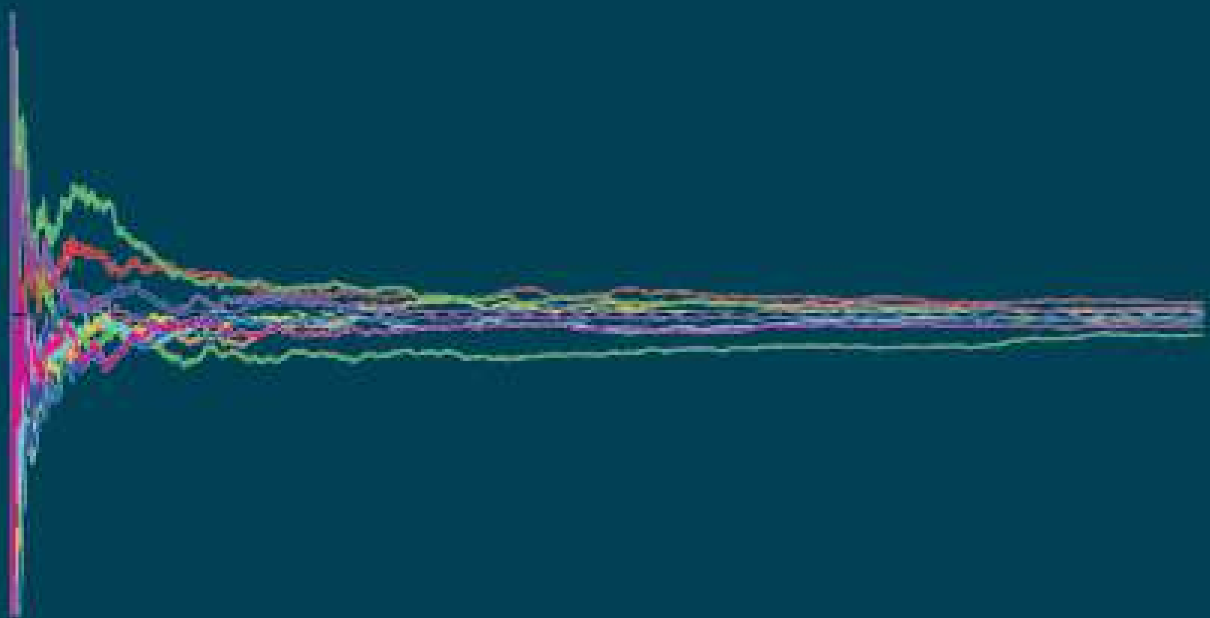
LE VAN THANH

LAWS OF LARGE NUMBERS FOR PAIRWISE INDEPENDENT RANDOM VARIABLES



NHÀ XUẤT BẢN ĐẠI HỌC VINH

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Laws of Large Numbers for Pairwise Independent Random Variables

– Monograph –

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Preface

The *Laws of Large Numbers* lie at the very foundation of statistical science and have a history dating back about 500 years. There is a significant amount of literature on the laws of large numbers, emphasizing its indispensable role in probability and statistical theory and their various applications. There are two types of laws of large numbers: strong laws of large numbers, which involve almost sure convergence, and weak laws of large numbers, which involve convergence in probability.

A special form of the law of large numbers was observed in the early sixteenth century by the Italian mathematician Gerolamo Cardano (1501-1575). He discovered that in the field of statistics, observations became more accurate with a higher number of trials (see [153]). This observation was first proved approximately 200 years later by the Swiss mathematician Jacob Bernoulli (1654–1705). Let $\{A_n, n \geq 1\}$ be a sequence of independent events with the same probability p, and let $\hat{p}_n = \sum_{i=1}^{n} \mathbf{1}(A_i)/n$ be the proportion of $\{A_1, \ldots, A_n\}$ to occur. Jacob Bernoulli proved the first weak law of large numbers:

$$\hat{p}_n \to p \text{ in probability.}$$

It took Jacob Bernoulli over twenty years to develop a rigorous mathematical proof of this result, and it was published in Bernoulli's renowned book *Ars Conjectandi* (*The Art of Conjecturing*) in 1713, eight years after his death. He referred to it as the "Golden Theorem", but it later became generally known as "Bernoulli's Theorem". In 1837, the French mathematician Siméon Poisson (1781-1840) extended Bernoulli's result to the case where the events $A_n, n \geq 1$ are independent but do not necessarily have the same probability and named his result "la loi des grandes nombres" ("the law of large numbers"). It has since been known by both names, but "law of large numbers" is more commonly used.

Over 200 years after Bernoulli proved his weak law of large numbers, in 1909, the French mathematician Émile Borel (1871–1956) improved the Bernoulli theorem by proving the strong law of large numbers:

$$\hat{p}_n \to p \ \text{ almost surely.}$$

The Borel strong law of large numbers is a key component in the proof of the Glivenko–Cantelli theorem which states that a population distribution function can be uniformly approximated by a sample distribution function as the sample size approaches infinity.

Rényi [187, Page 400] referred to this result as the *fundamental theorem of mathematical statistics* while Loève [141, Page 20] called it the *central statistical theorem*.

