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Mean convergence theorems for arrays of dependent random variables with applications to dependent bootstrap and non-homogeneous Markov chains

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Abstract

This paper provides sets of sufficient conditions for mean convergence theorems for arrays of dependent random variables. We expand and improve a number of particular cases in the literature including Theorem 2.1 in Sung (Appl Math Lett 26(1):18–24, 2013), Theorems 3.1–3.3 in Wu and Guan (J Math Anal Appl 377(2):613–623, 2011), and Theorem 3 in Lita da Silva (Results Math 74(1):1–11, 2019), among others. The proof is different from those in the aforementioned papers and the main results can be applied to obtain mean convergence results for arrays of functions of non-homogeneous Markov chains and dependent bootstrap.

Keywords Mean convergence · Weak law of large numbers · Negative dependence · Non-homogeneous Markov chain · Dependent bootstrap

Mathematics Subject Classification 60F05 · 60F25

1 Introduction

Weak laws of large numbers and mean convergence for arrays of dependent random variables were studied by many authors. We refer to Ordóñez Cabrera and Volodin (2005), Shen and Volodin (2017), Lita da Silva (2016), Lita da Silva (2019), Sung (2013), Wu and Guan (2011) and the references therein. Recently, Lita da Silva (2019) proved the following theorem.

Theorem 1.1 (Lita da Silva 2019, Theorem 3) *Let* $1 \le p < 2$ *and let* $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ *be a triangular array rowwise and pairwise negatively dependent random variables. Let* $\{b_n, n \ge 1\}$ *be a sequence of positive constants. Assume that for all* $\varepsilon > 0$,

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$$\sum_{i=1}^{n} \int_{0}^{\varepsilon b_n^p} \mathbb{P}(|X_{n,i}|^p > t)\, \mathrm{d}t = O(b_n^p) \text{ as } n \to \infty, \tag{1.1}$$

and

$$\sum_{i=1}^{n} \int_{\varepsilon b_n^p}^{\infty} \mathbb{P}(|X_{n,i}|^p > t)\, \mathrm{d}t = o(b_n^p) \text{ as } n \to \infty \text{ when } 1 < p < 2, \tag{1.2}$$

or

$$\sum_{i=1}^{n} \mathbb{E}\left(|X_{n,i}|\mathbf{1}(|X_{n,i}| > \varepsilon b_n)\right) = o(b_n) \text{ as } n \to \infty \text{ when } p = 1. \tag{1.3}$$

Then

$$\frac{1}{b_n} \sum_{i=1}^{n} \left(X_{n,i} - \mathbb{E}X_{n,i}\right) \xrightarrow{\mathcal{L}_p} 0 \text{ as } n \to \infty. \tag{1.4}$$

Another interesting direction is to study mean convergence for weighted sums which has many applications in statistics. In Sung (2013), Sung proved the following theorem.

Theorem 1.2 (Sung 2013, Theorem 2.1) *Let $\{u_n, n \geq 1\}$ and $\{v_n, n \geq 1\}$ be two sequences of integers such that $u_n < v_n$ for all $n \geq 1$ and $\lim_{n\to\infty}(v_n - u_n) = \infty$. Let $1 \leq p < 2$ and let $\{Y_{n,i}, u_n \leq i \leq v_n, n \geq 1\}$ be an array of rowwise and pairwise negatively dependent random variables. Let $\{a_{n,i}, n \geq 1\}$ be an array of constants. Suppose that*

$$\sup_{n\geq 1} \sum_{i=u_n}^{v_n} |a_{n,i}|^p \mathbb{E}|Y_{n,i}|^p < \infty \tag{1.5}$$

and

$$\lim_{n\to\infty} \sum_{i=u_n}^{v_n} |a_{n,i}|^p \mathbb{E}\left(|Y_{n,i}|^p \mathbf{1}\left(|a_{n,i}|^p |Y_{n,i}|^p > \varepsilon\right)\right) = 0 \text{ for all } \varepsilon > 0. \tag{1.6}$$

Then

$$\sum_{i=u_n}^{v_n} a_{n,i}\left(Y_{n,i} - \mathbb{E}Y_{n,i}\right) \xrightarrow{\mathcal{L}_p} 0 \text{ as } n \to \infty. \tag{1.7}$$

The initial objective of this note is to simplify the above result of Lita da Silva (2019) by replacing (1.3) by (1.2) with $p = 1$. But it turns out that we are able to establish a much more general result. The main results of the paper are a weak law of large numbers and a mean convergence theorem for arrays of random variables with a very general dependence structure which contains various well-known dependence structures such as pairwise negative dependence, extended negative dependence, functions of non-homogeneous Markov chains, and wide orthant dependence. Theorem 2.2 establishes a mean convergence result which covers both the non-weighted case and the weighted case. We unify and extend a number of particular cases in the literature including Theorem 1 of Lita da Silva (2016), Theorem 3 of Lita da Silva (2019), Theorem 2.1 of Sung (2013), Theorems 3.1 and 3.2 of Shen and Volodin (2017), and Theorems 3.1–3.3 of Wu and Guan (2011). The mean convergence result is obtained by making use of the weak law of large numbers and the Lebesgue dominated convergence theorem. This approach is different from those in the aforementioned papers. The main results are applied to obtain mean convergence theorems for dependent bootstrap and functions of non-homogeneous Markov chains.

The rest of the paper is organized as follows. In Sect. 2, we establish the main results of the paper. Section 3 presents some corollaries to the main result and three examples. We apply the main results to three special cases: (i) the case where the dominating coefficients are uniformly bounded, (ii) widely orthant dependent random variables, and (iii) functions of non-homogeneous Markov chains. Finally, a mean convergence theorem for dependent bootstrap is presented in Sect. 4 as an application of Corollary 3.1.

Throughout the paper, $\{u_n, n \geq 1\}$ and $\{v_n, n \geq 1\}$ denote two sequences in $\mathbb{Z} \cup \{-\infty, \infty\}$ such that $u_n < v_n$ for all $n \geq 1$ and $\lim(v_n - u_n) = \infty$. If $u_n = -\infty$ or $v_n = \infty$, we assume that the random series $\sum_{i=u_n}^{v_n} X_{n,i}$ converges almost surely (a.s.). The symbol C denotes a positive universal constant which is not necessarily the same in each appearance, and $\mathbf{1}(A)$ denotes the indicator function of the set A. For $x \in \mathbb{R}$, $\log x$ denotes the natural logarithm of $\max\{1, x\}$.

2 Main results

An array $\{X_{n,i}, u_n \leq i \leq v_n, n \geq 1\}$ of random variables is said to satisfy Condition (G_2) if for each $n \geq 1$, there exists $M_n \geq 1$ which may depend on n such that for all $a > 0$,

$$\mathbb{E}\left(\sum_{i=u_n}^{v_n}\left(X_{n,i}^{(a)} - \mathbb{E}X_{n,i}^{(a)}\right)\right)^2 \leq M_n \sum_{i=u_n}^{v_n} \mathbb{E}(X_{n,i}^{(a)})^2, \tag{2.1}$$

where

$$X_{n,i}^{(a)} = -a\mathbf{1}(X_{n,i} < -a) + X_{n,i}\mathbf{1}(|X_{n,i}| \leq a) + a\mathbf{1}(X_{n,i} > a). \tag{2.2}$$

The $M_n, n \geq 1$ are called the *dominating coefficients*.

A motivation of Condition (G_2) comes from array $\{X_{n,i}, u_n \leq i \leq v_n, n \geq 1\}$ of *rowwise and pairwise m_n-dependent* random variables. That is, for each $n \geq 1$, two random variables $X_{n,i}$ and $X_{n,j}$ (from the n-th row of the array) are independent whenever $|i-j|>m_n$, where m_n is a non-negative integer which may increase in n. For this type of dependence, one can prove that (2.1) is satisfied with $M_n=1+m_n$ [see, e.g., Lemma 2 in Thành (2005)]. Another motivation of Condition (G_2) comes from functions of non-homogeneous Markov chains with the maximal coefficient of correlation $\rho_{n,1}<1$ which was introduced by Peligrad in Peligrad (2012). Let $\{\xi_{n,i}, 1\le i\le n, n\ge 1\}$ be an array of random variables such that each row $\{\xi_{n,i}, 1\le i\le n\}$ is a non-homogeneous Markov chain taking values in a measurable space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ with the maximal coefficient of correlation $\rho_{n,1}<1$ but $\rho_{n,1}$ is allowed to converge to 1. Let $f_{n,i}:\mathcal{X}\to\mathbb{R}$ be Borel functions and let $X_{n,i}=f_{n,i}(\xi_{n,i})$, $1\le i\le n, n\ge 1$. Then by Proposition 13 of Peligrad (2012), (2.1) is satisfied with $M_n=(1+\rho_{n,1})/(1-\rho_{n,1})$. In Subsect. 3.2, we consider another dependence structure so-called *wide orthant dependence* [see Wang et al. (2013)] which also satisfies Condition (G_2) with unbounded dominating coefficients.

The following theorem establishes a weak law of large numbers for arrays of random variables satisfying Condition (G_2). Theorem 2.1 is new even when the dominating coefficients are uniformly bounded, i.e., $\sup_{n\ge1} M_n<\infty$. We note that Condition (2.4) of Theorem 2.1 when $M_n\equiv 1$ is strictly weaker than Condition (2.5) of Theorem 3.4 in Wu and Guan (2011) (see Example 3.8 in Sect. 3).

