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Some Mean Convergence Theorems for the Maximum of Normed Double Sums of Banach Space Valued Random Elements

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Abstract In this correspondence, we establish mean convergence theorems for the maximum of normed double sums of Banach space valued random elements. Most of the results pertain to random elements which are M-dependent. We expand and improve a number of particular cases in the literature on mean convergence of random elements in Banach spaces. One of the main contributions of the paper is to simplify and improve a recent result of Li, Presnell, and Rosalsky [*Journal of Mathematical Inequalities*, **16**, 117–126 (2022)]. A new maximal inequality for double sums of M-dependent random elements is proved which may be of independent interest. The sharpness of the results is illustrated by four examples.

Keywords Double sum, mean convergence, Rademacher type p Banach space, Banach space valued random element, M-dependent random elements

MR(2010) Subject Classification 60F25, 60B11, 60B12

1 Introduction

Mean convergence for sums of independent Banach space valued random elements was studied by many authors (see, e.g., Chen and Wang [1], Hoffmann-Jørgensen and Pisier [3], Hu et al. [4], Korzeniowski [5], Li et al. [6], Ordóñez Cabrera et al. [8], Parker and Rosalsky [9], Wang and Rao [14] and the references therein), but only a few of them consider mean convergence for the maximum of normed partial sums which is of special interest. Very recently, Li et al. [6] obtained a mean convergence theorem pertaining to an array of rowwise independent random elements in a real separable Rademacher type p $(1 < p \leq 2)$ Banach space. (Technical definitions such as Rademacher type p will be reviewed in Section 2.) Throughout, all random elements are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and take values in a real separable Banach

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space $\mathcal{X}$ with norm $\|\cdot\|$. The starting point of the current investigation is the aforementioned result of Li et al. [6] which is stated as follows.

Theorem 1.1 (Li et al. [6]) *Let $1 < p \leq 2$, let $\{\ell_n, n \geq 1\}$ be a sequence of positive integers with $\ell_n \to \infty$ as $n \to \infty$, and let $\{V_{n,j},\ 1 \leq j \leq \ell_n, n \geq 1\}$ be a triangular array of rowwise independent random elements in a real separable Rademacher type p Banach space $\mathcal{X}$ with $\mathbb{E}\|V_{n,j}\|^p < \infty$ for all $1 \leq j \leq \ell_n, n \geq 1$. Let $\{d_n, n \geq 1\}$ be a sequence of positive constants. If there exist a continuous function $h : [0,\infty) \to [0,\infty)$ and two sequences of positive constants $\{b_n, n \geq 1\}$ and $\{c_n, n \geq 1\}$ with $c_n < b_n, n \geq 1$ such that*

$$h(0) = 0, h(x) = O(x), \text{ and } h^p(x)/x \uparrow \quad \text{as } 0 < x \uparrow \infty, \tag{1.1}$$

$$\frac{\sum_{j=1}^{\ell_n} \mathbb{E}(X_{n,j}^p \mathbf{1}(X_{n,j} > b_n))}{d_n^p} \to 0 \quad \text{as } n \to \infty, \tag{1.2}$$

$$\frac{h^p(b_n) \sum_{j=1}^{\ell_n} \mathbb{E}(X_{n,j} \mathbf{1}(X_{n,j} > c_n))}{d_n^p b_n} \to 0 \quad \text{as } n \to \infty, \tag{1.3}$$

$$\frac{h^p(b_n) \sum_{j=1}^{\ell_n} \mathbb{E}X_{n,j}}{d_n^p b_n} = O(1), \tag{1.4}$$

and

$$\frac{h^p(c_n)}{c_n} = o\left(\frac{h^p(b_n)}{b_n}\right) \quad \text{as } n \to \infty, \tag{1.5}$$

where $X_{n,j} = h^{-1}(\|V_{n,j}\|)$, $1 \leq j \leq \ell_n, n \geq 1$, then

$$\frac{\sum_{j=1}^{\ell_n} (V_{n,j} - \mathbb{E}V_{n,j})}{d_n} \xrightarrow{\mathcal{L}_p} 0 \quad \text{as } n \to \infty. \tag{1.6}$$

In view of the renowned Markov law of large numbers (see, e.g., [2, p. 205]), a natural question to ask is whether or not the set of five conditions (1.1)–(1.5) can be simplified? The current work provides a positive answer to this question. A very special case of Theorem 3.1 in Section 3 is the following theorem.

Theorem 1.2 *Let $1 \leq p \leq 2$ and let $\{\ell_n, n \geq 1\}$, $\{V_{n,j},\ 1 \leq j \leq \ell_n, n \geq 1\}$, and $\{d_n, n \geq 1\}$ be as in Theorem 1.1. If*

$$\frac{\sum_{j=1}^{\ell_n} \mathbb{E}\|V_{n,j}\|^p}{d_n^p} \to 0 \quad \text{as } n \to \infty, \tag{1.7}$$

then

$$\frac{\max_{1 \leq \ell \leq \ell_n} \|\sum_{j=1}^{\ell} (V_{n,j} - \mathbb{E}V_{n,j})\|}{d_n} \xrightarrow{\mathcal{L}_p} 0 \quad \text{as } n \to \infty. \tag{1.8}$$

The conclusion (1.6) of Theorem 1.1 is weaker than the conclusion (1.8) of Theorem 1.2 whereas the hypotheses of the latter are structurally substantially simpler than those of the former. Quite surprisingly, we establish the following theorem which asserts that the two sets of hypotheses are indeed equivalent when $1 < p \leq 2$. (We note that $1 < p \leq 2$ in Theorem 1.1 whereas $1 \leq p \leq 2$ in Theorem 1.2.) Theorem 1.3 is one of the main contributions of this paper.

