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MAXIMAL INEQUALITIES FOR NORMED DOUBLE SUMS OF RANDOM ELEMENTS IN MARTINGALE TYPE p BANACH SPACES WITH APPLICATIONS TO DEGENERATE MEAN CONVERGENCE OF THE MAXIMUM OF NORMED SUMS

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Abstract. In this correspondence, we prove new maximal inequalities for normed double sums of random elements taking values in a real separable martingale type p Banach space. The result is then applied to establish mean convergence theorems for the maximum of normed and suitably centered double sums of Banach space-valued random elements.

1. **Introduction.** It is a great pleasure for us to contribute this paper to this special issue of *Numerical Algebra, Control and Optimization* in honor of Professor George Yin on the occasion of his 70$^{\text{th}}$ birthday.

Let $\{X_{m,n}, m \geq 1, n \geq 1\}$ be a double array of independent mean zero real-valued random variables, and let $S_{m,n} = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{i,j}$ denote the double sums. It was proved by Wichura [27] that

$$\mathbb{E}\left(\max_{1\leq k\leq m, 1\leq \ell\leq n} S_{k,\ell}^2\right) \leq 16\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{E}X_{i,j}^2,\ m\geq 1, n\geq 1. \tag{1}$$

Inequality (1) will be referred to as the Kolmogorov–Doob-type maximal inequality for double sums. This type of inequality plays a key role in proving many limit theorems.

In [18], Rosalsky and Thành established a Kolmogorov–Doob-type maximal inequality for normed double sums of independent random elements in a Rademacher type p Banach space. Dung et al. [5], Quang and Huan [17], and Son et al. [23] established a Kolmogorov–Doob-type maximal inequality for normed double sums of random elements taking values in a martingale type p Banach space. In this paper, we further generalize the Dung et al. [5], Quang and Huan [17], and Son et al. [23] results by considering the case where the moments are of higher order than p. We then use the obtained result to obtain a mean convergence theorem for the maximum of normed and suitably centered double sums of random elements

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taking values in a real separable martingale type p Banach space. The results in the current work are new even when the Banach space is the real line.

Throughout the rest of the paper, $\{k_m, m \geq 1\}$ and $\{\ell_n, n \geq 1\}$ are two sequences of positive integers satisfying $\lim_{m \vee n \to \infty} k_m \ell_n = \infty$ unless stated otherwise. All random elements are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and take values in a real separable Banach space $\mathcal{X}$ with norm $\|\cdot\|$. We consider an array $\{V_{m,n,i,j},\ 1 \leq i \leq k_m, 1 \leq j \leq \ell_n, m \geq 1, n \geq 1\}$ of $\mathcal{X}$-valued random elements, and prove that if $\mathcal{X}$ is of martingale type p, $1 \leq p \leq 2$, then for all $m \geq 1$ and $n \geq 1$, the inequality (3) holds (see Theorem 3.1 in Section 3). If $\mathbb{E}(V_{i,j}|\mathcal{F}_{i,j}) \equiv 0$ and $p = q = 2$, then inequality (3) reduces to the Kolmogorov–Doob-type maximal inequality for normed double sums of the form (1).

The maximal inequality (3) will be used to establish in Theorem 4.1 a mean convergence theorem for the maximum of normed double sums of the form

$$\frac{\max_{1 \leq k \leq k_m, 1 \leq \ell \leq \ell_n} \left\| \sum_{i=1}^{k_m} \sum_{j=1}^{\ell_n} (V_{m,n,i,j} - \mathbb{E}(V_{m,n,i,j}|\mathcal{F}_{m,n,i,j})) \right\|}{d_{m,n}}$$

where $\{d_{m,n}, m \geq 1, n \geq 1\}$ is a normalizing double array. It is worth noting that these normed double sums cover not only normed double sums from a double array but, also, normed single sums from a triangular array or from a sequence (see Rosalsky et al. [20]).

Mean convergence and laws of large numbers for sums of Banach space-valued random elements have enjoyed a wide literature of investigation (see, e.g., Adler et al. [1], Chen et al. [2], Chen and Wang [3], Hoffman-Jørgensen and Pisier [6], Hu et al. [7], Korzeniowski [10], Li et al. [11], Ordóñez Cabrera [12], Ordóñez Cabrera et al. [13], Parker and Rosalsky [14], Rosalsky and Thành [18], Rosalsky et al. [20], Son et al. [23], Thành and Yin [24], Wang and Rao [26], and the references therein). However, only a few of these investigations establish mean convergence for the *maximum* of normed sums which is of special interest. For example, Rosalsky et al. [20] established mean convergence for the maximum of normed sums assuming that the underlying Banach space is of Rademacher type p ($1 \leq p \leq 2$). Recently, Rosalsky and Thành [19] obtained mean convergence results for the maximum of normed double sums under compact uniform integrability conditions.

