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On Mean Convergence for the Partial Sums from Arrays of Rowwise and Pairwise M_N-Negatively Dependent Random Variables

Võ Thị Hông Vân[1*] and Phạm Nhu Ý[1]**

(Submitted by A. I. Volodin)

[1] *Department of Mathematics, Vinh University, Nghe An, Vietnam*

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Abstract—In this study, we prove a mean convergence theorem for the partial sums from triangular arrays of rowwise and pairwise m_n-negatively dependent random variables, where m_n may be unbounded. The main theorem extends Theorem 3.1 of Chen, Bai, and Sung (J. Math. Anal. Appl. **419**, 1290–1302 (2014)).

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1. INTRODUCTION

The concept of negative dependence of random variables was introduced by Lehmann [9] and further studied by Block et al. [2], and Ebrahimi and Ghosh [8]. Let $n \geq 2$ be an integer. A collection of random variables $\{X_1, \ldots, X_n\}$ is said to be *negatively dependent* if for all $x_1, \ldots, x_n \in \mathbb{R}$,

$$\mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n) \leq \mathbb{P}(X_1 \leq x_1) \ldots \mathbb{P}(X_n \leq x_n),$$

and

$$\mathbb{P}(X_1 > x_1, \ldots, X_n > x_n) \leq \mathbb{P}(X_1 > x_1) \ldots \mathbb{P}(X_n > x_n).$$

A sequence of random variables $\{X_i, i \geq 1\}$ is said to be negatively dependent if for any $n \geq 1$, the collection $\{X_1, \ldots, X_n\}$ is negatively dependent. A sequence of random variables $\{X_i, i \geq 1\}$ is said to be *pairwise negatively dependent* if for all $x, y \in \mathbb{R}$ and for all $i \neq j$,

$$\mathbb{P}(X_i \leq x, X_j \leq y) \leq \mathbb{P}(X_i \leq x)P(X_j \leq y).$$

It is easy to prove that $\{X_i, i \geq 1\}$ is pairwise negatively dependent if and only if for all $x, y \in \mathbb{R}$ and for all $i \neq j$,

$$\mathbb{P}(X_i > x, X_j > y) \leq \mathbb{P}(X_i > x)\mathbb{P}(X_j > y).$$

Let m be a nonnegative integer. Anh [1] and Wu and Rosalsky [19] introduced the following notation. A collection $\{X_i, 1 \leq i \leq n\}$ of random variables is said to be *pairwise m-negatively dependent* if either $n \leq m+1$ or $n > m+1$ and X_i and X_j are negatively dependent whenever $|i-j| > m$. When $m = 0$, this reduces to the concept of pairwise negative dependence. If $m' > m$, then it is easy to see that pairwise m-negative dependence implies pairwise m'-negative dependence.

Let $\{m_n, n \geq 1\}$ be a sequence of nonnegative integers. Following the idea in [14], we introduce the concept of rowwise and pairwise negative dependence of a triangular array of random variables as follows. A triangular array $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ of random variables is said to be *rowwise and*

*E-mail: vanvth@vinhuni.edu.vn
**E-mail: yphamnhu@gmail.com

pairwise m_n-negatively dependent if for each $n \geq 1$, the n-th row $\{X_{n,i}, 1 \leq i \leq k_n\}$ is pairwise m_n-negatively dependent.

