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On the notions of stochastic domination and uniform integrability in the Cesàro sense with applications to weak laws of large numbers for random fields*

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Abstract. In this paper, we simplify the notion of stochastic domination in the Cesàro sense for arrays of random variables and provide some sharp sufficient conditions for a multidimensional array of random variables stochastically dominated in the Cesàro sense. We establish relationships between stochastic domination in the Cesàro sense and uniform integrability in the Cesàro sense for a random field. We give applications to the weak law of large numbers for multidimensional arrays of random variables, extending a recent result [F. Boukhari, On a weak law of large numbers with regularly varying normalizing sequences, *J. Theor. Probab.*, 35:2068–2079, 2021] by a different method. We illustrate the sharpness of the results by two examples.

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1 Introduction

Let $\mathbb{Z}_+^d$, where d is a positive integer, denote the positive integer d-dimensional lattice points. The notation $\mathbf{m} \prec \mathbf{n}$ (or $\mathbf{n} \succ \mathbf{m}$) for $\mathbf{m} = (m_1, m_2, \ldots, m_d)$ and $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}_+^d$ means that $m_i \leqslant n_i$, $1 \leqslant i \leqslant d$. Let $\mathbf{1} = (1, \ldots, 1) \in \mathbb{Z}_+^d$. For $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}_+^d$, we denote $|\mathbf{n}| = \prod_{i=1}^d n_i$. A d-dimensional array of random variables $\{X_\mathbf{n}, \mathbf{n} \in \mathbb{Z}_+^d\}$ is said to be *stochastically dominated* by a random variable X if

$$\sup_{\mathbf{n} \succ \mathbf{1}} \mathrm{P}\big(|X_\mathbf{n}| > x\big) \leqslant \mathrm{P}\big(|X| > x\big) \quad \text{for all } x \in \mathbb{R}.$$

This notion is an extension of that for identically distributed $\{X_\mathbf{n}, \mathbf{n} \in \mathbb{Z}_+^d\}$. Fazekas [7], Gut [9], and Hu et al. [12] proved various strong laws of large numbers for multidimensional arrays of random variables under stochastic domination conditions.

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The notion of stochastic domination was extended to the notion of the so-called Cesàro stochastic domination by Gut [9] (for triangular arrays) and Fazekas and Tómács [8] (for d-dimensional arrays). A d-dimensional array of random variables $\{X_{\mathbf{n}},\ \mathbf{n} \in \mathbb{Z}_+^d\}$ is said to be *stochastically dominated in the Cesàro sense* (or *weakly mean dominated*) by a random variable Y if

$$\sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{1}\prec\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}| > x\big) \leqslant C\mathrm{P}\big(|Y| > x\big) \quad \text{for all } x \in \mathbb{R}. \tag{1.1}$$

In this paper, by using some techniques developed by Rosalsky and Thành [15] we prove that the constant C in (1.1) does not play any role, and the inequality in (1.1) can be replaced by equality. More precisely, we prove that the Gut–Fazekas–Tómács definition is equivalent to the following: A d-dimensional array of random variables $\{X_{\mathbf{n}},\ \mathbf{n} \in \mathbb{Z}_+^d\}$ is *stochastically dominated in the Cesàro sense* by a random variable X if

$$\sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{1}\prec\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}| > x\big) = \mathrm{P}\big(|X| > x\big) \quad \text{for all } x \in \mathbb{R}, \tag{1.2}$$

that is, we can choose $C = 1$ in (1.1). We also establish relationships between stochastic domination in the Cesàro sense and uniform integrability in the Cesàro sense (see Chandra [4, p. 309]) for multidimensional arrays of random variables. Then we apply these results to establish the weak law of large numbers for multidimensional arrays of random variables. A considerable extension of a recent result of Boukhari [3, Thm. 1.2] is proved by a completely different method.

The rest of the paper is arranged as follows. In Section 2, we establish the equivalence of the definitions of the Cesàro stochastic domination given by (1.1) and (1.2) and provide sharp sufficient conditions for a multidimensional array of random variables to be stochastically dominated in the Cesàro sense. In Section 3, we present and prove relationships between stochastic domination in the Cesàro sense and uniform integrability in the Cesàro sense. In Section 4, we give applications to the weak law of large numbers for multidimensional arrays of random variables. Finally, in the Appendix, we prove a technical result.

2 On stochastic domination in the Cesàro sense

In this section, we prove some new results on the notion of stochastic domination in the Cesàro sense. For a d-dimensional array of random variables $\{X_{\mathbf{n}},\ \mathbf{n} \in \mathbb{Z}_+^d\}$, the following proposition characterizes when the function

$$F(x) = 1 - \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}| > x\big), \quad x \in \mathbb{R},$$

is the distribution function of a random variable X such that $\{X_{\mathbf{n}},\ \mathbf{n} \in \mathbb{Z}_+^d\}$ is stochastically dominated by X in the Cesàro sense. The proof is similar to that of Theorem 2.1 in [15] (see also Theorem 2.1 in [18]) and is presented in the Appendix.

Proposition 1. *Let $\{X_{\mathbf{n}},\ \mathbf{n} \in \mathbb{Z}_+^d\}$ be a d-dimensional array of random variables, and let*

$$F(x) = 1 - \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}| > x\big), \quad x \in \mathbb{R}.$$

Then F is nondecreasing and right continuous, and $\lim_{x\to-\infty} F(x) = 0$. Moreover, F is the distribution function of a random variable X if and only if $\lim_{x\to+\infty} F(x) = 1$. In such a case, $\{X_{\mathbf{n}},\ \mathbf{n} \in \mathbb{Z}_+^d\}$ is stochastically dominated by X in the Cesàro sense.

Using Proposition 1, we can establish the equivalence of the definitions of the Cesàro stochastic domination given (1.1) and (1.2) as follows.