Theorem 2.1 *Let* $1\le p<2$ *and let* $\{X_{n,i}, u_n\le i\le v_n, n\ge 1\}$ *be an array of random variables such that for each* $n\ge 1$*, the collection* $\{X_{n,i}, u_n\le i\le v_n\}$ *satisfying* (2.1) *for all* $a>0$*. Let* $\{b_n, n\ge 1\}$ *be a sequence of positive constants. If*

$$\sup_{n\ge 1}\frac{M_n}{b_n^p}\sum_{i=u_n}^{v_n}\mathbb{E}|X_{n,i}|^p<\infty \tag{2.3}$$

and

$$\lim_{n\to\infty} M_n\sum_{i=u_n}^{v_n}\mathbb{P}\left(|X_{n,i}|>\varepsilon b_n\right)=0 \text{ for all } \varepsilon>0, \tag{2.4}$$

then

$$\frac{1}{b_n}\sum_{i=u_n}^{v_n}\left(X_{n,i}-\mathbb{E}Y_{n,i}\right)\xrightarrow{\mathbb{P}} 0 \ \text{ as } n\to\infty, \tag{2.5}$$

where

$$Y_{n,i}=-b_n\mathbf{1}(X_{n,i}<-b_n)+X_{n,i}\mathbf{1}(|X_{n,i}| \le b_n)+b_n\mathbf{1}(X_{n,i}>b_n),\ u_n\le i\le v_n, n\ge 1.$$

Proof Let $\varepsilon_1 > 0$ be arbitrary but fixed. Then

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{b_n}\left|\sum_{i=u_n}^{v_n}(X_{n,i}-\mathbb{E}Y_{n,i})\right|>\varepsilon_1\right)\\
&\quad\le\sum_{i=u_n}^{v_n}\mathbb{P}\left(|X_{n,i}|>b_n\right)+\mathbb{P}\left(\left|\sum_{i=u_n}^{v_n}(Y_{n,i}-\mathbb{E}Y_{n,i})\right|>\varepsilon_1 b_n\right):=I_{n,1}+I_{n,2}.
\end{aligned}$$

It follows from (2.4) that $\lim_{n\to\infty} I_{n,1}=0$. It thus remains to prove $\lim_{n\to\infty} I_{n,2}=0$. By using Markov's inequality and (2.1), we have

$$\begin{aligned}
I_{n,2}&=\mathbb{P}\left(\left|\sum_{i=u_n}^{v_n}(Y_{n,i}-\mathbb{E}Y_{n,i})\right|>\varepsilon_1 b_n\right)\\
&\le\frac{1}{\varepsilon_1^2 b_n^2}\mathbb{E}\left(\left|\sum_{i=u_n}^{v_n}(Y_{n,i}-\mathbb{E}Y_{n,i})\right|\right)^2\\
&\le\frac{M_n}{\varepsilon_1^2 b_n^2}\sum_{i=u_n}^{v_n}\mathbb{E}Y_{n,i}^2\\
&=\frac{M_n}{\varepsilon_1^2 b_n^2}\sum_{i=u_n}^{v_n}\left(\mathbb{E}X_{n,i}^2\mathbf{1}(|X_{n,i}|\le b_n)+b_n^2\mathbb{P}(|X_{n,i}|>b_n)\right)\\
&=\frac{M_n}{\varepsilon_1^2}\sum_{i=u_n}^{v_n}\frac{1}{b_n^2}\int_0^{b_n^2}\mathbb{P}\left(|X_{n,i}|>u^{1/2}\right)\mathrm{d}\,u. \qquad (2.6)
\end{aligned}$$

Let $0<\varepsilon<1/2$ be arbitrary. By using Markov's inequality (2.3), and (2.6), we have

$$\begin{aligned}
I_{n,2}&\le\frac{M_n}{\varepsilon_1^2}\sum_{i=u_n}^{v_n}\frac{1}{b_n^2}\int_0^{\varepsilon^2 b_n^2}\mathbb{P}\left(|X_{n,i}|>u^{1/2}\right)\mathrm{d}\,u\\
&\quad+\frac{M_n}{\varepsilon_1^2}\sum_{i=u_n}^{v_n}\frac{1}{b_n^2}\int_{\varepsilon^2 b_n^2}^{b_n^2}\mathbb{P}\left(|X_{n,i}|>u^{1/2}\right)\mathrm{d}\,u\\
&\le\frac{M_n}{\varepsilon_1^2}\sum_{i=u_n}^{v_n}\frac{\mathbb{E}|X_{n,i}|^p}{b_n^2}\int_0^{\varepsilon^2 b_n^2}\frac{1}{u^{p/2}}\mathrm{d}\,u\\
&\quad+\frac{M_n}{\varepsilon_1^2}\sum_{i=u_n}^{v_n}\frac{1}{b_n^2}\int_{\varepsilon^2 b_n^2}^{b_n^2}\mathbb{P}\left(|X_{n,i}|>\varepsilon b_n\right)\mathrm{d}\,u\\
&\le\frac{M_n}{\varepsilon_1^2}\sum_{i=u_n}^{v_n}\frac{\mathbb{E}|X_{n,i}|^p}{b_n^p}\varepsilon^{2-p}+\frac{M_n}{\varepsilon_1^2}\sum_{i=u_n}^{v_n}\mathbb{P}\left(|X_{n,i}|>\varepsilon b_n\right)
\end{aligned}$$

$$\leq C\varepsilon^{2-p} + \frac{M_n}{\varepsilon_1^2}\sum_{i=u_n}^{v_n}\mathbb{P}\left(|X_{n,i}| > \varepsilon b_n\right). \tag{2.7}$$

Since $0 < \varepsilon < 1/2$ is arbitrary and $1 \leq p < 2$, it follows from (2.4) and (2.7) that $\lim_{n\to\infty} I_{n,2} = 0$. The proof is completed. □

The following theorem establishes a mean convergence theorem for arrays of dependent random variables. It shows that in Theorem 2.1, if we assume further that (2.8) holds, then a mean convergence of order p is obtained. Examples in Sect. 3 show that (2.4) and (2.8) are independent conditions in the sense that non of them implies the other. Theorem 2.2 extends Theorem 1 of Lita da Silva (2016), Theorem 3 of Lita da Silva (2019), Theorems 3.1 and 3.2 of Shen and Volodin (2017), and Theorem 2.1 of Sung (2013). It also extends and improves Theorems 3.1–3.3 of Wu and Guan (2011) and Theorem 1 of Ordóñez Cabrera and Volodin (2005).

Theorem 2.2 *Let $1 \leq p < 2$. Let $\{X_{n,i}, u_n \leq i \leq v_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be as in Theorem 2.1. If (2.3) and (2.4) hold, and*

$$\lim_{n\to\infty} M_n \int_1^\infty \sum_{i=u_n}^{v_n}\mathbb{P}\left(|X_{n,i}| > b_n u^{1/p}\right)\mathrm{d}\,u = 0, \tag{2.8}$$

then

$$\frac{1}{b_n}\sum_{i=u_n}^{v_n}\left(X_{n,i} - \mathbb{E}X_{n,i}\right) \xrightarrow{\mathcal{L}_p} 0 \ \text{ as } n \to \infty. \tag{2.9}$$

Proof For $n \geq 1$, $u_n \leq i \leq v_n$ and $t > 0$, set

$$\begin{aligned} Y_{n,i,t} = &-b_n t^{1/p}\mathbf{1}(X_{n,i} < -b_n t^{1/p}) + X_{n,i}\mathbf{1}(|X_{n,i}| \leq b_n t^{1/p}) \\ &+ b_n t^{1/p}\mathbf{1}(X_{n,i} > b_n t^{1/p}). \end{aligned}$$

Then, it follows from (2.8) that

$$\begin{aligned} &\sup_{t\geq 1}\frac{1}{b_n t^{1/p}}\sum_{i=u_n}^{v_n}\mathbb{E}|X_{n,i} - Y_{n,i,t}| \\ &= \sup_{t\geq 1}\frac{1}{b_n t^{1/p}}\sum_{i=u_n}^{v_n}\mathbb{E}\left(\left(|X_{n,i}| - b_n t^{1/p}\right)\mathbf{1}\left(|X_{n,i}| > b_n t^{1/p}\right)\right) \\ &= \sup_{t\geq 1}\sum_{i=u_n}^{v_n}\left(\mathbb{E}\left(\frac{|X_{n,i}|}{b_n t^{1/p}}\mathbf{1}(|X_{n,i}| > b_n t^{1/p})\right)\right. \\ &\quad \left. -\mathbb{P}\left(|X_{n,i}| > b_n t^{1/p}\right)\right) \end{aligned}$$

$$= \sup_{t\geq 1} \sum_{i=u_n}^{v_n} \int_1^{\infty} \mathbb{P}(|X_{n,i}| > b_n t^{1/p} u) \, \mathrm{d}\, u$$
$$= \int_1^{\infty} \sum_{i=u_n}^{v_n} \mathbb{P}(|X_{n,i}| > b_n u) \, \mathrm{d}\, u$$
$$\to 0 \text{ as } n \to \infty. \tag{2.10}$$

For all $t \geq 1$, we have from (2.10) that for all large n,

$$\mathbb{P}\left(\frac{1}{b_n}\left|\sum_{i=u_n}^{v_n}(X_{n,i} - \mathbb{E}X_{n,i})\right| > t^{1/p}\right) \leq \sum_{i=u_n}^{v_n} \mathbb{P}\left(|X_{n,i}| > b_n t^{1/p}\right)$$
$$+ \mathbb{P}\left(\left|\sum_{i=u_n}^{v_n}(Y_{n,i,t} - \mathbb{E}Y_{n,i,t})\right| + \sum_{i=u_n}^{v_n} \mathbb{E}|X_{n,i} - Y_{n,i,t}| > b_n t^{1/p}\right)$$
$$\leq \sum_{i=u_n}^{v_n} \mathbb{P}\left(|X_{n,i}| > b_n t^{1/p}\right) + \mathbb{P}\left(\left|\sum_{i=u_n}^{v_n}(Y_{n,i,t} - \mathbb{E}Y_{n,i,t})\right| > b_n t^{1/p}/2\right).$$