Following the contributions of Bernoulli, Poisson, and Borel, other mathematicians such as Cantelli, Markov, Khintchine, and Kolmogorov established the laws of large numbers in more general forms. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables, Khintchine [120] proved that if $\mathbb{E}X_1 = c \in \mathbb{R}$, then the weak law of large numbers

$$\frac{X_1 + \cdots + X_n}{n} \to c \text{ in probability}$$

holds. It was proved by Kolmogorov that condition $\mathbb{E}X_1 = c \in \mathbb{R}$ is necessary and sufficient for the strong law of large numbers (see, e.g., Gut [89, Pages 295–296]), that is

$$\mathbb{E}X_1 = c \in \mathbb{R} \text{ if and only if } \frac{X_1 + \cdots + X_n}{n} \to c \text{ almost surely.}$$

The sufficient part of Kolmogorov strong law of large numbers asserts that the *sample mean* converges almost surely to the *population mean* as the *sample size* n approaches infinity provided the population mean exists and is finite. This result is of fundamental importance in statistical science. We refer to Rosalsky and Thành [200] for further discussions on the history and importance of the laws of large numbers.

Pairwise independence is an important concept in probability and statistics [80, 158] and has various applications in algorithm design and computer science [98, 119, 142, 165], offering a balance between mutual independence and dependence. In algorithm design and computer science, pairwise independence allows for efficient use of randomness in various applications, including randomized algorithms and derandomization techniques. A crucial tool in proving the strong law of large numbers is the Kolmogorov maximal inequality which is no longer valid for pairwise independent random variables. Etemadi [71] used the method of subsequences and proved that the Kolmogorov strong law of large numbers for the i.i.d. case still holds if the independence assumption is weakened to pairwise independence. This result was later extended to the so-called Marcinkiewicz–Zygmund strong law of large numbers by Rio [189]. Laws of large numbers and related limit theorems for sequences of pairwise independent random variables have also been developed in various other papers, we refer to [2, 13, 20, 60, 61, 113, 129, 147, 146, 191]. More recent contributions include those in [7, 27, 74, 73, 143, 172, 184, 219, 222] and the references cited therein.

Because of the rich history and ongoing interest in the law of large numbers and related limit theorems for pairwise independent random variables, it is desirable to have a comprehensive work that addresses these topics in a systematic and unified manner. This book aims to fill that gap by presenting a thorough treatment of the laws of large numbers, delving into both classical results and recent advancements. We desire to provide an enjoyable overview of the laws of large numbers for pairwise independent random variables, and to gather some recent developments on this topic. The main material is based on recent papers [225, 227, 229, 230, 231, 234, 236] by the author. The book is designed for Master's and PhD students, as well as for researchers, who are interested in the study of laws of large numbers and related limit theorems in probability. However, undergraduate students who have been taking courses in foundations of probability may also find it useful, as we provide a comprehensive exploration of the subject, offering both foundational insights and detailed discussions of recent advancements.

The book is organized as follows. Chapter 1 provides an introduction and some preliminaries which are needed in the subsequent chapters. The weak laws of large numbers and mean convergence theorems are presented in Chapter 2. For clarity and ease of presentation, we first recall the concepts of convergence in probability and convergence in mean of order p. Then we use Rio's method to provide some new maximal inequalities for pairwise independent random variables. These maximal inequalities play a key role in establishing a Feller-type weak law of large numbers and a Pyke–Root-type theorem on mean convergence for the maximum of partial sums from sequences of pairwise independent random variables. Results for the partial sums from triangular arrays of rowwise and pairwise independent random variables are also provided. In addition, we show by counterexamples that the conditions that we obtained are sharp. Chapter 3 focuses on strong laws of large numbers and related limit theorems such as a Hsu–Robbins–Erdös-type theorem and results on the (p, q)-type strong law of large numbers. Finally, Chapter 4 presents limit theorems for dependent random fields via Rio's method. This chapter covers more challenging concepts and delves into more complicated caculations. The author suggests that undergraduate and master's students may choose to skip it during their first readings of the book, possibly returning to it later when they have a stronger understanding about the methods and results presented in Chapters 2 and 3. At the end of each chapter, we include a section titled "Further Readings" to introduce some related results and possible generalizations, and to encourage the reader to use the methods presented in the chapter to explore new questions/problems and find solutions to them.

A glossary of symbols and conventions is provided for the reader's convenience. Results and equations are numbered by chapter and section. For instance, Theorem 3.3.1 is in Section 3.3 of Chapter 3, Equation (4.3.27) is the 27th Equation in Section 4.3 of Chapter 4, etc. Numerous references are given to stimulate further reading. The number(s) after each reference indicate the page(s) where that reference is cited.

I am grateful to Professor Nguyen Van Quang and Professor Andrew Rosalsky, from whom I have benefited substantially through many collaborations and stimulating discussions. I also would like to thank my former masters students, Ms. Vu Thi Ngoc Anh and Mr. Nguyen Chi Dzung, for their invaluable comments, suggestions, and corrections throughout the preparation of this book.

My greatest thank goes to my family, who are my constant source of happiness and inspiration. Without their encouragement and support, this work would not have been possible.

I appreciate readers sharing their thoughts, suggestions, and corrections as they read through this book. Your feedback is invaluable and will greatly contribute to future editions. Please reach out to me at levt@vinhuni.edu.vn with any insights you may have.

Vinh City, May 2024
Lê Văn Thành

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