TEST
https://doi.org/10.1007/s11749-022-00827-w

ORIGINAL PAPER

On a new concept of stochastic domination and the laws of large numbers

Lê Văn Thành[1]

Received: 4 March 2022 / Accepted: 28 July 2022

Abstract
Consider a sequence of positive integers $\{k_n, n \geq 1\}$, and an array of nonnegative real numbers $\{a_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ satisfying $\sup_{n\geq 1} \sum_{i=1}^{k_n} a_{n,i} = C_0 \in (0, \infty)$. This paper introduces the concept of $\{a_{n,i}\}$-stochastic domination. We develop some techniques concerning this concept and apply them to remove an assumption in a strong law of large numbers of Chandra and Ghosal (Acta Math Hung 71(4):327–336, 1996). As a by-product, a considerable extension of a recent result of Boukhari (J Theor Probab, 2021. https://doi.org/10.1007/s10959-021-01120-6) is established and proved by a different method. The results on laws of large numbers are new even when the summands are independent. Relationships between the concept of $\{a_{n,i}\}$-stochastic domination and the concept of $\{a_{n,i}\}$-uniform integrability are presented. Two open problems are also discussed.

Keywords Stochastic domination · Uniform integrability · Strong law of large numbers · Weak law of large numbers · Weighted sum · Cesàro stochastic domination

Mathematics Subject Classification 60E15 · 60F05 · 60F15

1 Introduction and motivation

Let $1 \leq p < 2$. The classical Marcinkiewicz–Zygmund strong law of large numbers (SLLN) states that for a sequence $\{X_n, n \geq 1\}$ of independent identically distributed mean zero random variables, condition $\mathbb{E}(|X_1|^p) < \infty$ is necessary and sufficient for

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n^{1/p}} = 0 \text{ almost surely (a.s.).} \tag{1.1}$$

✉ Lê Văn Thành
levt@vinhuni.edu.vn

[1] Department of Mathematics, Vinh University, Vinh, Nghe An, Vietnam

Published online: 15 August 2022

Now, let us recall a weak dependence structure introduced in Chandra and Ghosal (1996a) as follows. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be *asymptotically almost negatively associated* (AANA) if there exists a sequence of nonnegative real numbers $\{q_n, n \geq 1\}$ with $\lim_{n\to\infty} q_n = 0$ such that

$$\mathrm{Cov}(f(X_n), g(X_{n+1}, \ldots, X_{n+k})) \leq q_n \left(\mathrm{Var}(f(X_n))\,\mathrm{Var}(g(X_{n+1}, \ldots, X_{n+k}))\right)^{1/2},$$

for all $n \geq 1, k \geq 1$ and for all coordinatewise nondecreasing continuous functions f and g provided the right side of the above inequality is finite. The $q_n, n \geq 1$ are called *mixing coefficients*. The starting point of the current investigation is the following SLLN established by Chandra and Ghosal (1996a).

Theorem 1.1 (Chandra and Ghosal 1996a) *Let $1 \leq p < 2$ and let $\{X_n, n \geq 1\}$ be a sequence of AANA mean zero random variables with the sequence of mixing coefficients satisfying $\sum_{n=1}^{\infty} q_n^2 < \infty$. Let*

$$G(x) = \sup_{n\geq 1} \frac{1}{n} \sum_{i=1}^{n} \mathbb{P}(|X_i| > x),\ \ x \in \mathbb{R}.$$

If

$$\int_0^{\infty} x^{p-1} G(x)\,\mathrm{d}x < \infty, \tag{1.2}$$

and

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n|^p > n) < \infty, \tag{1.3}$$

then the Marcinkiewicz–Zygmund SLLN (1.1) is obtained.

The above result of Chandra and Ghosal (1996a) weakens the assumptions in the classical Marcinkiewicz–Zygmund SLLN not only by considering a weak dependence structure, but also by relaxing the identical distribution condition. We refer to (1.2) and (1.3) as the Chandra–Ghosal conditions. It is clear that for a sequence $\{X_n, n \geq 1\}$ of random variables with a common law, (1.2) and (1.3) are equivalent since each of them is equivalent to $\mathbb{E}(|X_1|^p) < \infty$. This leads to a natural question in this context is whether (1.2) or (1.3) can be removed. The current work is an attempt to answer this question. More precisely, we shall prove the following theorem.

Theorem 1.2 *Theorem 1.1 holds without Condition (1.3).*

To prove Theorem 1.2, we develop some results concerning a new concept of stochastic domination which leads to the concept of the Cesàro stochastic domination as a particular case.

Let $\{k_n, n \geq 1\}$ be a sequence of positive integers. An array $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ of random variables is said to be *stochastically dominated* by a random variable X if

$$\sup_{1\leq i\leq k_n, n\geq 1} \mathbb{P}(|X_{n,i}| > x) \leq \mathbb{P}(|X| > x), \quad \text{for all } x \in \mathbb{R}. \tag{1.4}$$

This concept was extended to the concept of the so-called Cesàro stochastic domination by Gut (1992) as follows. An array $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ of random variables is said to be *stochastically dominated in the Cesàro sense* (or *weakly mean dominated*) by a random variable X if

$$\sup_{n\geq 1} \frac{1}{k_n} \sum_{i=1}^{k_n} \mathbb{P}(|X_{n,i}| > x) \leq C\mathbb{P}(|X| > x), \quad \text{for all } x \in \mathbb{R}, \tag{1.5}$$

where $C > 0$ is a constant. It was shown by Gut [1992, Example 2.1] that (1.5) is strictly weaker than (1.4).

We will now introduce a new concept of stochastic domination. Let $\{a_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of nonnegative real numbers satisfying

$$\sup_{n\geq 1} \sum_{i=1}^{k_n} a_{n,i} = C_0 \in (0, \infty). \tag{1.6}$$

An array $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ of random variables is said to be $\{a_{n,i}\}$-*stochastically dominated* by a random variable X if

$$\sup_{n\geq 1} \sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(|X_{n,i}| > x) \leq C_0\mathbb{P}(|X| > x), \quad \text{for all } x \in \mathbb{R}. \tag{1.7}$$

In view of Gut's definition in (1.5), one may be tempted to give an apparently weaker definition of $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ being $\{a_{n,i}\}$-stochastically dominated by a random variable Y, namely that

$$\sup_{n\geq 1} \sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(|X_{n,i}| > x) \leq C\mathbb{P}(|Y| > x), \quad \text{for all } x \in \mathbb{R}, \tag{1.8}$$

for some finite constant $C > 0$. However, it will be shown in Theorem 2.2 that (1.7) and (1.8) are indeed equivalent. Therefore, concerning Gut's definition of the Cesàro stochastic domination, we can simply choose $C = 1$ in (1.5). If $a_{n,i} = 1/k_n, 1 \leq i \leq k_n,\ n \geq 1$, then it is obvious that $C_0 = 1$, and the concept of $\{a_{n,i}\}$-stochastic domination reduces to the concept of stochastic domination in the Cesàro sense.

If $0 < p < 1$ and $\{X_n, n \geq 1\}$ is a sequence of random variables satisfying (1.2) and (1.3), then the Marcinkiewicz–Zygmund SLLN (1.1) is valid irrespective of any dependence structure (see Remark 3 in Chandra and Ghosal 1996a). Boukhari (2021) recently used techniques from martingales theory to prove that a similar result holds true for the weak law of large numbers (WLLN) for maximal partial sums with general normalizing sequences. The tools developed in this paper also allow us to establish an extension of Theorem 1.2 of Boukhari (2021). A special case of our WLLNs in Sect. 4 is the following theorem.

Theorem 1.3 *Let* $\{X_n, n \geq 1\}$ *be a sequence of random variables,* $G(\cdot)$ *as in Theorem 1.1 and let* $\{b_n, n \geq 1\}$ *be a nondecreasing sequence of positive real numbers such that*

$$\sum_{i=1}^{n} \frac{b_i}{i^2} = O\left(\frac{b_n}{n}\right). \tag{1.9}$$

If

$$\lim_{k\to\infty} kG(b_k) = 0, \tag{1.10}$$

then the WLLN

$$\frac{1}{b_n} \max_{j\leq n} \left| \sum_{i=1}^{j} X_i \right| \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty \tag{1.11}$$

is obtained.

Remark 1.4 Boukhari [2021, Theorem 1.2] proved Theorem 1.3 under a stronger condition that the sequence $\{X_n, n \geq 1\}$ is stochastically dominated by a random variable X satisfying

$$\lim_{k\to\infty} k\mathbb{P}(|X| > b_k) = 0.$$

An example in Sect. 4 shows that for $0 < p < 1$ and $b_n = n^{1/p}$, there exists a sequence of random variables $\{X_n, n \geq 1\}$ with no stochastically dominating random variable, but (1.10) is satisfied and therefore the WLLN (1.11) is valid. Our proof of Theorem 1.3 is simpler than that of Theorem 1.2 of Boukhari (2021) in the sense that we do not use the Doob maximal inequality for martingales as was done in Boukhari (2021). The WLLN for dependent random variables and random vectors was also studied in Hien and Thành (2015), Kruglov (2011), Rosalsky and Thành (2009), among others.

The rest of the paper is organized as follows. In Sect. 2, we prove the equivalence between the definitions of $\{a_{n,i}\}$-stochastic domination given in (1.7) and (1.8). It is also shown that certain bounded moment conditions on an array of random variables $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ can accomplish the concept of $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ being $\{a_{n,i}\}$-stochastically dominated. Section 3 discusses about relationships between the concept of $\{a_{n,i}\}$-stochastic domination and the concept of $\{a_{n,i}\}$-uniform integrability. Strong and weak laws of large numbers for triangular arrays of random variables are presented in Sect. 4. From these general results, Theorems 1.2 and 1.3 follow. Section 5 contains further remarks and two open problems.

Notation: Throughout this paper, $\{k_n, n \geq 1\}$ is assumed to be a sequence of positive integers. For a set A, $\mathbf{1}(A)$ denotes the indicator function of A. For $x \geq 0$, let $\log x$ denote the logarithm base 2 of $\max\{2, x\}$. For $x \geq 0$ and for a fixed positive integer ν, let

$$\log_\nu(x) := (\log x)(\log\log x)\dots(\log\cdots\log x), \tag{1.12}$$

and

$$\log_\nu^{(2)}(x) := (\log x)(\log\log x)\dots(\log\cdots\log x)^2, \tag{1.13}$$

where in both (1.12) and (1.13), there are ν factors. For example, $\log_2(x) = (\log x)(\log\log x)$, $\log_3^{(2)}(x) = (\log x)(\log\log x)(\log\log\log x)^2$, and so on.

2 On the concept of $\{a_{n,i}\}$-stochastic domination

In this section, we employ some properties of slowly varying functions as well as techniques in Rosalsky and Thành (2021) to prove some results on the concept of $\{a_{n,i}\}$-stochastic domination. We note that all results in Sects. 2 and 3 are stated for a triangular array $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ of random variables but they still hold for a sequence of random variables $\{X_n, n \geq 1\}$ by considering $X_{n,i} = X_i$, $1 \leq i \leq k_n, n \geq 1$.