It thus follows that for all large n,

$$\mathbb{E}\left(\frac{1}{b_n}\left|\sum_{i=u_n}^{v_n}(X_{n,i} - \mathbb{E}X_{n,i})\right|\right)^p = \int_0^{\infty} \mathbb{P}\left(\frac{1}{b_n}\left|\sum_{i=u_n}^{v_n}(X_{n,i} - \mathbb{E}X_{n,i})\right| > t^{1/p}\right) \mathrm{d}\, t$$
$$= \int_0^{1} \mathbb{P}\left(\frac{1}{b_n}\left|\sum_{i=u_n}^{v_n}(X_{n,i} - \mathbb{E}X_{n,i})\right| > t^{1/p}\right) \mathrm{d}\, t$$
$$+ \int_1^{\infty} \mathbb{P}\left(\frac{1}{b_n}\left|\sum_{i=u_n}^{v_n}(X_{n,i} - \mathbb{E}X_{n,i})\right| > t^{1/p}\right) \mathrm{d}\, t$$
$$\leq \int_0^{1} \mathbb{P}\left(\frac{1}{b_n}\left|\sum_{i=u_n}^{v_n}(X_{n,i} - \mathbb{E}X_{n,i})\right| > t^{1/p}\right) \mathrm{d}\, t$$
$$+ \int_1^{\infty} \sum_{i=u_n}^{v_n} \mathbb{P}\left(|X_{n,i}| > b_n t^{1/p}\right) \mathrm{d}\, t$$
$$+ \int_1^{\infty} \mathbb{P}\left(\left|\sum_{i=u_n}^{v_n}(Y_{n,i,t} - \mathbb{E}Y_{n,i,t})\right| > b_n t^{1/p}/2\right) \mathrm{d}\, t$$
$$:= R_{n,1} + R_{n,2} + R_{n,3}. \tag{2.11}$$

By using Theorem 2.1 and (2.10) again, we obtain

$$\begin{aligned}
&\left|\frac{1}{b_n}\sum_{i=u_n}^{v_n}\left(X_{n,i}-\mathbb{E}X_{n,i}\right)\right| \\
&\quad \le \left|\frac{1}{b_n}\sum_{i=u_n}^{v_n}\left(X_{n,i}-\mathbb{E}Y_{n,i,1}\right)\right| + \frac{1}{b_n}\sum_{i=u_n}^{v_n}\mathbb{E}|X_{n,i}-Y_{n,i,1}| \\
&\quad \overset{\mathbb{P}}{\to} 0 \text{ as } n\to\infty.
\end{aligned} \tag{2.12}$$

Hence, by the Lebesgue dominated convergence theorem, we have from (2.12) that

$$\lim_{n\to\infty} R_{n,1} = 0. \tag{2.13}$$

By (2.8),

$$R_{n,2} \to 0 \text{ as } n \to \infty. \tag{2.14}$$

It thus remains to prove $\lim_{n\to\infty} R_{n,3} = 0$. By using Markov's inequality and (2.1), we have

$$\begin{aligned}
R_{n,3} &= \int_1^\infty \mathbb{P}\left(\left|\sum_{i=u_n}^{v_n}(Y_{n,i,t}-\mathbb{E}Y_{n,i,t})\right| > b_n t^{1/p}/2\right)\mathrm{d}t \\
&\le \int_1^\infty \frac{4}{b_n^2 t^{2/p}}\mathbb{E}\left(\left|\sum_{i=u_n}^{v_n}(Y_{n,i,t}-\mathbb{E}Y_{n,i,t})\right|\right)^2\mathrm{d}t \\
&\le 4M_n\sum_{i=u_n}^{v_n}\int_1^\infty \frac{\mathbb{E}Y_{n,i,t}^2}{b_n^2 t^{2/p}}\mathrm{d}t \\
&= 4M_n\sum_{i=u_n}^{v_n}\int_1^\infty \frac{1}{t^{2/p}}\left(\int_0^{t^{2/p}}\mathbb{P}\left(|b_n^{-1}X_{n,i}| > u^{1/2}\right)\mathrm{d}u\right)\mathrm{d}t \\
&:= 4(R_{n,3,1}+R_{n,3,2}),
\end{aligned} \tag{2.15}$$

where

$$R_{n,3,1} = M_n\sum_{i=u_n}^{v_n}\int_1^\infty \frac{1}{t^{2/p}}\left(\int_0^1 \mathbb{P}\left(|X_{n,i}| > b_n u^{1/2}\right)\mathrm{d}u\right)\mathrm{d}t,$$

and

$$R_{n,3,2} = M_n\sum_{i=u_n}^{v_n}\int_1^\infty \frac{1}{t^{2/p}}\left(\int_1^{t^{2/p}} \mathbb{P}\left(|X_{n,i}| > b_n u^{1/2}\right)\mathrm{d}u\right)\mathrm{d}t.$$

It is clear that

$$R_{n,3,1} \leq C \int_0^1 M_n \sum_{i=u_n}^{v_n} \mathbb{P}\left(|X_{n,i}| > b_n u^{1/2}\right) \mathrm{d}\, u. \tag{2.16}$$

By (2.3) and Markov's inequality, we have for all $u > 0$,

$$\sup_{n\geq 1} M_n \sum_{i=u_n}^{v_n} \mathbb{P}\left(|X_{n,i}| > b_n u^{1/2}\right) \leq \sup_{n\geq 1} \frac{M_n}{b_n^p u^{p/2}} \sum_{i=u_n}^{v_n} \mathbb{E}|X_{n,i}|^p \leq \frac{C}{u^{p/2}}. \tag{2.17}$$

By (2.4), we have for all $u > 0$,

$$\lim_{n\to\infty} M_n \sum_{i=u_n}^{v_n} \mathbb{P}\left(|X_{n,i}| > b_n u^{1/2}\right) = 0. \tag{2.18}$$

Since $p < 2$, the function $f(u) = C/u^{p/2}$ is integrable on $(0, 1)$. It thus follows from (2.17), (2.18), and the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty} \int_0^1 M_n \sum_{i=u_n}^{v_n} \mathbb{P}\left(|X_{n,i}| > b_n u^{1/2}\right) \mathrm{d}\, u = 0$$

which together with (2.16) imply

$$\lim_{n\to\infty} R_{n,3,1} = 0. \tag{2.19}$$

For $R_{n,3,2}$, we have

$$\begin{aligned}
R_{n,3,2} &= M_n \sum_{i=u_n}^{v_n} \int_1^\infty \mathbb{P}\left(|X_{n,i}| > b_n u^{1/2}\right) \int_{u^{p/2}}^\infty \frac{1}{t^{2/p}} \mathrm{d}\, t \, \mathrm{d}\, u \\
&\leq C M_n \sum_{i=u_n}^{v_n} \int_1^\infty u^{p/2-1} \mathbb{P}\left(|X_{n,i}| > b_n u^{1/2}\right) \mathrm{d}\, u \\
&= C M_n \int_1^\infty \sum_{i=u_n}^{v_n} \mathbb{P}\left(|X_{n,i}| > b_n x^{1/p}\right) \mathrm{d}\, x.
\end{aligned} \tag{2.20}$$

Combining (2.8) and (2.20) yields

$$\lim_{n\to\infty} R_{n,3,2} = 0. \tag{2.21}$$

By using (2.15), (2.19), and (2.21), we obtain

$$\lim_{n\to\infty} R_{n,3} = 0.$$

The proof of the theorem is completed. □

3 Corollaries and examples

In this section, we present some corollaries of Theorems 2.1 and 2.2. We apply these results to three special cases: (i) the case where the dominating coefficients are uniformly bounded, (ii) widely orthant dependent random variables, and (iii) functions of non-homogeneous Markov chains.

3.1 The case where the dominating coefficients are uniformly bounded

An array $\{X_{n,i}, u_n \le i \le v_n, n \ge 1\}$ of random variables is said to satisfy Condition (H_2) if for each $n \ge 1$, there exists a constant $M \ge 1$ which does not depend on n such that for all $a > 0$,

$$\mathbb{E}\left(\sum_{i=u_n}^{v_n}\left(X_{n,i}^{(a)} - \mathbb{E}X_{n,i}^{(a)}\right)\right)^2 \le M\sum_{i=u_n}^{v_n}\mathbb{E}(X_{n,i}^{(a)})^2,$$

where $X_{n,i}^{(a)}$ is defined as in (2.2) and $M \ge 1$ is a constant which does not depend on n. This is a special case of Condition (G_2) when the dominating coefficients are uniformly bounded. The authors in Adler and Matuła (2018), Dzung and Thành (2021), Rio (1995), Thành (2022) also used similar conditions to study complete convergence and strong and weak laws of large numbers.

Condition (H_2) includes various well known dependence structures such as arrays of rowwise and pairwise negative dependence [see (Lehmann 1966, Lemma 1 (ii) and Lemma 3)] and arrays of rowwise extended negative dependence [see, e.g., (Shen and Volodin 2017, Lemmas 2.1 and 2.3)]. A Reviewer so kindly noticed to us that many well known multivariate random vectors satisfy Condition (H_2) such as multinomial, multivariate hypergeometric, Dirichlet, negatively correlated normal, permutation distribution, and random sampling without replacement, etc. Indeed, these multivariate random vectors are proved to be negatively associated [see (Joag-Dev and Proschan 1983)], and since negative association is strictly stronger than pairwise negative dependence [see (Joag-Dev and Proschan 1983, Property P3 and Remark 2.5)], the aforementioned multivariate random vectors satisfy Condition (H_2). Moreover, by Example 3.1 of Hien and Thành (2015), we have that an array $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ of rowwise negatively associated random variables can be constructed such that each $X_{n,i}$ can have any specified marginal distributions.

The following corollary is a consequence of Theorems 2.1 and 2.2.

Corollary 3.1 *Let* $1 \le p < 2$ *and let* $\{X_{n,i}, u_n \le i \le v_n, n \ge 1\}$ *be an array of random variables satisfying Condition* (H_2). *Let* $\{b_n, n \ge 1\}$ *be a sequence of positive constants. If*

$$\sup_{n\ge 1}\frac{1}{b_n^p}\sum_{i=u_n}^{v_n}\mathbb{E}|X_{n,i}|^p < \infty \tag{3.1}$$

and

$$\lim_{n\to\infty}\sum_{i=u_n}^{v_n}\mathbb{P}\left(|X_{n,i}|>\varepsilon b_n\right)=0 \textit{ for all } \varepsilon>0, \tag{3.2}$$

then we obtain the weak law of large number (2.5).