Theorem 1.3 *The hypotheses of Theorem 1.1 and the hypotheses of Theorem 1.2 are equivalent for $1 < p \leq 2$.*

We postpone the proof of Theorems 1.2 and 1.3 to Section 4. Throughout the rest of the paper, $\{k_m\}$ and $\{\ell_n\}$ are two sequences of positive integers satisfying $\lim_{m \vee n \to \infty} k_m \ell_n = \infty$

unless stated otherwise. We consider an array $\{V_{m,n,i,j},\ 1 \le i \le k_m, 1 \le j \le \ell_n, m \ge 1, n \ge 1\}$ of $\mathcal{X}$-valued random elements, and establish mean convergence theorems for the maximum of normed double sums of the form

$$\frac{\max_{1\le k\le k_m, 1\le \ell\le \ell_n} \|\sum_{i=1}^{k_m}\sum_{j=1}^{\ell_n}(V_{m,n,i,j}-\mathbb{E}V_{m,n,i,j})\|}{d_{m,n}}$$

where $\{d_{m,n}, m \ge 1, n \ge 1\}$ is a normalizing double array. It turns out that these normed double sums cover not only the normed single sums from triangular arrays but also the normed double sums from double arrays, and Theorem 1.2 is a very special case of Theorem 3.1, which extends and improves several results in the literature including Theorem 3.10 of Parker and Rosalsky [9] and Theorem 1 of Li et al. [6]. The proof of Theorem 3.1 is based on Lemma 2.2 (ii) which is a maximal inequality for double sums of M-dependent random elements and may be of independent interest.

The plan of the paper is as follows. Notation, technical definitions, and lemmas which are used in proving the results are consolidated into Section 2. In Section 3, we establish the general mean convergence theorem (Theorem 3.1) for the maximum of normed double sums and its corollaries. Section 4 provides the proofs of Theorems 1.2 and 1.3. Section 5 contains four examples pertaining to the results.

2 Preliminaries

In this section, notation, technical definitions, and lemmas which are needed in connection with the main results will be presented.

The *expected value* or *mean* of an $\mathcal{X}$-valued random element V, denoted by $\mathbb{E}V$, is defined to be the *Pettis integral* provided it exists. That is, V has expected value $\mathbb{E}V \in \mathcal{X}$ if and only if $f(\mathbb{E}V) = \mathbb{E}f(V)$ for every $f \in \mathcal{X}^*$ where $\mathcal{X}^*$ denotes the (dual) space of all continuous linear functionals on $\mathcal{X}$.

Let $\{Y_n, n \ge 1\}$ be a symmetric Bernoulli sequence; that is, $\{Y_n, n \ge 1\}$ is a sequence of independent and identically distributed random variables with $\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = 1/2$. Let $\mathcal{X}^\infty = \mathcal{X} \times \mathcal{X} \times \mathcal{X} \times \cdots$ and define

$$\mathcal{C}(\mathcal{X}) = \left\{(v_1, v_2, \ldots) \in \mathcal{X}^\infty : \sum_{n=1}^{\infty} Y_n v_n \text{ converges in probability}\right\}.$$

Let $1 \le p \le 2$. Then $\mathcal{X}$ is said to be of Rademacher type p if there exists a finite constant $C > 0$ such that

$$\mathbb{E}\left\|\sum_{n=1}^{\infty} Y_n v_n\right\|^p \le C \sum_{n=1}^{\infty} \|v_n\|^p \quad \text{for all } (v_1, v_2, \ldots) \in \mathcal{C}(\mathcal{X}).$$

Throughout this paper, the symbol C will denote a generic constant $(0 < C < \infty)$ which is not necessarily the same one in each appearance. For $a, b \in \mathbb{R}$, $\min\{a, b\}$ and $\max\{a, b\}$ will be denoted, respectively, by $a \wedge b$ and $a \vee b$.

Let M be a nonnegative integer. A finite collection of random elements $\{V_{u,v}, 1 \le u \le m, 1 \le v \le n\}$ is said to be *M-dependent* if either $m \vee n \le M + 1$ or $m \vee n > M + 1$ and the sub-collection $\{V_{u,v}, 1 \le u \le i, 1 \le v \le j\}$ is independent of the sub-collection $\{V_{u,v}, k \le u \le m, \ell \le v \le n\}$ whenever $(k - i) \vee (\ell - j) > M$. A double array of random elements

$\{V_{u,v}, u \geq 1, v \geq 1\}$ is said to be *M-dependent* if for each $m \geq 1$, $n \geq 1$, the collection $\{V_{u,v}, 1 \leq u \leq m, 1 \leq v \leq n\}$ is M-dependent. When $M = 0$, the concept of M-dependence reduces to the concept of independence. Clearly, M-dependence implies M'-dependence for every nonnegative integer $M' > M$.

The first lemma is a well-known result and it is referred to as the c_p-inequality.

Lemma 2.1 *Let $a_1, \ldots, a_n$ be real numbers, and let $p > 0$. Then*

$$|a_1 + \cdots + a_n|^p \leq \max\{1, n^{p-1}\}(|a_1|^p + \cdots + |a_n|^p).$$

The following lemma plays an important role in proving Theorems 3.1 and 3.5. Part (ii) is proved by Parker and Rosalsky [9] when $M = 0$ and is proved by Quang, Thành, and Tien [10] when $q = p$.

Lemma 2.2 *Let $0 < p \leq 2$ and let $\{V_{i,j}, 1 \leq i \leq m, 1 \leq j \leq n\}$ be a collection of random elements in a real separable Banach space $\mathcal{X}$.*

(i) *If $0 < p \leq 1$, then for all $q \geq p$,*

$$\mathbb{E}\Bigg(\max_{\substack{1\leq k\leq m\\ 1\leq \ell\leq n}} \Bigg\|\sum_{i=1}^{k}\sum_{j=1}^{\ell} V_{i,j}\Bigg\|^q\Bigg) \leq (mn)^{q/p-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{E}\|V_{i,j}\|^q.$$

(ii) *Let M be a nonnegative integer. If $1 \leq p \leq 2$, $\mathcal{X}$ is of Rademacher type p, and $\{V_{i,j}, 1 \leq i \leq m, 1 \leq j \leq n\}$ is comprised of M-dependent mean zero random elements, then for all $q \geq p$,*

$$\mathbb{E}\Bigg(\max_{\substack{1\leq k\leq m\\ 1\leq \ell\leq n}} \Bigg\|\sum_{i=1}^{k}\sum_{j=1}^{\ell} V_{i,j}\Bigg\|^q\Bigg) \leq C(mn)^{q/p-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{E}\|V_{i,j}\|^q, \tag{2.1}$$

where C is a constant depending only on p, q and M.