The following theorem is a simple result and its proof is similar to that of Theorem 2.1 of Rosalsky and Thành (2021). It plays a useful role in proving the laws of large numbers in Sect. 4.

Theorem 2.1 *Let $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of random variables, let $\{a_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of nonnegative real numbers satisfying (1.6) and let*

$$F(x) = 1 - \frac{1}{C_0}\sup_{n\geq 1}\sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(|X_{n,i}| > x),\ x \in \mathbb{R}.$$

Then, $F(\cdot)$ is the distribution function of a random variable X if and only if $\lim_{x\to\infty} F(x) = 1$. In such a case, $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ is $\{a_{n,i}\}$-stochastically dominated by X.

Proof It is clear that $F(\cdot)$ is nondecreasing, and

$$\lim_{x\to-\infty} F(x) = 1 - \frac{1}{C_0}\sup_{n\geq 1}\sum_{i=1}^{k_n} a_{n,i} = 0.$$

Let $\varepsilon > 0$ be arbitrary. For $a \in \mathbb{R}$, let $n_0 \geq 1$ be such that

$$\frac{1}{C_0}\sum_{i=1}^{k_{n_0}} a_{n_0,i}\mathbb{P}(|X_{n_0,i}| > a) > \frac{1}{C_0}\sup_{n\geq 1}\sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(|X_{n,i}| > a) - \varepsilon/2,$$

or equivalently,

$$1 - \frac{1}{C_0}\sum_{i=1}^{k_{n_0}} a_{n_0,i}\mathbb{P}(|X_{n_0,i}| > a) < F(a) + \varepsilon/2. \tag{2.1}$$

Since the function

$$x \mapsto \frac{1}{C_0}\sum_{i=1}^{k_{n_0}} a_{n_0,i}\mathbb{P}(|X_{n_0,i}| > x),\ \ x \in \mathbb{R},$$

is nonincreasing and right continuous, there exists $\delta > 0$ such that

$$\begin{aligned}-\varepsilon/2 < \frac{1}{C_0}\sum_{i=1}^{k_{n_0}} a_{n_0,i}\mathbb{P}(|X_{n_0,i}| > x) - \frac{1}{C_0}\sum_{i=1}^{k_{n_0}} a_{n_0,i}\mathbb{P}(|X_{n_0,i}| > a) \\ \leq 0 \text{ for all } x \text{ such that } 0 \leq x - a < \delta.\end{aligned}$$

Therefore, for x satisfying $0 \leq x - a < \delta$, we have

$$\begin{aligned}F(x) - \varepsilon &= 1 - \frac{1}{C_0}\sup_{n\geq 1}\sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(|X_{n,i}| > x) - \varepsilon \\ &\leq 1 - \frac{1}{C_0}\sum_{i=1}^{k_{n_0}} a_{n_0,i}\mathbb{P}(|X_{n_0,i}| > x) - \varepsilon \\ &< 1 - \frac{1}{C_0}\sum_{i=1}^{k_{n_0}} a_{n_0,i}\mathbb{P}(|X_{n_0,i}| > a) - \varepsilon/2 \\ &< F(a) \ \ (\text{by } (2.1))\end{aligned}$$

and so $|F(x) - F(a)| < \varepsilon$. Thus, $\lim_{x\to a^+} F(x) = F(a)$. Since $a \in \mathbb{R}$ is arbitrary, this implies that F is right continuous on $\mathbb{R}$. Since $F(\cdot)$ is nondecreasing, right continuous and $\lim_{x\to -\infty} F(x) = 0$, it is the distribution function of a random variable X if and only if $\lim_{x\to\infty} F(x) = 1$. By definition of $F(\cdot)$, $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ is $\{a_{n,i}\}$-stochastically dominated by X. □

The next theorem establishes the equivalence between the definitions of $\{a_{n,i}\}$-stochastic domination given in (1.7) and (1.8). A similar result concerning the concept of stochastic domination was proved by Rosalsky and Thành (2021).

Theorem 2.2 *Let* $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ *be an array of random variables and let* $\{a_{n,i}, 1 \le i \le k_n, n \ge 1\}$ *be an array of nonnegative real numbers satisfying* (1.6). *Then, there exists a random variable* X *satisfying* (1.7) *if and only if there exist a random variable* Y *and a finite constant* $C > 0$ *satisfying* (1.8).

Moreover, (i) if $g : [0,\infty) \to [0,\infty)$ *is a measurable function with* $g(0) = 0$ *which is bounded on* $[0, A]$ *and differentiable on* $[A, \infty)$ *for some* $A \ge 0$ *or (ii) if* $g : [0,\infty) \to [0,\infty)$ *is a continuous function which is eventually nondecreasing with* $\lim_{x\to\infty} g(x) = \infty$, *then the condition* $\mathbb{E}(g(|Y|)) < \infty$ *where* Y *is as in* (1.8) *implies that* $\mathbb{E}(g(|X|)) < \infty$ *where* X *is as in* (1.7).

Proof The necessity half is immediate by taking $Y = X$ and $C = C_0$. Conversely, if there exist a nonnegative random variable Y and a finite constant $C > 0$ satisfying (1.8), then

$$\lim_{x\to\infty} \sup_{n\ge1} \sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(|X_{n,i}| > x) \le C \lim_{x\to\infty} \mathbb{P}(|Y| > x) = 0,$$

and so by Theorem 2.1, there exists a random variable X with distribution function

$$F(x) = 1 - \frac{1}{C_0} \sup_{n\ge1} \sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(|X_{n,i}| > x),\ x \in \mathbb{R}.$$

This implies

$$\sup_{n\ge1} \sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(|X_{n,i}| > x) = C_0(1 - F(x)) = C_0\mathbb{P}(X > x),\ x \in \mathbb{R}$$

thereby verifying (1.7).

The rest of the proof proceeds in a similar manner as that of Theorem 2.4 (i) and (ii) in Rosalsky and Thành (2021). The details will be omitted. □

The following result is a direct consequence of Theorem 2.2 by choosing $a_{n,i} = 1/k_n$ for all $1 \le i \le k_n$, $n \ge 1$. It says that in the Cesàro stochastic domination definition, (1.5) can be simplified to

$$\sup_{n\ge1} \frac{1}{k_n} \sum_{i=1}^{k_n} \mathbb{P}(|X_{n,i}| > x) \le \mathbb{P}(|Y| > x), \quad \text{for all } x \in \mathbb{R}, \tag{2.2}$$

for some random variable Y (surprisingly, this was not noticed by Gut 1992).

Corollary 2.3 *Let* $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ *be an array of random variables. Then, there exist a random variable* X *and a finite constant* $C > 0$ *satisfying* (1.5) *if and only if there exists a random variable* Y *satisfying* (2.2). *Moreover, if* g *is a measurable function satisfying assumptions in Theorem* 2.2, *then the condition* $\mathbb{E}(g(|X|)) < \infty$ *where* X *is as in* (1.5) *implies that* $\mathbb{E}(g(|Y|)) < \infty$ *where* Y *is as in* (2.2).

By using integration by parts, we have from (2.2) that for all $r > 0$ and $x \ge 0$

$$\sup_{n\ge 1} \frac{1}{k_n} \sum_{i=1}^{k_n} \mathbb{E}\left(|X_{n,i}|^r \mathbf{1}(|X_{n,i}| \le x)\right) \le \mathbb{E}\left(|Y|^r \mathbf{1}(|Y| \le x)\right) + x^r \mathbb{P}(|Y| > x), \quad (2.3)$$

and

$$\sup_{n\ge 1} \frac{1}{k_n} \sum_{i=1}^{k_n} \mathbb{E}\left(|X_{n,i}|^r \mathbf{1}(|X_{n,i}| > x)\right) \le \mathbb{E}\left(|Y|^r \mathbf{1}(|Y| > x)\right). \quad (2.4)$$

We will use (2.3) and (2.4) in our proofs without further mention.

The following consequence of Theorem 2.1 is also useful in proving the laws of large numbers.

Corollary 2.4 *Let $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ be an array of random variables and let*

$$F(x) = 1 - \sup_{n\ge 1} \frac{1}{k_n} \sum_{i=1}^{k_n} \mathbb{P}(|X_{n,i}| > x), \ x \in \mathbb{R}.$$

Then, F is the distribution function of a random variable X if and only if $\lim_{x\to\infty} F(x) = 1$. In such a case, $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ is stochastically dominated in the Cesàro sense by X.

In view of (2.2), if $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ is stochastically dominated in the Cesàro sense by a random variable Y, then it is $\{a_{n,i}\}$-stochastically dominated by Y with $a_{n,i} = 1/k_n$, $1 \le i \le k_n$, $n \ge 1$. The following example shows that the concept of $\{a_{n,i}\}$-stochastic domination is strictly weaker than the concept of stochastic domination in the Cesàro sense.

Example 2.5 Let $k_n \equiv n$ and let m_n be the greatest integer number which is less than or equal to $n/2$. Let $a_{n,i} = 1/m_n$ for $1 \le i \le m_n$, $n \ge 2$ and $a_{n,i} = 1/n^2$ for $m_n < i \le n, n \ge 1$. Let $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ be an array of random variables such that

$$\mathbb{P}(X_{n,i} = -1) = \mathbb{P}(X_{n,i} = 1) = 1/2, \ 1 \le i \le m_n, \ n \ge 2$$

and

$$\mathbb{P}(X_{n,i} = -n) = \mathbb{P}(X_{n,i} = n) = 1/2, \ m_n < i \le n, \ n \ge 1.$$

Then, for $x \ge 1$, we have

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{P}(|X_{n,i}| > x) = \begin{cases} 0 & \text{if } n \le x, \\ \dfrac{n - m_n}{n} & \text{if } n > x. \end{cases}$$

This implies

$$\sup_{n\geq 1}\frac{1}{n}\sum_{i=1}^{n}\mathbb{P}(|X_{n,i}|>x)\geq \frac{1}{2},\quad \text{for all } x\geq 1.$$

Thus, by Corollary 2.4, there is no random variable Y such that $\{X_{n,i}, 1\leq i\leq n, n\geq 1\}$ is stochastically dominated in the Cesàro sense by Y. Now, we have

$$1<C_0=\sup_{n\geq 1}\sum_{i=1}^{n}a_{n,i}=\sup_{n\geq 2}\left(1+\frac{n-m_n}{n^2}\right)\leq 2,$$

and for $x\geq 1$ and $n\geq 1$,

$$\sum_{i=1}^{n}a_{n,i}\mathbb{P}(|X_{n,i}|>x)=\sum_{i=m_n+1}^{n}\frac{1}{n^2}\mathbb{P}(|X_{n,i}|>x)=\begin{cases}0 & \text{if } n\leq x,\\ \dfrac{n-m_n}{n^2}\leq \dfrac{1}{n} & \text{if } n>x.\end{cases}$$

Thus, $\sum_{i=1}^{n}a_{n,i}\mathbb{P}(|X_{n,i}|>x)\to 0$ as $x\to\infty$. By Theorem 2.1, $\{X_{n,i}, 1\leq i\leq n, n\geq 1\}$ is $\{a_{n,i}\}$-stochastically dominated by a random variable X with distribution function

$$F(x)=1-\frac{1}{C_0}\sup_{n\geq 1}\sum_{i=1}^{n}a_{n,i}\mathbb{P}(|X_{n,i}|>x),\ x\in\mathbb{R}.$$

This shows that the concept of $\{a_{n,i}\}$-stochastic domination is strictly weaker than the concept of stochastic domination in the Cesàro sense.

