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Some Strong Laws of Large Numbers for Arrays of 2-Exchangeable Random Sets and Fuzzy Random Sets

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Abstract—The purpose of this paper is to give some strong laws of large numbers for arrays of 2-exchangeable random sets and fuzzy random sets in a separable Banach space. To get convergence theorems for multi-valued random variables, we also improve some results in the case of single-valued random variables taking values in a Banach space. Our results extend some related results in literature. Some typical examples illustrating this study are provided.

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1. INTRODUCTION

In recent years, the laws of large numbers, as well as other limit theorems, for random sets and fuzzy random sets, gave rise to applications in several fields, such as optimization and control, stochastic and integral geometry, mathematical economics, statistics and related fields.

The multivalued strong law of large numbers was first proved in 1975 by Artstein and Vitale [3] for independent and identically distributed (i.i.d.) random variables whose values are compact subsets of $\mathbf{R}^d$, with the Hausdorff metric convergence. It was extended in two directions: to random compact sets and to random closed (possibly unbounded) sets. For the first direction, we refer to Cressie [6], Hiai [15], Artstein and Hansen [1], Colubi et al. [5], Terán and Molchanov [25], Fu and Zhang [10], Giap, Quang, and Ngoc [11], etc. According to the second direction, the strong law of large numbers (SLLN) was first proved by Artstein and Hart [2] with the Kuratowski convergence for i.i.d. random variables having values in the closed subsets of $\mathbf{R}^d$ and applied it to a problem of optimal allocations; and later was extended by several authors with respect to the topologies Mosco and Wijsman (see Hiai [14], Hess [13]). However, most of the results for laws of large numbers was concerned with i.i.d. random sets. While it is not always possible to assume that the random sets are independent, they can be often dependent. In many statistical analysis some kind of dependency of random variables may be required, and exchangeability as an alternative to the random sample with i.i.d. random variables gives the study of asymptotic properties of random variables. On the other hand, statistical estimators are expressible of the linear form of random variables, which involves possibly random weights and possibly functions of dependent random variables which are permutation invariant with respect to distributions. Therefore, the SLLN for exchangeable random variables has been interested in studying by many authors and has been extended to random sets such as in [17–19]. Recently, the SLLN was proved in [23] for triangular array of row-wise exchangeable random sets and fuzzy random sets. The exchangeability is an extension of independency and identically distributed. In the classical strong law of large numbers, the assumption of independence can be weakened to that of pairwise independence. In [8], Etemadi et al. introduced the concept of 2-exchangeability which provided a unified treatment of the SLLN for both exchangeable and pairwise independent random variables. They also showed that, under 2-exchangeability, to preserve

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the Glivenko–Cantelli's theorem sometimes refereed to as the fundamental theorem of statistics—it is necessary that the random variables be pairwise independent. In this paper, we extend above result to random sets and fuzzy random sets. If one only uses the method as in Hiai [14, Theorems 3.2 and 3.3] or in Hess [13], then it is not available. We stress that the usual convexification technique developed in previous studies is no longer applicable because we deal the SLLN with double array. Thus, to give main results, we have to build structure of double array of selections to prove the "lim inf" part of Mosco convergence.

The organization of this paper is as follows. In Section 2, we summarize some basic concepts and related properties. Section 3 is concerned with the SLLN for arrays of 2-exchangeable random sets and fuzzy random sets with respect to the topologies Mosco and Wijsman. Some typical examples illustrating are provided in this section.

2. PRELIMINARIES

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a complete probability space, $(\mathfrak{X}, ||.||)$ be a separable Banach space and $\mathfrak{X}^*$ be its topological dual. In the present paper, $\mathbf{R}$ (resp., $\mathbf{N}$) will be denoted the set of all real numbers (resp., positive integers).

Let $c(\mathfrak{X})$ be the family of all nonempty closed subsets of $\mathfrak{X}$. For each $A, B \subset \mathfrak{X}$, cl$A$, $\overline{\text{co}}A$ denote the *norm-closure* and the *closed convex hull* of A, respectively; the *distance function* $d(., A)$ of A, the *norm* $||A||$ of A and the *support function* $s(., A)$ of A are defined by

$$d(x, A) = \inf\{||x-y|| : y \in A\}, (x \in \mathfrak{X}), \quad ||A|| = \sup\{||x|| : x \in A\},$$

$$s(x^*, A) = \sup\{\langle x^*, y\rangle : y \in A\}, (x^* \in \mathfrak{X}^*).$$

Let $\mathcal{P}(\mathfrak{X})$ be the family of all nonempty subsets of $\mathfrak{X}$. In $\mathcal{P}(\mathfrak{X})$, one defined Minkowski addition and scalar multiplication as follows: $A+B = \{a+b : a \in A, b \in B\}$, $\lambda A = \{\lambda a : a \in A\}$, where $A, B \in \mathcal{P}(\mathfrak{X})$, $\lambda \in \mathbf{R}$.

Let $\mathcal{B}_{\mathfrak{X}}$ be the Borel σ-field on $\mathfrak{X}$ and $\mathcal{B}_{c(\mathfrak{X})}$ be the σ-field on $c(\mathfrak{X})$ generated by the sets $U^- = \{C \in c(\mathfrak{X}) : C \cap U \neq \emptyset\}$ taken for all open subsets U of $\mathfrak{X}$. A mapping F from Ω to $c(\mathfrak{X})$ is said to be *measurable* if F is $(\mathcal{A}, \mathcal{B}_{c(\mathfrak{X})})$-measurable, i.e., for every open set U of $\mathfrak{X}$, the subset $F^{-1}(U^-) := \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\}$ belongs to $\mathcal{A}$. Such a mapping F is called a *random set* or *multivalued* (*closed-valued*) *random variable*.

Given the random set F, we define a sub-σ-field $\mathcal{A}_F$ of $\mathcal{A}$ by $\mathcal{A}_F = \{F^{-1}(\mathcal{U}) : \mathcal{U} \in \mathcal{B}_{c(\mathfrak{X})}\}$, where $F^{-1}(\mathcal{U}) = \{\omega \in \Omega : F(\omega) \in \mathcal{U}\}$, i.e., $\mathcal{A}_F$ is the smallest sub-σ-field of $\mathcal{A}$ with respect to which F is measurable. The *distribution* of F is a probability measure $\mathbf{P}_F$ on $\mathcal{B}_{c(\mathfrak{X})}$ defined by $\mathbf{P}_F(\mathcal{U}) = \mathbf{P}(F^{-1}(\mathcal{U}))$, $\mathcal{U} \in \mathcal{B}_{c(\mathfrak{X})}$. Random sets F_i, $i \in I$, are said to be *independent* if $\mathcal{A}_{F_i}$, $i \in I$, are independent, *identically distributed* if all $\mathbf{P}_{F_i}$ are identical, and *i.i.d.* if they are independent and identically distributed.

A random element $f : \Omega \to \mathfrak{X}$ is called a *selection* of the random set F if $f(\omega) \in F(\omega)$ for almost all $\omega \in \Omega$. An $F : \Omega \to c(\mathfrak{X})$ is measurable iff the *graph* $\text{Gr}(F) = \{(\omega, x) \in \Omega \times \mathfrak{X} : x \in F(\omega)\}$ of F is $\mathcal{A} \otimes \mathcal{B}_{\mathfrak{X}}$-measurable.