The plan of the paper is as follows. Technical definitions, notation, and lemmas which are used in establishing the main results are consolidated into Section 2. In Section 3, we establish maximal inequalities for normed double sums of random elements in a real separable martingale type p ($1 \leq p \leq 2$) Banach space. The mean convergence results for the maximum of normed sums are presented and proved in Section 4.

2. **Preliminaries.**

In this section, notation, technical definitions, and lemmas which are used in establishing the main results will be presented.

The *expected value* or *mean* of an $\mathcal{X}$-valued random element V, denoted $\mathbb{E}V$, is defined to be the *Pettis integral* provided it exists. That is, V has expected value $\mathbb{E}V \in \mathcal{X}$ if $f(\mathbb{E}V) = \mathbb{E}f(V)$ for every $f \in \mathcal{X}^*$ where $\mathcal{X}^*$ denotes the (dual) space of all continuous linear functionals on $\mathcal{X}$.

Throughout this paper, the symbol C will denote a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance. For $a, b \in \mathbb{R}$, $\min\{a, b\}$ and $\max\{a, b\}$ will be denoted, respectively, by $a \wedge b$ and $a \vee b$.

For a random element V and a sub-σ-field $\mathcal{G}$ of $\mathcal{F}$, the *conditional expectation* $\mathbb{E}(V|\mathcal{G})$ was introduced by Scalora [21] and is defined in an analogous manner to that in the real-valued random variable case and enjoys similar properties. A complete development of the notion of conditional expectation for random elements may be found in Scalora [21] including Banach space-valued martingales (which will be defined below) and martingale convergence theorems.

A sequence $\{S_n, n \geq 1\}$ of $\mathcal{X}$-valued random elements is said to be a *martingale* with respect to a non-decreasing sequence of sub-σ-fields $\{\mathcal{F}_n, n \geq 1\}$ of $\mathcal{F}$ (a *filtration*) if $\mathbb{E}S_n$ exists for all $n \geq 1$, S_n is $\mathcal{F}_n$-measurable for all $n \geq 1$, and

$$\mathbb{E}(S_{n+1}|\mathcal{F}_n) = S_n \text{ almost surely (a.s.)}, \ n \geq 1.$$

In this case, the sequence $\{X_n, \mathcal{F}_n, n \geq 1\}$ where $X_n = S_n - S_{n-1}$, $n \geq 1$, $S_0 = 0$ is said to be a *martingale difference sequence* and $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = 0$ a.s., $n \geq 1$ follows immediately.

A real separable Banach space $\mathcal{X}$ is said to be of *martingale type* p $(1 \leq p \leq 2)$ if there exists a constant $0 < C < \infty$ such that for all martingales $\{S_n, \mathcal{F}_n, n \geq 1\}$ with values in $\mathcal{X}$,

$$\sup_{n\geq 1} \mathbb{E}\,\|S_n\|^p \leq C \sum_{n=1}^{\infty} \mathbb{E}\|S_n - S_{n-1}\|^p$$

where $S_0 \equiv 0$. It can be shown that (see Pisier [16]) that $\mathcal{X}$ being of martingale type p is indeed equivalent to the apparently stronger condition that for all $q \geq 1$, there exists a constant $C_{p,q} < \infty$ depending only on p and q such that for all martingales $\{S_n, \mathcal{F}_n, n \geq 1\}$ with values in $\mathcal{X}$,

$$\mathbb{E}\left(\sup_{n\geq 1} \|S_n\|^q\right) \leq C_{p,q}\mathbb{E}\left(\sum_{n=1}^{\infty} \mathbb{E}\|S_n - S_{n-1}\|^p\right)^{q/p}, \ (S_0 \equiv 0). \tag{2}$$

It readily follows from (2) that if $\mathcal{X}$ is of martingale type p for some $p \in (1, 2]$, then it is of martingale type q for all $q \in [1, p]$.

Every real separable Banach space is of martingale type (at least) 1. For $1 \leq p < \infty$, the L_p-space and l_p-space are of martingale type $p \wedge 2$. We refer the reader to Pisier [15], Pisier [16], Schwartz [22], Woyczyński [28], Woyczyński [29] for detailed discussions of martingale type p Banach spaces including many interesting examples.

It follows from the Hoffmann-Jørgensen and Pisier [6] characterization of Rademacher type p Banach spaces that if a Banach space is of martingale type p, then it is of Rademacher type p. But the notion of martingale type p Banach spaces is only superficially similar to that of Rademacher type p Banach spaces. Indeed, a Banach space can be of Rademacher type 2 (and hence be of Rademacher type p for all $p \in [1, 2]$) yet be of martingale type p only for $p = 1$; for details see Pisier [15] and James [9].