Mean convergence theorems for pairwise negatively dependent random variables were studied by various authors. Ordóñez Cabrera and Volodin [11, Theorem 1] studied $\mathcal{L}_1$ convergence theorem for weighted sums from arrays $\{X_{n,i}, 1 \leq i \leq k_n, n \geq 1\}$ of rowwise and pairwise negatively dependent random variables satisfying the so-called h-integrability concerning the array of weights, where $\{h(n), n \geq 1\}$ is an increasing sequence of positive constants. Wu and Guan [17], and Sung [12] extended Theorem 1 of Ordóñez Cabrera and Volodin [11] to $\mathcal{L}_p$ convergence, $1 \leq p < 2$. In this current paper, we establish $\mathcal{L}_p$ convergence theorem for triangular arrays of rowwise and pairwise m_n-negatively dependent random variables. Laws of large numbers for m-negatively dependent and m-negatively associated random variables were studied by some authors where m is fixed (see, e.g., Wang et al. [16] and Wu et al. [18] and the references therein). As far as we know, Thành [14] is the first author who established laws of large numbers for the case where m is unbounded. When $m = 0$ and the underlying random variables are pairwise independent, our main theorem reduces to Theorem 3.1 of Chen et al. [5].

Remark 1.1. Pairwise negative dependence is much weaker than pairwise dependence and negative association. The strong laws of large numbers for sequences of pairwise independent random variables and negatively associated random variables with regular varying norming sequences b_n, $n \geq 1$ were recently studied in [13, 15]. It would be interesting if one could extend the norming sequences of the form $b_n = n^{1/p}(m_n + 1)^{1/2}$ in this paper to the case $b_n = n^{1/p}L^{1/p}(n)(m_n + 1)^{1/2}$, where $L(\cdot)$ is a slowly varying function.

Through out the paper, for $x \geq 0$, let $\lfloor x \rfloor$ denote the greatest integer that is not greater than x and let $\log x$ denote the natural logarithm of $(x + 2)$. The symbol C_p denotes a constant depending only on p and is not necessarily the same one in each appearance.

2. PRELIMINARIES

In this section, we state some important properties of pairwise negatively dependent random variables which are needed in the proof of the main result. The first lemma follows from Lemma 1 of Lehmann [9].

Lemma 2.1. *Let $\{X_i, i \geq 1\}$ be a sequence of pairwise negatively dependent random variables and let $f_i : \mathbb{R} \to \mathbb{R}$, $i \geq 1$ be measurable functions. If $\{f_i, i \geq 1\}$ consists of only nondecreasing functions or only nonincreasing functions, then $\{f_i(X_i), i \geq 1\}$ is also a sequence of pairwise negatively dependent random variables.*

The following result is Lemma 2.2 in Li at al. [10].

Lemma 2.2. *If $\{X_1, \ldots, X_n\}$ are pairwise negatively dependent random variables with finite variances, then*

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) \leq \sum_{i=1}^{n} \operatorname{Var}(X_i).$$

The following lemma is a special case of Theorem 2.1 of Chen and Sung [6]. We note that in the case where $p = 1$ or $p = 2$, we can choose $C_p = 1$ in (1).

Lemma 2.3. *Let $1 \leq p \leq 2$. Let $\{X_i, 1 \leq i \leq n\}$ be a collection of pairwise negatively dependent mean zero random variables satisfying $\mathbb{E}|X_i|^p < \infty$, $1 \leq i \leq n$. Then,*

$$\mathbb{E}\left|\sum_{i=1}^{n} X_i\right|^p \leq C_p \sum_{i=1}^{n} \mathbb{E}|X_i|^p. \tag{1}$$

A triangular array $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ of random variables is said to be *uniformly integrable in the Cesàro sense* (see Chandra [3]) if

$$\lim_{a\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(|X_{n,i}|\mathbf{1}(|X_{n,i}| > a)) = 0.$$

In [4], Chandra and Goswami presented the de La Vallée-Poussin criterion for uniform integrability in the Cesàro sense which reads as follows: a triangular array of random variables $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$

is uniformly integrable in the Cesàro sense if and only if there exists a measurable function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$, $g(x)/x \to \infty$ as $x \to \infty$, and

$$\sup_{n \geq 1} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(g(|X_{n,i}|)) < \infty.$$

Relationships between uniform integrability in the Cesàro sense and stochastic dominated in the Cesàro sense were established by Dat et al. [7] and Thành [15].

3. MAIN RESULT

In this section, we will present and prove the main theorem of the paper.