Theorem 1. *Let $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a d-dimensional array of random variables. Then there exists a random variable X satisfying* (1.2) *if and only if there exist a random variable Y and a finite constant $C > 0$ satisfying* (1.1).

Proof. If there exists a random variable X satisfying (1.2), then (1.1) is immediate by taking $Y = X$ and $C = 1$. Conversely, assume that there exist a random variable Y and a finite constant $C > 0$ satisfying (1.1). Then

$$\lim_{x\to+\infty} \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{1}\prec\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}| > x\big) \leqslant C \lim_{x\to+\infty} \mathrm{P}\big(|Y| > x\big) = 0,$$

and so by Proposition 1 there exists a random variable X with the distribution function

$$F(x) = 1 - \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{1}\prec\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}| > x\big), \quad x \in \mathbb{R},$$

satisfying (1.2). The proof of the theorem is completed. □

In view of Theorem 1, in the rest of the paper, we will use the definition of Cesàro stochastic domination as in (1.2). To provide useful sufficient conditions for the Cesàro stochastic domination, we will need some notation and definitions. We recall that a real-valued function R is said to be regularly varying with the index of regular variation $\rho \in \mathbb{R}$ if it is a positive and measurable function on $[A, +\infty)$ for some $A > 0$, and for each $\lambda > 0$,

$$\lim_{x\to+\infty} \frac{R(\lambda x)}{R(x)} = \lambda^\rho.$$

A regularly varying function with the index of regular variation $\rho = 0$ is called slowly varying. It is well known that a function R is regularly varying with the index of regular variation ρ if and only if it can be written in the form

$$R(x) = x^\rho L(x),$$

where L is a slowly varying function (see, e.g., [16, p. 2]). Let L be a slowly varying function. Then by [2, Thm. 1.5.13] there exists a slowly varying function $\tilde{L}$, unique up to asymptotic equivalence, satisfying

$$\lim_{x\to+\infty} L(x)\tilde{L}\big(xL(x)\big) = 1 \quad \text{and} \quad \lim_{x\to+\infty} \tilde{L}(x)L\big(x\tilde{L}(x)\big) = 1.$$

The function $\tilde{L}$ is called the de Bruijn conjugate of L (see, e.g., Bingham et al. [2, p. 29]). If $L(x) = \log^\gamma x$ for some $\gamma \in \mathbb{R}$, then $\tilde{L}(x) = 1/L(x)$, where $\log x = \log_2 \max\{2, x\}$, $x \geqslant 0$.

In the rest of the paper, by Anh et al. [1, Lemmas 2.2 and 2.3] we can assume without loss of generality that any regularly varying function with the index of regular variation $\rho > 0$ (resp., $\rho < 0$) is strictly increasing (resp., decreasing). We can also assume that all considered slowly varying functions L are differentiable and satisfy

$$\lim_{x\to+\infty} \frac{xL'(x)}{L(x)} = 0.$$

We will need the following simple lemma in the next main result. See Rosalsky and Thành [15] for a proof.

Lemma 1. *Let $g : [0, +\infty) \to [0, +\infty)$ be a measurable function with $g(0) = 0$ that is bounded on $[0, A]$ and differentiable on $[A, +\infty)$ for some $A \geqslant 0$. If ξ is a nonnegative random variable, then*

$$\mathrm{E}\big(g(\xi)\big) = \mathrm{E}\big(g(\xi)1(\xi \leqslant A)\big) + g(A) + \int_A^{+\infty} g'(x)\mathrm{P}(\xi > x)\,\mathrm{d}x.$$

The following theorem shows that uniformly bounded moment-type conditions on a d-dimensional array of random variables $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{Z}_+^d\}$ can guarantee stochastic domination in the Cesàro sense.

Theorem 2. *Let $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a d-dimensional array of random variables.*

(i) *Let $g : [0,+\infty) \to [0,+\infty)$ be a nondecreasing function on $[A,+\infty)$ for some $A > 0$ with $\lim_{x\to+\infty} g(x) = +\infty$. If*

$$\sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{E}\big(g\big(|X_{\mathbf{i}}|\big)\big) < +\infty, \tag{2.1}$$

then there exists a random variable X with the distribution function

$$F(x) = 1 - \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}| > x\big), \quad x \in \mathbb{R},$$

such that $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{Z}_+^d\}$ is stochastically dominated by X in the Cesàro sense.

(ii) *Let L be a slowly varying function. If*

$$\sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{E}\big(|X_{\mathbf{i}}|^p L\big(|X_{\mathbf{i}}|\big) \log\big(|X_{\mathbf{i}}|\big) \log^2\big(\log\big(|X_{\mathbf{i}}|\big)\big)\big) < +\infty \quad \textit{for some } p > 0, \tag{2.2}$$

then there exists a random variable X with the distribution function

$$F(x) = 1 - \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}| > x\big), \quad x \in \mathbb{R},$$

such that $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{Z}_+^d\}$ is stochastically dominated by X in the Cesàro sense and

$$\mathrm{E}\big(|X|^p L\big(|X|\big)\big) < +\infty.$$

Proof. (i) By the monotonicity of g and the Markov inequality we have

$$\sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}| > x\big) \leqslant \frac{1}{g(x)} \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{E}\big(g\big(|X_{\mathbf{i}}|\big)\big) \quad \text{for all } x \geqslant A.$$

Since $\lim_{x\to+\infty} g(x) = +\infty$, from (2.1) we have that

$$\lim_{x\to+\infty} \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}| > x\big) \leqslant \lim_{x\to+\infty} \left[\frac{1}{g(x)} \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{E}\big(g\big(|X_{\mathbf{i}}|\big)\big)\right] = 0.$$

By Proposition 1 the array $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{Z}_+^d\}$ is stochastically dominated in the Cesàro sense by a random variable X with distribution function

$$F(x) = 1 - \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}| > x\big), \quad x \in \mathbb{R}.$$

This completes the proof of (i).