For every sub-σ-field $\mathcal{F}$ of $\mathcal{A}$ and for $1 \le p < \infty$, $L^p(\Omega, \mathcal{F}, \mathbf{P}, \mathfrak{X})$ denotes the Banach space of (equivalence classes of) $\mathcal{F}$-measurable random elements $f : \Omega \to \mathfrak{X}$ such that the norm $||f||_p = (\mathbf{E}||f||^p)^{1/p}$ is finite. In special case, $L^p(\Omega, \mathcal{A}, \mathbf{P}, \mathfrak{X})$ (resp. $L^p(\Omega, \mathcal{A}, \mathbf{P}, \mathbf{R})$) is denoted by $L^p(\mathfrak{X})$ (resp., L^p). For each $\mathcal{F}$-measurable random set F, define the following closed subset of $L^p(\Omega, \mathcal{F}, \mathbf{P}, \mathfrak{X})$,

$$S_F^p(\mathcal{F}) = \{f \in L^p(\Omega, \mathcal{F}, \mathbf{P}, \mathfrak{X}) : f(\omega) \in F(\omega) \text{ a.s.}\}.$$

If $\mathcal{F} = \mathcal{A}$, then $S_F^p(\mathcal{F})$ is denoted for shortly by S_F^p. The *expectation* of F over Ω, with respect to $\mathcal{F}$, is defined by $\mathbf{E}(F, \mathcal{F}) = \{\mathbf{E}f : f \in S_F^1(\mathcal{F})\}$, where $\mathbf{E}f = \int_\Omega f d\mathbf{P}$ is the usual Bochner integral of f. Shortly, $\mathbf{E}(F, \mathcal{A})$ is denoted by $\mathbf{E}F$. We note that $\mathbf{E}F$ is not always closed (see [20, p. 1386]).

A random set $F : \Omega \to c(\mathfrak{X})$ is called *integrable* if the set S_F^1 is nonempty, and it is called *integrable bounded* if the random variable $||F||$ is in L^1. A random set F is integrable iff $d(0, F(\cdot))$ is in L^1.

Let $\mathbf{N}^d = \{\mathbf{n} = (n_1, n_2, ..., n_d) : n_i \in \mathbf{N}, 1 \le i \le d\}$ be the set of positive integer d-dimensional lattice points, where d is a positive integer. We will keep "$\preceq$" (or $\succeq$) for the usual partial ordering on $\mathbf{N}^d$, i.e., $\mathbf{m} \preceq \mathbf{n}$ if $m_i \le n_i, 1 \le i \le d$. Denote

$$|\mathbf{n}| = \prod_{i=1}^{d} n_i, \quad \mathbf{n}_{\max} = \max\{n_1, n_2, \ldots, n_d\}, \quad \mathbf{n}_{\min} = \min\{n_1, n_2, \ldots, n_d\}.$$

Also, $\mathbf{1}$ and $\mathbf{2}$ are assigned to $(1, 1, ..., 1)$ and $(2, 2, ..., 2)$, respectively.

Let t be a topology on $\mathfrak{X}$ and $\{A_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$ be an array in $c(\mathfrak{X})$. We put

$$t-\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = \left\{ x \in \mathfrak{X} : x = t-\lim_{\mathbf{n}_{\max}\to\infty} x_{\mathbf{n}},\ x_{\mathbf{n}} \in A_{\mathbf{n}}, \forall \mathbf{n} \in \mathbf{N}^d \right\},$$

$$t-\limsup_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = \left\{ x \in \mathfrak{X} : x = t-\lim_{\mathbf{K}_{\max}\to\infty} x_{\mathbf{K}},\ x_{\mathbf{K}} \in A_{\mathbf{n}_{\mathbf{K}}}, \forall \mathbf{K} \in \mathbf{N}^d \right\},$$

where $\{A_{\mathbf{n}_{\mathbf{K}}} : \mathbf{K} \in \mathbf{N}^d\}$ is a sub-array of $\{A_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$.

The subsets $t-\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}}$ and $t-\limsup_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}}$ are the *lower limit* and the *upper limit* of $\{A_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$, relative to topology t. We obviously have $t-\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} \subset t-\limsup_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}}$. An array $\{A_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$ converges to A, in the *sense of Kuratowski*, relatively to the topology t, if the two following equalities are satisfied

$$t-\limsup_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = t-\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = A.$$

In this case, we shall write $A = t-\lim_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}}$; this is true if and only if the next two inclusions hold

$$t-\limsup_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} \subset A \subset t-\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}}.$$

Let us denote by s (resp., w) the strong (resp., weak) topology of $\mathfrak{X}$. It is easily seen that

$$s-\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} \subset w-\limsup_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} \quad \text{and} \quad s-\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} \in c(\mathfrak{X})$$

unless it is empty. A subset A is said to be the *Mosco limit* of the array $\{A_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$ denoted by M-$\lim_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = A$ if $w-\limsup_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = s-\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = A$ which is true if and only if $w-\limsup_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} \subset A \subset s-\liminf_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}}$.

The Wijsman convergence on $c(\mathfrak{X})$ is the pointwise convergence of distance functions. This means that an array $\{A_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$ in $c(\mathfrak{X})$ converges to $A \in c(\mathfrak{X})$ with respect to Wijsman convergence, denoted by W-$\lim_{\mathbf{n}_{\max}\to\infty} A_{\mathbf{n}} = A$ as $\mathbf{n}_{\max} \to \infty$ if, for every $x \in \mathfrak{X}$, one has $d(x, A) = \lim_{\mathbf{n}_{\max}\to\infty} d(x, A_{\mathbf{n}})$.

The convergence relatively to $\mathbf{n}_{\min} \to \infty$ is stated similarly. The corresponding definitions of pointwise convergence and almost sure convergence for an array $\{F_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$ of multivalued functions defined on Ω are clear. In fact, in the above definitions, it suffices to replace $A_{\mathbf{n}}$ by $F_{\mathbf{n}}(\omega)$ and A by $F(\omega)$ for almost surely $\omega \in \Omega$.

Concerning expectations, conditional expectations, martingales, Mosco convergence and Wijsman convergence of random sets we refer to Hess [13], Hiai and Umegaki [16].

An array $\{f_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$ of random elements is called *uniformly integrable* if and only if $\mathbf{E}\left(||f_{\mathbf{n}}|| I_{(||f_{\mathbf{n}}||>a)}\right) \to 0$ as $a \to \infty$ uniformly in $\mathbf{n}$.

Next, we introduce some concepts of 2-exchangeability. An array of random sets $\{F_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$ in d-dimensional time is said to be 2-*exchangeable in distributions*, or simply 2-*exchangeable*, if $(F_{\mathbf{i}}, F_{\mathbf{j}})$ has the same distribution as $(F_{\mathbf{1}}, F_{\mathbf{2}})$ for all different $\mathbf{i}$'s and $\mathbf{j}$'s in $\mathbf{N}^d$, that is, $\mathbf{P}\{F_{\mathbf{i}} \in B_1, F_{\mathbf{j}} \in B_2\} = \mathbf{P}\{F_{\mathbf{1}} \in B_1, F_{\mathbf{2}} \in B_2\}$, for all $B_1, B_2 \in \mathcal{B}_{c(\mathfrak{X})}$ and for every $\mathbf{i} \neq \mathbf{j}$.