The following lemma is an immediate consequence of (2). A similar result appears in Hu et al. [8] where the authors considered the case $p = q$.

Lemma 2.1. *Suppose that the real separable Banach space $\mathcal{X}$ is of martingale type p $(1 \leq p \leq 2)$. Then for $q \geq 1$, there exists a constant $C_{p,q}$ depending only on p and q such that for all martingales $\{S_n, \mathcal{F}_n, n \geq 1\}$ with values in $\mathcal{X}$,*

$$\mathbb{E}\left(\max_{1\leq j\leq n} \|S_j\|^q\right) \leq C_{p,q}\mathbb{E}\left(\sum_{i=1}^{n} \|S_i - S_{i-1}\|^p\right)^{q/p}, \ n \geq 1$$

where $S_0 \equiv 0$.

The next lemma is a well-known result and it is referred to as the c_p-inequality.

Lemma 2.2. *Let $a_1, \ldots, a_n$ be real numbers, and let $p > 0$. Then*

$$|a_1 + \cdots + a_n|^p \leq \max\{1, n^{p-1}\} \left(|a_1|^p + \cdots + |a_n|^p\right).$$

3. **Maximal inequalities for normed double sums of random elements in martingale type p Banach spaces.** The following theorem is a maximal inequality for normed double sums of random elements in martingale type p Banach spaces. The proof is based on ideas from Lemma 2.3 of Rosalsky and Thành [18] which establishes a maximal inequality for normed double sums of independent mean zero random elements in Rademacher type p Banach spaces. In the papers [18, 23], the authors considered the p-th moment of the maximum partial sums. In some cases, it may be necessary to bound moments of order higher than p for either the partial sums or the maximum of the partial sums. For example, when proving the Hsu–Robbins–Erdös theorem, we need to bound the 4-th moment of the partial sums (see, e.g., [25]). Theorem 3.1 considers the case where the moments are of order q for $q \geq p$.

Theorem 3.1. *Let $1 \leq p \leq 2$, $q \geq p$ and let $\{V_{i,j}, i \geq 1, j \geq 1\}$ be a double array of random elements taking values in a real separable martingale type p Banach space $\mathcal{X}$ such that $\mathbb{E}\|V_{i,j}\|^q < \infty$ for all $i \geq 1, j \geq 1$. For $k \geq 1, \ell \geq 1$, define the σ-fields $\mathcal{F}_{k,\ell}$ as follows:*

$$\mathcal{F}_{k,\ell} = \begin{cases} \{\emptyset, \Omega\} & \text{if } k = \ell = 1, \\ \sigma\left(\{V_{i,j} : i \geq 1, 1 \leq j < \ell\}\right) & \text{if } k = 1,\ \ell \geq 2, \\ \sigma\left(\{V_{i,j} : 1 \leq i < k, j \geq 1\}\right) & \text{if } k \geq 2,\ \ell = 1, \\ \sigma\left(\{V_{i,j} : 1 \leq i < k \text{ or } 1 \leq j < \ell\}\right) & \text{if } k \wedge \ell \geq 2. \end{cases}$$

Then for all $m \geq 1, n \geq 1$,

$$\begin{aligned} &\mathbb{E}\left(\max_{\substack{1\leq k\leq m\\1\leq \ell\leq n}} \left\|\sum_{i=1}^{k}\sum_{j=1}^{\ell}(V_{i,j} - \mathbb{E}(V_{i,j}|\mathcal{F}_{i,j}))\right\|^q\right) \\ &\qquad \leq C_{p,q}(mn)^{q/p-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{E}\|V_{i,j} - \mathbb{E}(V_{i,j}|\mathcal{F}_{i,j})\|^q, \end{aligned} \tag{3}$$

where $C_{p,q}$ is a constant depending only on p and q.

Proof. Let $m \geq 1$ and $n \geq 1$ be fixed. If $q = p = 1$, then

$$\begin{aligned} \mathbb{E}\left(\max_{\substack{1\leq k\leq m\\1\leq \ell\leq n}} \left\|\sum_{i=1}^{k}\sum_{j=1}^{\ell}(V_{i,j} - \mathbb{E}(V_{i,j}|\mathcal{F}_{i,j}))\right\|\right) &\leq \mathbb{E}\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\|V_{i,j} - \mathbb{E}(V_{i,j}|\mathcal{F}_{i,j})\|\right)\right) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{E}\|V_{i,j} - \mathbb{E}(V_{i,j}|\mathcal{F}_{i,j})\|, \end{aligned}$$

thereby establishing (3).