(ii) Let

$$g(x) = x^p L(x) \log(x) \log^2(\log x), \quad h(x) = x^p L(x), \quad x \geqslant 0.$$

Since L is a slowly varying function, we have $\lim_{x\to+\infty} xL'(x)/L(x) = 0$. Then there exists B large enough such that g and h are strictly increasing on $[B, +\infty)$ and $|xL'(x)/L(x)| \leqslant p/2$, $x > B$. Therefore

$$h'(x) = px^{p-1}L(x) + x^pL'(x) = x^{p-1}L(x)\left(p + \frac{xL'(x)}{L(x)}\right) \leqslant \frac{3px^{p-1}L(x)}{2}, \quad x > B. \tag{2.3}$$

Since g is strictly increasing on $[B, +\infty)$, it follows from (2.2) and Theorem 2(i) that $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{Z}^d_+\}$ is stochastically dominated in the Cesàro sense by a random variable X with distribution function

$$F(x) = 1 - \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}| > x\big), \quad x \in \mathbb{R}.$$

Moreover, by the Markov inequality we have

$$\sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}| > x\big) \leqslant \frac{1}{g(x)} \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{E}\big(g\big(|X_{\mathbf{i}}|\big)\big) \quad \text{for all } x \geqslant B,$$

which is equivalent to

$$g(x)\mathrm{P}\big(|X| > x\big) \leqslant \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{E}\big(g\big(|X_{\mathbf{i}}|\big)\big) \quad \text{for all } x \geqslant B. \tag{2.4}$$

By Lemma 1, (2.2), (2.3), and (2.4) there exists a constant C_1 such that

$$\begin{aligned}
\mathrm{E}\big(h\big(|X|\big)\big) &= \mathrm{E}\big(h\big(|X|\big)1\big(|X| \leqslant B\big)\big) + h(B) + \int_B^{+\infty} h'(x)\mathrm{P}\big(|X| > x\big)\,\mathrm{d}x \\
&\leqslant C_1 + \frac{3p}{2} \int_B^{+\infty} x^{p-1}L(x)\mathrm{P}\big(|X| > x\big)\,\mathrm{d}x \\
&= C_1 + \frac{3p}{2} \int_B^{+\infty} g(x) \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}| > x\big) x^{-1} \log^{-1}(x) \log^{-2}(\log x)\,\mathrm{d}x \\
&\leqslant C_1 + \frac{3p}{2} \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{E}\big(g\big(|X_{\mathbf{i}}|\big)\big) \int_B^{+\infty} x^{-1} \log^{-1}(x) \log^{-2}(\log x)\,\mathrm{d}x \\
&< +\infty.
\end{aligned}$$

The proof of (ii) is completed. $\square$

3 Relationships between stochastic domination and uniform integrability in the Cesàro sense

The notion of uniform integrability in the Cesàro sense for sequences of random variables was introduced by Chandra [4]. Following Chandra [4], we say that a d-dimensional array of random variables $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{Z}^d_+\}$ is *uniformly integrable in the Cesàro sense* if

$$\lim_{a\to+\infty} \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{E}\big(|X_{\mathbf{i}}|1\big(|X_{\mathbf{i}}| > a\big)\big) = 0.$$

The de la Vallée Poussin criterion for sequences of random variables was established by Chandra and Goswami [5, p. 228]. For d-dimensional arrays, it is formulated as follows. The proof is the same as that of Chandra and Goswami [5, p. 228] in the case $d = 1$ and can be found in [6].

Lemma 2. *Let $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a d-dimensional array of random variables. Consider the following statement*:

(A) *There exists a measurable function ϕ defined on $[0, +\infty)$ with $\phi(0) = 0$ such that $\phi(t)/t \to +\infty$ as $t \to +\infty$ and*

$$\sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{E}\big(\phi\big(|X_{\mathbf{i}}|\big)\big) < +\infty.$$

Then the following statements hold:

(i) *If* (A) *holds, then $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{Z}_+^d\}$ is uniformly integrable in the Cesàro sense.*

(ii) *If $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{Z}_+^d\}$ is uniformly integrable in the Cesàro sense, then* (A) *holds. Moreover, ϕ can be chosen convex and such that $\phi(t)/t$ is increasing.*

We will now establish relationships between stochastic domination in the Cesàro sense and uniform integrability in the Cesàro sense for d-dimensional arrays of random variables. The main result of this section is the following theorem.

Theorem 3. *Let $p > 0$, let $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a d-dimensional array of random variables, and let L be a slowly varying function.*

(i) *If $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{Z}_+^d\}$ is stochastically dominated in the Cesàro sense by a random variable X satisfying*

$$\mathrm{E}\big(|X|^p L\big(|X|^p\big)\big) < +\infty,$$

then $\{|X_{\mathbf{n}}|^p L(|X_{\mathbf{n}}|^p), \mathbf{n} \in \mathbb{Z}_+^d\}$ is uniformly integrable in the Cesàro sense.

(ii) *If $\{|X_{\mathbf{n}}|^p L(|X_{\mathbf{n}}|^p), \mathbf{n} \in \mathbb{Z}_+^d\}$ is uniformly integrable in the Cesàro sense, then there exists a random variable X with distribution function*

$$F(x) = 1 - \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}| > x\big), \quad x \in \mathbb{R},$$

such that $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{Z}_+^d\}$ is stochastically dominated in the Cesàro sense by X and

$$\mathrm{E}\left(\frac{|X|^p L(|X|^p)}{\log(|X|)\log^2(\log(|X|))}\right) < +\infty. \tag{3.1}$$

Moreover,

$$\lim_{n\to+\infty} n\mathrm{P}\big(|X| > b_n\big) = 0, \tag{3.2}$$

where $b_n = n^{1/p}\tilde{L}^{1/p}(n)$, $n \geqslant 1$.