In addition, if

$$\lim_{n\to\infty}\int_1^\infty\sum_{i=u_n}^{v_n}\mathbb{P}\left(|X_{n,i}|>b_n u^{1/p}\right)\mathrm{d}\,u=0, \tag{3.3}$$

then we obtain (2.9).

Proof Since $\{X_{n,i}, u_n\le i\le v_n, n\ge 1\}$ satisfies Condition (H_2), (2.1) is fulfilled with $M_n=M<\infty$ for all $n\ge 1$. Corollary 3.1 thus follows from Theorems 2.1 and 2.2. □

In the following remark, we will make some comments on Conditions (1.1)–(1.3) in Theorem 1.1, and Conditions (3.1)–(3.3) in Corollary 3.1.

Remark 3.2 (i) It is clear that (1.3) implies (1.2) with $p=1$. Conversely, assume that (1.2) holds with $p=1$. For any fixed $\varepsilon>0$, we then have

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^n\varepsilon b_n\mathbb{P}(|X_{n,i}|>\varepsilon b_n)&\le\sum_{i=1}^n\int_{\varepsilon b_n/2}^{\varepsilon b_n}\mathbb{P}(|X_{n,i}|>x)\,\mathrm{d}\,x\\&\le\sum_{i=1}^n\int_{\varepsilon b_n/2}^{\infty}\mathbb{P}(|X_{n,i}|>x)\,\mathrm{d}\,x\\&=o(b_n)\text{ as }n\to\infty.\end{aligned}$$

It thus follows that

$$\begin{aligned}\sum_{i=1}^n\mathbb{E}(|X_{n,i}|\mathbf{1}(|X_{n,i}|>\varepsilon b_n)&=\sum_{i=1}^n\int_{\varepsilon b_n}^{\infty}\mathbb{P}(|X_{n,i}|>x)\,\mathrm{d}\,x\\&\quad+\sum_{i=1}^n\varepsilon b_n\mathbb{P}(|X_{n,i}|>\varepsilon b_n)\\&=o(b_n)\text{ as }n\to\infty\end{aligned}$$

establishing (1.3). Therefore, (1.3) is indeed equivalent to (1.2) with $p=1$.

(ii) A Reviewer so kindly pointed out to us that for the case $u_n\equiv 1$ and $v_n\equiv n$, Conditions (3.1)–(3.3) are equivalent to the pair of Conditions (1.1) and (1.2) (with $1\le p<2$). To see this, we note that (1.1) and (1.2) imply

$$\int_0^\infty\sum_{i=1}^n\mathbb{P}(|X_{n,i}|>b_n u^{1/p})\,\mathrm{d}\,u=O(1)$$

which, in turn, implies (3.1). On the other hand, for any fixed $\varepsilon > 0$, by letting $t = \varepsilon b_n^p u$, Condition (1.2) (with $1 \le p < 2$) can be rewritten as

$$\lim_{n\to\infty} \int_1^\infty \sum_{i=1}^n \mathbb{P}\left(|X_{n,i}| > \varepsilon^{1/p} b_n u^{1/p}\right) \mathrm{d}\,u = 0. \tag{3.4}$$

It is clear that (3.4) implies (3.3) by letting $\varepsilon = 1$. It also follows from (3.4) that for all $\varepsilon > 0$

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_1^\infty \sum_{i=1}^n \mathbb{P}\left(|X_{n,i}| > \varepsilon b_n u^{1/p}/2\right) \mathrm{d}\,u \\ &\ge \lim_{n\to\infty} \int_1^2 \sum_{i=1}^n \mathbb{P}\left(|X_{n,i}| > \varepsilon b_n u^{1/p}/2\right) \mathrm{d}\,u \\ &\ge \lim_{n\to\infty} \int_1^2 \sum_{i=1}^n \mathbb{P}\left(|X_{n,i}| > \varepsilon b_n\right) \mathrm{d}\,u \\ &= \lim_{n\to\infty} \sum_{i=1}^n \mathbb{P}\left(|X_{n,i}| > \varepsilon b_n\right) \end{aligned}$$

implying (3.2). Therefore, the pair of Conditions (1.1) and (1.2) (with $1 \le p < 2$) implies (3.1), (3.2) and (3.3). Conversely, it is clear that (3.1) ensures (1.1), and (3.3) ensures (3.4) (i.e., (1.2) with $1 \le p < 2$) if $\varepsilon \ge 1$. If $0 < \varepsilon < 1$, then

$$\begin{aligned} b_n^{-p} \int_{\varepsilon b_n^p}^\infty \sum_{i=1}^n \mathbb{P}\left(|X_{n,i}|^p > t\right) \mathrm{d}\,t &= \int_\varepsilon^\infty \sum_{i=1}^n \mathbb{P}\left(|X_{n,i}| > b_n u^{1/p}\right) \mathrm{d}\,u \\ &\le (1-\varepsilon) \sum_{i=1}^n \mathbb{P}\left(|X_{n,i}| > b_n \varepsilon^{1/p}\right) \\ &\quad + \int_1^\infty \sum_{i=1}^n \mathbb{P}\left(|X_{n,i}| > b_n u^{1/p}\right) \mathrm{d}\,u, \end{aligned}$$

hence (3.2) and (3.3) ensure that (1.2) (with $1 \le p < 2$) is satisfied for $0 < \varepsilon < 1$. □

From Remark 3.2, we immediately have the following corollary which simplifies Theorem 1.1.

Corollary 3.3 *In Theorem 1.1, (1.3) can be replaced by (1.2) with $p = 1$.*

Remark 3.4 (i) When each row of the array $\{X_{n,i}, u_n \le i \le v_n, n \ge 1\}$ is comprised of pairwise negatively dependent random variables (Wu and Guan 2011, Theorem 3.4) obtained weak law of large numbers (2.5) under (3.1) and

$$\lim_{n\to\infty} \frac{1}{b_n^p} \sum_{i=u_n}^{v_n} \sup_{y \ge h_n^p} y \mathbb{P}\left(|X_{n,i}|^p > y\right) = 0, \tag{3.5}$$

where $\{h_n, n \geq 1\}$ is a sequence of positive constants satisfying $h_n \uparrow \infty$ and $\lim_{n\to\infty} h_n/b_n = 0$.

We will show that (3.5) implies (3.2). To see this, let $\varepsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} h_n/b_n = 0$, there exists n_0 such that $h_n < \varepsilon b_n$ for all $n \geq n_0$. It thus follows from (3.5) that

$$0 = \lim_{n\to\infty} \frac{1}{b_n^p} \sum_{i=u_n}^{v_n} \sup_{y \geq h_n^p} y\mathbb{P}\left(|X_{n,i}|^p > y\right) \geq \lim_{n\to\infty} \frac{1}{b_n^p} \sum_{i=u_n}^{v_n} \sup_{y \geq \varepsilon^p b_n^p} y\mathbb{P}\left(|X_{n,i}|^p > y\right)$$
$$\geq \varepsilon^p \lim_{n\to\infty} \sum_{i=u_n}^{v_n} \mathbb{P}\left(|X_{n,i}| > \varepsilon b_n\right)$$

thereby implying (3.2). We will see in Example 3.8 that (3.5) is strictly stronger than (3.2). □

(ii) When each row of the array $\{X_{n,i}, u_n \leq i \leq v_n, n \geq 1\}$ is comprised of extended negatively dependent random variables (Shen and Volodin 2017, Theorem 3.1) obtained mean convergence (2.9) under (3.1) and

$$\lim_{n\to\infty} \frac{1}{b_n^p} \sum_{i=u_n}^{v_n} \mathbb{E}\left((|X_{n,i}| - h_n^{1/p})^p \mathbf{1}\left(|X_{n,i}|^p > h_n\right)\right) = 0, \tag{3.6}$$

where $\{h_n, n \geq 1\}$ is a sequence of positive constant satisfying $h_n \uparrow \infty$ and $\lim_{n\to\infty} h_n/b_n = 0$.

We will show that (3.6) implies both (3.2) and (3.3). To see this, let $\varepsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} h_n/b_n = 0$, there exists n_0 such that $h_n^{1/p} < (\varepsilon b_n)^{1/p}/2 < (\varepsilon b_n)^{1/p}$ for all $n \geq n_0$. It thus follows from (3.6) that

$$0 = \lim_{n\to\infty} \frac{1}{b_n^p} \sum_{i=u_n}^{v_n} \mathbb{E}\left(\left(|X_{n,i}| - h_n^{1/p}\right)^p \mathbf{1}\left(|X_{n,i}|^p > h_n\right)\right)$$
$$\geq \lim_{n\to\infty} \frac{1}{b_n^p} \sum_{i=u_n}^{v_n} \mathbb{E}\left(\left(|X_{n,i}| - (\varepsilon b_n)^{1/p}/2\right)^p \mathbf{1}\left(|X_{n,i}|^p > \varepsilon b_n\right)\right)$$
$$\geq \lim_{n\to\infty} \frac{1}{2^p b_n^p} \sum_{i=u_n}^{v_n} \mathbb{E}\left(|X_{n,i}|^p \mathbf{1}\left(|X_{n,i}|^p > \varepsilon b_n\right)\right).$$

This implies

$$\lim_{n\to\infty} \int_1^\infty \sum_{i=1}^{n} \mathbb{P}\left(|X_{n,i}| > \varepsilon b_n u^{1/p}\right) \mathrm{d}\,u = 0 \text{ for all } \varepsilon > 0$$

which, in turn, implies both (3.2) and (3.3). By applying Lemmas 2.1 and 2.3 of Shen and Volodin (2017), we have that extended negatively dependent random variables satisfy condition (H_2). Therefore, Corollary 3.1 extends Theorem 3.1 of Shen and Volodin (2017). □

By letting $X_{n,i} \equiv b_n a_{n,i} Y_{n,i}$, we obtain "weighted form" of Corollary 3.1. The following corollary extends Theorem 1.2 [i.e., Theorem 2.1 of Sung (2013)].