Proof Firstly, we prove Part (i). We have

$$\begin{aligned}
\mathbb{E}\Bigg(\max_{\substack{1\leq k\leq m\\ 1\leq \ell\leq n}} \Bigg\|\sum_{i=1}^{k}\sum_{j=1}^{\ell} V_{i,j}\Bigg\|^q\Bigg) &= \mathbb{E}\Bigg(\max_{\substack{1\leq k\leq m\\ 1\leq \ell\leq n}} \Bigg\|\sum_{i=1}^{k}\sum_{j=1}^{\ell} V_{i,j}\Bigg\|^p\Bigg)^{q/p} \\
&\leq \mathbb{E}\Bigg(\max_{\substack{1\leq k\leq m\\ 1\leq \ell\leq n}} \sum_{i=1}^{k}\sum_{j=1}^{\ell} \|V_{i,j}\|^p\Bigg)^{q/p} \\
&= \mathbb{E}\Bigg(\sum_{i=1}^{m}\sum_{j=1}^{n} \|V_{i,j}\|^p\Bigg)^{q/p} \\
&\leq (mn)^{q/p-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{E}\|V_{i,j}\|^q,
\end{aligned}$$

where we have used Lemma 2.1 in both the first and second inequalities.

Next, we prove Part (ii). Since the case where $M = 0$ was proved by Parker and Rosalsky [9], we may assume $M \geq 1$. If $m \vee n \leq M + 1$, then (2.1) follows readily from Lemma 2.1. Now, we consider the case where $m \wedge n > M + 1$. Let $1 \leq k \leq m, 1 \leq \ell \leq n$ be fixed. Since we can introduce additional terms $V_{i,j} = 0$ if $k \leq M + 1$ or $\ell \leq M + 1$, we may assume that

$k \wedge \ell > M+1$. Then

$$\Big\|\sum_{i=1}^{k}\sum_{j=1}^{\ell} V_{i,j}\Big\|^q \le \Big(\sum_{s,t=1}^{M+1}\Big\|\sum_{\substack{0\le u(M+1)\le k-s\\ 0\le v(M+1)\le \ell-t}} V_{u(M+1)+s,v(M+1)+t}\Big\|\Big)^q$$
$$\le (M+1)^{2(q-1)}\sum_{s,t=1}^{M+1}\Big\|\sum_{\substack{0\le u(M+1)\le k-s\\ 0\le v(M+1)\le \ell-t}} V_{u(M+1)+s,v(M+1)+t}\Big\|^q,$$

where we have used the triangle inequality in the first inequality, and Lemma 2.1 (with $n = (M+1)^2$) in the second inequality. It thus follows that

$$\mathbb{E}\Big(\max_{\substack{1\le k\le m\\ 1\le \ell\le n}}\Big\|\sum_{i=1}^{k}\sum_{j=1}^{\ell} V_{i,j}\Big\|^q\Big)$$
$$\le (M+1)^{2(q-1)}\sum_{s,t=1}^{M+1}\mathbb{E}\Big(\max_{\substack{1\le k\le m\\ 1\le \ell\le n}}\Big\|\sum_{\substack{0\le u(M+1)\le k-s\\ 0\le v(M+1)\le \ell-t}} V_{u(M+1)+s,v(M+1)+t}\Big\|^q\Big)$$
$$\le C\sum_{s,t=1}^{M+1}(mn)^{q/p-1}\Big(\sum_{\substack{0\le u(M+1)+s\le m\\ 0\le v(M+1)+t\le n}}\mathbb{E}\|V_{u(M+1)+s,v(M+1)+t}\|^q\Big)$$
$$\le C(mn)^{q/p-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{E}\|V_{i,j}\|^q,$$

where we have used the M-dependence and Rademacher type p hypotheses and Lemma 2.4 of Parker and Rosalsky [9] in the second inequality.

Finally, we consider the case where $m \vee n > M+1$ and $m \wedge n \le M+1$. Without loss of generality, we can assume that $m \le M+1$ and $n > M+1$. By introducing additional terms $V_{i,j} = 0$ for $m+1 \le i \le M+2$ and $1 \le j \le n$, the proof of this case follows from the case where $m \wedge n > M+1$. □

3 A General Mean Convergence Theorem and Its Corollaries

In this section, we will establish a very general mean convergence theorem for the maximum of normed double sums from arrays of M-dependent random elements in a real separable Rademacher type p ($1 \le p \le 2$) Banach space $\mathcal{X}$. Its proof follows directly from Lemma 2.2 (ii), and it covers results concerning both single sums from triangular arrays and double sums from double arrays. We present three corollaries of Theorem 3.1 which extend and improve several results from the literature.

Theorem 3.1 *Let M be a nonnegative integer. Let $1 \le p \le 2$ and let $\{V_{m,n,i,j}, 1 \le i \le k_m, 1 \le j \le \ell_n, m \ge 1, n \ge 1\}$ be an array of random elements in a real separable Rademacher type p Banach space $\mathcal{X}$ such that for fixed $m \ge 1$ and $n \ge 1$, the double array $\{V_{m,n,i,j}, 1 \le i \le k_m, 1 \le j \le \ell_n\}$ is comprised of M-dependent random elements. Let $q \ge p$, and let $\{d_{m,n}, m \ge 1, n \ge 1\}$ be an array of positive constants such that $\mathbb{E}\|V_{m,n,i,j}\|^q < \infty$ for all*