Proof. Let $f(x) = x^p L(x^p)$ and $g(x) = x^{1/p}\tilde{L}^{1/p}(x)$, $x \geqslant 0$. Recall that we assume f and g to be strictly increasing on $[0, +\infty)$.

(i) Since $\mathrm{E}(|X|^p L(|X|^p)) < +\infty$, it follows from the de la Vallée Poussin criterion for uniform integrability that there exists a continuous strictly increasing function $h : [0, +\infty) \to [0, +\infty)$ such that $h(0) = 0$,

$\lim_{x\to+\infty} h(x)/x = +\infty$, and $\mathrm{E}(h(|X|^p L(|X|^p))) < +\infty$. Since $f(x)$ is strictly increasing on $[0,+\infty)$, the Cesàro stochastic domination assumption ensures that

$$\sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(f\big(|X_{\mathbf{i}}|\big) > h^{-1}(x)\big) = \mathrm{P}\big(f\big(|X|\big) > h^{-1}(x)\big), \quad x \in \mathbb{R},$$

or, equivalently,

$$\sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}|^p L\big(|X_{\mathbf{i}}|^p\big) > h^{-1}(x)\big) = \mathrm{P}\big(|X|^p L\big(|X|^p\big) > h^{-1}(x)\big), \quad x \in \mathbb{R},$$

which in turn is equivalent to

$$\sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(h\big(|X_{\mathbf{i}}|^p L\big(|X_{\mathbf{i}}|^p\big)\big) > x\big) = \mathrm{P}\big(h\big(|X|^p L\big(|X|^p\big)\big) > x\big), \quad x \in \mathbb{R}.$$

It follows that

$$\begin{aligned}
\sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{E}\big(h\big(|X_{\mathbf{i}}|^p L\big(|X_{\mathbf{i}}|^p\big)\big)\big) &= \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \int_0^{+\infty} \mathrm{P}\big(h\big(|X_{\mathbf{i}}|^p L\big(|X_{\mathbf{i}}|^p\big)\big) > x\big)\, \mathrm{d}x \\
&= \int_0^{+\infty} \mathrm{P}\big(h\big(|X|^p L\big(|X|^p\big)\big) > x\big)\, \mathrm{d}x \\
&= \mathrm{E}\big(h\big(|X|^p L\big(|X|^p\big)\big)\big) < +\infty.
\end{aligned}$$

By Lemma 2(i), $\{|X_{\mathbf{n}}|^p L(|X_{\mathbf{n}}|^p),\ \mathbf{n} \in \mathbb{Z}_+^d\}$ is uniformly integrable in the Cesàro sense.

(ii) We now prove (ii). By Lemma 2(ii) there exists a nondecreasing function h on $[0,+\infty)$ with $h(0) = 0$ such that

$$\lim_{x\to+\infty} \frac{h(x)}{x} = +\infty \tag{3.3}$$

and

$$\sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{E}\big(h\big(f\big(|X_{\mathbf{i}}|\big)\big)\big) = \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{E}\big(h\big(|X_{\mathbf{i}}|^p L\big(|X_{\mathbf{i}}|^p\big)\big)\big) < +\infty, \tag{3.4}$$

and so by Theorem 2(i), (3.4) implies that $\{X_{\mathbf{n}},\ \mathbf{n} \in \mathbb{Z}_+^d\}$ is stochastically dominated in the Cesàro sense by a random variable X with distribution function

$$F(x) = 1 - \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}| > x\big), \quad x \in \mathbb{R}.$$

Since $\{|X_{\mathbf{n}}|^p L(|X_{\mathbf{n}}|^p), \mathbf{n} \in \mathbb{Z}_+^d\}$ is uniformly integrable in the Cesàro sense, (3.1) follows from Theorem 2(ii).

We will now prove (3.2). By Anh et al. [1, Lemma 2.1] we have $\lim_{x\to+\infty} f(g(x))/x = 1$, and therefore

$$f\big(g(n)\big) > \frac{n}{2} \quad \text{for all large } n. \tag{3.5}$$

Thus by (3.3), (3.4), (3.5), and Markov's inequality we have that

$$\begin{aligned}
\lim_{n\to+\infty} n\mathrm{P}\big(|X|>b_n\big) &= \lim_{n\to+\infty} n \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|}\sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}|>g(n)\big) \\
&= \lim_{n\to+\infty} n \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|}\sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(f\big(|X_{\mathbf{i}}|\big)\geqslant f\big(g(n)\big)\big) \\
&\leqslant \lim_{n\to+\infty} n \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|}\sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\bigg(f\big(|X_{\mathbf{i}}|\big)\geqslant \frac{n}{2}\bigg) \\
&= \lim_{n\to+\infty} n \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|}\sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\bigg(h\big(f\big(|X_{\mathbf{i}}|\big)\big)\geqslant h\bigg(\frac{n}{2}\bigg)\bigg) \\
&\leqslant \lim_{n\to+\infty} n \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|}\sum_{\mathbf{i}\prec\mathbf{n}} \frac{\mathrm{E}(h(f(|X_{\mathbf{i}}|)))}{h(\frac{n}{2})} \\
&= 2\sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|}\sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{E}\big(h\big(f\big(|X_{\mathbf{i}}|\big)\big)\big) \lim_{n\to+\infty}\frac{\frac{n}{2}}{h(\frac{n}{2})} = 0,
\end{aligned}$$

verifying (3.2). The proof of the theorem is completed. □

As we will see in Section 4, (3.2) is a sufficient condition for the weak laws of large numbers in the case $0<p<2$. See [3, 5, 10, 11, 13, 14] and the references therein for other results of this type. We close this section by presenting an example showing that in Theorem 3(ii) the assumption that $\{|X_{\mathbf{n}}|^p L(|X_{\mathbf{n}}|^p),\ \mathbf{n}\in\mathbb{Z}_+^d\}$ is uniformly integrable in the Cesàro sense cannot be weakened to the assumption that

$$\sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|}\sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{E}\big(|X_{\mathbf{n}}|^p L\big(|X_{\mathbf{n}}|^p\big)\big) < +\infty. \tag{3.6}$$