It remains to consider the case where $q > 1$. Suppose now that $m \wedge n \geq 2$. By following the proof of Lemma 2.3 of Rosalsky and Thành [18], we set

$$S_{k,\ell} = \sum_{i=1}^{k}\sum_{j=1}^{\ell}(V_{i,j} - \mathbb{E}(V_{i,j}|\mathcal{F}_{i,j})),\ 1 \leq k \leq m, 1 \leq \ell \leq n,$$

$$\mathcal{G}_k = \sigma\left(\{V_{i,j}, 1 \le i \le k, 1 \le j \le n\}\right), \ 1 \le k \le m,$$

and

$$\mathcal{F}_\ell = \sigma\left(\{V_{i,j}, 1 \le i \le m, 1 \le j \le \ell\}\right), \ 1 \le \ell \le n.$$

Then $\sigma\left(\{S_{k,\ell-1}\}\right) \subset \mathcal{F}_{\ell-1} \subset \mathcal{F}_{k,\ell}$ for all $\ell \ge 2$ and $k \ge 1$. For each $1 \le k \le m$ and $2 \le \ell \le n$, we have

$$\begin{aligned}\mathbb{E}\left(S_{k,\ell}|\mathcal{F}_{\ell-1}\right) &= \mathbb{E}\left(S_{k,\ell-1} + \sum_{i=1}^{k}(V_{i,\ell} - \mathbb{E}(V_{i,\ell}|\mathcal{F}_{i,\ell}))\big|\mathcal{F}_{\ell-1}\right) \\ &= \mathbb{E}\left(S_{k,\ell-1}\big|\mathcal{F}_{\ell-1}\right) \\ &= S_{k,\ell-1} \text{ a.s.}\end{aligned}$$

and so $\{S_{k,\ell}, \mathcal{F}_\ell, 1 \le \ell \le n\}$ is a martingale for each $1 \le k \le m$. We also have $\sigma\left(\{S_{k-1,n}\}\right) \subset \mathcal{G}_{k-1} \subset \mathcal{F}_{k,j}$ for all $k \ge 2$ and $j \ge 1$. For each $2 \le k \le m$, we thus have

$$\begin{aligned}\mathbb{E}\left(S_{k,n}|\mathcal{G}_{k-1}\right) &= \mathbb{E}\left(S_{k-1,n} + \sum_{j=1}^{n}(V_{k,j} - \mathbb{E}(V_{k,j}|\mathcal{F}_{k,j}))\big|\mathcal{G}_{k-1}\right) \\ &= \mathbb{E}\left(S_{k-1,n}\big|\mathcal{G}_{k-1}\right) \\ &= S_{k-1,n} \text{ a.s.}\end{aligned}$$

and so $\{S_{k,n}, \mathcal{G}_k, 1 \le k \le m\}$ is a martingale. By proceeding in the same manner as in (2.6) and (2.7) of Rosalsky and Thành [18], we obtain

$$\mathbb{E}\left(\max_{\substack{1\le k\le m\\ 1\le \ell\le n}} \|S_{k,\ell}\|^q\right) \le \left(\frac{q}{q-1}\right)^{2q} \mathbb{E}\|S_{m,n}\|^q. \tag{4}$$

Since $\{S_{k,n}, \mathcal{G}_k, 1 \le k \le m\}$ is a martingale, it follows from Lemmas 2.1 and 2.2 that

$$\begin{aligned}\mathbb{E}\|S_{m,n}\|^q &\le C_{p,q}\mathbb{E}\left(\sum_{i=1}^{m}\left\|\sum_{j=1}^{n}(V_{i,j} - \mathbb{E}(V_{i,j}|\mathcal{F}_{i,j}))\right\|^p\right)^{q/p} \\ &\le C_{p,q} m^{q/p-1}\sum_{i=1}^{m}\mathbb{E}\left(\left\|\sum_{j=1}^{n}(V_{i,j} - \mathbb{E}(V_{i,j}|\mathcal{F}_{i,j}))\right\|^q\right).\end{aligned} \tag{5}$$

Here and hereafter, $C_{p,q}$ is a constant depending only on p and q and is not necessarily the same one in each appearance. For $1 \le i \le m, 1 \le \ell \le n$, set

$$T_{i,\ell} = \sum_{j=1}^{\ell}(V_{i,j} - \mathbb{E}(V_{i,j}|\mathcal{F}_{i,j})).$$