$1 \le i \le k_m, 1 \le j \le \ell_n, m \ge 1, n \ge 1$. *If*

$$\frac{(k_m \ell_n)^{q/p-1} \sum_{i=1}^{k_m} \sum_{j=1}^{\ell_n} \mathbb{E}\|V_{m,n,i,j}\|^q}{d_{m,n}^q} \to 0 \quad \text{as } m \vee n \to \infty, \tag{3.1}$$

then

$$\frac{\max_{1\le k\le k_m, 1\le \ell\le \ell_n} \|\sum_{i=1}^{k} \sum_{j=1}^{\ell} (V_{m,n,i,j} - \mathbb{E}V_{m,n,i,j})\|}{d_{m,n}} \xrightarrow{\mathcal{L}_q} 0 \quad \text{as } m \vee n \to \infty. \tag{3.2}$$

Proof For $m \ge 1, n \ge 1$,

$$\begin{aligned}
&\frac{\mathbb{E}(\max_{1\le k\le k_m, 1\le \ell\le \ell_n} \|\sum_{i=1}^{k} \sum_{j=1}^{\ell} (V_{m,n,i,j} - \mathbb{E}V_{m,n,i,j})\|^q)}{d_{m,n}^q} \\
&\le \frac{C(k_m \ell_n)^{q/p-1} \sum_{i=1}^{k_m} \sum_{j=1}^{\ell_n} \mathbb{E}\|V_{m,n,i,j} - \mathbb{E}V_{m,n,i,j}\|^q}{d_{m,n}^q} \text{ (by Lemma 2.2 (ii))} \\
&\le \frac{C(k_m \ell_n)^{q/p-1} \sum_{i=1}^{k_m} \sum_{j=1}^{\ell_n} \mathbb{E}\|V_{m,n,i,j}\|^q}{d_{m,n}^q} \text{ (by Lemma 2.1 with } n=2) \\
&\to 0 \quad \text{as } m \vee n \to \infty \text{ (by (3.1))}
\end{aligned}$$

thereby establishing (3.2). □

Next, we will present three corollaries which demonstrate the generality of Theorem 3.1. The first corollary indicates that Theorem 3.1 covers the "rowwise independent triangular arrays" case. The special case $M = 0$ and $q = p$ of Corollary 3.2 is Theorem 1.2 and it generalizes and improves Theorem 1 of Li et al. [6].

Corollary 3.2 *Let M be a nonnegative integer. Let $1 \le p \le 2$, $q \ge p$, let $\{\ell_n, n \ge 1\}$ be a sequence of positive integers with $\ell_n \to \infty$ as $n \to \infty$, and let $\{V_{n,j},\ 1 \le j \le \ell_n, n \ge 1\}$ be a triangular array of rowwise M-dependent random elements in a real separable Rademacher type p Banach space $\mathcal{X}$ with $\mathbb{E}\|V_{n,j}\|^q < \infty$ for all $1 \le j \le \ell_n, n \ge 1$. Let $\{d_n, n \ge 1\}$ be a sequence of positive constants such that*

$$\frac{(\ell_n)^{q/p-1} \sum_{j=1}^{\ell_n} \mathbb{E}\|V_{n,j}\|^q}{d_n^q} \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

Then

$$\frac{\max_{1\le \ell\le \ell_n} \|\sum_{j=1}^{\ell} (V_{n,j} - \mathbb{E}V_{n,j})\|}{d_n} \xrightarrow{\mathcal{L}_q} 0 \quad \text{as } n \to \infty.$$

Proof Consider $k_m \equiv 1$, and an array $\{V_{m,n,i,j},\ 1 \le i \le k_m, 1 \le j \le \ell_n, m \ge 1, n \ge 1\}$ of $\mathcal{X}$-valued random elements defined by

$$V_{1,n,1,j} = V_{n,j}, \quad 1 \le j \le \ell_n, n \ge 1,$$

and

$$V_{m,n,1,j} = 0, \quad 1 \le j \le \ell_n, m \ge 2, n \ge 1.$$

Consider an array $\{d_{m,n}, m \ge 1, n \ge 1\}$ of positive constants such that

$$d_{m,n} = d_n, \quad m \ge 1, n \ge 1.$$

We thus have from (3.3) that for $m = 1$,

$$\frac{(k_m \ell_n)^{q/p-1} \sum_{i=1}^{k_m} \sum_{j=1}^{\ell_n} \mathbb{E}\|V_{m,n,i,j}\|^q}{d_{m,n}^q}$$

$$= \frac{(\ell_n)^{q/p-1} \sum_{j=1}^{\ell_n} \mathbb{E}\|V_{n,j}\|^q}{d_n^q} \to 0 \quad \text{as } n \to \infty,$$

and for $m \geq 2$,

$$\frac{(k_m \ell_n)^{q/p-1} \sum_{i=1}^{k_m} \sum_{j=1}^{\ell_n} \mathbb{E}\|V_{m,n,i,j}\|^q}{d_{m,n}^q} = 0.$$

Therefore, all assumptions of Theorem 3.1 are satisfied, and so we obtain from (3.2) that

$$\begin{aligned} &\frac{\max_{1 \leq \ell \leq \ell_n} \| \sum_{j=1}^{\ell} (V_{n,j} - \mathbb{E}V_{n,j}) \|}{d_n} \\ &= \frac{\max_{1 \leq k \leq k_1, 1 \leq \ell \leq \ell_n} \| \sum_{i=1}^{k} \sum_{j=1}^{\ell} (V_{1,n,i,j} - \mathbb{E}V_{1,n,i,j}) \|}{d_{1,n}} \\ &\xrightarrow{\mathcal{L}_q} 0 \quad \text{as } n \to \infty \end{aligned}$$

completing the proof. □

The next corollary indicates that Theorem 3.1 covers the "double arrays" case. In the case where $M = 0$, $k_m \equiv m$, and $\ell_n \equiv n$, Corollary 3.3 reduces to Theorem 3.10 of Parker and Rosalsky [9].