Example 1. Let $p>0$, and let $\{X_{\mathbf{n}},\ \mathbf{n}\in\mathbb{Z}_+^d\}$ be a d-dimensional array of random variables such that

$$\mathrm{P}(X_{\mathbf{n}}=0) = 1-\frac{1}{|\mathbf{n}|},\quad \mathrm{P}\big(X_{\mathbf{n}}=|\mathbf{n}|^{1/p}\big) = \frac{1}{|\mathbf{n}|},\quad \mathbf{n}\succ\mathbf{1}.$$

For the slowly varying function $L(x)\equiv 1$, we have

$$\sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|}\sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{E}\big(|X_{\mathbf{n}}|^p L\big(|X_{\mathbf{n}}|^p\big)\big) = \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|}\sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{E}\big(|X_{\mathbf{n}}|^p\big) = 1,$$

verifying (3.6). By part (i) of Theorem 2, $\{X_{\mathbf{n}},\ \mathbf{n}\in\mathbb{Z}_+^d\}$ is stochastically dominated in the Cesàro sense by a random variable X with distribution function

$$F(x) = 1-\sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|}\sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}\big(|X_{\mathbf{i}}|>x\big),\quad x\in\mathbb{R}.$$

On the other hand, for all $a>0$, we have

$$\sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|}\sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{E}\big(|X_{\mathbf{n}}|^p 1\big(|X_{\mathbf{n}}|>a\big)\big) = 1,$$

which implies

$$\lim_{a\to+\infty}\sup_{\mathbf{n}\succ\mathbf{1}}\frac{1}{|\mathbf{n}|}\sum_{\mathbf{i}\prec\mathbf{n}}\mathrm{E}\big(|X_{\mathbf{n}}|^p 1\big(|X_{\mathbf{n}}|>a\big)\big)=1.$$

Therefore $\{|X_{\mathbf{n}}|^p, \mathbf{n}\in\mathbb{Z}_+^d\}$ is not uniformly integrable in the Cesàro sense.

Now, for $x\in\mathbb{R}$, let $\lceil x\rceil$ be the smallest integer that is greater than x. Then

$$\mathrm{P}(X>x)=\begin{cases}1 & \text{if } x<1,\\ \frac{1}{\lceil x^p\rceil} & \text{if } x\geqslant 1.\end{cases}\tag{3.7}$$

It follows from (3.7) that

$$\lim_{n\to+\infty} n\mathrm{P}\big(X>n^{1/p}\big)=\lim_{n\to+\infty}\frac{n}{n+1}=1,$$

thereby showing that (3.2) fails.

4 Applications to weak laws of large numbers

In this section, we establish WLLNs for d-dimensional arrays of random variables under the Cesàro stochastic domination condition without any dependence structure. Throughout this section, by C we denote a positive universal constant, not necessarily the same in each appearance.

Firstly, we will present a general WLLN for d-dimensional arrays of random variables without any dependence structure.

Theorem 4. *Let $0<p\leqslant 1$. Let $\{X_{\mathbf{n}}, \mathbf{n}\in\mathbb{Z}_+^d\}$ be a d-dimensional array of random variables, and let $\{b_n, n\geqslant 1\}$ be a nondecreasing sequence of positive constants. If*

$$\sum_{\mathbf{i}\prec\mathbf{n}}\mathrm{P}\big(X_{\mathbf{i}}\neq Y_{\mathbf{n},\mathbf{i}}\big)\to 0\quad \textit{as } |\mathbf{n}|\to+\infty\tag{4.1}$$

and

$$\frac{\sum_{\mathbf{i}\prec\mathbf{n}}\mathrm{E}(|Y_{\mathbf{n},\mathbf{i}}|^p)}{b_{|\mathbf{n}|}^p}\to 0\quad \textit{as } |\mathbf{n}|\to+\infty,\tag{4.2}$$

where

$$Y_{\mathbf{n},\mathbf{i}}=X_{\mathbf{i}}1\big(|X_{\mathbf{i}}|\leqslant b_{|\mathbf{n}|}\big),\quad \mathbf{1}\prec\mathbf{i}\prec\mathbf{n},\mathbf{n}\succ 1,$$

then we have the WLLN

$$\frac{\max_{\mathbf{k}\prec\mathbf{n}}|\sum_{\mathbf{i}\prec\mathbf{k}}X_{\mathbf{i}}|}{b_{|\mathbf{n}|}}\xrightarrow{\mathrm{P}} 0\quad \textit{as } |\mathbf{n}|\to+\infty.\tag{4.3}$$

Proof. Let $\varepsilon>0$ be arbitrary. Firstly, by (4.1) we have

$$\begin{aligned}\mathrm{P}\bigg(\frac{\max_{\mathbf{k}\prec\mathbf{n}}|\sum_{\mathbf{i}\prec\mathbf{k}}(X_{\mathbf{i}}-Y_{\mathbf{n},\mathbf{i}})|}{b_{|\mathbf{n}|}}>\varepsilon\bigg)&\leqslant\mathrm{P}\bigg(\sum_{\mathbf{i}\prec\mathbf{n}}X_{\mathbf{i}}\neq\sum_{\mathbf{i}\prec\mathbf{n}}Y_{\mathbf{n},\mathbf{i}}\bigg)\leqslant\mathrm{P}\bigg(\bigcup_{\mathbf{i}\prec\mathbf{n}}\big(|X_{\mathbf{i}}|>b_{|\mathbf{n}|}\big)\bigg)\\&\leqslant\sum_{\mathbf{i}\prec\mathbf{n}}\mathrm{P}\big(|X_{\mathbf{i}}|>b_{|\mathbf{n}|}\big)\to 0\quad\text{as } |\mathbf{n}|\to+\infty.\end{aligned}$$