An array of real-valued random variables $\{f_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$ is said to be 2-*exchangeable up to second moments* if $\mathbf{E}(f_{\mathbf{i}} f_{\mathbf{j}}) = \mathbf{E}(f_{\mathbf{1}} f_{\mathbf{2}})$ for all $\mathbf{i} \neq \mathbf{j}$, $\mathbf{E}(f_{\mathbf{i}}^2) = \mathbf{E}(f_{\mathbf{1}}^2)$ and $\mathbf{E}(f_{\mathbf{i}}) = \mathbf{E}(f_{\mathbf{1}})$ for all $\mathbf{i} \in \mathbf{N}^d$.

The array $\{f_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$ of random elements in a Banach space is called *converges in r-mean* (or: *in* $\mathbf{L}^r$) $(r > 0)$ to the random element f as $\mathbf{n}_{\max} \to \infty$ (resp., $\mathbf{n}_{\min} \to \infty$) iff $\mathbf{E}||f_{\mathbf{n}} - f||^r \to 0$ as $\mathbf{n}_{\max} \to \infty$ (resp., $\mathbf{n}_{\min} \to \infty$).

In the following, we describe some basic concepts of fuzzy random variables.

A *fuzzy set* in $\mathfrak{X}$ is a function $u : \mathfrak{X} \to [0,1]$. For each fuzzy set u, the *α-level set* is denoted by $L_\alpha u = \{x \in \mathfrak{X} : u(x) \geq \alpha\}$, $0 \leq \alpha \leq 1$. It is easy to see that, for every $\alpha \in (0,1]$, $L_\alpha u = \bigcap_{\beta<\alpha} L_\beta u$. We also define $L_{\alpha^+} u = \{x \in \mathfrak{X} : u(x) > \alpha\}$, $0 \leq \alpha < 1$.

Let $F(\mathfrak{X})$ denote the space of fuzzy sets $u : \mathfrak{X} \to [0,1]$ such that

(1)u is normal, i.e., the 1-level set $L_1 u \neq \emptyset$,

(2) u is upper semicontinuous, that is, for each $\alpha \in (0,1]$, the α-level set $L_\alpha u$ is a closed subset of $\mathfrak{X}$.

A linear structure in $F(\mathfrak{X})$ is defined by the following operations

$$(u+v)(x) = \sup_{y+z=x} \min\{u(y), v(z)\}, \quad (\lambda u)(x) = \begin{cases} u(\lambda^{-1}x), & \text{if} \quad \lambda \neq 0, \\ I_{\{0\}}(x), & \text{if} \quad \lambda = 0, \end{cases}$$

where $u, v \in F(\mathfrak{X})$, $\lambda \in \mathbf{R}$. Then, for each $\alpha \in (0,1]$, $L_\alpha(u+v) = \text{cl}\{L_\alpha(u) + L_\alpha(v)\}$ and $L_\alpha(\lambda u) = \lambda L_\alpha(u)$.

A function u in $F(\mathfrak{X})$ is called *convex* if it satisfies $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for every $x, y \in \mathfrak{X}$ and $\lambda \in [0,1]$. It is known that u is convex in the above sense iff, for any $\alpha \in (0,1]$, the level set $L_\alpha u$ is a convex subset of $\mathfrak{X}$.

The *closed convex hull* $\overline{\text{co}}\, u$ of $u \in F(\mathfrak{X})$ is defined as follows

$$\overline{\text{co}}\, u(x) = \sup\{\alpha \in [0,1] : x \in \overline{\text{co}}\, L_\alpha u\},$$

so that $L_\alpha(\overline{\text{co}}\, u) = \overline{\text{co}}\, L_\alpha u$ for all $\alpha \in [0,1]$.

The concept of fuzzy random set as a generalization for a random set was extensively studied by Puri and Ralescu [22]. A *fuzzy-valued random variable* (or *fuzzy random set*) is a mapping $\widetilde{F} : \Omega \to F(\mathfrak{X})$ such that $L_\alpha \widetilde{F}$ is a random set for every $\alpha \in (0,1]$.

The *expected value* of any fuzzy random set $\widetilde{F}$, denoted by $\mathbf{E}\widetilde{F}$, is a fuzzy set such that, for every $\alpha \in (0,1]$, $L_\alpha\left(\mathbf{E}\widetilde{F}\right) = \mathbf{E}\left(L_\alpha \widetilde{F}\right)$. A fuzzy random set $\widetilde{F}$ is called *integrable* if it has expected value.

The 2-exchangeability of array $\left\{\widetilde{F}_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\right\}$ of fuzzy random sets is defined to be one of array $\left\{L_\alpha(\widetilde{F}_{\mathbf{n}}) : \mathbf{n} \in \mathbf{N}^d\right\}$ of random sets, for each $\alpha \in [0,1]$.

For notational convenience, the logarithms are to the base 2, for $a \in \mathbf{R}$, $\log(\max\{a, 1\})$ will be denoted by $\log^+ a$.

3. MAIN RESULTS

To prove main results, we need the following some lemmas.

Lemma 3.1. *Let $\{F_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$ be an array of 2-exchangeable random sets in $c(\mathfrak{X})$ and $\varphi : c(\mathfrak{X}) \to \mathfrak{X}$ be a $(\mathcal{B}_{c(\mathfrak{X})}, \mathcal{B}_{\mathfrak{X}})$-measurable function. Then, $\{\varphi(F_{\mathbf{n}}) : \mathbf{n} \in \mathbf{N}^d\}$ be an array of 2-exchangeable random elements.*

Proof. Since the 2-exchangeability of $\{F_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$, we have that for any $\mathbf{i} \neq \mathbf{j}$ in $\mathbf{N}^d$ and any Borel subsets $\{B_1, B_2\}$ of $\mathcal{B}_{\mathfrak{X}}$,

$$\begin{aligned} \mathbf{P}\left(\varphi(F_{\mathbf{i}}) \in B_1, \varphi(F_{\mathbf{j}}) \in B_2\right) &= \mathbf{P}\left(F_{\mathbf{i}} \in \varphi^{-1}(B_1), F_{\mathbf{j}} \in \varphi^{-1}(B_2)\right) \\ &= \mathbf{P}\left(F_{\mathbf{1}} \in \varphi^{-1}(B_1), F_{\mathbf{2}} \in \varphi^{-1}(B_2)\right) \quad (\text{by } \varphi^{-1}(B_i) \in \mathcal{B}_{c(\mathfrak{X})}, i = 1, 2) \\ &= \mathbf{P}\left(\varphi(F_{\mathbf{1}}) \in B_1, \varphi(F_{\mathbf{2}}) \in B_2\right). \end{aligned}$$

Therefore, $\{\varphi(F_{\mathbf{n}}) : \mathbf{n} \in \mathbf{N}^d\}$ is an array of 2-exchangeable random elements. □

Lemma 3.2. (1) *Let $\{F_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$ be an array of 2-exchangeable random sets. For each $f_{\mathbf{1}} \in S^1_{F_{\mathbf{1}}}(\mathcal{A}_{F_{\mathbf{1}}})$, there exists an array $\{f_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$ of $f_{\mathbf{n}} \in S^1_{F_{\mathbf{n}}}(\mathcal{A}_{F_{\mathbf{n}}})$ such that $\{f_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$ is an array of 2-exchangeable random elements.*

(2) Let $\{F_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$ be an array of 2-exchangeable random sets with $S^1_{F_{\mathbf{1}}} \neq \emptyset$. Then, $\mathbf{E}(F_{\mathbf{i}}, \mathcal{A}_{F_{\mathbf{i}}}) = \mathbf{E}(F_{\mathbf{1}}, \mathcal{A}_{F_{\mathbf{1}}})$, for all $\mathbf{i} \in \mathbf{N}^d$.