Corollary 3.3 *Let M be a nonnegative integer. Let $1 \leq p \leq 2$, $q \geq p$, and let $\{V_{i,j}, i \geq 1, j \geq 1\}$ be a double array of M-dependent random elements in a real separable Rademacher type p Banach space $\mathcal{X}$ with $\mathbb{E}\|V_{i,j}\|^q < \infty$ for all $i \geq 1, j \geq 1$. Let $\{d_{m,n}, m \geq 1, n \geq 1\}$ be a double array of positive constants such that*

$$\frac{(k_m \ell_n)^{q/p-1} \sum_{i=1}^{k_m} \sum_{j=1}^{\ell_n} \mathbb{E}\|V_{i,j}\|^q}{d_{m,n}^q} \to 0 \quad \text{as } m \vee n \to \infty. \tag{3.4}$$

Then

$$\frac{\max_{1 \leq k \leq k_m, 1 \leq \ell \leq \ell_n} \| \sum_{i=1}^{k} \sum_{j=1}^{\ell} (V_{i,j} - \mathbb{E}V_{i,j}) \|}{d_{m,n}} \xrightarrow{\mathcal{L}_q} 0 \quad \text{as } m \vee n \to \infty.$$

Proof Consider an array $\{V_{m,n,i,j}, 1 \leq i \leq k_m, 1 \leq j \leq \ell_n, m \geq 1, n \geq 1\}$ of $\mathcal{X}$-valued random elements defined by

$$V_{m,n,i,j} = V_{i,j}, \quad 1 \leq i \leq k_m, 1 \leq j \leq \ell_n, m \geq 1, n \geq 1.$$

We thus have from (3.4) that

$$\begin{aligned} &\frac{(k_m \ell_n)^{q/p-1} \sum_{i=1}^{k_m} \sum_{j=1}^{\ell_n} \mathbb{E}\|V_{m,n,i,j}\|^q}{d_{m,n}^q} \\ &= \frac{(k_m \ell_n)^{q/p-1} \sum_{i=1}^{k_m} \sum_{j=1}^{\ell_n} \mathbb{E}\|V_{i,j}\|^q}{d_{m,n}^q} \to 0 \quad \text{as } m \vee n \to \infty. \end{aligned}$$

Therefore, all assumptions of Theorem 3.1 are satisfied, and so we obtain from (3.2) that

$$\begin{aligned} &\frac{\max_{1 \leq k \leq k_m, 1 \leq \ell \leq \ell_n} \| \sum_{i=1}^{k} \sum_{j=1}^{\ell} (V_{i,j} - \mathbb{E}V_{i,j}) \|}{d_{m,n}} \\ &= \frac{\max_{1 \leq k \leq k_m, 1 \leq \ell \leq \ell_n} \| \sum_{i=1}^{k} \sum_{j=1}^{\ell} (V_{m,n,i,j} - \mathbb{E}V_{m,n,i,j}) \|}{d_{m,n}} \\ &\xrightarrow{\mathcal{L}_q} 0 \quad \text{as } m \vee n \to \infty \end{aligned}$$

completing the proof. □

A double array of constants $\{d_{m,n}, m \geq 1, n \geq 1\}$ is said to be increasing if $d_{k,\ell} \leq d_{m,n}$ whenever $k \leq m, \ell \leq n$. The following corollary provides an exact characterization of Rademacher type p Banach spaces. The implication ((i)⇒(ii)) was proved by Parker and Rosalsky [9, Theorem 3.1] for the independence (or $M = 0$) case.

Corollary 3.4 *Let $1 \leq p \leq 2$, and let $\mathcal{X}$ be a real separable Banach space. Then the following two statements are equivalent.*

(i) *$\mathcal{X}$ is of Rademacher type p.*

(ii) *For any increasing double array $\{d_{m,n}, m \geq 1, n \geq 1\}$ of positive constants satisfying $\lim_{m\vee n\to\infty} d_{m,n} = \infty$, and for any double array $\{V_{m,n}, m \geq 1, n \geq 1\}$ of M-dependent $\mathcal{X}$-valued random elements, the condition*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\mathbb{E}\|V_{i,j}\|^p}{d_{i,j}^p} < \infty \tag{3.5}$$

implies

$$\frac{\max_{1\leq k\leq m,1\leq \ell\leq n}\|\sum_{i=1}^{k}\sum_{j=1}^{\ell}(V_{i,j}-\mathbb{E}V_{i,j})\|}{d_{m,n}} \xrightarrow{\mathcal{L}_p} 0 \quad \text{as } m\vee n\to\infty \tag{3.6}$$

irrespective of the value of $M \in \{0, 1, 2, \ldots\}$.

Proof Firstly, we prove ((i)⇒(ii)). Let $\{V_{m,n}, m \geq 1, n \geq 1\}$ be a double array of M-dependent $\mathcal{X}$-valued random elements and let $\{d_{m,n}, m \geq 1, n \geq 1\}$ be an increasing double array of positive constants such that (3.5) holds and $\lim_{m\vee n\to\infty} d_{m,n} = \infty$. By applying the Kronecker lemma for double sums with nonnegative terms (see, Móricz [7]), we obtain from (3.5) that

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\mathbb{E}\|V_{i,j}\|^p}{d_{m,n}^p} \to 0 \quad \text{as } m\vee n\to\infty. \tag{3.7}$$

By applying Corollary 3.3 with $q = p$, (3.6) follows from (3.7) thereby verifying (ii).

Next, we prove ((ii)⇒(i)). We apply (ii) with $d_{m,n} = mn$, $m \geq 1, n \geq 1$ and $M = 0$. Let $\{V_{m,n}, m \geq 1, n \geq 1\}$ be a double array of independent mean zero $\mathcal{X}$-valued random elements such that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\mathbb{E}\|V_{i,j}\|^p}{(ij)^p} < \infty. \tag{3.8}$$

By (ii), we obtain

$$\frac{\max_{1\leq k\leq m,1\leq \ell\leq n}\|\sum_{i=1}^{k}\sum_{j=1}^{\ell}V_{i,j}\|}{mn} \xrightarrow{\mathcal{L}_p} 0 \quad \text{as } m\vee n\to\infty,$$

and so

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}V_{i,j}}{mn} \xrightarrow{\mathbb{P}} 0 \quad \text{as } m\vee n\to\infty. \tag{3.9}$$

By Theorem 3.1 of [13], we obtain from (3.8) and (3.9) that

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}V_{i,j}}{mn} \to 0 \quad \text{almost surely (a.s.) as } m\vee n\to\infty.$$

By Theorem 3.1 of [11], we have ((ii)⇒(i)). This completes the proof of the corollary. □

In the next theorem, we show for $0 < p \leq 1$ that (3.10) implies (3.11) without imposing any geometric conditions on the Banach space and with no dependence type of conditions and with no mean zero conditions imposed on the random elements comprising the array. Theorem 3.5 may be compared with Theorem 3.1 wherein $1 \le p \le 2$.