This implies

$$\frac{\max_{\mathbf{k}\prec\mathbf{n}}|\sum_{\mathbf{i}\prec\mathbf{k}}(X_{\mathbf{i}}-Y_{\mathbf{n},\mathbf{i}})}{b_{|\mathbf{n}|}}\xrightarrow{\mathrm{P}}0\quad\text{as } |\mathbf{n}|\to+\infty. \tag{4.4}$$

Next, by Markov's inequality, the C_p-inequality, and (4.2) we have

$$\begin{aligned}\mathrm{P}\biggl(\frac{\max_{\mathbf{k}\prec\mathbf{n}}|\sum_{\mathbf{i}\prec\mathbf{k}}Y_{\mathbf{n},\mathbf{i}}|}{b_{|\mathbf{n}|}}>\varepsilon\biggr)&\leqslant\frac{\mathrm{E}(\max_{\mathbf{k}\prec\mathbf{n}}|\sum_{\mathbf{i}\prec\mathbf{k}}Y_{\mathbf{n},\mathbf{i}}|)^p}{\varepsilon^p b^p_{|\mathbf{n}|}}\\&\leqslant\frac{\mathrm{E}(\sum_{\mathbf{i}\prec\mathbf{n}}|Y_{\mathbf{n},\mathbf{i}}|)^p}{\varepsilon^p b^p_{|\mathbf{n}|}}\leqslant\frac{\mathrm{E}(\sum_{\mathbf{i}\prec\mathbf{n}}|Y_{\mathbf{n},\mathbf{i}}|^p)}{\varepsilon^p b^p_{|\mathbf{n}|}}\\&=\frac{\sum_{\mathbf{i}\prec\mathbf{n}}\mathrm{E}(|Y_{\mathbf{n},\mathbf{i}}|^p)}{\varepsilon^p b^p_{|\mathbf{n}|}}\to0\quad\text{as } |\mathbf{n}|\to+\infty.\end{aligned}$$

This implies

$$\frac{\max_{\mathbf{k}\prec\mathbf{n}}|\sum_{\mathbf{i}\prec\mathbf{k}}Y_{\mathbf{n},\mathbf{i}}|}{b_{|\mathbf{n}|}}\xrightarrow{\mathrm{P}}0\quad\text{as } |\mathbf{n}|\to+\infty. \tag{4.5}$$

Combining (4.4) and (4.5) yields (4.3). $\square$

The following theorem is the main result of this section. The particular case $d=1$ of Theorem 5 considerably extends Theorem 1.2 of Boukhari [3]. We note that our proof is completely different from that of Theorem 1.2 of Boukhari [3].

Theorem 5. *Let $\{X_{\mathbf{n}},\,\mathbf{n}\in\mathbb{Z}^d_+\}$ be a d-dimensional array of random variables that is stochastically dominated in the Cesàro sense by a random variable X. Let $\{b_n,\,n\geqslant1\}$ be a nondecreasing sequence of positive constants satisfying*

$$\sum_{k=1}^{n}\frac{b_k}{k^2}=O\biggl(\frac{b_n}{n}\biggr). \tag{4.6}$$

If

$$\lim_{k\to+\infty}k\mathrm{P}\bigl(|X|>b_k\bigr)=0, \tag{4.7}$$

then we have the WLLN

$$\frac{\max_{\mathbf{k}\prec\mathbf{n}}|\sum_{\mathbf{i}\prec\mathbf{k}}X_{\mathbf{i}}|}{b_{|\mathbf{n}|}}\xrightarrow{\mathrm{P}}0\quad\textit{as } |\mathbf{n}|\to+\infty. \tag{4.8}$$

Proof. For $\mathbf{n}\succ\mathbf{1}$, set

$$Y_{\mathbf{n},\mathbf{i}}=X_{\mathbf{i}}1\bigl(|X_{\mathbf{i}}|\leqslant b_{|\mathbf{n}|}\bigr),\quad \mathbf{1}\prec\mathbf{i}\prec\mathbf{n}.$$

We will verify conditions (4.1) and (4.2) of Theorem 4.

Firstly, by the Cesàro stochastic domination assumption and (4.7) we have

$$\sum_{\mathbf{i}\prec\mathbf{n}}\mathrm{P}\bigl(|X_{\mathbf{i}}|>b_{|\mathbf{n}|}\bigr)\leqslant|\mathbf{n}|\mathrm{P}\bigl(|X|>b_{|\mathbf{n}|}\bigr)\to0\quad\text{as } |\mathbf{n}|\to+\infty,$$

verifying (4.1).

Secondly, from the triangular inequality and the Cesàro stochastic domination assumption we obtain that

$$\begin{aligned}\frac{\sum_{\mathbf{i}\prec\mathbf{n}}\mathrm{E}(|Y_{\mathbf{n},\mathbf{i}}|)}{\varepsilon b_{|\mathbf{n}|}} &\leqslant \frac{1}{\varepsilon b_{|\mathbf{n}|}}\sum_{\mathbf{i}\prec\mathbf{n}}\int_0^{b_{|\mathbf{n}|}}\mathrm{P}\big(|X_{\mathbf{i}}|>t\big)\,\mathrm{d}t \leqslant \frac{|\mathbf{n}|}{\varepsilon b_{|\mathbf{n}|}}\int_0^{b_{|\mathbf{n}|}}\mathrm{P}\big(|X|>t\big)\,\mathrm{d}t\\ &= \frac{|\mathbf{n}|}{\varepsilon b_{|\mathbf{n}|}}\sum_{k=1}^{|\mathbf{n}|}\int_{b_{k-1}}^{b_k}\mathrm{P}\big(|X|>t\big)\,\mathrm{d}t \leqslant \frac{|\mathbf{n}|}{\varepsilon b_{|\mathbf{n}|}}\sum_{k=1}^{|\mathbf{n}|}(b_k-b_{k-1})\mathrm{P}\big(|X|>b_{k-1}\big)\\ &= \frac{|\mathbf{n}|}{\varepsilon b_{|\mathbf{n}|}}\sum_{k=1}^{|\mathbf{n}|}\frac{b_k-b_{k-1}}{k}k\mathrm{P}\big(|X|>b_{k-1}\big).\end{aligned} \tag{4.9}$$