Corollary 3.5 *Let $1 \le p < 2$ and let $\{Y_{n,i}, u_n \le i \le v_n, n \ge 1\}$ be a triangular array of random variables such that for each $n \ge 1$, the collection $\{Y_{n,i}, u_n \le i \le v_n\}$ satisfies condition (H_2). Let $\{a_{n,i}, n \ge 1\}$ be an array of constants. Suppose that (1.5) holds, and*

$$\lim_{n\to\infty} \sum_{i=u_n}^{v_n} \mathbb{P}\left(|a_{n,i}||Y_{n,i}| > \varepsilon\right) = 0 \text{ for all } \varepsilon > 0, \tag{3.7}$$

and

$$\lim_{n\to\infty} \sum_{i=u_n}^{v_n} \int_1^\infty \mathbb{P}\left(|a_{n,i}||Y_{n,i}| > u^{1/p}\right) \mathrm{d}\,u = 0. \tag{3.8}$$

Then we obtain (1.7).

Remark 3.6 (i) As noted by Sung [see Corollaries 2.1–2.3 in Sung (2013)], Theorem 1.2 (and hence Corollary 3.5) extends and improves Theorems 3.1–3.3 of Wu and Guan (2011). Corollary 3.5 also extends and improves Theorem 1 of Ordóñez Cabrera and Volodin (2005) [see Remark 3.2 in Wu and Guan (2011)].

(ii) Shen and Volodin (2017, Theorem 3.2) established Theorem 1.2 for the case where the pairwise negative dependence assumption is replaced by extended negative dependence. Therefore, Corollary 3.5 also extends Theorem 3.2 of Shen and Volodin (2017). □

We now present three examples to illustrate the sharpness of Corollary 3.1. The first example shows that in Corollary 3.1, (3.2) cannot be dispensed with. In Example 3.7, (3.2) fails while both (3.1) and (3.3) hold. It also shows that (3.3) is strictly weaker than

$$\lim_{n\to\infty} \frac{1}{b_n^p} \sum_{i=u_n}^{v_n} \mathbb{E}\left(|X_{n,i}|^p \mathbf{1}(|X_{n,i}| > b_n)\right) = 0. \tag{3.9}$$

Example 3.7 Let $1 \le p < 2$, $0 < \alpha < 1$, $u_n \equiv 1$, $v_n \equiv n$, $b_n \equiv n^{1/p}$. Let $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ be an array of independent symmetric random variables such that for all $1 \le i \le n, n \ge 1$,

$$\mathbb{P}(X_{n,i} = 0) = 1 - \frac{1}{n},\ \mathbb{P}\left(X_{n,i} = -(n + i^\alpha)^{1/p}\right) = \mathbb{P}\left(X_{n,i} = (n + i^\alpha)^{1/p}\right) = \frac{1}{2n}.$$

Then (3.1) is satisfied since

$$\sup_{n\geq 1}\frac{1}{b_n^p}\sum_{i=1}^{n}\mathbb{E}|X_{n,i}|^p=\sup_{n\geq 1}\frac{1}{n}\sum_{i=1}^{n}\frac{n+i^\alpha}{n}\leq 2<\infty.$$

Let $0<\varepsilon<1$. Since

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\mathbb{P}\left(|X_{n,i}|>\varepsilon n^{1/p}\right)&=\lim_{n\to\infty}\sum_{i=1}^{n}\mathbb{P}\left(|X_{n,i}|=(n+i^\alpha)^{1/p}\right)\\&=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}=1,\end{aligned}\tag{3.10}$$

we have (3.2) fails. Applying Theorem 1 (ii) of Etemadi (1985), we have for all $n\geq 1$,

$$\begin{aligned}&\left(1-\mathbb{P}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}X_{n,i}\right|>\varepsilon n^{1/p}/2\right)\right)\\&\sum_{i=1}^{n}\mathbb{P}(|X_i|>\varepsilon n^{1/p})\leq\mathbb{P}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}X_{n,i}\right|>\varepsilon n^{1/p}/2\right).\end{aligned}\tag{3.11}$$

Combining (3.10) and (3.11) yields

$$\frac{1}{n^{1/p}}\max_{1\leq k\leq v_n}\left|\sum_{i=1}^{k}X_{n,i}\right|\overset{\mathbb{P}}{\nrightarrow}0.\tag{3.12}$$

Applying Theorem 1 (i) of Etemadi (1985), we have

$$\mathbb{P}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}X_{n,i}\right|>4\varepsilon n^{1/p}\right)\leq 4\max_{1\leq k\leq n}\mathbb{P}\left(\left|\sum_{i=1}^{k}X_{n,i}\right|>\varepsilon n^{1/p}\right).\tag{3.13}$$

From (3.12) and (3.13), we conclude that

$$\max_{1\leq k\leq n}\mathbb{P}\left(\left|\sum_{i=1}^{k}X_{n,i}\right|>\varepsilon n^{1/p}\right)\nrightarrow 0$$

and therefore (2.5) fails. That is, in Corollary 3.1, (3.2) cannot be dispensed with.

Finally, since $\alpha < 1$, we have

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{n}\int_{1}^{\infty}\mathbb{P}\left(|X_{n,i}|>n^{1/p}u^{1/p}\right)\mathrm{d}\,u\\
&=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{n}\mathbb{E}\left(|X_{n,i}|^{p}\mathbf{1}(|X_{n,i}|>n^{1/p})\right)-\mathbb{P}(|X_{n,i}|>n^{1/p})\right)\\
&=\lim_{n\to\infty}\frac{1}{n^{2}}\sum_{i=1}^{n}i^{\alpha}=0
\end{aligned}$$

thereby establishing (3.3). However, (3.9) fails since

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left(|X_{n,i}|^{p}\mathbf{1}(|X_{n,i}|>n^{1/p})\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{n+i^{\alpha}}{n}\geq 1.$$

This shows that (3.3) is strictly weaker than (3.9). □

The next example shows that there exists an array of independent random variables satisfying (3.1) and (3.2) of Corollary 3.1 but (3.5) [i.e., Condition (2.5) of Theorem 3.4 in Wu and Guan (2011)] is not fulfilled. In Example 3.8, (3.3) and (2.9) fail. That is, it shows that in Corollary 3.1, (2.9) can fail if (3.3) is dispensed with.

Example 3.8 Let $p = 1$, $u_n \equiv 1$, $v_n \equiv n$, $b_n \equiv n^{1/p}$. Let $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ be an array of random variables with $X_{n,i} = 0$ for $1 \leq i < n$, $n \geq 1$ and

$$\mathbb{P}(X_{n,n}=0)=1-\frac{1}{n},\ \mathbb{P}\left(X_{n,n}=-(n+1)^{2}\right)=\mathbb{P}\left(X_{n,n}=(n+1)^{2}\right)=\frac{1}{2n},\ n\geq 1.$$

Then both (3.1) and (3.2) are satisfied since

$$\sup_{n\geq 1}\frac{1}{b_n^{p}}\sum_{i=1}^{n}\mathbb{E}|X_{n,i}|^{p}=\sup_{n\geq 1}\frac{1}{n}\times\frac{(n+1)^{2}}{n}<\infty$$

and for all $\varepsilon > 0$,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\mathbb{P}\left(|X_{n,i}|>\varepsilon b_n\right)=\lim_{n\to\infty}\mathbb{P}\left(|X_{n,n}|=(n+1)^{2}\right)=0.$$

Now, for all sequence $\{h_n, n \geq 1\}$ satisfying $\lim_{n\to\infty} h_n/b_n = 0$, we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{b_n^{p}}\sum_{i=1}^{n}\sup_{y\geq h_n^{p}}y\mathbb{P}\left(|X_{n,i}|^{p}>y\right)&\geq\lim_{n\to\infty}\frac{1}{n}\sup_{y\geq n^{2}}y\mathbb{P}\left(|X_{n,n}|>y\right)\\
&\geq\lim_{n\to\infty}\frac{1}{n}\times n^{2}\times\frac{1}{n}=1
\end{aligned}$$

thereby showing that (3.5) fails. Hence, we cannot apply Theorem 3.4 of Wu and Guan (2011) for this example.

Finally, by noting $p = 1$, we have

$$\begin{aligned}\lim_{n\to\infty}\int_1^{\infty}\sum_{i=1}^{n}\mathbb{P}\left(|X_{n,i}| > b_n u^{1/p}\right)\mathrm{d}u &= \lim_{n\to\infty}\int_1^{\infty}\mathbb{P}\left(|X_{n,n}| > nu\right)\mathrm{d}u\\ &\geq \lim_{n\to\infty}\int_1^{n}\mathbb{P}\left(|X_{n,n}| > nu\right)\mathrm{d}u\\ &= \lim_{n\to\infty}\int_1^{n}\frac{1}{n}\,\mathrm{d}u = 1\end{aligned}$$

thereby showing that (3.3) is not satisfied. We also have

$$\mathbb{E}\left|\frac{1}{b_n}\sum_{i=1}^{n}\left(X_{n,i} - \mathbb{E}X_{n,i}\right)\right| = \frac{1}{n}\mathbb{E}|X_{n,n}| = \frac{1}{n}\times(n+1)^2\times\frac{1}{n}\to 1,$$

and so (2.9) fails. □

Remark 3.9 (i) From Examples 3.7 and 3.8, we see that (3.2) and (3.3) are independent conditions in the sense that neither of them implies the other. Both (3.2) and (3.3) follows from

$$\lim_{n\to\infty}\int_1^{\infty}\sum_{i=u_n}^{v_n}\mathbb{P}\left(|X_{n,i}| > \varepsilon b_n u^{1/p}\right)\mathrm{d}u = 0 \text{ for all } \varepsilon > 0. \tag{3.14}$$

We note that Condition (3.14) coincides with (1.2) when $u_n \equiv 1$ and $v_n \equiv n$. Example 3.7 shows that (3.3) is strictly weaker than (3.14) with $\varepsilon = 1$ and Example 3.8 shows that (3.2) is strictly weaker than (3.14). In the case where $u_n \equiv 1$ and $v_n \equiv n$, (3.14) coincides with (1.2). By proceeding in exactly the same manner as Remark 3.2, we have (3.14) is equivalent to the pair of Conditions (3.2) and (3.3).