Theorem 3.5 *Let $0 < p \le 1$ and let $\{V_{m,n,i,j},\ 1 \le i \le k_m, 1 \le j \le \ell_n, m \ge 1, n \ge 1\}$ be an array of random elements in a real separable Banach space $\mathcal{X}$. Let $q \ge p$, and let $\{d_{m,n}, m \ge 1, n \ge 1\}$ be an array of positive constants such that $\mathbb{E}\|V_{m,n,i,j}\|^q < \infty$ for all $1 \le i \le k_m, 1 \le j \le \ell_n, m \ge 1, n \ge 1$. If*

$$\frac{(k_m\ell_n)^{q/p-1}\sum_{i=1}^{k_m}\sum_{j=1}^{\ell_n}\mathbb{E}\|V_{m,n,i,j}\|^q}{d_{m,n}^q} \to 0 \quad \text{as } m \vee n \to \infty, \tag{3.10}$$

then

$$\frac{\max_{1\le k\le k_m, 1\le \ell\le \ell_n}\|\sum_{i=1}^{k}\sum_{j=1}^{\ell}V_{m,n,i,j}\|}{d_{m,n}} \xrightarrow{\mathcal{L}_q} 0 \quad \text{as } m \vee n \to \infty. \tag{3.11}$$

Proof The proof is similar to that of Theorem 3.1. For $m \ge 1, n \ge 1$, we have

$$\begin{aligned}
&\frac{\mathbb{E}(\max_{1\le k\le k_m, 1\le \ell\le \ell_n}\|\sum_{i=1}^{k}\sum_{j=1}^{\ell}V_{m,n,i,j}\|^q)}{d_{m,n}^q}\\
&\le \frac{(k_m\ell_n)^{q/p-1}\sum_{i=1}^{k_m}\sum_{j=1}^{\ell_n}\mathbb{E}\|V_{m,n,i,j}\|^q}{d_{m,n}^q} \text{ (by Lemma 2.2 (i))}\\
&\to 0 \quad \text{as } m \vee n \to \infty \text{ (by (3.10))}
\end{aligned}$$

thereby establishing (3.11). □

4 Proofs of Theorems 1.2 and 1.3

In this section, we will present the proofs of Theorems 1.2 and 1.3.

Proof of Theorem 1.2 Theorem 1.2 follows immediately from Corollary 3.2 by letting $M = 0$ and $q = p$. □

Proof of Theorem 1.3 Let $1 < p \le 2$. We first verify that the hypotheses of Theorem 1.1 imply those of Theorem 1.2. Let the function $h : [0,\infty) \to [0,\infty)$ and the two sequences of positive constants $\{b_n, n \ge 1\}$ and $\{c_n, n \ge 1\}$ satisfy the hypotheses of Theorem 1.1. It only needs to be shown that (1.7) holds. For $1 \le j \le \ell_n, n \ge 1$, let

$$U_{n,j} = V_{n,j}\mathbf{1}(\|V_{n,j}\| \le h(b_n)) \quad \text{and} \quad W_{n,j} = V_{n,j}\mathbf{1}(\|V_{n,j}\| > h(b_n)).$$

In Li et al. [6] the proof of Theorem 1.1, it is shown that its hypotheses ensure

$$\frac{\sum_{j=1}^{\ell_n}\mathbb{E}\|U_{n,j}\|^p}{d_n^p} \to 0 \quad \text{and} \quad \frac{\sum_{j=1}^{\ell_n}\mathbb{E}\|W_{n,j}\|^p}{d_n^p} \to 0 \quad \text{as } n \to \infty. \tag{4.1}$$

Note that for $1 \le j \le \ell_n, n \ge 1$, we have

$$\|V_{n,j}\|^p = \|U_{n,j}\|^p + \|W_{n,j}\|^p,$$

and so

$$\mathbb{E}\|V_{n,j}\|^p = \mathbb{E}\|U_{n,j}\|^p + \mathbb{E}\|W_{n,j}\|^p. \tag{4.2}$$

Thus (1.7) follows from (4.1) and (4.2).

Next, we verify that the hypotheses of Theorem 1.2 imply those of Theorem 1.1. Assume that (1.7) holds. It needs to be shown that there exist a continuous function $h : [0,\infty) \to [0,\infty)$

and two sequences of positive constants $\{b_n, n \geq 1\}$ and $\{c_n, n \geq 1\}$ with $c_n < b_n, n \geq 1$ satisfying (1.1)–(1.5).

Let $h(x) = x, x \geq 0$. Then (1.1) is satisfied. Since $X_{n,j} = h^{-1}(\|V_{n,j}\|) = \|V_{n,j}\|$, $1 \leq j \leq \ell_n, n \geq 1$, it is clear that (1.2) follows from (1.7). Since $\mathbb{E}\|V_{n,j}\|^p < \infty$ for all $1 \leq j \leq \ell_n, n \geq 1$, we have from Jensen's inequality that $\mathbb{E}\|V_{n,j}\| < \infty$ for all $1 \leq j \leq \ell_n, n \geq 1$. Therefore

$$\sum_{j=1}^{\ell_n} \mathbb{E}\|V_{n,j}\| < \infty, \quad n \geq 1.$$

For $n \geq 1$, set

$$b_n = \begin{cases} \left(\dfrac{d_n^p}{n \sum_{j=1}^{\ell_n} \mathbb{E}\|V_{n,j}\|} \right)^{1/(p-1)}, & \text{if } \displaystyle\sum_{j=1}^{\ell_n} \mathbb{E}\|V_{n,j}\| > 0, \\ 1, & \text{if } \displaystyle\sum_{j=1}^{\ell_n} \mathbb{E}\|V_{n,j}\| = 0 \end{cases}$$