Theorem 3.1. *Let $1 \leq p < 2$, let $\{m_n, n \geq 1\}$ be a sequence of nonnegative integers and let $\{X_{n,i}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of rowwise and pairwise m_n-negatively dependent random variables such that $\{|X_{n,i}|^p, 1 \leq i \leq n, n \geq 1\}$ is uniformly integrable in the Cesàro sense, that is,*

$$\lim_{a \to \infty} \sup_{n \geq 1} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left(|X_{n,i}|^p \mathbf{1}(|X_{n,i}| > a)\right) = 0.$$

Then,

$$\frac{1}{n^{1/p}(m_n + 1)^{1/2}} \sum_{i=1}^{n} (X_{n,i} - \mathbb{E}X_{n,i}) \xrightarrow{\mathcal{L}_p} 0 \text{ as } n \to \infty. \tag{2}$$

Proof. Let $\varepsilon > 0$ be arbitrary. Since $\{|X_{n,i}|^p, 1 \leq i \leq n, n \geq 1\}$ is uniformly integrable in the Cesàro sense, there exists $M > 0$ such that

$$\sup_{n \geq 1} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left(|X_{n,i}|^p \mathbf{1}(|X_{n,i}| > M)\right) < \varepsilon. \tag{3}$$

For $n \geq 1, 1 \leq i \leq n$, set

$$X_{n,i}^{(1)} = -M\mathbf{1}(X_{n,i} < -M) + X_{n,i}\mathbf{1}(|X_{n,i}| \leq M) + M\mathbf{1}(X_{n,i} > M), \quad X_{n,i}^{(2)} = X_{n,i} - X_{n,i}^{(1)},$$

and

$$Y_{n,i} = X_{n,i}^{(1)} - \mathbb{E}X_{n,i}^{(1)}, \quad Z_{n,i} = X_{n,i}^{(2)} - \mathbb{E}X_{n,i}^{(2)}.$$

Applying the C_p-inequality, we have

$$\mathbb{E}\left|\sum_{i=1}^{n} (X_{n,i} - \mathbb{E}X_{n,i})\right|^p \leq 2\left(\mathbb{E}\left|\sum_{i=1}^{n} Y_{n,i}\right|^p + \mathbb{E}\left|\sum_{i=1}^{n} Z_{n,i}\right|^p\right) := 2(I_1 + I_2). \tag{4}$$

The proof of (2) will be completed if we can show that

$$\frac{I_1}{n(m_n + 1)^{p/2}} \to 0 \text{ as } n \to \infty, \tag{5}$$

and

$$\frac{I_2}{n(m_n + 1)^{p/2}} \to 0 \text{ as } n \to \infty. \tag{6}$$

Step 1: Proof of (5). By Lemma 2.1, we have that for all $n \geq 1$, $\{Y_{n,i}, 1 \leq i \leq n\}$ and $\{Z_{n,i}, 1 \leq i \leq n\}$ are comprised of pairwise m_n-negatively dependent random variables. We will prove that for all $n \geq 1$,

$$\mathbb{E}\left|\sum_{i=1}^{n} Y_{n,i}\right|^p \leq C_p(m_n + 1)^{p-1} \sum_{i=1}^{n} \mathbb{E}|Y_{n,i}|^p. \tag{7}$$

If $n \le m_n + 1$, then (7) follows immediately from the C_p-inequality. Suppose that $n > m_n + 1$. Then,

$$\begin{aligned}\mathbb{E}\left|\sum_{i=1}^{n} Y_{n,i}\right|^p &= \mathbb{E}\left|\sum_{k=1}^{m_n+1} \sum_{0\le i(m_n+1)\le n-k} Y_{n,i(m_n+1)+k}\right|^p \\ &\le (m_n+1)^{p-1}\sum_{k=1}^{m_n+1}\mathbb{E}\left|\sum_{0\le i(m_n+1)\le n-k} Y_{n,i(m_n+1)+k}\right|^p \\ &\le C_p(m_n+1)^{p-1}\sum_{k=1}^{m_n+1}\sum_{0\le i(m_n+1)\le n-k}\mathbb{E}\left|Y_{n,i(m_n+1)+k}\right|^p \\ &= C_p(m_n+1)^{p-1}\sum_{i=1}^{n}\mathbb{E}|Y_{n,i}|^p,\end{aligned}$$