Recall that a real-valued function $L(\cdot)$ is said to be *slowly varying* (at infinity) if it is a positive and measurable function on $[A,\infty)$ for some $A\geq 0$, and for each $\lambda>0$,

$$\lim_{x\to\infty}\frac{L(\lambda x)}{L(x)}=1.$$

If $L(\cdot)$ is a slowly varying function, then there exists a slowly varying function $\tilde{L}(\cdot)$, unique up to an asymptotic equivalence, satisfying

$$\lim_{x\to\infty}L(x)\tilde{L}\left(xL(x)\right)=1 \text{ and } \lim_{x\to\infty}\tilde{L}(x)L\left(x\tilde{L}(x)\right)=1. \tag{2.5}$$

The function $\tilde{L}(\cdot)$ is called the de Bruijn conjugate of $L(\cdot)$ (see Bingham et al. [1989, p. 29]). For many "nice" slowly varying functions $L(\cdot)$, we can choose $\tilde{L}(x)=1/L(x)$. Especially, if $L(x)=(\log x)^{\gamma}$ or $L(x)=(\log\log x)^{\gamma}$ for some $\gamma\in\mathbb{R}$, then $\tilde{L}(x)=1/L(x)$.

Let $L(\cdot)$ be a slowly varying function and let $\alpha>0$. By using a suitable asymptotic equivalence version (see Lemmas 2.2 and 2.3 (i) in Anh et al. 2021), we can firstly assume that $L(\cdot)$ is positive and differentiable on $[a, \infty)$, and $x^\alpha L(x)$ is strictly increasing on $[a, \infty)$ for some large a. Next, let $L_1(\cdot)$ be a slowly varying function satisfying $L_1(x) = L(a)x/a$ if $0 \le x < a$ and $L_1(x) = L(x)$ if $x \ge a$ (i.e., $L_1(0) = 0$ with a linear growth to $L(a)$ over $[0, a)$, and $L_1(x) \equiv L(x)$ on $[a, \infty)$). Then, (i) $L_1(x)$ is continuous on $[0, \infty)$ and differentiable on $[a, \infty)$, and (ii) $x^\alpha L_1(x)$ is strictly increasing on $[0, \infty)$. In this paper, we will assume, without loss of generality, that these properties are fulfilled for the underlying slowly varying functions.

The next theorem shows that bounded moment conditions on an array of random variables $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ with respect to weights $\{a_{n,i}, 1 \le i \le k_n, n \ge 1\}$ can accomplish $\{a_{n,i}\}$-stochastic domination.

Theorem 2.6 *Let $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ be an array of random variables, $\{a_{n,i}, 1 \le i \le k_n, n \ge 1\}$ an array of nonnegative real numbers satisfying (1.6). Let $p > 0$ and let ν be a fixed positive integer. Let $L(\cdot)$ be a slowly varying function. If*

$$\sup_{n\ge1} \sum_{i=1}^{k_n} a_{n,i} \mathbb{E}\left(|X_{n,i}|^p L(|X_{n,i}|) \log_\nu^{(2)}(|X_{n,i}|)\right) < \infty, \tag{2.6}$$

then there exists a random variable X with distribution function

$$F(x) = 1 - \frac{1}{C_0} \sup_{n\ge1} \sum_{i=1}^{k_n} a_{n,i} \mathbb{P}(|X_{n,i}| > x), \ x \in \mathbb{R} \tag{2.7}$$

such that $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ is $\{a_{n,i}\}$-stochastically dominated by X, and

$$\mathbb{E}\left(|X|^p L(|X|)\right) < \infty. \tag{2.8}$$

Remark 2.7 A weaker version of Theorem 2.6 for stochastic domination (without the appearance of the slowly varying function $L(\cdot)$) was proved by Rosalsky and Thành (2021) (see Theorem 2.5 (ii) and (iii) in Rosalsky and Thành 2021). Typical examples of slowly varying functions $L(\cdot)$ for (2.6) are $L(x) \equiv 1$ and $L(x) \equiv L_1(x)(\log_\nu^{(2)}(x))^{-1}$, where $L_1(\cdot)$ is another slowly varying function. Theorem 2.6 is proved by employing an idea from Galambos and Seneta (1973).

Before proving Theorem 2.6, we recall a simple result on the expectation of a nonnegative random variable, see Rosalsky and Thành (2021) for a proof.

Lemma 2.8 *Let $h : [0, \infty) \to [0, \infty)$ be a measurable function with $h(0) = 0$ which is bounded on $[0, A]$ and differentiable on $[A, \infty)$ for some $A \ge 0$. If ξ is a nonnegative random variable, then*

$$\mathbb{E}(h(\xi)) = \mathbb{E}(h(\xi)\mathbf{1}(\xi \le A)) + h(A) + \int_A^\infty h'(x)\mathbb{P}(\xi > x)\,\mathrm{d}x.$$

Proof of Theorem 2.6. Set

$$g(x)=x^pL(x)\log_\nu^{(2)}(x),\quad \text{and } h(x)=x^pL(x),\ x\ge 0.$$

Since $\lim_{x\to\infty} g(x)=\infty$ and $g(\cdot)$ is strictly increasing on $[0,\infty)$ as we have assumed before, we have from Markov's inequality and (2.6) that

$$0\le \lim_{x\to\infty}\sup_{n\ge 1}\sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(|X_{n,i}|>x)\le \lim_{x\to\infty}\frac{1}{g(x)}\sup_{n\ge 1}\sum_{i=1}^{k_n} a_{n,i}\mathbb{E}(g(|X_{n,i}|))=0.$$

By Theorem 2.1, the array $\{X_{n,i},\ 1\le i\le k_n, n\ge 1\}$ is $\{a_{n,i}\}$-stochastically dominated by a random variable X with distribution function $F(\cdot)$ given in (2.7).

Next, we prove (2.8). We firstly consider the case where the slowly varying function $L(\cdot)$ is differentiable on an infinite interval far enough from 0, and

$$\lim_{x\to\infty}\frac{xL'(x)}{L(x)}=0. \tag{2.9}$$

By (2.9), there exists $B\ge 0$ such that

$$\left|\frac{xL'(x)}{L(x)}\right|\le \frac{p}{2},\ x>B.$$

It follows that

$$\begin{aligned} h'(x)&=px^{p-1}L(x)+x^pL'(x)=x^{p-1}L(x)\left(p+\frac{xL'(x)}{L(x)}\right)\\ &\le \frac{3px^{p-1}L(x)}{2},\ x\ge B. \end{aligned}\tag{2.10}$$

Therefore, there exists a constant C_1 such that

$$\begin{aligned} \mathbb{E}(h(X))&=\mathbb{E}(h(X)\mathbf{1}(X\le B))+h(B)+\int_B^\infty h'(x)\mathbb{P}(X>x)\mathrm{d}x\\ &\le C_1+\frac{3p}{2}\int_B^\infty x^{p-1}L(x)\mathbb{P}(X>x)\mathrm{d}x\\ &= C_1+\frac{3p}{2C_0}\int_B^\infty x^{p-1}L(x)\sup_{n\ge 1}\sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(|X_{n,i}|>x)\mathrm{d}x\\ &\le C_1+\frac{3p}{2C_0}\int_B^\infty \frac{1}{x\log_\nu^{(2)}(x)}\sup_{n\ge 1}\sum_{i=1}^{k_n} a_{n,i}\mathbb{E}\left(g(|X_{n,i}|)\right)\mathrm{d}x\\ &= C_1+\frac{3p}{2C_0}\sup_{n\ge 1}\sum_{i=1}^{k_n} a_{n,i}\mathbb{E}\left(g(|X_{n,i}|)\right)\int_B^\infty \frac{\mathrm{d}x}{x\log_\nu^{(2)}(x)}<\infty, \end{aligned}$$

where we have applied Lemma 2.8 in the first equality, (2.10) in the first inequality, Markov's inequality in the second inequality, and (2.6) in the last inequality. Thus, we obtain (2.8) in this case.

For general slowly varying function $L(\cdot)$, by a result from page 111 of Galambos and Seneta (1973), there exists a slowly varying function $L_1(\cdot)$ which is differentiable on $[B_1, \infty)$ for some B_1 large enough, and satisfies

$$\lim_{x\to\infty} \frac{L_1(x)}{L(x)} = 1 \tag{2.11}$$

and

$$\lim_{x\to\infty} \frac{xL_1'(x)}{L_1(x)} = 0.$$

For $n \geq 1$, $1 \leq i \leq n$, we have from (2.11) that for all B_2 large enough

$$\begin{aligned} &\mathbb{E}\left(|X_{n,i}|^p L_1(|X_{n,i}|)\log_\nu^{(2)}(|X_{n,i}|)\right) \\ &= \mathbb{E}\left(|X_{n,i}|^p L_1(|X_{n,i}|)\log_\nu^{(2)}(|X_{n,i}|)\mathbf{1}(|X_{n,i}| \leq B_2)\right) \\ &\quad + \mathbb{E}\left(|X_{n,i}|^p L_1(|X_{n,i}|)\log_\nu^{(2)}(|X_{n,i}|)\mathbf{1}(|X_{n,i}| > B_2)\right) \\ &\leq C_2 + 2\mathbb{E}\left(|X_{n,i}|^p L(|X_{n,i}|)\log_\nu^{(2)}(|X_{n,i}|)\mathbf{1}(|X_{n,i}| > B_2)\right), \end{aligned} \tag{2.12}$$

where C_2 is a finite constant. Combining (2.6) and (2.12) yields

$$\sup_{n\geq 1} \sum_{i=1}^{k_n} a_{n,i}\mathbb{E}\left(|X_{n,i}|^p L_1(|X_{n,i}|)\log_\nu^{(2)}(|X_{n,i}|)\right) < \infty. \tag{2.13}$$

Proceeding exactly the same manner as the first case with $L(\cdot)$ is replaced by $L_1(\cdot)$, we obtain (2.8). The proof of the theorem is completed. □

The following corollary follows immediately from Theorem 2.6.

