tụ theo trung bình; các biến ngẫu nhiên độc lập đôi một; tính khả tích đều.