(ii) A Reviewer so kindly pointed out to us that there are cases in which (3.14) tends to zero more rapidly than (3.2) [see, e.g., (Lita da Silva 2016, Page 350) for the case where $|X_{n,i}|^p$ has the exponential distribution Expo(1), $1 \leq i \leq n$, $n \geq 1$].

The following example, which is inspired by Example 4.3 in Rosalsky and Thành (2021) and Example 5 in Thành (2023), shows that in Corollary 3.1, we cannot obtain a.s. convergence in (2.9).

Example 3.10 Let $1 \leq p < 2$, $u_n \equiv 1$, $v_n \equiv n$, $b_n \equiv n^{1/p}$ and let $\{X_i, i \geq 1\}$ be a sequence of pairwise negatively dependent random variables such that for all $i \geq 1$,

$$\mathbb{P}(X_{n,i} = 0) = 1 - \frac{1}{(i+1)\log(i+1)},\ \mathbb{P}\left(X_{n,i} = \pm(i+1)^{1/p}\right) = \frac{1}{2(i+1)\log(i+1)}.$$

Let $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise independent random variables with

$$X_{n,i} = X_i, 1 \leq i \leq n, n \geq 1.$$

Then

$$\sup_{1\leq i\leq n, n\geq 1} \mathbb{E}(|X_{n,i}|^p \log |X_{n,i}|) = \frac{1}{p} < \infty.$$

By the classical de la Vallée Poussin theorem, $\{|X_{n,i}|^p, 1 \leq i \leq n, n \geq 1\}$ is uniformly integrable, so that the hypotheses of Corollary 3.1 are fulfilled. Therefore, we obtain (2.9). However, by proceeding in exactly the same manner as Example 4.3 of Rosalsky and Thành (2021), we obtain

$$\frac{\sum_{i=1}^n X_{n,i}}{n^{1/p}} = \frac{\sum_{i=1}^n X_i}{n^{1/p}} \nrightarrow 0 \text{ a.s. as } n \to \infty,$$

that is, a.s. convergence does not prevail in (2.9). □

3.2 Widely orthant dependent random variables

As mentioned in Sect. 2, arrays of pairwise m_n-dependent random variables, widely orthant dependent random variables, and non-homogeneous Markov chains are typical examples of Condition (G_2) where the dominating sequence $\{M_n, n \geq 1\}$ can be unbounded.

The concept of widely orthant dependent random variables was introduced by Wang et al. (2013). A collection $\{X_i, 1 \leq i \leq n\}$ of random variables is said to be *widely orthant dependent* (WOD) if there exists a positive constant g_n which may depend on n such that for all $x_i \in \mathbb{R}$, $1 \leq i \leq n$,

$$\mathbb{P}(X_1 > x_1, \ldots, X_n > x_n) \leq g_n \mathbb{P}(X_1 > x_1) \ldots \mathbb{P}(X_n > x_n),$$

and

$$\mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n) \leq g_n \mathbb{P}(X_1 \leq x_1) \ldots \mathbb{P}(X_n \leq x_n).$$

The $g_n, n \geq 1$ are called the dominating coefficients. If there exists a positive constant M such that $\sup_{n\geq 1} g_n \leq M$, then this reduces to the concept of extended negative dependence. We refer to Wang et al. (2013, Section 3) and Wu et al. (2018, Example 1.2) for examples of sequences of WOD random variables with the dominating coefficients g_n satisfying $\lim_{n\to\infty} g_n = \infty$.

The following corollary establishes a weak law of large numbers and a mean convergence result for arrays of rowwise WOD random variables. For simplicity and since it is just meant to be an illustration, we only state the result for the case $u_n \equiv 1$ and $v_n \equiv n$.

Corollary 3.11 *Let $1 \leq p < 2$ and let $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array such that for each $n \geq 1$, the collection $\{X_{n,i}, 1 \leq i \leq n\}$ is comprised of WOD random variables with the dominating coefficient g_n. Let $\{b_n, n \geq 1\}$ be a sequence of positive constants. If*

$$\sup_{n\geq 1}(1+g_n)b_n^{-p}\sum_{i=1}^{n}\mathbb{E}|X_{n,i}|^p < \infty \tag{3.15}$$

and

$$\lim_{n\to\infty}(1+g_n)\sum_{i=1}^{n}\mathbb{P}\left(|X_{n,i}| > \varepsilon b_n\right) = 0 \text{ for all } \varepsilon > 0, \tag{3.16}$$

then we obtain the weak law of large number

$$\frac{1}{b_n}\sum_{i=1}^{n}\left(X_{n,i} - \mathbb{E}Y_{n,i}\right) \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty,$$

where $Y_{n,i}$ is as in Theorem 2.1. In addition, if

$$\lim_{n\to\infty}(1+g_n)\int_1^{\infty}\sum_{i=1}^{n}\mathbb{P}\left(|X_{n,i}| > b_n u^{1/p}\right)\mathrm{d}u = 0, \tag{3.17}$$

then we obtain (1.4).

Proof By Lemma 2.1 (i) and Corollary 2.3 in Wang et al. (2014) [see also Lemma 2 of Lita da Silva (2016)], we have for each $n \geq 1$,

$$\mathbb{E}\left(\left|\sum_{i=1}^{n}\left(X_i^{(a)} - \mathbb{E}X_i^{(a)}\right)\right|\right)^2 \leq C(1+g_n)\sum_{i=1}^{n}\mathbb{E}(X_i^{(a)})^2,$$

that is, (2.1) is satisfied with $M_n \equiv C(1+g_n)$. Applying Theorems 2.1 and 2.2, we obtain the conclusions of the corollary. □

Lita da Silva (2016, Theorem 1) established the following result.

Proposition 3.12 (Theorem 1 of Lita da Silva (2016)) *Let $1 \leq p < 2$ and let $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be as in Corollary 3.11. If*

$$\sum_{i=1}^{n}\int_0^{\varepsilon b_n^p}\mathbb{P}(|X_{n,i}|^p > t)\,\mathrm{d}t = O\left(\frac{b_n^p}{1+g_n}\right) \text{ as } n \to \infty \text{ for all } \varepsilon > 0, \tag{3.18}$$

and

$$\sum_{i=1}^{n}\int_{\varepsilon b_n^p}^{\infty}\mathbb{P}(|X_{n,i}|^p>t)\,\mathrm{d}t$$
$$=o\left(\frac{b_n^p}{1+g_n}\right)\ \textit{as } n\to\infty \textit{ for all } \varepsilon>0 \textit{ when } 1<p<2, \tag{3.19}$$

or

$$\sum_{i=1}^{n}\int_{\varepsilon b_n}^{\infty}\mathbb{P}(|X_{n,i}|>t)\,\mathrm{d}t=o\left(\frac{b_n}{1+g_n}\right)\ \textit{as } n\to\infty, \tag{3.20}$$

and

$$\sum_{i=1}^{n}\mathbb{P}\left(|X_{n,i}|>\varepsilon b_n\right)=o(1)\ \textit{as } n\to\infty \textit{ for all } \varepsilon>0 \textit{ when } p=1,$$

then we obtain (1.4).

Similar to Corollary 3.3, we obtain the following simplification of Theorem 1 of Lita da Silva (2016).

Corollary 3.13 *Let* $1\le p<2$ *and let* $\{X_{n,i},1\le i\le n,n\ge 1\}$ *and* $\{b_n,n\ge 1\}$ *be as in Corollary* 3.11. *If* (3.18) *and* (3.19) *holds with* $1\le p<2$, *then we obtain* (1.4).

Proof We see that (3.18) and (3.19) imply

$$\sum_{i=1}^{n}\int_{0}^{\infty}\mathbb{P}(|X_{n,i}|^p>t)\,\mathrm{d}t=O\left(\frac{b_n^p}{1+g_n}\right)\ \text{as } n\to\infty$$

which yields (3.15). On the other hand, by letting $u=\varepsilon b_n^p$, Condition (3.19) with $1\le p<2$ can be rewritten as

$$\lim_{n\to\infty}(1+g_n)\int_{1}^{\infty}\sum_{i=1}^{n}\mathbb{P}\left(|X_{n,i}|>\varepsilon^{1/p}b_n u^{1/p}\right)\mathrm{d}u=0 \text{ for all } \varepsilon>0$$

which, in turn, implies both (3.16) and (3.17). The proof of Corollary 3.13 thus follows by applying Corollary 3.11. □

3.3 Non-homogeneous Markov chains

In this subsection, we will present applications of Theorems 2.1 and 2.2 to non-homogeneous Markov chains. Let $\mathcal{A}$ and $\mathcal{B}$ be two σ-fields. Define the maximal coefficient of correlation

$$\rho(\mathcal{A}, \mathcal{B}) = \sup_{f \in \mathcal{L}_2(\mathcal{A}), g \in \mathcal{L}_2(\mathcal{B})} |\mathrm{corr}(f, g)|,$$

where $\mathcal{L}_2(\mathcal{A})$ is the space of random variables that are $\mathcal{A}$-measurable and square integrable. For a collection $\{\xi_{n,i}, 1 \le i \le n\}$ of random variables, we define

$$\rho_{n,k} = \max_{1 \le s, s+k \le n} \rho(\sigma(\xi_{n,i}, i \le s), \sigma(\xi_{n,j}, j \ge s+k)). \tag{3.21}$$

The following result establishes mean convergence for functions of non-homogeneous Markov chains.