Proof. (1) Since $\mathfrak{X}$ is separable and $f_{\mathbf{1}}$ is $\mathcal{A}_{F_{\mathbf{1}}}$-measurable, there exists a $(\mathcal{B}_{c(\mathfrak{X})}, \mathcal{B}_{\mathfrak{X}})$-measurable function $\varphi : c(\mathfrak{X}) \to \mathfrak{X}$ satisfying $f_{\mathbf{1}}(\omega) = \varphi(F_{\mathbf{1}}(\omega))$ for every $\omega \in \Omega$. Define $f_{\mathbf{i}}(\omega) = \varphi(F_{\mathbf{i}}(\omega))$ for every $\omega \in \Omega$ and $\mathbf{i} \in \mathbf{N}^d$. Since 2-exchangeability of $\{F_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$ and by virtue of Lemma 3.1, the array $\{f_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$ is also 2-exchangeable, and so they are identically distributed.

Because the function $(x, A) \mapsto d(x, A)$ of $\mathfrak{X} \times c(\mathfrak{X})$ into $\mathbf{R}$ is $\mathcal{B}_{\mathfrak{X}} \otimes \mathcal{B}_{c(\mathfrak{X})}$-measurable, the array of random variables $\{d(f_{\mathbf{n}}(\cdot), F_{\mathbf{n}}(\cdot)) : \mathbf{n} \in \mathbf{N}^d\}$ is identically distributed. Hence, $d(f_{\mathbf{1}}(\omega), F_{\mathbf{1}}(\omega)) = 0$ a.s. implies $d(f_{\mathbf{n}}(\omega), F_{\mathbf{n}}(\omega)) = 0$ a.s. for all $\mathbf{n} \in \mathbf{N}^d$. Combining this with

$$\int_{\Omega} ||f_{\mathbf{n}}(\omega)|| d\mathbf{P} = \int_{c(\mathfrak{X})} ||\varphi(X)|| d\mathbf{P}_{F_{\mathbf{n}}} = \int_{c(\mathfrak{X})} ||\varphi(X)|| d\mathbf{P}_{F_{\mathbf{1}}} = \int_{\Omega} ||f_{\mathbf{1}}(\omega)|| d\mathbf{P} < \infty,$$

we obtain $f_{\mathbf{n}} \in S^1_{F_{\mathbf{n}}}(\mathcal{A}_{F_{\mathbf{n}}})$ for every $\mathbf{n} \in \mathbf{N}^d$. (2) follows immediately from (1). □

Remark. Hiai [15, Lemma 6] showed that if F is an integrable random set and $\mathbf{E}F = \{x\}$ ($x \in \mathfrak{X}$), then $F(\omega) = \{f(\omega)\}$ a.s., where $f \in L^1(\mathfrak{X})$. Thus, the condition "mean zero" in the single-valued random variable case is usually extended by $0 \in \mathbf{E}(F, \mathcal{A}_F)$. This is used by Ezzaki (see [9]) to define the multivalued martingale difference, and is also used by Quang and Thuan to obtain the SLLN for adapted arrays of fuzzy-valued random variables in Banach space (see [24]). The following example will show that the condition $0 \in \mathbf{E}F$ is not equivalent to the condition $0 \in \mathbf{E}(F, \mathcal{A}_F)$ where F is a random set.

Example 3.3. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be nonatomic and $\Omega = \Omega_1 \cup \Omega_2$, where $\mathbf{P}(\Omega_1) = \mathbf{P}(\Omega_2) = 1/2$. Define $F : \Omega \to c(\mathbf{R})$ by

$$F(\omega) = \begin{cases} \{-1/2; 1\}, & \text{if} \quad \omega \in \Omega_1, \\ \{-1/2\}, & \text{if} \quad \omega \in \Omega_2. \end{cases}$$

Therefore, $f \in S^1_F$ iff f only takes the values in $\{-1/2; 1\}$ such that $\mathbf{P}(f = 1) = p$ and $\mathbf{P}(f = -1/2) = 1 - p$ with some arbitrary $p \in [0, 1/2]$. Then, $\mathbf{E}f = (3/2)p - 1/2$ and so $\mathbf{E}F = [-1/2, 1/4]$. Thus, $0 \in \mathbf{E}F$.

Furthermore, since $\mathcal{A}_F = \{\emptyset, \Omega_1, \Omega_2, \Omega\}$, for each $f \in S^1_F(\mathcal{A}_F)$, there is there is only one of the following situations: $f(\omega) = -1/2$ for all $\omega \in \Omega$ which implies $\mathbf{E}f = -1/2$, or

$$f(\omega) = \begin{cases} -1/2, & \text{if} \quad \omega \in \Omega_2 \\ 1, & \text{if} \quad \omega \in \Omega_1 \end{cases}$$

which yields $\mathbf{E}f = 1/4$. Hence, $\mathbf{E}(F, \mathcal{A}_F) = \{-1/2; 1/4\}$ and so $0 \notin \mathbf{E}(F, \mathcal{A}_F)$.

In [8], Etemadi et al. proved the SLLN for 2-exchangeable real-valued random variables. Later, in [7], Etemadi extended an important part of [8, Theorem 2] to the case where random variables are taking values in a separable Banach space by using the appropriate modification given in Padgett and Taylor [21, pp. 42–44]. In the following, we complete the extension of [8, Theorem 2] for Banach space valued random variables case.

Theorem 1. *Let $\{f_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$ be an array of 2-exchangeable random elements in d-dimensional time, $d \geq 1$, taking values in the separable Banach space $\mathfrak{X}$, and $S_{\mathbf{n}} = \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}} f_{\mathbf{i}}$. Then,*

$$\mathbf{E}\left(||f_{\mathbf{1}}||(\log^+ ||f_{\mathbf{1}}||)^{d-1}\right) < \infty \Rightarrow \lim_{\mathbf{n}_{\max} \to \infty} \frac{S_{\mathbf{n}}}{|\mathbf{n}|} = f \ \text{ a.s. and in } \mathbf{L}^1,$$

where f is a random element with $\mathbf{E}f = \mathbf{E}f_{\mathbf{1}}$.

Proof. By virtue of [7, Corollary 1], there exists a random element f such that $\lim_{\mathbf{n}_{\max}\to\infty} \frac{S_{\mathbf{n}}}{|\mathbf{n}|} = f$ a.s.