and

$$c_n = \frac{b_n}{2n}.$$

Then $c_n < b_n$ for all $n \geq 1$ and

$$\begin{aligned} \frac{h^p(b_n)}{d_n^p b_n} \sum_{j=1}^{\ell_n} \mathbb{E}X_{n,j}\mathbf{1}(X_{n,j} > c_n) &\leq \frac{h^p(b_n)}{d_n^p b_n} \sum_{j=1}^{\ell_n} \mathbb{E}X_{n,j} \\ &= \frac{b_n^{p-1} \sum_{j=1}^{\ell_n} \mathbb{E}\|V_{n,j}\|}{d_n^p} \\ &= \begin{cases} \dfrac{1}{n}, & \text{if } \displaystyle\sum_{j=1}^{\ell_n} \mathbb{E}\|V_{n,j}\| > 0, \\ 0, & \text{if } \displaystyle\sum_{j=1}^{\ell_n} \mathbb{E}\|V_{n,j}\| = 0 \end{cases} \\ &\to 0 \quad \text{as } n \to \infty \end{aligned}$$

verifying (1.3) and (1.4). We also have

$$\frac{h^p(c_n)}{c_n} = c_n^{p-1} = \left(\frac{b_n}{2n} \right)^{p-1} \quad \text{and} \quad \frac{h^p(b_n)}{b_n} = b_n^{p-1}, \quad n \geq 1.$$

Therefore (1.5) also holds.

The proof of the theorem is completed. □

5 Some Interesting Examples

In this section, we present four examples which illustrate the results in Section 3.

The first example, which is inspired by Example 4.3 of Rosalsky and Thành [12], shows that in Theorem 3.1, we cannot obtain a.s. convergence in (3.2).

Example 5.1 Let $1 \leq p = q < 2$ and $\{X_n, n \geq 1\}$ be a sequence of independent real-valued random variables such that

$$\mathbb{P}(X_n = 0) = 1 - \frac{1}{(n+1)\log(n+1)}, \quad \mathbb{P}(X_n = \pm(n+1)^{1/p}) = \frac{1}{2(n+1)\log(n+1)}, \quad n \geq 1.$$

Then

$$\mathbb{E}|X_n|^p = \frac{1}{\log(n+1)}, \quad n \geq 1.$$

Let $k_m \equiv 1$, $\ell_n \equiv n$, and define an array $\{d_{m,n}, m \geq 1, n \geq 1\}$ of positive constants, and an array $\{V_{m,n,i,j}, 1 \leq i \leq k_m, 1 \leq j \leq \ell_n, m \geq 1, n \geq 1\}$ of random variables by

$$d_{m,n} = n^{1/p}, \quad m \geq 1, n \geq 1,$$
$$V_{1,n,1,j} = X_j, \quad 1 \leq j \leq n, n \geq 1,$$

and

$$V_{m,n,1,j} = 0, \quad 1 \leq j \leq n, m \geq 2, n \geq 1.$$

Then for $m = 1$ and $n \geq 1$,

$$\begin{aligned} \frac{(k_m \ell_n)^{q/p-1} \sum_{i=1}^{k_m} \sum_{j=1}^{\ell_n} \mathbb{E}|V_{m,n,i,j}|^q}{d_{m,n}^q} &= \frac{\sum_{j=1}^{n} \mathbb{E}|X_j|^p}{n} \\ &= \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\log(j+1)} \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

and for $m \geq 2$ and $n \geq 1$,

$$\frac{(k_m \ell_n)^{q/p-1} \sum_{i=1}^{k_m} \sum_{j=1}^{\ell_n} \mathbb{E}|V_{m,n,i,j}|^q}{d_{m,n}^q} = 0.$$

By Theorem 3.1 with any $M \geq 0$, we have

$$\frac{\max_{1 \leq k \leq k_m, 1 \leq \ell \leq \ell_n} |\sum_{i=1}^{k} \sum_{j=1}^{\ell} (V_{m,n,i,j} - \mathbb{E}V_{m,n,i,j})|}{d_{m,n}} \xrightarrow{\mathcal{L}_p} 0 \quad \text{as } m \vee n \to \infty.$$

We also have

$$\frac{\sum_{i=1}^{k_1} \sum_{j=1}^{\ell_n} V_{1,n,i,j}}{d_{1,n}} = \frac{\sum_{j=1}^{n} X_j}{n^{1/p}}.$$

Rosalsky and Thành [12, Example 4.3] showed that

$$\frac{\sum_{j=1}^{n} X_j}{n^{1/p}} \nrightarrow 0 \quad \text{a.s. as } n \to \infty.$$

Therefore

$$\begin{aligned} &\frac{\max_{1 \leq k \leq k_m, 1 \leq \ell \leq \ell_n} |\sum_{i=1}^{k} \sum_{j=1}^{\ell} (V_{m,n,i,j} - \mathbb{E}V_{m,n,i,j})|}{d_{m,n}} \\ &\geq \frac{|\sum_{i=1}^{k_1} \sum_{j=1}^{\ell_n} V_{1,n,i,j}|}{d_{m,n}} \nrightarrow 0 \quad \text{a.s. as } m \vee n \to \infty \end{aligned}$$

and so a.s. convergence does not hold in (3.2).

The second example shows that in Theorem 3.1, the M-dependence hypothesis cannot be dispensed with.