Now, when $|\mathbf{n}|\geqslant 2$, by (4.6) we have

$$\begin{aligned}\frac{|\mathbf{n}|}{b_{|\mathbf{n}|}}\sum_{k=1}^{|\mathbf{n}|}\frac{b_k-b_{k-1}}{k} &= \frac{|\mathbf{n}|}{b_{|\mathbf{n}|}}\left(\sum_{k=1}^{|\mathbf{n}|-1}\frac{b_k}{k(k+1)}+\frac{b_{|\mathbf{n}|}}{|\mathbf{n}|}\right) \leqslant \frac{|\mathbf{n}|}{b_{|\mathbf{n}|}}\left(\sum_{k=1}^{|\mathbf{n}|-1}\frac{b_k}{k^2}+\frac{b_{|\mathbf{n}|}}{|\mathbf{n}|}\right)\\ &\leqslant O(1).\end{aligned} \tag{4.10}$$

Boukhari proved that (4.6) implies $b_n/n\to+\infty$ as $n\to+\infty$ (see, e.g., [3, Remark 2.4(i)]). Therefore, for all fixed $k\geqslant 1$, by (4.7) we have

$$\frac{|\mathbf{n}|}{b_{|\mathbf{n}|}}\left(\frac{b_k-b_{k-1}}{k}\right)\to 0 \quad \text{as } |\mathbf{n}|\to+\infty \quad \text{and} \quad k\mathrm{P}\big(|X|>b_{k-1}\big)\to 0 \quad \text{as } k\to+\infty. \tag{4.11}$$

Using (4.10) and (4.11) and applying the Toeplitz lemma, we have

$$\frac{|\mathbf{n}|}{\varepsilon b_{|\mathbf{n}|}}\sum_{k=1}^{|\mathbf{n}|}\frac{b_k-b_{k-1}}{k}k\mathrm{P}\big(|X|>b_{k-1}\big)\to 0 \quad \text{as } |\mathbf{n}|\to+\infty. \tag{4.12}$$

Combining (4.9) and (4.12) yields

$$\frac{\sum_{\mathbf{i}\prec\mathbf{n}}\mathrm{E}(|Y_{\mathbf{n},\mathbf{i}}|)}{b_{|\mathbf{n}|}}\to 0 \quad \text{as } |\mathbf{n}|\to+\infty,$$

thereby proving (4.2) with $p=1$.

The WLLN (4.8) thus follows by applying Theorem 4. □

The case $d=1$ of Theorem 5 was proved by Boukhari [3, Thm. 1.2] under the condition that the array $\{X_{\mathbf{n}},\,\mathbf{n}\in\mathbb{Z}_+^d\}$ is stochastically dominated by a random variable X satisfying (4.7). The following example illustrates that Theorem 1.2 of Boukhari [3] is strictly weaker than Theorem 5.

Example 2. Let $0<p<1$ and $b_n\equiv n^{1/p}$, and let $\{X_{\mathbf{n}},\,\mathbf{n}\in\mathbb{Z}_+^d\}$ be a d-dimensional array of random variables such that

$$\mathrm{P}(X_{\mathbf{n}}=0)=1 \quad \text{for } |\mathbf{n}|\neq 2^m,\ m\geqslant 0,$$

and

$$\mathrm{P}\big(X_{\mathbf{n}}=-m^{1/(2p)}\big)=\mathrm{P}\big(X_{\mathbf{n}}=m^{1/(2p)}\big)=\frac{1}{2} \quad \text{for } |\mathbf{n}|=2^m,\ m\geqslant 0. \tag{4.13}$$

For $\mathbf{n} \succ \mathbf{1}$, let $m \geqslant 0$ be such that $2^m \leqslant |\mathbf{n}| < 2^{m+1}$. Then

$$\sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{E}\big(|X_{\mathbf{i}}|^{2p}\big) \leqslant \sup_{\mathbf{n}\succ\mathbf{1}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{i}\prec\mathbf{n},\ \log(|\mathbf{i}|)\in\mathbb{Z}_+^1} \log\big(|\mathbf{i}|\big) \leqslant C < +\infty.$$

Therefore $\{X_{\mathbf{n}},\ \mathbf{n} \in \mathbb{Z}_+^d\}$ is uniformly integrable in the Cesàro sense by the de la Vallée Poussin criterion for the Cesàro uniform integrability (Lemma 2). Then by Theorem 3(ii) the array $\{X_{\mathbf{n}},\ \mathbf{n} \in \mathbb{Z}_+^d\}$ is stochastically dominated in the Cesàro sense by a random variable X, and condition (4.7) (with $b_n \equiv n^{1/p}$) is satisfied. We can easily check that condition (4.6) is also satisfied. It thus follows from Theorem 5 that the WLLN (4.8) holds.

On the other hand, we have from (4.13) that

$$\sup_{\mathbf{n}\succ\mathbf{1}} \mathrm{P}\big(|X_{\mathbf{n}}| > x\big) = 1 \quad \text{for all } x \in \mathbb{R}.$$

This implies that there is no random variable X such that the d-dimensional array $\{X_{\mathbf{n}},\ \mathbf{n} \in \mathbb{Z}_+^d\}$ is stochastically dominated by X, and so we cannot apply Theorem 1.2 of Boukhari [3] even in the case $d = 1$.

Combining Theorems 3 and 5, we have the following result. Note that in Corollary 1, if $L(x) \equiv 1$, then we obtain the L_p convergence in (4.14) (see [17, Thm. 2.1]). However, Corollary 1 is a new result for general slowly varying functions L.