Next, by applying [8, Theorem 2] for array $\{||f_{\mathbf{n}}|| : \mathbf{n} \in \mathbf{N}^d\}$ of 2-exchangeable random variables, we have that the array $\{\frac{1}{|\mathbf{n}|}\sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}} ||f_{\mathbf{i}}|| : \mathbf{n} \in \mathbf{N}^d\}$ of integrable random variables converges almost sure and in $\mathbf{L}^1$ to a random variable g as $\mathbf{n}_{\max} \to \infty$ with $\mathbf{E}g = \mathbf{E}||f_{\mathbf{1}}|| \le 2 + \mathbf{E}\left(||f_{\mathbf{1}}||(\log^+ ||f_{\mathbf{1}}||)^{d-1}\right) < \infty$. Thus, by [12, Theorem 5.2], the array $\{\frac{1}{|\mathbf{n}|}\sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}} ||f_{\mathbf{i}}|| : \mathbf{n} \in \mathbf{N}^d\}$ of random variables is uniformly integrable. This implies that the array $\{\frac{S_{\mathbf{n}}}{|\mathbf{n}|} : \mathbf{n} \in \mathbf{N}^d\}$ of random elements is uniformly integrable. Moreover, the array $\{\frac{S_{\mathbf{n}}}{|\mathbf{n}|} : \mathbf{n} \in \mathbf{N}^d\}$ converges a.s. to f as $\mathbf{n}_{\max} \to \infty$. Hence, by applying [12, Theorem 5.2] again, we get that the array $\{\frac{S_{\mathbf{n}}}{|\mathbf{n}|} : \mathbf{n} \in \mathbf{N}^d\}$ converges in $\mathbf{L}^1$ to f. Consequently, $\mathbf{E}f = \lim_{\mathbf{n}_{\max}\to\infty} \mathbf{E}\frac{S_{\mathbf{n}}}{|\mathbf{n}|} = \mathbf{E}f_{\mathbf{1}}$. The theorem is proved completely. □

Combining above theorem with [8, Corollary 1], we obtain the following theorem that the limit of the average will be a constant.

Theorem 2. *Suppose that the array $\{f_{\mathbf{n}} : \mathbf{n} \in \mathbf{N}^d\}$ of 2-exchangeable random elements in $L^2(\mathfrak{X})$ with the separable dual space $\mathfrak{X}^*$ satisfying $Cov(\langle x^*, f_{\mathbf{1}}\rangle, \langle x^*, f_{\mathbf{2}}\rangle) = 0$ for every $x^* \in \mathfrak{X}^*$. Then,*

$$\frac{S_{\mathbf{n}}}{|\mathbf{n}|} \to \mathbf{E}f_{\mathbf{1}} \quad \text{a.s. and in} \quad \mathbf{L}^1 \quad \text{as} \quad \mathbf{n}_{\max} \to \infty.$$

Proof. For each $x^* \in \mathfrak{X}^*$, set $S^*_{\mathbf{n}} = \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}} \langle x^*, f_{\mathbf{i}}\rangle$. It is not hard to prove that there exists a positive constant C such that $(\log^+ x)^{d-1} \le Cx$ for all $x \ge 0$, which is suffices to show that the assumption $f_{\mathbf{1}} \in L^2(\mathfrak{X})$ yields $\mathbf{E}\left(||f_{\mathbf{1}}||(\log^+ ||f_{\mathbf{1}}||)^{d-1}\right) < \infty$. Therefore, the array $\{f_{\mathbf{i}} : \mathbf{i} \in \mathbf{N}^d\}$ satisfies all conditions of Theorem 1. By using this theorem, we get $\lim_{\mathbf{n}_{\max}\to\infty} \frac{S_{\mathbf{n}}}{|\mathbf{n}|} = f$ a.s. and in $\mathbf{L}^1$. It implies

$$\lim_{\mathbf{n}_{\max}\to\infty} \frac{S^*_{\mathbf{n}}}{|\mathbf{n}|} = \langle x^*, f\rangle \quad \text{a.s. and in} \quad \mathbf{L}^1.$$

It follows from $f_{\mathbf{i}} \in L^2(\mathfrak{X})$ that $\langle x^*, f_{\mathbf{i}}\rangle \in L^2$, for every $x^* \in \mathfrak{X}^*$. Thus, the array $\{\langle x^*, f_{\mathbf{n}}\rangle : \mathbf{n} \in \mathbf{N}^d\}$ is 2-exchangeable up to second moments. Using [8, Corollary 1], we have $\langle x^*, f\rangle = \mathbf{E}\langle x^*, f_{\mathbf{1}}\rangle$ a.s., it is equivalent to $\langle x^*, f - \mathbf{E}f_{\mathbf{1}}\rangle = 0$ a.s. for every $x^* \in \mathfrak{X}^*$. Since $\mathfrak{X}^*$ is separable, there exists a dense subset $\{x^*_j : j \ge 1\}$ of $\mathfrak{X}^*$. It follows that $\langle x^*_j, f - \mathbf{E}f_{\mathbf{1}}\rangle = 0$ a.s. for all $j \ge 1$. So there exists a negligible set $N \in \mathcal{A}$ such that for each $\omega \in \Omega\backslash N$, $\langle x^*_j, f(\omega) - \mathbf{E}f_{\mathbf{1}}\rangle = 0$, for all $j \ge 1$. Let $x = f(\omega) - \mathbf{E}f_{\mathbf{1}}$. If $x \ne 0$, then $||x|| > 0$. Then, by Hahn–Banach's theorem, there exists $x^* \in \mathfrak{X}^*$ such that $\langle x^*, x\rangle \ne 0$, that is

$$|\langle x^*, x\rangle| > 0. \tag{3.1}$$

For every $\varepsilon > 0$, there exists k, $||x^* - x^*_k|| < \frac{\varepsilon}{||x||}$. Therefore,

$$|\langle x^*, x\rangle| \le |\langle x^*, x\rangle - \langle x^*_k, x\rangle| + |\langle x^*_k, x\rangle| = |\langle x^* - x^*_k, x\rangle| \le ||x^* - x^*_k||||x|| < \varepsilon.$$

This is inconsistency with (3.1), and so $x = 0$. This means that $f = \mathbf{E}f_{\mathbf{1}}$ a.s. □

The following theorem is a generalization of Theorem 1 for multivalued random variables case. To obtain the desired result, we need to use the method as in Hiai [14] and some other calculations.

Theorem 3. *Suppose that $\{F_{ij} : i \ge 1, j \ge 1\}$ is a double array of 2-exchangeable random sets with $S^1_{F_{11}} \ne \emptyset$ and $\mathbf{E}\left(||F_{11}|| \log^+ ||F_{11}||\right) < \infty$. Let $S_{mn} = \mathrm{cl}\sum_{i=1}^m \sum_{j=1}^n F_{ij}$. Then, we obtain the following conclusions:*

(a) $\overline{\mathrm{co}}\mathbf{E}F_{11} \subset s\text{–}\liminf_{\min\{m,n\}\to\infty} \mathrm{cl}\mathbf{E}\frac{S_{mn}}{mn}$ *and* $\overline{\mathrm{co}}\mathbf{E}F_{11} \subset \mathrm{cl}\mathbf{E}F$, *where* $F(\omega) = s\text{–}\liminf_{\min\{m,n\}\to\infty} \frac{S_{mn}(\omega)}{mn}$;