Corollary 2.9 *Let $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of random variables, and let $L(\cdot)$ be a slowly varying function. Let $p > 0$ and let ν be a fixed positive integer. If*

$$\sup_{n\geq 1} \frac{1}{k_n} \sum_{i=1}^{k_n} \mathbb{E}\left(|X_{n,i}|^p L(|X_{n,i}|)\log_\nu^{(2)}(|X_{n,i}|)\right) < \infty,$$

then there exists a random variable X with distribution function

$$F(x) = 1 - \sup_{n\geq 1} \frac{1}{k_n} \sum_{i=1}^{k_n} \mathbb{P}(|X_{n,i}| > x),\ x \in \mathbb{R}$$

such that $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ *is stochastically dominated in the Cesàro sense by* X, *and*

$$\mathbb{E}\left(|X|^p L(|X|)\right) < \infty.$$

3 Relationships between $\{a_{n,i}\}$-stochastic domination and $\{a_{n,i}\}$-uniform integrability

The concept of $\{a_{n,i}\}$-uniform integrability was introduced by Ordóñez Cabrera (1994). Let $\{a_{n,i}, 1 \le i \le k_n, n \ge 1\}$ be an array of nonnegative real numbers satisfying (1.6). An array $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ of random variables is said to be $\{a_{n,i}\}$-*uniformly integrable* if

$$\lim_{a\to\infty} \sum_{i=1}^{k_n} a_{n,i}\mathbb{E}(|X_{n,i}|\mathbf{1}(|X_{n,i}| > a)) = 0.$$

Similar to the classical characterization of the uniform integrability, it was proved by Ordóñez Cabrera (1994) that an array of random variables $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ is $\{a_{n,i}\}$-uniformly integrable if and only if

$$\sup_{n\ge 1} \sum_{i=1}^{k_n} a_{n,i}\mathbb{E}(|X_{n,i}|) < \infty$$

and for each $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $\{A_{n,i}, 1 \le i \le k_n, n \ge 1\}$ is an array of events satisfying

$$\sup_{n\ge 1} \sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(A_{n,i}) < \delta,$$

then

$$\sup_{n\ge 1} \sum_{i=1}^{k_n} a_{n,i}\mathbb{E}(|X_{n,i}|\mathbf{1}(A_{n,i})) < \varepsilon.$$

If $a_{n,i} = 1/k_n, 1 \le i \le k_n,\ n \ge 1$, then it reduces to the concept of $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ being *uniformly integrable in the Cesàro sense* which was introduced in Chandra (1989). The de La Vallée–Poussin criterion for uniform integrability in the Cesàro sense, and for $\{a_{n,i}\}$-uniform integrability was proved, respectively, by Chandra and Goswami (1992), and Ordóñez Cabrera (1994). The former is a special case of the latter, which reads as follows: An array of random variables $\{X_{n,i}, 1 \le i \le k_n, n \ge 1\}$ is $\{a_{n,i}\}$-uniformly integrable if and only if there exists a measurable function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$, $g(x)/x \to \infty$ as $x \to \infty$, and

$$\sup_{n\geq 1}\sum_{i=1}^{k_n} a_{n,i}\mathbb{E}(g(|X_{n,i}|)) < \infty.$$

Moreover, g can be selected to be convex and such that $g(x)/x$ is nondecreasing.

The next theorem establishes relationships between the concept of $\{a_{n,i}\}$-stochastic domination and the concept of $\{a_{n,i}\}$-uniform integrability.

Theorem 3.1 *Let $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of random variables, and let $\{a_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of nonnegative real numbers satisfying (1.6). Let $p > 0$ and let $\tilde{L}(\cdot)$ be the Bruijn conjugate of a slowly varying function $L(\cdot)$.*

- (i) *If $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ is $\{a_{n,i}\}$-stochastically dominated by a random variable X with $\mathbb{E}(|X|^p L(|X|^p)) < \infty$, then $\{|X_{n,i}|^p L(|X_{n,i}|^p), 1 \leq i \leq k_n, n \geq 1\}$ is $\{a_{n,i}\}$-uniformly integrable.*
- (ii) *If $\{|X_{n,i}|^p L(|X_{n,i}|^p), 1 \leq i \leq k_n, n \geq 1\}$ is $\{a_{n,i}\}$-uniformly integrable, then there exists a random variable X with distribution function*

$$F(x) = 1 - \frac{1}{C_0}\sup_{n\geq 1}\sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(|X_{n,i}| > x), x \in \mathbb{R},$$

such that $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ is $\{a_{n,i}\}$-stochastically dominated by X,

$$\mathbb{E}\left(|X|^p L(|X|^p)(\log_\nu^{(2)}(|X|))^{-1}\right) < \infty \text{ for all fixed positive integer } \nu, \quad (3.1)$$

and

$$\lim_{x\to\infty} x\mathbb{P}\left(|X| > x^{1/p}\tilde{L}^{1/p}(x)\right) = 0. \quad (3.2)$$

Proof Let $f(x) = x^p L(x^p),\ g(x) = x^{1/p}\tilde{L}^{1/p}(x),\ x \geq 0$. Recalling that we assume, without loss of generality, that f and g are strictly increasing on $[0, \infty)$.

- (i) Since $\mathbb{E}(|X|^p L(|X|^p)) < \infty$, it follows from the classical de La Vallée Poussin criterion for uniform integrability that there exists a continuous and strictly increasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$, $\lim_{x\to\infty} h(x)/x = \infty$, and $\mathbb{E}(h(|X|^p L(|X|^p))) < \infty$. Since $f(x)$ is strictly increasing on $[0, \infty)$, the $\{a_{n,i}\}$-stochastic domination assumption ensures that for all $n \geq 1$,

$$\sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(|X_{n,i}| > f^{-1}(h^{-1}(x))) \leq C_0\mathbb{P}(|X| > f^{-1}(h^{-1}(x))),\ x \in \mathbb{R}$$

or, equivalently,

$$\sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(|X_{n,i}|^p L(|X_{n,i}|^p) > h^{-1}(x)) \leq C_0\mathbb{P}(|X|^p L(|X|^p) > h^{-1}(x)),\ x \in \mathbb{R}$$

which, in turn, is equivalent to

$$\sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(h(|X_{n,i}|^p L(|X_{n,i}|^p)) > x) \leq C_0\mathbb{P}(h(|X|^p L(|X|^p)) > x),\ \ x \in \mathbb{R}.$$

It follows that

$$\begin{aligned}
&\sup_{n\geq 1}\sum_{i=1}^{k_n} a_{n,i}\mathbb{E}(h(|X_{n,i}|^p L(|X_{n,i}|^p)))\\
&\quad= \sup_{n\geq 1}\sum_{i=1}^{k_n} a_{n,i}\int_0^{\infty}\mathbb{P}(h(|X_{n,i}|^p L(|X_{n,i}|^p)) > x)\,\mathrm{d}\,x\\
&\quad\leq C_0\int_0^{\infty}\mathbb{P}(h(|X|^p L(|X|^p)) > x)\,\mathrm{d}\,x\\
&\quad= C_0\mathbb{E}(h(|X|^p L(|X|^p))) < \infty.
\end{aligned}$$

By the de La Vallée Poussin criterion for $\{a_{n,i}\}$-uniform integrability (Ordóñez Cabrera (1994)), $\{|X_{n,i}|^p L(|X_{n,i}|^p), 1 \leq i \leq k_n, n \geq 1\}$ is $\{a_{n,i}\}$-uniformly integrable.

(ii) Since $\{|X_{n,i}|^p L(|X_{n,i}|^p), 1 \leq i \leq k_n, n \geq 1\}$ is $\{a_{n,i}\}$-uniformly integrable,

$$\sup_{n\geq 1}\sum_{i=1}^{k_n} a_{n,i}\mathbb{E}\left(|X_{n,i}|^p L(|X_{n,i}|^p)\right) < \infty,$$

and so by Theorem 2.6, $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ is $\{a_{n,i}\}$-stochastically dominated by a random variable X with distribution function

$$F(x) = 1 - \frac{1}{C_0}\sup_{n\geq 1}\sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(|X_{n,i}| > x),\ \ x \in \mathbb{R},$$

and (3.1) holds.

Finally, by using the de La Vallée Poussin criterion for $\{a_{n,i}\}$-uniform integrability again, there exists a nondecreasing function h defined on $[0, \infty)$ with $h(0) = 0$ such that

$$\lim_{x\to\infty}\frac{h(x)}{x} = \infty, \tag{3.3}$$

and

$$\sup_{n\geq 1}\sum_{i=1}^{k_n} a_{n,i}\mathbb{E}(h(|X_{n,i}|^p L(|X_{n,i}|^p))) < \infty. \tag{3.4}$$

By applying Lemma 2.1 in Anh et al. (2021), we have $f(g(x))/x \to 1$ as $x \to \infty$, and therefore

$$f(g(x)) > x/2 \text{ for all large } x. \tag{3.5}$$

We thus have from (3.3), (3.4), (3.5) and Markov's inequality that

$$\begin{aligned}
\lim_{x\to\infty} x\mathbb{P}(|X| > g(x)) &= \frac{1}{C_0} \lim_{x\to\infty} x \sup_{n\geq 1} \sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(|X_{n,i}| > g(x)) \\
&\leq \frac{1}{C_0} \lim_{x\to\infty} x \sup_{n\geq 1} \sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(f(|X_{n,i}|) \geq f(g(x))) \\
&\leq \frac{1}{C_0} \lim_{x\to\infty} x \sup_{n\geq 1} \sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(f(|X_{n,i}|) \geq x/2) \\
&\leq \frac{1}{C_0} \lim_{x\to\infty} x \sup_{n\geq 1} \sum_{i=1}^{k_n} a_{n,i}\mathbb{P}(h(f(|X_{n,i}|)) \geq h(x/2)) \\
&\leq \frac{1}{C_0} \lim_{x\to\infty} x \sup_{n\geq 1} \sum_{i=1}^{k_n} a_{n,i} \frac{\mathbb{E}(h(f(|X_{n,i}|)))}{h(x/2)} \\
&= \frac{2}{C_0} \sup_{n\geq 1} \sum_{i=1}^{k_n} a_{n,i}\mathbb{E}(h(f(|X_{n,i}|))) \lim_{x\to\infty} \frac{x/2}{h(x/2)} = 0,
\end{aligned}$$

thereby proving (3.2). □

The following corollary is a direct consequence of Theorem 3.1. It plays an important role in establishing the weak laws of large numbers with general normalizing sequences under the Cesàro uniform integrability condition in Sect. 4.

Corollary 3.2 *Let $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of random variables. Let $p > 0$ and let $L(\cdot)$ be a slowly varying function.*

(i) *If $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ is stochastically dominated in the Cesàro sense by a random variable X with $\mathbb{E}(|X|^p L(|X|^p)) < \infty$, then $\{|X_{n,i}|^p L(|X_{n,i}|^p), 1 \leq i \leq k_n, n \geq 1\}$ is uniformly integrable in the Cesàro sense.*
(ii) *If $\{|X_{n,i}|^p L(|X_{n,i}|^p), 1 \leq i \leq k_n, n \geq 1\}$ is uniformly integrable in the Cesàro sense, then there exists a random variable X with distribution function*

$$F(x) = 1 - \sup_{n\geq 1} \frac{1}{k_n} \sum_{i=1}^{k_n} \mathbb{P}(|X_{n,i}| > x), x \in \mathbb{R},$$

such that $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ is stochastically dominated in the Cesàro sense by X, and (3.1) and (3.2) hold.