Example 5.2 Let $q = p > 1$, $k_m \equiv m, \ell_n \equiv n$, $d_{m,n} \equiv mn$, and let X be a real-valued random variable with

$$\mathbb{P}(X = -1) = \mathbb{P}(X = 1) = \frac{1}{2}.$$

Let $\{V_{m,n,i,j},\ 1 \le i \le m, 1 \le j \le n, m \ge 1, n \ge 1\}$ be an array of random variables with

$$V_{m,n,i,j} = X \quad \text{a.s., } 1 \le i \le m, 1 \le j \le n, m \ge 1, n \ge 1.$$

Then for no $M \ge 0$ is the double array $\{V_{m,n,i,j},\ 1 \le i \le m, 1 \le j \le n\}$ comprised of M-dependent random variables for $m \vee n > M+1$. Now

$$\frac{(k_m \ell_n)^{q/p-1} \sum_{i=1}^{k_m} \sum_{j=1}^{\ell_n} \mathbb{E}\|V_{m,n,i,j}\|^q}{d_{m,n}^q} = \frac{mn}{(mn)^q} \to 0 \quad \text{as } m \vee n \to \infty$$

and so (3.1) holds. However,

$$\begin{aligned} &\frac{\max_{1 \le k \le k_m, 1 \le \ell \le \ell_n} \|\sum_{i=1}^{k} \sum_{j=1}^{\ell} (V_{m,n,i,j} - \mathbb{E}V_{m,n,i,j})\|}{d_{m,n}} \\ &= \frac{\max_{1 \le k \le k_m, 1 \le \ell \le \ell_n} |k\ell X|}{mn} \\ &= \frac{mn|X|}{mn} = 1 \quad \text{a.s.} \end{aligned}$$

for all $m \ge 1, n \ge 1$ and so (3.2) fails.

Apropos of Examples 5.3 and 5.4, for $1 \le p \le 2$, we consider the real separable Banach space l_p consisting of absolute p-th power summable real sequences $v = \{v_r, r \ge 1\}$ with norm $\|v\| = (\sum_{r=1}^{\infty} |v_r|^p)^{1/p}$. The element of l_p having 1 in its r-th position and 0 elsewhere will be denoted by $v^{(r)}$, $r \ge 1$. Let $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a one-to-one and onto map, and let $\{V_{i,j}, i \ge 1, j \ge 1\}$ be a double array of random elements in l_p by requiring the $V_{i,j}, i \ge 1, j \ge 1$ to be independent with

$$\mathbb{P}(V_{i,j} = v^{(\varphi(i,j))}) = \mathbb{P}(V_{i,j} = -v^{(\varphi(i,j))}) = \frac{1}{2}, \quad i \ge 1, j \ge 1.$$

The third example, which was inspired by Example 5.1 of Rosalsky and Thành [11], shows that Theorem 3.1 can fail if the hypothesis that $\mathcal{X}$ is of Rademacher type p is dispensed with.

Example 5.3 Let $\mathcal{X} = l_1$, $1 < p \le 2$. It is well known that l_1 is not of Rademacher type p for every $1 < p \le 2$. Consider an array $\{V_{m,n,i,j},\ 1 \le i \le m, 1 \le j \le n, m \ge 1, n \ge 1\}$ of l_1-valued random elements defined by

$$V_{m,n,i,j} = V_{i,j}, \quad 1 \le i \le m, 1 \le j \le n, m \ge 1, n \ge 1.$$

Let

$$q = p, \quad k_m \equiv m, \quad \ell_n \equiv n, \quad \text{and} \quad d_{m,n} \equiv mn.$$

Now

$$\begin{aligned} \frac{(k_m \ell_n)^{q/p-1} \sum_{i=1}^{k_m} \sum_{j=1}^{\ell_n} \mathbb{E}\|V_{m,n,i,j}\|^q}{d_{m,n}^q} &= \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} 1}{(mn)^p} \\ &= \frac{mn}{(mn)^p} \to 0 \quad \text{as } m \vee n \to \infty \end{aligned}$$

and so all of the hypotheses of Theorem 3.1 (except for the Rademacher type p hypothesis) are satisfied with any $M \ge 0$. However,

$$\frac{\max_{1 \le k \le k_m, 1 \le \ell \le \ell_n} \|\sum_{i=1}^{k} \sum_{j=1}^{\ell} (V_{m,n,i,j} - \mathbb{E}V_{m,n,i,j})\|}{d_{m,n}}$$

$$= \frac{\max_{1\le k\le m, 1\le \ell\le n} k\ell}{mn}$$
$$= \frac{mn}{mn}$$
$$= 1 \quad \text{a.s.}$$

for all $m \ge 1, n \ge 1$ and so (3.2) fails.

Apropos of Corollary 3.4, the fourth example shows that for a double array of M-dependent random elements in a real separable Rademacher type p Banach space and an increasing double array $\{d_{m,n}, m \ge 1, n \ge 1\}$ of positive constants satisfying $\lim_{m\vee n\to\infty} d_{m,n} = \infty$, (3.6) can fail if the condition (3.5) is weakened to

$$\lim_{i\vee j\to\infty} \frac{\mathbb{E}\|V_{i,j}\|^p}{d_{i,j}^p} = 0. \tag{5.1}$$

Example 5.4 was inspired by Example 5.1 of Rosalsky, Thành, and Thuy [13].

Example 5.4 Let $1 \le p \le 2$ and $\mathcal{X} = l_p$. It is well known that l_p is of Rademacher type p. Let $d_{m,n} = (mn)^{1/p}, m \ge 1, n \ge 1$. Now

$$\frac{\mathbb{E}\|V_{i,j}\|^p}{d_{i,j}^p} = \frac{1}{ij}, \quad i \ge 1, j \ge 1$$

and so (5.1) holds but (3.5) fails. All of the conditions of Corollary 3.4 (ii) are satisfied with any $M \ge 0$ except for (3.5). Moreover,

$$\frac{\max_{1\le k\le m,1\le \ell\le n} \|\sum_{i=1}^{k}\sum_{j=1}^{\ell}(V_{m,n,i,j} - \mathbb{E}V_{m,n,i,j})\|}{d_{m,n}} = \frac{\max_{1\le k\le m,1\le \ell\le n}(k\ell)^{1/p}}{(mn)^{1/p}}$$
$$= \frac{(mn)^{1/p}}{(mn)^{1/p}}$$
$$= 1 \quad \text{a.s.}$$

for all $m \ge 1, n \ge 1$ and so (3.6) also fails.

Conflict of Interest The authors declare no conflict of interest.

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