For each $1 \le i \le m$ and $2 \le \ell \le n$, by noting that $\sigma(\{T_{i,\ell-1}\}) \subset \mathcal{F}_{\ell-1} \subset \mathcal{F}_{i,\ell}$, we have

$$\begin{aligned}\mathbb{E}\left(T_{i,\ell}|\mathcal{F}_{\ell-1}\right) &= \mathbb{E}\left(T_{i,\ell-1} + V_{i,\ell} - \mathbb{E}(V_{i,\ell}|\mathcal{F}_{i,\ell})\big|\mathcal{F}_{\ell-1}\right) \\ &= \mathbb{E}\left(T_{i,\ell-1}\big|\mathcal{F}_{\ell-1}\right) \\ &= T_{i,\ell-1} \text{ a.s.}\end{aligned}$$

and so $\{T_{i,\ell}, \mathcal{F}_\ell, 1 \leq \ell \leq n\}$ is a martingale for each $1 \leq i \leq m$. Therefore, by using Lemmas 2.1 and 2.2 again, we have for all $1 \leq i \leq m$,

$$\begin{aligned}\mathbb{E}\left(\left\|\sum_{j=1}^{n}(V_{i,j} - \mathbb{E}(V_{i,j}|\mathcal{F}_{i,j}))\right\|^q\right) &= \mathbb{E}\|T_{i,n}\|^q \\ &\leq C_{p,q}\mathbb{E}\left(\sum_{j=1}^{n}\|V_{i,j} - \mathbb{E}(V_{i,j}|\mathcal{F}_{i,j})\|^p\right)^{q/p} \qquad (6) \\ &\leq C_{p,q}n^{q/p-1}\sum_{j=1}^{n}\mathbb{E}\|V_{i,j} - \mathbb{E}(V_{i,j}|\mathcal{F}_{i,j})\|^q.\end{aligned}$$

Combining (4)–(6) yields

$$\mathbb{E}\left(\max_{\substack{1\leq k\leq m\\1\leq \ell\leq n}}\|S_{k,\ell}\|^q\right) \leq C_{p,q}(mn)^{q/p-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{E}\left\|V_{i,j} - \mathbb{E}(V_{i,j}|\mathcal{F}_{i,j})\right\|^q$$

again establishing (3).

Finally, if $m \wedge n = 1$, then (3) follows as in the $m \wedge n \geq 2$ case, *mutatis mutandis*. □

Remark 3.2. As was mentioned in Section 1, the case $\mathbb{E}(V_{i,j}|\mathcal{F}_{i,j}) \equiv 0$ is of independent interest. In this case, inequality (3) with $q = p = 2$ reduces to the Kolmogorov–Doob-type maximal inequality for normed double sums of the form (1). It is clear that if $\{V_{i,j}, i \geq 1, j \geq 1\}$ is a double array of independent mean zero random elements, then this condition is satisfied. Another example, which is inspired by a remark in Choi and Klass [4, p. 811], is as follows. Let $\{X_{i,j}, i \geq 1, j \geq 1\}$ be a double array of integrable real-valued random variables. Let $\{Y_{i,j}, i \geq 1, j \geq 1\}$ be a double array of independent mean zero real-valued random variables such that $\{Y_{i,j}, i \geq 1, j \geq 1\}$ is independent of $\{X_{i,j}, i \geq 1, j \geq 1\}$. Let $V_{i,j} = X_{i,j}Y_{i,j}$, $i \geq 1, j \geq 1$. Then

$$\begin{aligned}\mathbb{E}(V_{i,j}|\mathcal{F}_{i,j}) &= \mathbb{E}\left(\mathbb{E}(X_{i,j}Y_{i,j}|\mathcal{F}_{i,j}, X_{i,j})|\mathcal{F}_{i,j}\right) \\ &= \mathbb{E}\left(X_{i,j}\mathbb{E}(Y_{i,j}|\mathcal{F}_{i,j}, X_{i,j})|\mathcal{F}_{i,j}\right) \\ &= \mathbb{E}\left(X_{i,j}\mathbb{E}Y_{i,j}|\mathcal{F}_{i,j}\right) = 0.\end{aligned}$$

The next result is an immediate consequence of Theorem 3.1. It plays an important role in proving Theorem 4.1. For the special case $p = q$, Dung et al. [5], Quang and Huan [17], and Son et al. [23] obtained a similar result.