(b) *If $\mathfrak{X}$ is reflexive and $\sup_{m,n\ge 1} ||F_{mn}(\omega)|| < \infty$ a.s., then $\mathrm{cl}\mathbf{E}G \subset \overline{\mathrm{co}}\mathbf{E}F_{11}$ where $G(\omega) = w\text{–}\limsup_{\max\{m,n\}\to\infty} \frac{S_{mn}(\omega)}{mn}$.*

Proof. (a) Let $x \in \overline{\text{co}}\mathbf{E}F_{11}$ and $\varepsilon > 0$, by [14, Lemma 3.1(1)] and [4, Lemma 3.6], there exists $f_j \in S^1_{F_{11}}(\mathcal{A}_{F_{11}})$, $1 \leq j \leq k$ such that $\left|\left|\frac{1}{k}\sum_{j=1}^k \mathbf{E}f_j - x\right|\right| < \varepsilon$. By Lemma 3.2, we can choose $f_{ij} \in S^1_{F_{ij}}(\mathcal{A}_{F_{ij}})$, $1 \leq i, j \leq k$ such that

$$\mathbf{E}f_{ij} = \begin{cases} \mathbf{E}f_{i+j-1}, & \text{if} \quad i+j \leq k+1, \\ \mathbf{E}f_{i+j-1-k}, & \text{if} \quad i+j > k+1. \end{cases}$$

Let $x_j = \mathbf{E}f_j, 1 \leq j \leq k$ and let $x_{ij} = \mathbf{E}f_{ij}, 1 \leq i \leq k, 1 \leq j \leq k$. It is easy to check that

$$\frac{1}{k}\sum_{i=1}^k x_i = \frac{1}{k^2}\sum_{i=1}^k\sum_{j=1}^k x_{ij}. \tag{3.2}$$

By Lemma 3.2(1), there exists a double array $\{f_{ij} : i \geq 1, j \geq 1\}$ of $f_{ij} \in S^1_{F_{ij}}(\mathcal{A}_{F_{ij}})$ such that $\{f_{(s-1)k+i,(t-1)k+j} : s \geq 1, t \geq 1\}$ is 2-exchangeable for each $i, j = 1, 2, \ldots, k$.

For every $i, j \geq 1$, we get

$$\mathbf{E}\left(||f_{ij}|| \log^+ ||f_{ij}||\right) \leq \mathbf{E}\left(||F_{ij}|| \log^+ ||F_{ij}||\right) = \mathbf{E}\left(||F_{11}|| \log^+ ||F_{11}||\right) < \infty.$$

Hence, for each $i, j = 1, \ldots, k$, applying Theorem 1 for double array $\{f_{(s-1)k+i,(t-1)k+j} : s \geq 1, t \geq 1\}$ of integrable random elements, we have

$$\frac{1}{st}\sum_{l=1}^s\sum_{r=1}^t f_{(l-1)k+i,(r-1)k+j}(\omega) \to g_{ij}(\omega) \quad \text{a.s. and in} \quad \mathbf{L}^1 \quad \text{as} \quad \max\{s,t\} \to \infty, \tag{3.3}$$

where g_{ij} is a random element with $\mathbf{E}g_{ij} = \mathbf{E}f_{ij} = x_{ij}$.

If $m = (s-1)k + p$ and $n = (t-1)k + q$, where $1 \leq p, q \leq k$, then the following estimation holds

$$\begin{aligned} \left|\left|\frac{1}{mn}\sum_{i=1}^m\sum_{j=1}^n f_{ij}(\omega) - \frac{1}{k^2}\sum_{i=1}^k\sum_{j=1}^k g_{ij}(\omega)\right|\right| &\leq \frac{st}{mn}\sum_{i=1}^k\sum_{j=1}^k \left|\left|\frac{1}{st}\sum_{l=1}^s\sum_{r=1}^t f_{(l-1)k+i,(r-1)k+j}(\omega) - g_{ij}(\omega)\right|\right| \\ + \frac{t}{mn}\sum_{i=p+1}^k\sum_{j=1}^k \frac{1}{t}\sum_{r=1}^t \left|\left|f_{(s-1)k+i,(r-1)k+j}(\omega)\right|\right| &+ \frac{s}{mn}\sum_{i=1}^k\sum_{j=q+1}^k \frac{1}{s}\sum_{l=1}^s \left|\left|f_{(l-1)k+i,(t-1)k+j}(\omega)\right|\right| \\ &+ \left(\frac{st}{mn} - \frac{1}{k^2}\right)\left|\left|\sum_{i=1}^k\sum_{j=1}^k g_{ij}(\omega)\right|\right|. \end{aligned} \tag{3.4}$$

We obtain from (3.3) that

$$\frac{st}{mn}\sum_{i=1}^k\sum_{j=1}^k \left|\left|\frac{1}{st}\sum_{l=1}^s\sum_{r=1}^t f_{(l-1)k+i,(r-1)k+j}(\omega) - g_{ij}(\omega)\right|\right| \to 0$$
$$\text{a.s. and in} \quad \mathbf{L}^1 \quad \text{as} \quad \max\{m,n\} \to \infty. \tag{3.5}$$

Since $\{f_{(s-1)k+i,(r-1)k+j} : r \geq 1\}$ is a sequence of 2-exchangeable random elements for each $s \geq 1$, by virtue of Theorem 1, it follows that $\frac{1}{t}\sum_{r=1}^t \left|\left|f_{(s-1)k+i,(r-1)k+j}(\omega) - g_{ij}(\omega)\right|\right| \to 0$ a.s. and in $\mathbf{L}^1$ as $t \to \infty$. Hence,

$$\begin{aligned} &\frac{t}{mn}\sum_{i=p+1}^k\sum_{j=1}^k \frac{1}{t}\sum_{r=1}^t \left|\left|f_{(s-1)k+i,(r-1)k+j}(\omega)\right|\right| \\ &\leq \frac{t}{mn}\sum_{i=p+1}^k\sum_{j=1}^k \frac{1}{t}\sum_{r=1}^t \left|\left|f_{(s-1)k+i,(r-1)k+j}(\omega) - g_{ij}(\omega)\right|\right| + \frac{t}{mn}\sum_{i=p+1}^k\sum_{j=1}^k \frac{1}{t}\sum_{r=1}^t ||g_{ij}(\omega)|| \end{aligned}$$

$$\le \frac{t}{mn}\sum_{i=p+1}^{k}\sum_{j=1}^{k}\frac{1}{t}\sum_{r=1}^{t}\left|\left|f_{(s-1)k+i,(r-1)k+j}(\omega)-g_{ij}(\omega)\right|\right|+\frac{t}{mn}\sum_{i=p+1}^{k}\sum_{j=1}^{k}||g_{ij}(\omega)||$$
$$\to 0 \quad \text{a.s. and in} \quad \mathbf{L}^1 \quad \text{as} \quad \min\{m,n\}\to\infty. \tag{3.6}$$

Similarly, we get

$$\frac{s}{mn}\sum_{i=1}^{k}\sum_{j=q+1}^{k}\frac{1}{s}\sum_{l=1}^{s}\left|\left|f_{(l-1)k+i,(t-1)k+j}(\omega)\right|\right|\to 0 \quad \text{a.s. and in} \quad \mathbf{L}^1 \quad \text{as} \quad \min\{m,n\}\to\infty. \tag{3.7}$$

Further, in view of

$$\lim_{\min\{m,n\}\to\infty}\frac{st}{mn}=\lim_{\min\{m,n\}\to\infty}\left(\frac{1}{k}+\frac{1}{m}-\frac{p}{mk}\right)\left(\frac{1}{k}+\frac{1}{n}-\frac{q}{nk}\right)=\frac{1}{k^2}, \tag{3.8}$$

we have

$$\left(\frac{st}{mn}-\frac{1}{k^2}\right)\left|\left|\sum_{i=1}^{k}\sum_{j=1}^{k}g_{ij}(\omega)\right|\right|\to 0 \quad \text{a.s. and in} \quad \mathbf{L}^1 \quad \text{as} \quad \min\{m,n\}\to\infty. \tag{3.9}$$

Combining (3.4)–(3.7) and (3.9), we get

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}f_{ij}(\omega)\to\frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=1}^{k}g_{ij}(\omega) \quad \text{a.s. and in} \quad \mathbf{L}^1 \quad \text{as} \quad \min\{m,n\}\to\infty.$$

Combine this with (3.2), we obtain (a).