4 Laws of large numbers for triangular arrays and proofs of Theorems 1.2 and 1.3

In this section, we establish strong and weak laws of large numbers for triangular arrays of random variables. We say that a collection $\{X_i, 1 \le i \le N\}$ of random variables satisfies condition (H) if for all $a > 0$, there exists a constant C such that

$$\mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=m+1}^{m+k}\left(X_i^{(a)} - \mathbb{E}X_i^{(a)}\right)\right|\right)^2 \le C\sum_{i=m+1}^{m+n}\mathbb{E}(X_i^{(a)})^2,\ m \ge 0, n \ge 1, m+n \le N, \tag{4.1}$$

where

$$X_i^{(a)} = -a\mathbf{1}(X_i < -a) + X_i\mathbf{1}(|X_i| \le a) + a\mathbf{1}(X_i > a),\ 1 \le i \le N.$$

An infinite sequence of random variables $\{X_i, i \ge 1\}$ is said to satisfy condition (H) if every finite subsequence satisfies condition (H). Many dependence structures meet this condition. For example, condition (H) holds for negatively associated sequences, negatively superadditive dependent sequences, AANA sequences with the sequence of mixing coefficients is in ℓ_2 (the mixing coefficients $q_n, n \ge 1$ satisfying $\sum_{n=1}^{\infty} q_n^2 < \infty$). In Adler and Matuła (2018), the authors used a similar condition to establish exact SLLNs (see Theorems 3.2 and 4.1 in Adler and Matuła 2018).

Throughout this section, the symbol C denotes a positive universal constant which is not necessarily the same in each appearance. We shall let the indices k_n in the previous sections be $k_n \equiv n$.

The following theorem establishes the rate of convergence in SLLN for maximal partial sums from triangular arrays of dependent random variables under the Chandra–Ghosal-type condition (see Condition (4.2)).

Theorem 4.1 *Let $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ be an array of mean zero random variables such that for each $n \ge 1$ fixed, the collection $\{X_{n,i}, 1 \le i \le n\}$ satisfies condition (H), and let*

$$G(x) = \sup_{n\ge 1}\frac{1}{n}\sum_{i=1}^{n}\mathbb{P}(|X_{n,i}| > x),\ x \in \mathbb{R}.$$

Let $L(\cdot)$ be a slowly varying function and let $1 \le p < 2$. When $p = 1$, we further assume that $L(\cdot)$ is nondecreasing and $L(x) \ge 1$ for all $x \ge 0$. If

$$\int_0^{\infty} x^{p-1}L^p(x)G(x)\,\mathrm{d}x < \infty, \tag{4.2}$$

then

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_{n,i}\right| > \varepsilon n^{1/p}\tilde{L}(n^{1/p})\right) < \infty \textit{ for all } \varepsilon > 0, \tag{4.3}$$

where $\tilde{L}(\cdot)$ *is the Bruijn conjugate of* $L(\cdot)$.

Proof Since $L(\cdot)$ is a slowly varying function and $L(x) \ge 1$ for all $x \ge 0$ when $p = 1$, it follows from (4.2) that $\lim_{x\to\infty} G(x) = 0$. By Corollary 2.4, the array $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ is stochastically dominated in the Cesàro sense by a random variable X with distribution function $F(x) = 1 - G(x),\ x \in \mathbb{R}$. Thus,

$$G(x) = \sup_{n\ge 1}\frac{1}{n}\sum_{i=1}^{n}\mathbb{P}(|X_{n,i}| > x) = \mathbb{P}(|X| > x),\ x \in \mathbb{R}. \tag{4.4}$$

Using the same arguments as in the proof of Theorem 2.6, we can assume, without loss of generality, that the function $L(\cdot)$ satisfies

$$\lim_{x\to\infty}\frac{xL'(x)}{L(x)} = 0. \tag{4.5}$$

Let $h(x) = x^p L^p(x)$, $x \ge 0$. Then, it follows from (4.5) that for all large x,

$$h'(x) = px^{p-1}L^p(x)\left(1 + \frac{xL'(x)}{L(x)}\right) \le \frac{3}{2}px^{p-1}L^p(x). \tag{4.6}$$

Applying Lemma 2.8, it thus follows from (4.2), (4.4) and (4.6) that

$$\mathbb{E}(h(|X|)) = \mathbb{E}\left(|X|^p L^p(|X|)\right) < \infty. \tag{4.7}$$

We have proved that the array $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ is stochastically dominated in the Cesàro sense by a random variable X satisfying (4.7). For $n \ge 1$, set $b_n = n^{1/p}\tilde{L}(n^{1/p})$,

$$Y_{n,i} = -b_n\mathbf{1}(X_{n,i} < -b_n) + X_{n,i}\mathbf{1}(|X_{n,i}| \le b_n) + b_n\mathbf{1}(X_{n,i} > b_n), 1 \le i \le n,$$

and

$$S_{n,k} = \sum_{i=1}^{k}(Y_{n,i} - \mathbb{E}(Y_{n,i})),\ 1 \le k \le n.$$

We will now follow the proof of the implication ((i)⇒(ii)) of Theorem 3.1 in Anh et al. (2021). Let $\varepsilon > 0$ be arbitrary. For $n \geq 1$,

$$\begin{aligned} &\mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_{n,i}\right| > \varepsilon b_n\right) \\ &\le \mathbb{P}\left(\max_{1\le k\le n}|X_{n,k}| > b_n\right) + \mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} Y_{n,i}\right| > \varepsilon b_n\right) \\ &\le \mathbb{P}\left(\max_{1\le k\le n}|X_{n,k}| > b_n\right) + \mathbb{P}\left(\max_{1\le k\le n}|S_{n,k}| > \varepsilon b_n - \sum_{i=1}^{n}\left|\mathbb{E}(Y_{n,i})\right|\right). \end{aligned} \tag{4.8}$$

Since the array $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ is stochastically dominated in the Cesàro sense by a random variable X satisfying (4.7), we have from Proposition 2.6 in Anh et al. (2021) that

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1}\mathbb{P}\left(\max_{1\le k\le n}|X_{n,k}| > b_n\right) &\le \sum_{n=1}^{\infty} n^{-1}\sum_{k=1}^{n}\mathbb{P}\left(|X_{n,k}| > b_n\right) \\ &\le \sum_{n=1}^{\infty}\mathbb{P}(|X| > b_n) < \infty. \end{aligned} \tag{4.9}$$

For $n \ge 1$, it follows from the assumption $\mathbb{E}(X_{n,i}) \equiv 0$ and the Cesàro stochastic domination condition that

$$\begin{aligned} \frac{\sum_{i=1}^{n}|\mathbb{E}(Y_{n,i})|}{b_n} &\le \frac{\sum_{i=1}^{n}\left(\left|\mathbb{E}(X_{n,i}\mathbf{1}(|X_{n,i}| \le b_n))\right| + b_n\mathbb{P}(|X_{n,i}| > b_n)\right)}{b_n} \\ &= \frac{\sum_{i=1}^{n}\left(\left|\mathbb{E}(X_{n,i}\mathbf{1}(|X_{n,i}| > b_n))\right| + b_n\mathbb{P}(|X_{n,i}| > b_n)\right)}{b_n} \\ &\le \frac{2\sum_{i=1}^{n}\mathbb{E}(|X_{n,i}|\mathbf{1}(|X_{n,i}| > b_n))}{b_n} \\ &\le \frac{2n\mathbb{E}\left(|X|\mathbf{1}(|X| > b_n)\right)}{b_n}. \end{aligned} \tag{4.10}$$

For n large enough and for $\omega \in (|X| > b_n)$, we have (see (3.10) in Anh et al. 2021)

$$\frac{n}{b_n} \le C|X(\omega)|^{p-1}L^p(|X(\omega)|). \tag{4.11}$$

Applying (4.10), (4.11), (4.7) and the dominated convergence theorem, we have

$$\frac{\sum_{i=1}^{n}|\mathbb{E}(Y_{n,i})|}{b_n} \le C\mathbb{E}\left(|X|^{\alpha}L^p(|X|)\mathbf{1}\left(|X| > b_n\right)\right) \to 0 \text{ as } n \to \infty. \tag{4.12}$$

From (4.8), (4.9) and (4.12), to obtain (4.3), it remains to show that

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}\left(\max_{1\le k\le n}|S_{n,k}|>b_n\varepsilon/2\right)<\infty. \tag{4.13}$$

Applying Markov's inequality, condition (H), and the Cesàro stochastic domination condition yields

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{P}\left(\max_{1\le k\le n}|S_{n,k}|>b_n\varepsilon/2\right) &\le \sum_{n=1}^{\infty}\frac{4}{\varepsilon^2 n b_n^2}\mathbb{E}\left(\max_{1\le k\le n}|S_{n,k}|\right)^2\\
&\le \sum_{n=1}^{\infty}\frac{C}{nb_n^2}\sum_{i=1}^{n}\mathbb{E}\left(Y_{n,i}^2\right)\\
&\le C\sum_{n=1}^{\infty}\frac{\left(\mathbb{E}\left(X^2\mathbf{1}(|X|\le b_n)\right)+b_n^2\mathbb{P}(|X|>b_n)\right)}{b_n^2}\\
&= C\sum_{n=1}^{\infty}\left(\mathbb{P}(|X|>b_n)+\frac{\mathbb{E}(X^2\mathbf{1}(|X|\le b_n))}{b_n^2}\right).
\end{aligned} \tag{4.14}$$

Using the last four lines of (3.13) of Anh et al. (2021) and (3.14) of Anh et al. (2021), we have

$$\sum_{n=1}^{\infty}\left(\mathbb{P}(|X|>b_n)+\frac{\mathbb{E}(X^2\mathbf{1}(|X|\le b_n))}{b_n^2}\right)\le C+C\mathbb{E}(|X|^pL^p(|X|)). \tag{4.15}$$

Combining (4.14), (4.15), and (4.7) yields (4.13). □

The following corollary establishes rate of convergence in a Marcinkiewicz–Zygmund-type SLLN for arrays of random variables under a uniformly bounded moment condition.

Corollary 4.2 *Let $1\le p<2$ and let ν be a fixed positive integer. Let $\{X_{n,i},1\le i\le n,n\ge 1\}$ be an array of mean zero random variables such that for each $n\ge 1$ fixed, the collection $\{X_{n,i},1\le i\le n\}$ satisfies condition (H). If*

$$\sup_{n\ge 1}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(|X_{n,i}|^pL^p(|X_{n,i}|)\log_\nu^{(2)}(|X_{n,i}|)\right)<\infty, \tag{4.16}$$

then (4.3) is obtained.