Corollary 3.14 *Let $1 \le p < 2$, let $\{b_n, n \ge 1\}$ be a sequence of positive constants, and let $\{\xi_{n,i}, 1 \le i \le n, n \ge 1\}$ be an array of random variables such that such that each row $\{\xi_{n,i}, 1 \le i \le n\}$ is a non-homogeneous Markov chain taking values in a measurable space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ with the maximal coefficient of correlation $\rho_{n,1}$ defined as in (3.21). Let $f_{n,i} : \mathcal{X} \to \mathbb{R}$ be Borel functions and let $X_{n,i} = f_{n,i}(\xi_{n,i})$, $1 \le i \le n, n \ge 1$. Assume that $\rho_{n,1} < 1$ for all $n \ge 1$. If*

$$\sup_{n \ge 1} \frac{1}{(1-\rho_{n,1}) b_n^p} \sum_{i=1}^{n} \mathbb{E}|X_{n,i}|^p < \infty, \tag{3.22}$$

$$\lim_{n \to \infty} \frac{1}{1-\rho_{n,1}} \sum_{i=1}^{n} \mathbb{P}\left(|X_{n,i}| > \varepsilon b_n\right) = 0 \text{ for all } \varepsilon > 0, \tag{3.23}$$

and

$$\lim_{n \to \infty} \frac{1}{1-\rho_{n,1}} \int_1^\infty \sum_{i=1}^{n} \mathbb{P}\left(|X_{n,i}| > b_n u^{1/p}\right) \mathrm{d}u = 0, \tag{3.24}$$

then we obtain

$$\frac{1}{b_n} \sum_{i=1}^{n} \left(X_{n,i} - \mathbb{E}X_{n,i}\right) \xrightarrow{\mathcal{L}_p} 0 \ \text{ as } n \to \infty.$$

Proof By Proposition 13 of Peligrad (2012), we have

$$\begin{aligned} \mathrm{Var}\left(\sum_{i=1}^{n} X_{n,i}\right)^2 &\le \frac{1+\rho_{n,1}}{1-\rho_{n,1}} \sum_{i=1}^{n} \mathbb{E}X_{n,i}^2 \\ &\le \frac{2}{1-\rho_{n,1}} \sum_{i=1}^{n} \mathbb{E}X_{n,i}^2. \end{aligned}$$

Applying Theorem 2.2 with $u_n \equiv 1$, $v_n \equiv n$, and $M_n \equiv \frac{2}{1-\rho_{n,1}}$, we obtain the conclusion of the corollary. □

Remark 3.15 The central limit theorem for $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ was established by Peligrad (2012). The main theme in Peligrad (2012) is central limit theorems for non-homogeneous Markov chains with the maximal coefficients of correlation, $\rho_{n,1}$, are allowed to converge to 1. For example, for the uniformly bounded case $\sup_{1\le i\le n, n\ge 1} |X_{n,i}| \le C_1 < \infty$ a.s., with the variance of individual summands satisfying $\mathrm{Var}(X_{n,i}) > c > 0$, a sufficient condition for the central limit theorem [see Equation (7) and Corollary 3 in Peligrad (2012)] is

$$\frac{(1-\rho_{n,1})n^{1/3}}{\log^{2/3} n} \to \infty. \tag{3.25}$$

For mean convergence, Corollary 3.14 also allows $\rho_{n,1}$ approaching 1 whereas it imposes very weak conditions on the moment. For example, if we only assume $\{|X_{n,i}|^p, 1 \le i \le n, n \ge 1\}$ is uniformly integrable with $1 < p < 2$, then by letting $b_n \equiv n$, we obtain by Corollary 3.14 that

$$\frac{1}{n}\sum_{i=1}^{n}\left(X_{n,i} - \mathbb{E}X_{n,i}\right) \stackrel{\mathcal{L}_p}{\to} 0 \text{ as } n \to \infty$$

provided

$$(1-\rho_{n,1})n^{p-1} \to \infty. \tag{3.26}$$

It is clear that if $4/3 \le p < 2$, (3.26) allows $\rho_{n,1}$ approaching 1 faster than that in (3.25). Finally, it is worth noting that if $\sup_{n\ge 1}\rho_{n,1} < 1$ and $\{|X_{n,i}|^p, 1 \le i \le n, n \ge 1\}$ is uniformly integrable, then three conditions (3.22), (3.23) and (3.24) are all fulfilled with $b_n \equiv n^{1/p}$, and thus Corollary 3.14 can be applied. □

Remark 3.16 Bradley (2011) (see Theorem 1 in Bradley (2011) and the paragraph before that theorem) showed that for any sequence $\{a_n, n \ge 1\} \subset (0, 1)$, there exists an array $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ of row-wise stationary Markov chains such that $\rho_{n,1} = a_n$ and each $X_{n,i}$ is uniformly distributed on a finite set. Using this result of Bradley, for example, with

$$\rho_{n,1} = a_n = 1 - \left(\frac{\log n}{2n}\right)^{p-1}, \ 1 < p < 2, \ n \ge 1,$$

then (3.26) is fulfilled. □

4 Mean convergence result for dependent bootstrap

In this section, we will apply Corollary 3.1 to obtain mean convergence for dependent bootstrap.

The notion of the dependent bootstrap procedure was introduced by Smith and Taylor (2001) for a sequence of independent identically distributed (i.i.d.) random variables. However, the dependent bootstrap procedure can be defined for an arbitrary sequence of random variables as remarked by Ahmed et al. (2005). Let $\{X_n, n \geq 1\}$ be a sequence of arbitrary random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{m_n, n \geq 1\}$ and $\{k_n, n \geq 1\}$ be two sequences of positive integers such that $m_n \leq nk_n$ for all $n \geq 1$. For $\omega \in \Omega$ and $n \geq 1$, the dependent bootstrap is defined to be the sample of size m_n, denoted by $\{\hat{X}_{n,j}^{(\omega)}, 1 \leq j \leq m_n\}$, drawn without replacement from the collection of nk_n items made up of k_n copies each of the sample observations $X_1(\omega), \ldots, X_n(\omega)$.

Smith and Taylor (2001) proposed the dependent bootstrap as a procedure to reduce variation of estimators and obtain better confidence intervals than those obtained using the classical Efron resampling (with replacement) bootstrap. Ahmed et al. (2005) pointed out that if we take $k_n = \infty$ for all $n \geq 1$, then the dependent bootstrap reduces to the classical Efron independent bootstrap. Therefore, we may consider the dependent bootstrap procedure as a more general procedure than the classical Efron independent bootstrap. The result of this section, Theorem 4.4, does not require any assumptions on k_n. Therefore, it is also true for the classical Efron independent bootstrap.

From the above definition, for each of the m_n selections, each $X_i(\omega)$ has probability $1/n$ of being chosen. Hence, for each $\omega \in \Omega$ and $n \geq 1$, $\hat{X}_{n,j}^{(\omega)}, 1 \leq j \leq m_n$ are identically distributed random variables with distribution

$$\hat{\mathbb{P}}\left(\hat{X}_{n,1}^{(\omega)} = X_i(\omega)\right) = \frac{1}{n}, \ 1 \leq i \leq n, \tag{4.1}$$

where $\hat{\mathbb{P}}$ is the conditional probability measure given by $\{X_j, 1 \leq j \leq n\}$ carrying for each $n \geq 1$, the uniform distribution on $\{X_1(\omega), \ldots, X_n(\omega)\}$ of each resampled $\{\hat{X}_{n,j}^{(\omega)}, 1 \leq j \leq m_n\}$. We refer to Smith and Taylor (2001) for more details.

Let $\hat{\mathbb{E}}$ denote the expectation with respect to $\hat{\mathbb{P}}$. In the sequel, we will need the following lemmas. The first lemma follows immediately from (4.1).

Lemma 4.1 *For each $\omega \in \Omega$ and $n \geq 1$, we have for any positive integer k and for any nonnegative function g defined on $[0, \infty)$ that*

$$\hat{\mathbb{E}}\left(\hat{X}_{n,1}^{(\omega)}\right)^k = \frac{1}{n}\sum_{j=1}^{n}(X_j(\omega))^k, \tag{4.2}$$

and

$$\hat{\mathbb{E}}g\left(\left|\hat{X}_{n,1}^{(\omega)}\right|\right) = \frac{1}{n}\sum_{j=1}^{n} g\left(|X_j(\omega)|\right). \tag{4.3}$$

Remark 4.2 By applying (4.2) with $k = 1$ and $k = 2$, we have

$$\hat{\mathbb{E}}\hat{X}_{n,1}^{(\omega)} = \overline{X}_n(\omega) \tag{4.4}$$

and

$$\hat{\mathrm{Var}}\hat{X}_{n,1}^{(\omega)} = S_n^2(\omega), \tag{4.5}$$

where $\hat{\mathrm{Var}}$ denote the variance with respect to $\hat{\mathbb{P}}$, $\overline{X}_n = n^{-1}\sum_{j=1}^n X_j$ is the sample mean and $S_n^2 = n^{-1}\sum_{j=1}^n (X_j - \overline{X}_n)^2$ is the biased version of the sample variance. Both (4.4) and (4.5) were established by Smith and Taylor (2001) for the case where the random variables $X_n, n \geq 1$ are i.i.d. □

The next lemma was established by Ahmed et al. (2005).

Lemma 4.3 *For each $\omega \in \Omega$ and $n \geq 1$, the dependent bootstrap random variables $\{\hat{X}_{n,j}^{(\omega)}, 1 \leq j \leq m_n\}$ are negatively dependent and exchangeable.*

We recall that a sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if

$$\sup_{n\geq 1} \mathbb{P}(|X_n| > t) \leq \mathbb{P}(|X| > t),\ t > 0.$$

Stochastic domination and uniform integrability have interesting relationships. We refer to (Rosalsky and Thành 2021; Thành 2022) for recent developments on this topic.