(b) Let $X=\overline{\text{co}}\,\mathbf{E}F_{11}$. Invoking the fact that $\mathfrak{X}$ has the Lindeloff property: given a family of open sets, there is a countable family having the same union. Consequently, there is a sequence $\{x_j^*: j\ge 1\}$ such that $X=\bigcap_{j=1}^{\infty}\left\{x\in\mathfrak{X}:\langle x_j^*,x\rangle\le s(x_j^*,X)\right\}$. The function $X\mapsto s(x_j^*,X)$ of $c(\mathfrak{X})$ into $(-\infty,\infty]$ is $(\mathcal{B}_{c(\mathfrak{X})},\mathcal{B}_{\mathbf{R}})$-measurable and $\mathbf{E}(s(x_j^*,F_{11}))=s(x_j^*,X)<\infty$, $j\ge 1$.

By using Lemma 3.1, the double array $\{s(x_j^*,F_{mn}): m\ge 1, n\ge 1\}$ of 2-exchangeable random variables satisfies all conditions of Theorem 1 for real-valued random variables case for each $j\ge 1$, so, by applying this theorem, there exists a negligible set $N\in\mathcal{A}$ such that for every $\omega\in\Omega\backslash N$ and $j\ge 1$,

$$s\left(x_j^*,\frac{S_{mn}(\omega)}{mn}\right)=\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n}s\left(x_j^*,F_{kl}(\omega)\right)\to Y_j(\omega) \quad \text{as} \quad \max\{m,n\}\to\infty,$$

where Y_j is some random variable satisfying $\mathbf{E}Y_j=s(x_j^*,X)$. By [14, Theorem 2.2(2)], we get that $Y=w-\limsup_{\max\{m,n\}\to\infty}\frac{S_{mn}}{mn}$ is a random set. If $f\in S_Y^1$, then $f_{rs}(\omega)\overset{w}{\to}f(\omega)$ as $\max\{r,s\}\to\infty$ for some $f_{rs}(\omega)\in\frac{S_{m_rn_s}(\omega)}{m_rn_s}$ and, hence,

$$\langle x_j^*,f(\omega)\rangle=\lim_{\max\{r,s\}\to\infty}\langle x_j^*,f_{rs}(\omega)\rangle\le\lim_{\max\{r,s\}\to\infty}s\left(x_j^*,\frac{S_{m_rn_s}(\omega)}{m_rn_s}\right)=Y_j(\omega), \quad j\ge 1.$$

This yields $\langle x_j^*,\mathbf{E}f\rangle=\mathbf{E}\langle x_j^*,f\rangle\le\mathbf{E}Y_j=s(x_j^*,X)$, $j\ge 1$, which implies $\mathbf{E}f\in X$. Thus, we obtain $\text{cl}(\mathbf{E}Y)\subset X$ a.s. □

Next, we establish the multivalued SLLN for double array of 2-exchangeable random sets that the limit of the average is a non-random set. The idea behind the proof is to utilize Theorem 2 and combine it with extending the convexification technique to double array case to establish the "lim inf" part of Mosco convergence.

Theorem 4. *Assume that $\{F_{ij}: i\ge 1, j\ge 1\}$ is a double array of 2-exchangeable random sets in $c(\mathfrak{X})$ with the separable dual space $\mathfrak{X}^*$ and $S^1_{F_{11}}\ne\emptyset$. If the following conditions are satisfied:*

(a) for every $x^* \in \mathfrak{X}^*$, *Cov* $(\langle x^*, g(F_{11})\rangle, \langle x^*, g(F_{22})\rangle) = 0$,
(b) $\mathbf{E}||F_{11}||^2 < \infty$,
then

$$\frac{1}{mn} cl \sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}(\omega) \to \overline{co}\, \mathbf{E}F_{11} \quad \text{a.s. as} \quad \max\{m,n\} \to \infty,$$

with respect to the topologies Mosco and Wijsman, where $g : c(\mathfrak{X}) \to \mathfrak{X}$ *is some measurable function.*

Proof. Let $X = \overline{\text{co}}\, \mathbf{E}F_{11}$ and $G_{mn}(\omega) = \frac{1}{mn}\text{cl}\sum_{i=1}^{m}\sum_{j=1}^{n} F_{ij}(\omega), \omega \in \Omega, m \geq 1, n \geq 1$. At first, we prove that $X \subset s-\liminf_{\max\{m,n\}\to\infty} G_{mn}(\omega)$ a.s. To do this, we will use [4, Proposition 3.5]. For each $x \in X$ and $\varepsilon > 0$, the random elements f_i, f_{ij} and the constant elements x_i, x_{ij} $(i, j \in \{1, 2, \ldots, k\})$ defined as in the proof of Theorem 3, it is easy to check that

$$\frac{1}{k}\sum_{i=1}^{k} x_i = \begin{cases} \frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=1}^{k} x_{ij}, \\ \frac{1}{k}\sum_{i=1}^{k} x_{ij} \quad \text{for each} \quad j = 1, 2, \ldots, k, \\ \frac{1}{k}\sum_{j=1}^{k} x_{ij} \quad \text{for each} \quad i = 1, 2, \ldots, k. \end{cases} \tag{3.10}$$

We will show that

$$\frac{1}{k}\sum_{i=1}^{k} x_i \in s-\liminf_{\max\{m,n\}\to\infty} G_{mn}(\omega) \quad \text{a.s.} \tag{3.11}$$

is enough to prove the "lim inf" part of Mosco convergence.