Theorem 3.3. *Let $1 \leq p \leq 2$, $q \geq p$ and let $\{V_{i,j}, 1 \leq i \leq m, 1 \leq j \leq n\}$ be a collection of random elements taking values in a real separable martingale type p Banach space $\mathcal{X}$ such that $\mathbb{E}\|V_{i,j}\|^q < \infty$ for all $1 \leq i \leq m, 1 \leq j \leq n$. For $1 \leq k \leq m, 1 \leq \ell \leq n$, define the σ-fields $\mathcal{F}_{k,\ell}$ as follows:*

$$\mathcal{F}_{k,\ell} = \begin{cases}\{\emptyset, \Omega\} & \text{if } k = \ell = 1, \\ \sigma\left(\{V_{i,j} : 1 \leq i \leq m, 1 \leq j < \ell\}\right) & \text{if } k = 1,\ n \geq \ell \geq 2, \\ \sigma\left(\{V_{i,j} : 1 \leq i < k, 1 \leq j \leq n\}\right) & \text{if } m \geq k \geq 2,\ \ell = 1, \\ \sigma\left(\{V_{i,j} : 1 \leq i < k \text{ or } 1 \leq j < \ell\}\right) & \text{if } m \wedge n \geq k \wedge \ell \geq 2.\end{cases}$$

Then

$$\mathbb{E}\left(\max_{\substack{1\le k\le m\\1\le \ell\le n}}\left\|\sum_{i=1}^{k}\sum_{j=1}^{\ell}(V_{i,j}-\mathbb{E}(V_{i,j}|\mathcal{F}_{i,j}))\right\|^q\right)$$
$$\le C_{p,q}(mn)^{q/p-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{E}\|V_{i,j}-\mathbb{E}(V_{i,j}|\mathcal{F}_{i,j})\|^q,$$

where $C_{p,q}$ is a constant depending only on p and q.

Proof. By introducing additional terms $V_{i,j}=0$ for $i\ge m+1$ or $j\ge n+1$, the proof of Theorem 3.3 follows from Theorem 3.1. □

4. **Applications to mean convergence theorems.** In this section, we will establish a very general mean convergence theorem for the maximum of normed and suitably centered double sums of random elements in a real separable martingale type p ($1\le p\le 2$) Banach space $\mathcal{X}$. Its proof follows directly from Theorem 3.3, and it covers results concerning double sums from double arrays as will be seen in Corollary 4.4. Note that there are no independence or mean zero conditions imposed on the random elements comprising the arrays.

Theorem 4.1. *Let $1\le p\le 2$, $q\ge p$ and let $\{V_{m,n,i,j},\ 1\le i\le k_m, 1\le j\le \ell_n, m\ge 1, n\ge 1\}$ be an array of random elements in a real separable martingale type p Banach space $\mathcal{X}$ such that $\mathbb{E}\|V_{m,n,i,j}\|^q<\infty$ for all $1\le i\le k_m, 1\le j\le \ell_n, m\ge 1, n\ge 1$. Let $\{d_{m,n}, m\ge 1, n\ge 1\}$ be an array of positive constants. For $m\ge 1, n\ge 1, 1\le k\le m, 1\le \ell\le n$, let*

$$\mathcal{F}_{m,n,k,\ell}=\begin{cases}\{\emptyset,\Omega\} & \text{if } k=\ell=1,\\ \sigma\left(\{V_{i,j}:1\le i\le m, 1\le j<\ell\}\right) & \text{if } k=1,\ n\ge\ell\ge 2,\\ \sigma\left(\{V_{i,j}:1\le i<k, 1\le j\le n\}\right) & \text{if } m\ge k\ge 2,\ \ell=1,\\ \sigma\left(\{V_{i,j}:1\le i<k \text{ or } 1\le j<\ell\}\right) & \text{if } m\wedge n\ge k\wedge\ell\ge 2,\end{cases}$$

and

$$S_{m,n,k,\ell}=\sum_{i=1}^{k}\sum_{j=1}^{\ell}\left(V_{m,n,i,j}-\mathbb{E}(V_{m,n,i,j}|\mathcal{F}_{m,n,i,j})\right).$$

If

$$\frac{(k_m\ell_n)^{q/p-1}\sum_{i=1}^{k_m}\sum_{j=1}^{\ell_n}\mathbb{E}\|V_{m,n,i,j}\|^q}{d_{m,n}^q}\to 0 \text{ as } m\vee n\to\infty, \tag{7}$$

then

$$\frac{\max_{1\le k\le k_m, 1\le\ell\le\ell_n}\|S_{m,n,k,\ell}\|}{d_{m,n}}\xrightarrow{\mathcal{L}_q}0 \text{ as } m\vee n\to\infty. \tag{8}$$