Proof By applying Corollary 2.9, we have from (4.16) that the array $\{X_{n,i},1\le i\le n,n\ge 1\}$ is stochastically dominated in the Cesàro sense by a random variable X with $\mathbb{E}(|X|^pL^p(|X|))<\infty$, that is, (4.2) is satisfied. Applying Theorem 4.1, we obtain (4.3). □

The moment condition (4.16) is almost optimal. The following example shows that there exists an array of random variables $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ such that

$$\sup_{n\ge1} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left(|X_{n,i}|^p \log_\nu(|X_{n,i}|)\right) < \infty \tag{4.17}$$

for every fixed positive integer ν, but (4.3) fails with $\tilde{L}(x) \equiv L(x) \equiv 1$.

Example 4.3 Let ν be an arbitrary fixed positive integer and let $1 \le p < 2$. Let $\{X_n, n \ge 1\}$ be a sequence of independent random variables with

$$\mathbb{P}(X_n = 0) = 1 - \frac{1}{n\log_\nu(n)}, \ \mathbb{P}\left(X_n = \pm(n+1)^{1/p}\right) = \frac{1}{2n\log_\nu(n)}, n \ge 1$$

and let $X_{n,i} = X_i$, $1 \le i \le n, n \ge 1$. Then, (4.17) is satisfied, and

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > n^{1/p}) = \sum_{n=1}^{\infty} \frac{1}{n\log_\nu(n)} = \infty. \tag{4.18}$$

If (4.3) (with $\tilde{L}(x) \equiv L(x) \equiv 1$) holds, then

$$\sum_{n=1}^{\infty} \frac{1}{n}\mathbb{P}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} X_i\right| > \varepsilon n^{1/p}\right) < \infty \text{ for all } \varepsilon > 0.$$

This implies that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n X_i}{n^{1/p}} = 0 \text{ a.s,}$$

and thus

$$\lim_{n\to\infty} \frac{X_n}{n^{1/p}} = 0 \text{ a.s.} \tag{4.19}$$

Applying the Borel–Cantelli lemma, we have from (4.19) that

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > n^{1/p}) < \infty$$

contradicting (4.18). Therefore, (4.3) (with $\tilde{L}(x) \equiv L(x) \equiv 1$) must fail.

If we consider sequences of random variables instead of triangular arrays, we obtain the following Marcinkiewicz–Zygmund-type SLLN with general normalizing sequences.

Corollary 4.4 *Let $\{X_n, n \ge 1\}$ be a sequence of mean zero random variables satisfying condition (H). Let $L(\cdot)$ be a slowly varying function and let $1 \le p < 2$. When $p = 1$, we further assume that $L(\cdot)$ is nondecreasing and $L(x) \ge 1$ for all $x \ge 0$. If*

$$\int_0^\infty x^{p-1} L^p(x) \left(\sup_{n\ge 1} \frac{1}{n} \sum_{i=1}^n \mathbb{P}(|X_i| > x) \right) \mathrm{d}x < \infty, \tag{4.20}$$

then

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n X_i}{n^{1/p} \tilde{L}(n^{1/p})} = 0 \text{ a.s.}, \tag{4.21}$$

where $\tilde{L}(\cdot)$ is the Bruijn conjugate of $L(\cdot)$.

Proof Set

$$X_{n,i} = X_i, 1 \le i \le n, n \ge 1.$$

Then, (4.20) coincides with (4.2). Applying Theorem 4.1, we have

$$\sum_{n=1}^\infty \frac{1}{n} \mathbb{P}\left(\max_{1\le j\le n} \left| \sum_{i=1}^j X_i \right| > \varepsilon n^{1/p} \tilde{L}(n^{1/p}) \right) < \infty \text{ for all } \varepsilon > 0. \tag{4.22}$$

The Marcinkiewicz–Zygmund-type SLLN (4.21) follows from (4.22). □

Proof of Theorem 1.2. Since $\{X_n, n \ge 1\}$ is a sequence of AANA random variables with the sequence of mixing coefficients is in ℓ_2, it satisfies condition (H) (see Lemmas 2.1 and 2.2 of Ko et al. (2005)). Thus, Theorem 1.2 follows from Corollary 4.4 by taking $L(x) \equiv 1$. □

The following theorem is a significant extension of Theorem 1.3. It establishes a WLLN for weighted sums from arrays of random variables.

Theorem 4.5 *Let $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ be an array of random variables. Let $G(\cdot)$ be as in Theorem 4.1 and let $\{b_n, n \ge 1\}$ be a nondecreasing sequence of positive real numbers satisfying (1.9). Let $\{c_{n,i}, 1 \le i \le n\}$ be an array of nonnegative real numbers satisfying*

$$0 < A_n := \sum_{i=1}^n c_{n,i} \le Cn, \ n \ge 1 \tag{4.23}$$

and let

$$\hat{G}(x) = \sup_{n\ge 1} \sum_{i=1}^n a_{n,i} \mathbb{P}(|X_{n,i}| > x), \ x \in \mathbb{R},$$

where $a_{n,i} = A_n^{-1} c_{n,i},\ 1 \le i \le n, n \ge 1$. *If*

$$\lim_{k\to\infty} kG(b_k) = 0 \ \text{ and } \ \lim_{k\to\infty} k\hat{G}(b_k) = 0, \tag{4.24}$$

then the WLLN

$$\frac{1}{b_n} \max_{j \le n} \left| \sum_{i=1}^{j} c_{n,i} X_{n,i} \right| \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty \tag{4.25}$$

is obtained.

Remark 4.6 It is clear that in the unweighted case, i.e., $c_{n,i} \equiv 1$, then $\hat{G}(x) \equiv G(x)$. It is also easy to see that if $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ is stochastically dominated by a random variable X with $\lim_{k\to\infty} k\mathbb{P}(|X| > b_k) = 0$, then both halves of (4.24) are fulfilled.

At the first look, the second half of (4.24) may seem to be a technical condition. However, the following example shows that it cannot be dispensed with.

Example 4.7 Let $0 < p < 1$, $b_n = n^{1/p},\ n \ge 1$ and let $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ be an array of random variables such that

$$\mathbb{P}(X_{n,i} = -1) = \mathbb{P}(X_{n,i} = 1) = 1/2 \ \text{ for } 1 \le i < n, n \ge 2,$$

and

$$\mathbb{P}(X_{n,n} = -n^{1/p} \log^{-1/p}(n)) = \mathbb{P}(X_{n,n} = n^{1/p} \log^{-1/p}(n)) = 1/2 \ \text{ for } n \ge 1.$$

Let $\{c_{n,i}, 1 \le i \le n, n \ge 1\}$ be an array of real numbers such that

$$c_{n,i} = 0 \text{ for } 1 \le i < n, n \ge 2, \ \text{ and } c_{n,n} = n \text{ for } n \ge 1$$

and let

$$A_n = \sum_{i=1}^{n} c_{n,i},\ a_{n,i} = \frac{c_{n,i}}{A_n},\ 1 \le i \le n, n \ge 1.$$

Then, (4.23) is satisfied since $A_n \equiv n$. Let

$$G(x) = \sup_{n\ge1} \frac{1}{n} \sum_{i=1}^{n} \mathbb{P}(|X_{n,i}| > x),\ x \in \mathbb{R},$$

and

$$\hat{G}(x) = \sup_{n\ge1} \sum_{i=1}^{n} a_{n,i} \mathbb{P}(|X_{n,i}| > x),\ x \in \mathbb{R}.$$

For $n \geq 1$, we have

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(|X_{n,i}|^p\log(|X_{n,i}|)\right) &\leq 1+\frac{1}{n}\mathbb{E}(|X_{n,n}|^p\log(|X_{n,n}|))\\ &= 1+\frac{1}{\log(n)}\left(\frac{\log(n)-\log(\log(n))}{p}\right)\\ &\leq 1+\frac{1}{p}<\infty.\end{aligned}$$

Therefore, $\{|X_{n,i}|^p, 1 \leq i \leq n, n \geq 1\}$ is uniformly integrable in the Cesàro sense by the de La Vallée Poussin criterion for the Cesàro uniform integrability. Then, by Corollary 3.2, $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ is stochastically dominated in the Cesàro sense by a random variable X with distribution function

$$F(x)=1-\sup_{n\geq 1}\frac{1}{n}\sum_{i=1}^{n}\mathbb{P}(|X_{n,i}|>x)=1-G(x),\ x\in\mathbb{R},$$

and the first half of (4.24) (with $b_n \equiv n^{1/p}$) is satisfied. However, the second half of (4.24) (with $b_n \equiv n^{1/p}$) fails since

$$\hat{G}(x)=\sup_{n\geq 1}\sum_{i=1}^{n}a_{n,i}\mathbb{P}(|X_{n,i}|>x)=\sup_{n\geq 1}\mathbb{P}(|X_{n,n}|>x)=1 \text{ for all } x\in\mathbb{R}.$$

For $n \geq 1$, we have with probability 1,

$$\begin{aligned}\frac{1}{b_n}\max_{j\leq n}\left|\sum_{i=1}^{j}c_{n,i}X_{n,i}\right| &= \frac{1}{n^{1/p}}c_{n,n}|X_{n,n}|\\ &= \frac{n}{\log^{1/p}(n)}\to\infty\end{aligned}$$

therefore, the WLLN (4.25) also fails.

Proof of Theorem 4.5. From (1.9), we have $b_n \to \infty$ (see (4.31)). Since $G(x)$ and $\hat{G}(x)$ are nonincreasing, it follows from (4.24) that $\lim_{x\to\infty} G(x) = 0$ and $\lim_{x\to\infty} \hat{G}(x) = 0$. By Corollary 2.4, $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ is stochastically dominated in the Cesàro sense by a random variable X, and by Theorem 2.1, $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ is $\{a_{n,i}\}$-stochastically dominated by a random variable Y. The distribution functions of X and Y, respectively, are

$$F_X(x)=1-G(x)\ \text{ and }\ F_Y(x)=1-\hat{G}(x),\ x\in\mathbb{R}.$$

Thus,

$$\sup_{n\geq 1}\frac{1}{n}\sum_{i=1}^{n}\mathbb{P}(|X_{n,i}|>x)=\mathbb{P}(|X|>x),\ x\in\mathbb{R},\tag{4.26}$$

and

$$\sup_{n\geq 1}\sum_{i=1}^{n}a_{n,i}\mathbb{P}(|X_{n,i}|>x)=\mathbb{P}(|Y|>x),\ x\in\mathbb{R},\tag{4.27}$$

and so (4.24) becomes

$$\lim_{k\to\infty}k\mathbb{P}(|X|>b_k)=0\ \text{ and }\ \lim_{k\to\infty}k\mathbb{P}(|Y|>b_k)=0.\tag{4.28}$$

For $n\geq 1$, set

$$Y_{n,i}=X_{n,i}\mathbf{1}(|X_{n,i}|\leq b_n),\ 1\leq i\leq n.$$

We first verify that

$$\frac{\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}c_{n,i}\left(X_{n,i}-Y_{n,i}\right)\right|}{b_n}\xrightarrow{\mathbb{P}}0\text{ as }n\to\infty.\tag{4.29}$$

To see this, let $\varepsilon>0$ be arbitrary. Then, we have from (4.26) and the first half of (4.28) that

$$\begin{aligned}\mathbb{P}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}c_{n,i}\left(X_{n,i}-Y_{n,i}\right)\right|>b_n\varepsilon\right)&\leq\mathbb{P}\left(\bigcup_{i=1}^{n}(X_{n,i}\neq Y_{n,i})\right)\\&\leq\sum_{i=1}^{n}\mathbb{P}(|X_{n,i}|>b_n)\\&\leq n\mathbb{P}(|X|>b_n)\to 0\text{ as }n\to\infty\end{aligned}$$

thereby proving (4.29).