where we have applied the C_p-inequality in the first inequality, Lemma 2.3 in the second inequality. The proof of (7) is completed. Applying Jensen's inequality and (7), we have

$$\begin{aligned}I_1 &\le \left(\mathbb{E}\left(\sum_{i=1}^{n}(Y_{n,i} - \mathbb{E}Y_{n,i})\right)^2\right)^{p/2} \le \left((m_n+1)\sum_{i=1}^{n}\mathbb{E}(Y_{n,i}-\mathbb{E}Y_{n,i})^2\right)^{p/2} \\ &\le \left((m_n+1)\sum_{i=1}^{n}\mathbb{E}Y_{n,i}^2\right)^{p/2} \le (m_n+1)^{p/2}n^{p/2}M^p. \qquad (8)\end{aligned}$$

Since $1 \le p < 2$, (5) follows immediately from (8).

Step 2: Proof of (6). Similar to (7), we can also prove that

$$\mathbb{E}\left|\sum_{i=1}^{n} Z_{n,i}\right|^p \le C_p(m_n+1)^{p-1}\sum_{i=1}^{n}\mathbb{E}|Z_{n,i}|^p. \qquad (9)$$

By definition of $X_{n,i}^{(2)}$, we have $X_{n,i}^{(2)} = (X_{n,i}+M)\mathbf{1}(X_{n,i} < -M) + (X_{n,i}-M)\mathbf{1}(X_{n,i} > M)$. Therefore,

$$|X_{n,i}^{(2)}| \le |X_{n,i}|\mathbf{1}(|X_{n,i}| > M). \qquad (10)$$

Combining (3), (9) and (10) yields

$$\begin{aligned}I_2 &\le (m_n+1)^{p-1}C_p\sum_{i=1}^{n}\mathbb{E}|Z_{n,i}|^p \le (m_n+1)^{p-1}C_p\sum_{i=1}^{n}\mathbb{E}|X_{n,i}^{(2)}|^p \\ &\le (m_n+1)^{p-1}C_p\sum_{i=1}^{n}\mathbb{E}\left(|X_{n,i}|^p\mathbf{1}(|X_{n,i}| > M)\right) \le (m_n+1)^{p-1}C_p n\varepsilon. \qquad (11)\end{aligned}$$

It follows from (11) that

$$\frac{I_2}{n(m_n+1)^{p/2}} \le (m_n+1)^{p-1-p/2}C_p\varepsilon \le C_p\varepsilon \qquad (12)$$

thereby proving (6) since $\varepsilon > 0$ is arbitrary.

The proof of the theorem is completed. □

We close the paper by considering a case, where $m_n \to \infty$ as $n \to \infty$.

Corollary 3.2. *Let $\alpha > 0$, $1 \le p < 2$ and let $\{X_{n,i}, 1 \le i \le n, n \ge 1\}$ be a triangular array of rowwise and pairwise $\lfloor \log^\alpha n \rfloor$-dependent random variables such that $\{|X_{n,i}|^p, 1 \le i \le n, n \ge 1\}$ is uniformly integrable in the Cesàro sense. Then,*

$$\frac{1}{n^{1/p}\log^{\alpha/2}(n)}\sum_{i=1}^{n}(X_{n,i} - \mathbb{E}X_{n,i}) \xrightarrow{\mathcal{L}_p} 0 \text{ as } n \to \infty.$$

Proof. Applying Theorem 3.1 for the case where $m_n \equiv \lfloor \log^\alpha n \rfloor$, we immediately obtain the conclusion of the corollary. □

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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