Proof. For $m\ge 1, n\ge 1$, we have

$$\begin{aligned}&\frac{\mathbb{E}\left(\max_{1\le k\le k_m,1\le\ell\le\ell_n}\|S_{m,n,k,\ell}\|^q\right)}{d_{m,n}^q}\\ &\le \frac{C_{p,q}(k_m\ell_n)^{q/p-1}\sum_{i=1}^{k_m}\sum_{j=1}^{\ell_n}\mathbb{E}\left\|V_{m,n,i,j}-\mathbb{E}(V_{m,n,i,j}|\mathcal{F}_{m,n,i,j})\right\|^q}{d_{m,n}^q} \qquad (9)\\ &\le \frac{C_{p,q}(k_m\ell_n)^{q/p-1}\sum_{i=1}^{k_m}\sum_{j=1}^{\ell_n}\left(\mathbb{E}\left\|V_{m,n,i,j}\right\|^q+\mathbb{E}\left\|\mathbb{E}(V_{m,n,i,j}|\mathcal{F}_{m,n,i,j})\right\|^q\right)}{d_{m,n}^q},\end{aligned}$$

where we have applied Theorem 3.3 in the first inequality and Lemma 2.2 with $n = 2$ in the second inequality. Applying Jensen's inequality for conditional expectations, we have for all $1 \le i \le k_m, 1 \le j \le \ell_n, m \ge 1, n \ge 1$ that

$$\mathbb{E} \left\| \mathbb{E}(V_{m,n,i,j} | \mathcal{F}_{m,n,i,j}) \right\|^q \le \mathbb{E} \left(\mathbb{E}(\|V_{m,n,i,j}\|^q | \mathcal{F}_{m,n,i,j}) \right) = \mathbb{E} \left\| V_{m,n,i,j} \right\|^q . \qquad (10)$$

Combining (7), (9) and (10) yields

$$\begin{aligned} &\frac{\mathbb{E}\left(\max_{1 \le k \le k_m, 1 \le \ell \le \ell_n} \|S_{m,n,k,\ell}\|^q\right)}{d_{m,n}^q} \\ &\le \frac{C_{p,q}(k_m \ell_n)^{q/p-1} \sum_{i=1}^{k_m} \sum_{j=1}^{\ell_n} \mathbb{E} \left\| V_{m,n,i,j} \right\|^q}{d_{m,n}^q} \to 0 \text{ as } m \vee n \to \infty \end{aligned}$$

establishing (8). □

Remark 4.2. It is clear from the proof of Theorem 4.1 that the result remains valid if the assumption (7) is replaced by

$$\frac{(k_m \ell_n)^{q/p-1} \sum_{i=1}^{k_m} \sum_{j=1}^{\ell_n} \mathbb{E}\|V_{m,n,i,j}\|^q}{d_{m,n}^q} \to 0 \text{ as } m \wedge n \to \infty$$

and the conclusion (8) is replaced by

$$\frac{\max_{1 \le k \le k_m, 1 \le \ell \le \ell_n} \|S_{m,n,k,\ell}\|}{d_{m,n}} \xrightarrow{\mathcal{L}_q} 0 \text{ as } m \wedge n \to \infty.$$

A similar remark pertains to Corollary 4.4.

Remark 4.3. Rosalsky et al. [20] studied mean convergence theorem for double arrays of M-dependent random variables. They also gave various interesting examples to illustrate their results. By Example 5.1 of Rosalsky et al. [20], we have that in Theorem 4.1, almost sure convergence does not necessarily hold in (8). By Example 5.3 of Rosalsky et al. [20], we have that Theorem 4.1 can fail if the hypothesis that $\mathcal{X}$ is of martingale type p is dispensed with. Finally, inspired by Example 1 of Adler et al. [1], we have that Theorem 4.1 can fail if the condition (7) is weakened to

$$\sum_{i=1}^{k_m} \sum_{j=1}^{\ell_n} \mathbb{E}\|V_{i,j}\|^p = O(d_{m,n}^p) \quad \text{as} \quad m \vee n \to \infty$$

and by Example 5.4 of Rosalsky et al. [20], we have that Theorem 4.1 can fail if the condition (7) is weakened to

$$\lim_{i \vee j \to \infty} \frac{\mathbb{E}\|V_{i,j}\|^p}{d_{i,j}^p} = 0.$$

The next corollary indicates that Theorem 4.1 covers the "double array" case.