Next, it will be shown that

$$\frac{\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}c_{n,i}Y_{n,i}\right|}{b_n}\xrightarrow{\mathbb{P}}0\text{ as }n\to\infty.\tag{4.30}$$

To accomplish this, we first recall that (1.9) implies (see Remark 2.4 (i) in Boukhari (2021))

$$\frac{n}{b_n}\to 0\text{ as }n\to\infty.\tag{4.31}$$

Set $b_0 = 0$. Again, let $\varepsilon > 0$ be arbitrary. Then,

$$\begin{aligned}
\mathbb{P}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} c_{n,i}Y_{n,i}\right| > b_n\varepsilon\right) &\le \frac{1}{b_n\varepsilon}\mathbb{E}\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} c_{n,i}Y_{n,i}\right|\right)\\
&\le \frac{1}{b_n\varepsilon}\sum_{i=1}^{n} c_{n,i}\mathbb{E}(|Y_{n,i}|)\\
&\le \frac{1}{b_n\varepsilon}\sum_{i=1}^{n} c_{n,i}\int_0^{b_n}\mathbb{P}(|X_{n,i}| > x)\,\mathrm{d}x\\
&= \frac{A_n}{b_n\varepsilon}\sum_{i=1}^{n} a_{n,i}\int_0^{b_n}\mathbb{P}(|X_{n,i}| > x)\,\mathrm{d}x \qquad (4.32)\\
&\le \frac{A_n}{b_n\varepsilon}\int_0^{b_n}\mathbb{P}(|Y| > x)\,\mathrm{d}x\\
&= \frac{A_n}{b_n\varepsilon}\sum_{k=1}^{n}\int_{b_{k-1}}^{b_k}\mathbb{P}(|Y| > x)\,\mathrm{d}x\\
&\le \frac{Cn}{b_n\varepsilon}\sum_{k=1}^{n}\frac{b_k - b_{k-1}}{k}k\mathbb{P}(|Y| > b_{k-1}),
\end{aligned}$$

where we have applied Markov's inequality in the first inequality, (4.27) in the fourth inequality, and (4.23) in the last inequality. Now when $n \ge 2$,

$$\begin{aligned}
\frac{n}{b_n}\sum_{k=1}^{n}\frac{b_k - b_{k-1}}{k} &= \frac{n}{b_n}\left(\sum_{k=1}^{n-1}\frac{b_k}{k(k+1)} + \frac{b_n}{n}\right)\\
&\le \frac{n}{b_n}\left(\sum_{k=1}^{n-1}\frac{b_k}{k^2} + \frac{b_n}{n}\right)\\
&\le C \ \ (\text{by } 1.9),
\end{aligned}$$

for all fixed k,

$$\frac{n}{b_n}\left(\frac{b_k - b_{k-1}}{k}\right) \to 0 \text{ as } n \to \infty \ \ (\text{by } (4.31)),$$

and for $k \ge 2$,

$$\begin{aligned}
&k\mathbb{P}(|Y| > b_{k-1})\\
&\quad \le 2(k-1)\mathbb{P}(|Y| > b_{k-1}) \to 0 \text{ as } k \to \infty \ \ (\text{by the second half of } 4.28).
\end{aligned}$$

Thus, by the Toeplitz lemma

$$\frac{n}{b_n}\sum_{k=1}^{n}\frac{b_k-b_{k-1}}{k}k\mathbb{P}(|Y|>b_{k-1})\to 0 \text{ as } n\to\infty$$

and (4.30) then follows from (4.32). Combining (4.29) and (4.30) yields (4.25). □

Proof of Theorem 1.3. Set

$$c_{n,i}=1,\ X_{n,i}=X_i,\, 1\le i\le n, n\ge 1.$$

Then, (1.10) coincides with (4.24). Theorem 1.3 follows from Theorem 4.5. □

We will now present an example to illustrate Theorem 1.3. This example shows that for $0<p<1$ and $b_n=n^{1/p}$, there exists a sequence of random variables $\{X_n, n\ge 1\}$ with no stochastically dominating random variable and condition (1.10) is satisfied. In this example, we also show that (1.3) (with $0<p<1$) holds but (1.2) (with $0<p<1$) does not.

Example 4.8 Let $0<p<1$, $b_n=n^{1/p},\ n\ge 1$ and let $\{X_n, n\ge 1\}$ be a sequence of random variables such that

$$\mathbb{P}(X_n=-1)=\mathbb{P}(X_n=1)=1/2 \text{ for } n\ne 2^m,\ m\ge 0, \tag{4.33}$$

and

$$\mathbb{P}\left(X_{2^m}=-2^{m/p}/m^{1/p}\right)=\mathbb{P}\left(X_{2^m}=2^{m/p}/m^{1/p}\right)=1/2 \text{ for } m\ge 0. \tag{4.34}$$

Then,

$$\sup_{n\ge1}\mathbb{P}(|X_n|>x)=\sup_{m\ge1}\mathbb{P}(|X_{2^m}|>x)=1 \text{ for all } x\ge 1.$$

Thus, there is no random variable X such that the sequence $\{X_n, n\ge 1\}$ is stochastically dominated by X, and so we cannot apply Theorem 2.1 of Boukhari (2021).

Now, for $n\ge 1$, let $m\ge 0$ be such that $2^m\le n<2^{m+1}$. Then,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(|X_i|^p\log(|X_i|)\right)&\le 1+\frac{1}{2^m}\sum_{i=0}^{m}\mathbb{E}\left(|X_{2^i}|^p\log(|X_{2^i}|)\right)\\&=1+\frac{1}{2^m}\sum_{i=0}^{m}\frac{2^i}{i}\left(\frac{i-\log(i)}{p}\right)\\&\le 1+\frac{2}{p}<\infty.\end{aligned} \tag{4.35}$$

Therefore, $\{|X_n|^p, n \geq 1\}$ is uniformly integrable in the Cesàro sense by the de La Vallée Poussin criterion for the Cesàro uniform integrability. Then, by Corollary 3.2, the sequence $\{X_n, n \geq 1\}$ is stochastically dominated in the Cesàro sense by a random variable X with distribution function

$$F(x) = 1 - \sup_{n\geq 1} \frac{1}{n} \sum_{i=1}^{n} \mathbb{P}(|X_i| > x) := 1 - G(x),\ x \in \mathbb{R}, \tag{4.36}$$

and condition (1.10) (with $b_n \equiv n^{1/p}$) is satisfied. It thus follows from Theorem 1.3 that the WLLN (1.11) holds.

Finally, it is clear that $\mathbb{P}(|X_n|^p > n) = 0$ for all $n \geq 1$ so that (1.3) holds. We will show that (1.2) (with $0 < p < 1$) is not satisfied. To see this, for $x \geq 1$, let n_x be the smallest integer such that

$$\frac{2^{n_x}}{n_x} > x^p.$$

Then, for $x \geq 1$,

$$\frac{2^{n_x}}{n_x} > x^p \geq \frac{2^{n_x - 1}}{n_x - 1} \tag{4.37}$$

and it follows from (4.37) that there exists $\varepsilon_0 > 0$ such that

$$\frac{1}{2^{n_x}} > \frac{\varepsilon_0}{x^p \log(x)} \quad \text{for all large } x. \tag{4.38}$$

Combining (4.33)–(4.36) and (4.38), we have

$$\begin{aligned} G(x) &= \sup_{n\geq 1} \frac{1}{n} \sum_{i=1}^{n} \mathbb{P}(|X_i| > x) \geq \frac{1}{2^{n_x}} \sum_{i=1}^{2^{n_x}} \mathbb{P}(|X_i| > x) \\ &= \frac{1}{2^{n_x}} \geq \frac{\varepsilon_0}{x^p \log(x)} \quad \text{for all large } x \end{aligned} \tag{4.39}$$

and it follows from (4.39) that (1.2) (with $0 < p < 1$) fails. Thus, we cannot apply Remark 3 of Chandra and Ghosal (1996a) to obtain the Marcinkiewicz–Zygmund SLLN for the case $0 < p < 1$.

Now, we discuss about the normalizing sequences in the WLLN in Theorem 1.3. We note that for $0 < p < 2$ and $b_n \equiv n^{1/p}$, (1.9) is fulfilled if $0 < p < 1$ but it fails to hold if $1 \leq p < 2$. For the case where $1 \leq p < 2$, we also have to require some dependence structures to obtain the WLLN (see Boukhari 2021 for a counterexample). Kruglov (2011) established a Kolmogorov–Feller-type WLLN for sequences of negatively associated identically distributed random variables with normalizing sequences $b_n, n \geq 1$ satisfying

$$\sum_{i=1}^{n} \frac{b_i^2}{i^2} = O\left(\frac{b_n^2}{n}\right). \tag{4.40}$$

It was observed by Kruglov (2011) that (4.40) holds for the case $b_n \equiv n^{1/p}L(n)$, where $1 \leq p < 2$ and $L(\cdot)$ is a slowly varying function. The following theorem appears to be new even when the underlying random variables are independent. It also extends the sufficient part of Theorem 1 of Kruglov (2011). The proof of Theorem 4.9 can be obtained by proceeding in a similar manner as that of Theorem 4.5: We firstly use the nondecreasing truncation as in the proof of Theorem 4.1, and use the maximal inequality (4.1) instead of the triangular inequality (in the place of the second inequality in (4.32)), and then change the other places accordingly. We leave the details to the interested reader.

Theorem 4.9 *Let $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ be an array of random variables such that for each $n \geq 1$ fixed, the collection $\{X_{n,i}, 1 \leq i \leq n\}$ satisfies condition (H) and let $G(\cdot)$ be as in Theorem 4.1. Let $\{b_n, n \geq 1\}$ be a nondecreasing sequence of positive real numbers satisfying (4.40), and let $\{c_{n,i}, 1 \leq i \leq n, n \geq 1\}$ be an array of nonnegative real numbers satisfying*

$$0 < A_n := \sum_{i=1}^{n} c_{n,i}^2 \leq Cn, \ n \geq 1. \tag{4.41}$$

Let

$$\hat{G}(x) = \sup_{n\geq 1} \sum_{i=1}^{n} a_{n,i}\mathbb{P}(|X_{n,i}| > x), \ x \in \mathbb{R},$$

where $a_{n,i} = A_n^{-1}c_{n,i}^2, \ 1 \leq i \leq n, n \geq 1$. If

$$\lim_{k\to\infty} kG(b_k) = 0 \ \text{and} \ \lim_{k\to\infty} k\hat{G}(b_k) = 0, \tag{4.42}$$

then the WLLN

$$\frac{1}{b_n} \max_{j\leq n} \left| \sum_{i=1}^{j} c_{n,i} \left(X_{n,i} - \mathbb{E}(X_{n,i}\mathbf{1}(|X_{n,i}| \leq b_n))\right) \right| \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty$$

is obtained.