The strong and weak laws of large numbers and complete convergence for dependent bootstrap were studied by some authors [see, e.g., (Ahmed et al. 2005; Smith and Taylor 2001; Volodin et al. 2006) and the references therein]. In the following theorem, we use Corollary 3.1 to establish a mean convergence theorem for the sums $\sum_{j=1}^{m_n} \hat{X}_{n,j}^{(\omega)}$ of the dependent bootstrap samples $\{\hat{X}_{n,j}^{(\omega)}, 1 \leq j \leq m_n, n \geq 1\}$ for the case where $\{X_n, n \geq 1\}$ is comprised of pairwise independent random variables and stochastically dominated by a random variable X.

Theorem 4.4 *Let $\{X_n, n \geq 1\}$ be a sequence of pairwise independent random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{m_n, n \geq 1\}$ and $\{k_n, n \geq 1\}$ be two sequences of positive integers such that $m_n \leq nk_n$ for all $n \geq 1$. For $\omega \in \Omega$ and $n \geq 1$, let $\{\hat{X}_{n,j}^{(\omega)}, 1 \leq j \leq m_n\}$ be the corresponding sequence of dependent bootstrap samples. Let $1 \leq p < 2$ and let $\{b_n, n \geq 1\}$ be a sequence of positive constants such that $\lim_{n\to\infty} b_n = \infty$ and $m_n \leq Cb_n^p$ for all $n \geq 1$. If the sequence $\{X_n, n \geq 1\}$ is stochastically dominated by a random variable X satisfying*

$$\mathbb{E}|X|^p < \infty, \tag{4.6}$$

then for almost every $\omega \in \Omega$, we have

$$\frac{1}{b_n}\sum_{j=1}^{m_n}\left(\hat{X}_{n,j}^{(\omega)} - \overline{X}_n(\omega)\right) \overset{\mathcal{L}_p}{\to} 0 \text{ as } n \to \infty, \tag{4.7}$$

where $\overline{X}_n = n^{-1}\sum_{j=1}^{n} X_j$, $n \geq 1$.

Proof By (4.6) and the de La Vallée Poussin criterion for uniform integrability, there exists a nondecreasing function g defined on $[0,\infty)$ with $g(0) = 0$ such that $\lim_{x\to\infty} g(x)/x^p = \infty$ and

$$\mathbb{E}(g(|X|^p)) < \infty. \tag{4.8}$$

Since the sequence $\{X_n, n \geq 1\}$ is stochastically dominated by a random variable X satisfying (4.8), we can apply the Kolmogorov strong law of large numbers for sequences of pairwise independent random variables [see, e.g., Théorèm 1 in Rio (1995) or Theorem 2.1 in Dzung and Thành (2021)] to obtain

$$\frac{1}{n}\sum_{j=1}^{n}\left(g(|X_j|^p) - \mathbb{E}g(|X_j|^p)\right) \to 0 \text{ a.s. as } n \to \infty. \tag{4.9}$$

By applying the stochastic domination and (4.8) again, we have

$$\sup_{n\geq 1} \mathbb{E}g(|X_n|^p) \leq \mathbb{E}g(|X|^p) < \infty. \tag{4.10}$$

Combining (4.3), (4.9), and (4.10) yields for almost $\omega \in \Omega$,

$$\sup_{n\geq 1} \hat{\mathbb{E}}g\left(\left|\hat{X}_{n,1}^{(\omega)}\right|^p\right) = \sup_{n\geq 1}\frac{1}{n}\sum_{j=1}^{n} g\left(|X_j(\omega)|^p\right) < \infty. \tag{4.11}$$

Let $u_n \equiv 1$ and $v_n \equiv m_n$. Since for every $n \geq 1$ and $\omega \in \Omega$, the random variables $\hat{X}_{n,j}^{(\omega)}$, $1 \leq j \leq m_n$ are identically distributed and $m_n \leq C b_n^p$, we claim that (4.11) implies that for almost every $\omega \in \Omega$, the array of random variables $\{\hat{X}_{n,j}^{(\omega)}, 1 \leq j \leq m_n, n \geq 1\}$ satisfies all three conditions (3.1), (3.2) and (3.3). To see this, let $\varepsilon > 0$ be arbitrary. By using (4.11) and the de La Vallée Poussin criterion for uniform integrability, we get that $\{|\hat{X}_{n,1}^{(\omega)}|^p, n \geq 1\}$ is uniformly integrable for almost every $\omega \in \Omega$. Therefore, we have for almost every $\omega \in \Omega$,

$$\sup_{n\geq 1}\frac{1}{b_n^p}\sum_{j=1}^{m_n}\hat{\mathbb{E}}|\hat{X}_{n,j}^{(\omega)}|^p = \sup_{n\geq 1}\frac{1}{b_n^p}m_n\hat{\mathbb{E}}|\hat{X}_{n,1}^{(\omega)}|^p \leq C\sup_{n\geq 1}\hat{\mathbb{E}}|\hat{X}_{n,1}^{(\omega)}|^p < \infty$$

establishing (3.1). Secondly, for almost every $\omega \in \Omega$, by using Markov's inequality and the uniform integrability of $\{|\hat{X}_{n,1}^{(\omega)}|^p, n \geq 1\}$ again, we have

$$\begin{aligned}
\lim_{n\to\infty} \sum_{j=1}^{m_n} \hat{\mathbb{P}}\left(|\hat{X}_{n,j}^{(\omega)}| > \varepsilon b_n\right) &= \lim_{n\to\infty} m_n \hat{\mathbb{P}}\left(|\hat{X}_{n,1}^{(\omega)}| > \varepsilon b_n\right) \\
&\leq \lim_{n\to\infty} m_n \frac{\hat{\mathbb{E}}\left(|\hat{X}_{n,1}^{(\omega)}|^p \mathbf{1}\left(|\hat{X}_{n,1}^{(\omega)}| > \varepsilon b_n\right)\right)}{\varepsilon^p b_n^p} \\
&\leq \frac{C}{\varepsilon^p} \lim_{n\to\infty} \hat{\mathbb{E}}\left(|\hat{X}_{n,1}^{(\omega)}|^p \mathbf{1}\left(|\hat{X}_{n,1}^{(\omega)}| > \varepsilon b_n\right)\right) \\
&\leq \frac{C}{\varepsilon^p} \lim_{n\to\infty} \left(\sup_{m\geq 1} \hat{\mathbb{E}}\left(|\hat{X}_{m,1}^{(\omega)}|^p \mathbf{1}\left(|\hat{X}_{m,1}^{(\omega)}| > \varepsilon b_n\right)\right)\right) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} \int_1^\infty \sum_{j=1}^{m_n} \hat{\mathbb{P}}\left(|\hat{X}_{n,j}^{(\omega)}| > b_n u^{1/p}\right) \mathrm{d}u &= \lim_{n\to\infty} m_n \int_1^\infty \hat{\mathbb{P}}\left(|\hat{X}_{n,j}^{(\omega)}| > b_n u^{1/p}\right) \mathrm{d}u \\
&\leq \lim_{n\to\infty} m_n \int_0^\infty \hat{\mathbb{P}}\left(|\hat{X}_{n,j}^{(\omega)}| > b_n u^{1/p}\right) \mathrm{d}u \\
&= \lim_{n\to\infty} m_n \frac{\hat{\mathbb{E}}\left(|\hat{X}_{n,1}^{(\omega)}|^p \mathbf{1}\left(|\hat{X}_{n,1}^{(\omega)}| > b_n\right)\right)}{b_n^p} \\
&\leq C \lim_{n\to\infty} \hat{\mathbb{E}}\left(|\hat{X}_{n,1}^{(\omega)}|^p \mathbf{1}\left(|\hat{X}_{n,1}^{(\omega)}| > \varepsilon b_n\right)\right) = 0
\end{aligned}$$

establishing (3.2) and (3.3), respectively.

On the other hand, by Lemma 4.3, for every $\omega \in \Omega$, each row of the array $\{\hat{X}_{n,j}^{(\omega)}, 1 \leq j \leq m_n, n \geq 1\}$ is comprised of negatively dependent random variables, so it satisfies Condition (H_2). Applying Corollary 3.1 with $u_n \equiv 1$ and $v_n \equiv m_n$ and recalling (4.4), we have for almost $\omega \in \Omega$

$$\frac{1}{b_n} \sum_{j=1}^{m_n} \left(\hat{X}_{n,j}^{(\omega)} - \overline{X}_n(\omega)\right) = \frac{1}{b_n} \sum_{j=1}^{m_n} \left(\hat{X}_{n,j}^{(\omega)} - \hat{\mathbb{E}}\hat{X}_{n,j}^{(\omega)}\right) \overset{\mathcal{L}_p}{\to} 0 \text{ as } n \to \infty$$

establishing (4.7). The proof of the theorem is completed. □

Remark 4.5 (i) Under the same stochastic domination assumption and (4.6), Ahmed et al. (2005), Corollary 1 obtained the weak law of large numbers

$$\frac{1}{n^{1/p}} \sum_{j=1}^{n} \left(\hat{X}_{n,j}^{(\omega)} - \overline{X}_n(\omega)\right) \overset{\mathbb{P}}{\to} 0 \text{ as } n \to \infty \text{ for almost } \omega \in \Omega. \tag{4.12}$$

It is clear that (4.7) is stronger and more general than (4.12). However, Ahmed et al. (2005) did not require any specific dependence structure of the sequence $\{X_n, n \geq 1\}$.

(ii) If the sequence $\{X_n, n \geq 1\}$ is comprised of pairwise negatively dependent (or extended negative dependent) random variables, then by applying Theorem 2.1 in Dzung and Thành (2021), we can show that the strong law of large numbers (4.9) still holds. Therefore, Theorem 4.4 still holds for the case where the sequence $\{X_n, n \geq 1\}$ is comprised of pairwise negatively dependent (or extended negative dependent) random variables.

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