Corollary 4.4. *Let $1 \le p \le 2$, $q \ge p$, and let $\{V_{i,j}, i \ge 1, j \ge 1\}$ be a double array of random elements in a real separable martingale type p Banach space $\mathcal{X}$ such that $\mathbb{E}\|V_{i,j}\|^q < \infty$ for all $i \ge 1, j \ge 1$. Let $\{d_{m,n}, m \ge 1, n \ge 1\}$ be an array of positive constants. For $m \ge 1, n \ge 1, 1 \le k \le m, 1 \le \ell \le n$, let*

$$\mathcal{F}_{m,n,k,\ell} = \begin{cases} \{\emptyset, \Omega\} & \text{if } k = \ell = 1, \\ \sigma\left(\{V_{i,j} : 1 \le i \le m, 1 \le j < \ell\}\right) & \text{if } k = 1, \ n \ge \ell \ge 2, \\ \sigma\left(\{V_{i,j} : 1 \le i < k, 1 \le j \le n\}\right) & \text{if } m \ge k \ge 2, \ \ell = 1, \\ \sigma\left(\{V_{i,j} : 1 \le i < k \text{ or } 1 \le j < \ell\}\right) & \text{if } m \wedge n \ge k \wedge \ell \ge 2, \end{cases}$$

and

$$S_{m,n,k,\ell}=\sum_{i=1}^{k}\sum_{j=1}^{\ell}\left(V_{i,j}-\mathbb{E}(V_{i,j}|\mathcal{F}_{m,n,i,j})\right).$$

If

$$\frac{(k_m\ell_n)^{q/p-1}\sum_{i=1}^{k_m}\sum_{j=1}^{\ell_n}\mathbb{E}\|V_{i,j}\|^q}{d_{m,n}^q}\to 0 \text{ as } m\vee n\to\infty, \tag{11}$$

then

$$\frac{\max_{1\le k\le k_m,1\le\ell\le\ell_n}\|S_{m,n,k.\ell}\|}{d_{m,n}}\overset{\mathcal{L}_q}{\to}0 \text{ as } m\vee n\to\infty.$$

Proof. Consider an array $\{V_{m,n,i,j},\ 1\le i\le k_m, 1\le j\le \ell_n, m\ge 1, n\ge 1\}$ of $\mathcal{X}$-valued random elements defined by

$$V_{m,n,i,j}=V_{i,j},\ 1\le i\le k_m, 1\le j\le\ell_n, m\ge 1, n\ge 1.$$

We thus have from (11) that

$$\begin{aligned}&\frac{(k_m\ell_n)^{q/p-1}\sum_{i=1}^{k_m}\sum_{j=1}^{\ell_n}\mathbb{E}\|V_{m,n,i,j}\|^q}{d_{m,n}^q}\\&=\frac{(k_m\ell_n)^{q/p-1}\sum_{i=1}^{k_m}\sum_{j=1}^{\ell_n}\mathbb{E}\|V_{i,j}\|^q}{d_{m,n}^q}\to 0 \text{ as } m\vee n\to\infty.\end{aligned}$$

Therefore, all assumptions of Theorem 4.1 are satisfied, and so we obtain from (8) that

$$\begin{aligned}&\frac{\max_{1\le k\le k_m,1\le\ell\le\ell_n}\|S_{m,n,k,\ell}\|}{d_{m,n}}\\&=\frac{\max_{1\le k\le k_m,1\le\ell\le\ell_n}\left\|\sum_{i=1}^{k}\sum_{j=1}^{\ell}\left(V_{m,n,i,j}-\mathbb{E}(V_{m,n,i,j}|\mathcal{F}_{m,n,i,j})\right)\right\|}{d_{m,n}}\\&\overset{\mathcal{L}_q}{\to}0 \text{ as } m\vee n\to\infty\end{aligned}$$

completing the proof. □

By using Lemma 2.1 and a similar argument as in the proof of Theorem 4.1, we obtain the following result which establishes mean convergence for the maximum of normed and suitably centered row sums from a triangular array of random elements. We omit the details.

Theorem 4.5. *Let $1\le p\le 2$, $q\ge p$ and let $\{V_{n,j},\ 1\le j\le\ell_n, n\ge 1\}$ be a triangular array of random elements in a real separable martingale type p Banach space $\mathcal{X}$ with $\mathbb{E}\|V_{n,j}\|^q<\infty$ for all $1\le j\le\ell_n, n\ge 1$. Let $\{d_n, n\ge 1\}$ be a sequence of positive constants such that*

$$\frac{(\ell_n)^{q/p-1}\sum_{j=1}^{\ell_n}\mathbb{E}\|V_{n,j}\|^q}{d_n^q}\to 0 \text{ as } n\to\infty.$$

Then

$$\frac{\max_{1\le\ell\le\ell_n}\left\|\sum_{j=1}^{\ell}\left(V_{n,j}-v_{n,j}\right)\right\|}{d_n}\overset{\mathcal{L}_q}{\to}0 \text{ as } n\to\infty,$$

where for $n\ge 1$, $v_{n,1}=\mathbb{E}V_{n,1}$, $v_{n,j}=\mathbb{E}(V_{n,j}|V_{n,1},\dots,V_{n,j-1})$, $2\le j\le\ell_n$, $\ell_n\ge 2$.

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