We now apply Corollary 3.2 and Theorem 4.9 to obtain a WLLN for arrays of random variables under the Cesàro uniform integrability condition. For simplicity and since it is just meant to be an illustration, we only consider the unweighted case, i.e., the case where $c_{n,i} \equiv 1$.

Corollary 4.10 *Let $1 \le p < 2$ and let $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ be an array of random variables such that for each $n \ge 1$ fixed, the collection $\{X_{n,i}, 1 \le i \le n\}$ satisfies condition (H). Let $G(\cdot)$ be as in Theorem 4.1 and $L(\cdot)$ be a slowly varying function. Let $\tilde{L}(\cdot)$ be the Bruijn conjugate of $L(\cdot)$. In the case $p = 1$, we further assume that $L(x)$ is nondecreasing and $L(x) \ge 1$ for all $x \ge 0$. If $\{|X_{n,i}|^p L(|X_{n,i}|^p), 1 \le i \le n, n \ge 1\}$ is uniformly integrable in the Cesàro sense, that is,*

$$\lim_{a\to\infty} \sup_{n\ge 1} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left(|X_{n,i}|^p L(|X_{n,i}|^p)\mathbf{1}(|X_{n,i}| > a)\right) = 0, \tag{4.43}$$

then the WLLN

$$\frac{1}{n^{1/p}\tilde{L}^{1/p}(n)} \max_{j\le n} \left| \sum_{i=1}^{j} \left(X_{n,i} - \mathbb{E}(X_{n,i})\right) \right| \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty \tag{4.44}$$

is obtained.

Proof In Theorem 4.9, if we choose $c_{n,i} \equiv 1$, then (4.41) is automatic, and $\hat{G}(x) \equiv G(x)$. By Corollary 3.2, it follows from (4.43) that (4.42) holds with $b_n \equiv n^{1/p}\tilde{L}^{1/p}(n)$. Applying Theorem 4.9, we obtain

$$\frac{1}{b_n} \max_{j\le n} \left| \sum_{i=1}^{j} \left(X_{n,i} - \mathbb{E}(X_{n,i}\mathbf{1}(|X_{n,i}| \le b_n))\right) \right| \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty.$$

To obtain (4.44), it remains to show that

$$\frac{1}{b_n} \max_{j\le n} \left| \sum_{i=1}^{j} \mathbb{E}(X_{n,i}\mathbf{1}(|X_{n,i}| > b_n)) \right| \to 0 \text{ as } n \to \infty. \tag{4.45}$$

Since the function $x^{p-1}L(x^p)$ is nondecreasing, we have

$$\begin{aligned} &\frac{1}{b_n} \max_{j\le n} \left| \sum_{i=1}^{j} \mathbb{E}(X_{n,i}\mathbf{1}(|X_{n,i}| > b_n)) \right| \\ &\le \frac{1}{b_n} \sum_{i=1}^{n} \mathbb{E}(|X_{n,i}|\mathbf{1}(|X_{n,i}| > b_n)) \\ &\le \frac{1}{b_n} \sum_{i=1}^{n} \frac{\mathbb{E}(|X_{n,i}|^p L(|X_{n,i}|^p)\mathbf{1}(|X_{n,i}| > b_n))}{b_n^{p-1} L(b_n^p)} \\ &= \left(\frac{1}{\tilde{L}(n) L\left(n\tilde{L}(n)\right)} \right) \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(|X_{n,i}|^p L(|X_{n,i}|^p)\mathbf{1}(|X_{n,i}| > b_n)). \end{aligned} \tag{4.46}$$

Now, from the second half of (2.5) we have $\lim_{n\to\infty} \tilde{L}(n)L\left(n\tilde{L}(n)\right) = 1$, and from (4.43) we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(|X_{n,i}|^p L(|X_{n,i}|^p)\mathbf{1}(|X_{n,i}| > b_n)) = 0.$$

Therefore, (4.45) follows from (4.46). The proof of the corollary is completed. □

5 Conclusions and open problems

In Sect. 4, our results on the concept of $\{a_{n,i}\}$-stochastic domination are applied to obtain the WLLNs for weighted sums. The results on the Cesàro stochastic domination case are applied to obtain rate of convergence in the SLLN with general normalizing sequences under the Chandra–Ghosal-type condition, and these results help us to remove an assumption of a SLLN established by Chandra and Ghosal (1996a). The results on the concept of $\{a_{n,i}\}$-stochastic domination may also be useful in proving weighted SLLNs of Chandra and Ghosal (1996b) as we will describe as follows.

Let $1 \le p < 2$ and let $\{a_n, n \ge 1\}$ be a sequence of positive real numbers with

$$A_n := \sum_{i=1}^{n} a_i \to \infty \text{ as } n \to \infty.$$

Let $\{X_n, n \ge 1\}$ be a sequence of mean zero random variables which satisfies suitable dependence conditions. Let $G(x)$ be as in Theorem 1.1 and let

$$\tilde{G}(x) = \sup_{n\ge 1}\left(\sum_{i=1}^{n} a_i^{1/p}\right)^{-1} \sum_{i=1}^{n} a_i^{1/p}\mathbb{P}(|X_i| > x), x \in \mathbb{R}.$$

Chandra and Ghosal (1996b) considered the following three conditions (see (2.14)–(2.16) in Chandra and Ghosal 1996b):

$$\int_0^\infty x^{p-1}G(x)\,\mathrm{d}x < \infty, \tag{5.1}$$

$$\int_0^\infty x^{p-1}\tilde{G}(x)\,\mathrm{d}x < \infty, \tag{5.2}$$

and

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n|^p > A_n/a_n) < \infty. \tag{5.3}$$

Chandra and Ghosal (see Theorems 2.6 and 2.7 in Chandra and Ghosal 1996b) proved that if (5.1), (5.2) and (5.3) are all satisfied, then the weighted Marcinkiewicz–Zygmund SLLN

$$\frac{\sum_{i=1}^{n} a_i^{1/p} X_i}{A_n^{1/p}} \to 0 \text{ a.s. as } n \to \infty$$

is obtained. In view of Theorems 1.2 and 4.5, we state an open problem as to whether or not the Chandra and Ghosal result mentioned above still holds without Condition (5.3).

For $n \geq 1$, let

$$a_{n,i} = \left(\sum_{i=1}^{n} a_i^{1/p}\right)^{-1} a_i^{1/p}, 1 \leq i \leq n.$$

Since $\tilde{G}(x)$ is nonincreasing, it follows from (5.2) that $\lim_{x\to\infty} \tilde{G}(x) = 0$. By Theorem 2.1, $\{X_n, n \geq 1\}$ is $\{a_{n,i}\}$-stochastically dominated by a random variable X with distribution function $F(x) = 1 - \tilde{G}(x)$, and (5.2) becomes $\mathbb{E}(|X|^p) < \infty$. In view of the proof of Theorem 4.5, the results on the concept of $\{a_{n,i}\}$-stochastic domination established in Sects. 2 and 3 may help in answering the above open problem.

Finally, we present an open problem concerning Corollary 4.10. For the case where $L(x) \equiv \tilde{L}(x) \equiv 1$, we can obtain convergence in mean of order p in (4.44) (see, e.g., Theorem 1 in Chandra 1989, Theorem 4 in Ordóñez Cabrera 1994, Theorem 2.1 in Thành 2005). However, the methods in Chandra (1989), Ordóñez Cabrera (1994) and Thành (2005) do not seem to work for general slowly varying function $L(\cdot)$, even with assumption that the underlying random variables are independent. It is an open problem as to whether or not convergence in mean of order p prevails in the conclusion (4.44).

Acknowledgements The author would like to thank the Associate Editor and two anonymous referees for carefully reading the manuscript and for offering useful comments and suggestions which enabled him to improve the paper. The author is also grateful to Professor Andrew Rosalsky and Mr. Nguyen Chi Dzung for some helpful and important suggestions and remarks.

Funding The author did not receive support from any organization for this work.

Declarations

Conflict of interest The author has no conflict of interest.

References

Adler A, Matuła P (2018) On exact strong laws of large numbers under general dependence conditions. Probab Math Stat 38(1):103–121

Anh VTN, Hien NTT, Thành LV, Van VTH (2021) The Marcinkiewicz–Zygmund-type strong law of large numbers with general normalizing sequences. J Theor Probab 34(1):331–348

Bingham NH, Goldie CM, Teugels JL (1989) Regular variation, vol 27. Cambridge University Press, Cambridge

Boukhari F (2021) On a weak law of large numbers with regularly varying normalizing sequences. J Theor Probab. https://doi.org/10.1007/s10959-021-01120-6

Chandra TK (1989) Uniform integrability in the Cesàro sense and the weak law of large numbers. Sankhyā Indian J Stat Ser A 51(3):309–317

Chandra TK, Ghosal S (1996) Extensions of the strong law of large numbers of Marcinkiewicz and Zygmund for dependent variables. Acta Math Hung 71(4):327–336

Chandra TK, Ghosal S (1996) The strong law of large numbers for weighted averages under dependence assumptions. J Theor Probab 9(3):797–809

Chandra TK, Goswami A (1992) Cesàro uniform integrability and the strong law of large numbers. Sankhyā Indian J Stat Ser A 54(2):215–231

Galambos J, Seneta E (1973) Regularly varying sequences. Proc Am Math Soc 41(1):110–116

Gut A (1992) Complete convergence for arrays. Period Math Hung 25(1):51–75

Hien NTT, Thành LV (2015) On the weak laws of large numbers for sums of negatively associated random vectors in Hilbert spaces. Stat Probab Lett 107:236–245

Ko MH, Kim TS, Lin Z (2005) The Hájeck-Renyi inequality for the AANA random variables and its applications. Taiwan J Math 9(1):111–122

Kruglov VM (2011) A generalization of weak law of large numbers. Stoch Anal Appl 29(4):674–683

Ordóñez Cabrera M (1994) Convergence of weighted sums of random variables and uniform integrability concerning the weights. Collectanea Math 45(2):121–132

Rosalsky A, Thành LV (2009) Weak laws of large numbers for double sums of independent random elements in Rademacher type p and stable type p Banach spaces. Nonlinear Anal Theory Methods Appl 71(12):e1065–e1074

Rosalsky A, Thành LV (2021) A note on the stochastic domination condition and uniform integrability with applications to the strong law of large numbers. Stat Probab Lett 178:109181

Thành LV (2005) On the L_p-convergence for multidimensional arrays of random variables. Int J Math Math Sci 2005(8):1317–1320