Corollary 1. *Let $0 < p < 1$. Let $\{X_{\mathbf{n}},\ \mathbf{n} \in \mathbb{Z}_+^d\}$ be a d-dimensional array of random variables, and L be a slowly varying function. If $\{|X_{\mathbf{n}}|^p L(|X_{\mathbf{n}}|^p),\ \mathbf{n} \in \mathbb{Z}_+^d\}$ is uniformly integrable in the Cesàro sense, then*

$$\frac{\max_{\mathbf{k}\prec\mathbf{n}} |\sum_{\mathbf{i}\prec\mathbf{k}} X_{\mathbf{i}}|}{b_{|\mathbf{n}|}} \xrightarrow{\mathrm{P}} 0 \quad \text{as } |\mathbf{n}| \to +\infty, \tag{4.14}$$

where $b_n = n^{1/p}\tilde{L}^{1/p}(n)$, $n \geqslant 1$.

Proof. Since $\{|X_{\mathbf{n}}|^p L(|X_{\mathbf{n}}|^p),\ \mathbf{n} \in \mathbb{Z}_+^d\}$ is uniformly integrable in the Cesàro sense, it follows from Theorem 3 that $\{X_{\mathbf{n}},\ \mathbf{n} \in \mathbb{Z}_+^d\}$ is stochastically dominated in the Cesàro sense by a random variable X satisfying

$$\lim_{k\to+\infty} k\mathrm{P}\big(|X| > b_k\big) = 0. \tag{4.15}$$

By [3, Remark 2.4 (ii)] we have

$$\sum_{k=1}^{n} \frac{b_k}{k^2} = O\left(\frac{b_n}{n}\right). \tag{4.16}$$

From (4.15) and (4.16) we obtain (4.14) by applying Theorem 5. $\square$

Appendix

Proof of Proposition 1. It is clear that F is nondecreasing. Since $\mathrm{P}(|X_{\mathbf{i}}| > x) = 1$ for all $\mathbf{1} \prec \mathbf{i} \prec \mathbf{n}$ and $x < 0$, we have $\lim_{x\to-\infty} F(x) = 0$.

Let $G(x) = \sup_{\mathbf{n}\succ\mathbf{1}}(1/|\mathbf{n}|)\sum_{\mathbf{i}\prec\mathbf{n}} \mathrm{P}(|X_{\mathbf{i}}| > x)$, $x \in \mathbb{R}$. To show that F is right continuous, we will show that

$$\lim_{x\to a^+} G(x) = G(a) \quad \text{for all } a \in \mathbb{R}.$$

Let $\varepsilon > 0$ and $a \in \mathbb{R}$. Since $G(a) = \sup_{\mathbf{n}\succ\mathbf{1}}(1/|\mathbf{n}|)\sum_{\mathbf{i}\prec\mathbf{n}}\mathrm{P}(|X_{\mathbf{i}}| > a)$, there exists $\mathbf{n_0} \succ \mathbf{1}$ such that

$$\frac{1}{|\mathbf{n_0}|}\sum_{\mathbf{i}\prec\mathbf{n_0}}\mathrm{P}\big(|X_{\mathbf{i}}| > a\big) > G(a) - \frac{\varepsilon}{2}.$$

Since the function

$$x \mapsto \frac{1}{|\mathbf{n_0}|}\sum_{\mathbf{i}\prec\mathbf{n_0}}\mathrm{P}\big(|X_{\mathbf{i}}| > x\big), \quad x \in \mathbb{R},$$

is nonincreasing and right continuous, there exists $\delta > 0$ such that

$$-\frac{\varepsilon}{2} < \frac{1}{|\mathbf{n_0}|}\sum_{\mathbf{i}\prec\mathbf{n_0}}\mathrm{P}\big(|X_{\mathbf{i}}| > x\big) - \frac{1}{|\mathbf{n_0}|}\sum_{\mathbf{i}\prec\mathbf{n_0}}\mathrm{P}\big(|X_{\mathbf{i}}| > a\big) \leqslant 0 \quad \text{for all } x \text{ such that } 0 \leqslant x - a < \delta.$$

Therefore, for x such that $0 \leqslant x - a < \delta$, we have

$$\begin{aligned} G(x) + \varepsilon = \sup_{\mathbf{n}\succ\mathbf{1}}\frac{1}{|\mathbf{n}|}\sum_{\mathbf{i}\prec\mathbf{n}}\mathrm{P}\big(|X_{\mathbf{i}}| > x\big) + \varepsilon &\geqslant \frac{1}{|\mathbf{n_0}|}\sum_{\mathbf{i}\prec\mathbf{n_0}}\mathrm{P}\big(|X_{\mathbf{i}}| > x\big) + \varepsilon \\ &> \frac{1}{|\mathbf{n_0}|}\sum_{\mathbf{i}\prec\mathbf{n_0}}\mathrm{P}\big(|X_{\mathbf{i}}| > a\big) + \frac{\varepsilon}{2} > G(a), \end{aligned}$$

and so $|G(x) - G(a)| < \varepsilon$. Thus $\lim_{x\to a^+} G(x) = G(a)$. Since F is nondecreasing and right continuous with $\lim_{x\to-\infty} F(x) = 0$, it is the distribution function of a random variable X if and only if

$$\lim_{x\to+\infty} F(x) = 1.$$

Since $\mathrm{P}(|X_{\mathbf{i}}| > x) = 1$ for all $\mathbf{1} \prec \mathbf{i} \prec \mathbf{n}$ and $x < 0$, we have $F(x) = 0$ for all $x < 0$, that is, $X \geqslant 0$ almost surely. By the definition of F it is clear that (1.2) holds, that is, $\{X_{\mathbf{n}},\, \mathbf{n} \in \mathbb{Z}_+^d\}$ is stochastically dominated by X in the Cesàro sense. □

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