Indeed, since $\mathfrak{X}$ is separable, there exists a countable dense set D_X of X. For each fixed $x^{(j)} \in D_X$ and for every $\varepsilon_s = \frac{1}{s}$ $(s \geq 1)$, by (3.11), there exists z_s of $\mathfrak{X}$, which depends on $x^{(j)}$ and ε_s, such that $z_s \in s-\liminf_{\max\{m,n\}\to\infty} G_{mn}(\omega)$ a.s. Therefore, for each $s \geq 1$, there exists a negligible set $N_s \in \mathcal{A}$ such that $z_s \in s-\liminf_{\max\{m,n\}\to\infty} G_{mn}(\omega)$ for all $\omega \in \Omega \backslash N_s$. Letting $N = \bigcup_{s=1}^{\infty} N_s$, then $\mathbf{P}(N) = 0$. For each $\omega \in N$, it follows from the set $s-\liminf_{\max\{m,n\}\to\infty} G_{mn}(\omega)$ is closed, $z_s \in s-\liminf_{\max\{m,n\}\to\infty} G_{mn}(\omega)$ for all s and $z_s \to x^{(j)}$ as $s \to \infty$, that $x^{(j)} \in s-\liminf_{\max\{m,n\}\to\infty} G_{mn}(\omega)$. This means that $x^{(j)} \in s-\liminf_{\max\{m,n\}\to\infty} G_{mn}(\omega)$ a.s., for each fixed $j \geq 1$. Noting that D_X is a countable set, we obtain $D_X \subset s-\liminf_{\max\{m,n\}\to\infty} G_{mn}(\omega)$ a.s. Since the set $s-\liminf_{\max\{m,n\}\to\infty} G_{mn}(\omega)$ is closed for each ω, by taking the closure of both sides of the above relation, we have $X \subset s-\liminf_{\max\{m,n\}\to\infty} G_{mn}(\omega)$ a.s. Therefore, the above statement is proved.

By Lemma 3.2(1), there exists a double array $\{f_{ij} : i \geq 1, j \geq 1\}$ of $f_{ij} \in S^1_{F_{ij}}(\mathcal{A}_{F_{ij}})$ such that for each $i, j \in \{1, 2, \ldots, k\}$, $\left\{f_{(s-1)k+i,(t-1)k+j} : s \geq 1, t \geq 1\right\}$ is 2-exchangeable.

If $m = (s-1)k + p, n = (t-1)k + q$, where $1 \leq p, q \leq k$, then the following estimations hold

$$\left|\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(\omega) - \frac{1}{k}\sum_{i=1}^{k} x_i\right|\right| = \left|\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(\omega) - \frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=1}^{k} x_{ij}\right|\right| \quad (\text{by}(3.10))$$

$$\leq \frac{st}{mn}\sum_{i=1}^{k}\sum_{j=1}^{k}\left|\left|\frac{1}{st}\sum_{l=1}^{s}\sum_{r=1}^{t} f_{(l-1)k+i,(r-1)k+j}(\omega) - x_{ij}\right|\right| \tag{3.12}$$

$$+\frac{t}{mn}\sum_{i=p+1}^{k}\sum_{j=1}^{k}\frac{1}{t}\sum_{r=1}^{t}\left|\left|f_{(s-1)k+i,(r-1)k+j}(\omega)\right|\right| + \frac{s}{mn}\sum_{i=1}^{k}\sum_{j=q+1}^{k}\frac{1}{s}\sum_{l=1}^{s}\left|\left|f_{(l-1)k+i,(t-1)k+j}(\omega)\right|\right| \tag{3.13}$$

$$+\left(\frac{st}{mn}-\frac{1}{k^2}\right)\left|\left|\sum_{i=1}^{k}\sum_{j=1}^{k}x_{ij}\right|\right|. \qquad (3.14)$$

Now we prove the desired estimations that constitute the main technical part of the proof.

For (3.12), it follows from the condition (b) that the double array $\{f_{(s-1)k+i,(t-1)k+j} : s \geq 1, t \geq 1\}$ is in $L^2(\mathfrak{X})$ for each $i, j \in \{1, 2, \ldots, k\}$. Combining the above statements and by Theorem 2 to each array $\{f_{(s-1)k+i,(t-1)k+j} : s \geq 1, t \geq 1\}$, $1 \leq i, j \leq k$, we get

$$\left|\left|\frac{1}{st}\sum_{l=1}^{s}\sum_{r=1}^{t}f_{(l-1)k+i,(r-1)k+j}(\omega)-x_{ij}\right|\right| \to 0 \quad \text{a.s. as} \quad \max\{s,t\}\to\infty.$$

Thus,

$$\frac{st}{mn}\sum_{i=1}^{k}\sum_{j=1}^{k}\left|\left|\frac{1}{st}\sum_{l=1}^{s}\sum_{r=1}^{t}f_{(l-1)k+i,(r-1)k+j}(\omega)-x_{ij}\right|\right| \to 0 \quad \text{a.s.} \quad \text{as} \quad \max\{m,n\}\to\infty.$$

For (3.13), for each $s \geq 1$, since $\{f_{(s-1)k+i,(r-1)k+j} : r \geq 1\}$ is a sequence of 2-exchangeable random elements in $L^2(\mathfrak{X})$, it follows from Theorem 2 that $\frac{1}{t}\sum_{r=1}^{t}||f_{(s-1)k+i,(r-1)k+j}(\omega)-x_{ij}|| \to 0$ a.s. as $t\to\infty$. Hence,

$$\frac{t}{mn}\sum_{i=p+1}^{k}\sum_{j=1}^{k}\frac{1}{t}\sum_{r=1}^{t}||f_{(s-1)k+i,(r-1)k+j}(\omega)||$$

$$\leq \frac{t}{mn}\sum_{i=p+1}^{k}\sum_{j=1}^{k}\frac{1}{t}\sum_{r=1}^{t}||f_{(s-1)k+i,(r-1)k+j}(\omega)-x_{ij}|| + \frac{t}{mn}\sum_{i=p+1}^{k}\sum_{j=1}^{k}\frac{1}{t}\sum_{r=1}^{t}||x_{ij}||$$

$$\leq \frac{t}{mn}\sum_{i=p+1}^{k}\sum_{j=1}^{k}\frac{1}{t}\sum_{r=1}^{t}||f_{(s-1)k+i,(r-1)k+j}(\omega)-x_{ij}||$$

$$+\frac{t}{mn}\sum_{i=p+1}^{k}\sum_{j=1}^{k}||x_{ij}|| \to 0 \quad \text{a.s.} \quad \text{as} \quad \min\{m,n\}\to\infty.$$

Similarly, we get $\frac{s}{mn}\sum_{i=1}^{k}\sum_{j=q+1}^{k}\frac{1}{s}\sum_{l=1}^{s}||f_{(l-1)k+i,(t-1)k+j}(\omega)|| \to 0$ a.s. as $\min\{m,n\}\to\infty$.

For (3.14), by (3.8), we obtain $\left(\frac{st}{mn}-\frac{1}{k^2}\right)\left|\left|\sum_{i=1}^{k}\sum_{j=1}^{k}x_{ij}\right|\right| \to 0$ as $\min\{m,n\}\to\infty$. Combining the above limits, we get

$$\left|\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}f_{ij}(\omega)-\frac{1}{k}\sum_{i=1}^{k}x_i\right|\right| \to 0 \quad \text{a.s. as} \quad \min\{m,n\}\to\infty.$$

Next, for each $n=(t-1)k+j$, $1\leq j\leq k$, if $m=(s-1)k+p$, $1\leq p\leq k$, then

$$\left|\left|\frac{1}{m}\sum_{i=1}^{m}f_{in}(\omega)-\frac{1}{k}\sum_{i=1}^{k}x_i\right|\right| = \left|\left|\frac{1}{m}\sum_{i=1}^{m}f_{in}(\omega)-\frac{1}{k}\sum_{i=1}^{k}x_{ij}\right|\right| \quad \text{(by (3.10))}$$

$$\leq \frac{s}{m}\sum_{i=1}^{k}\left|\left|\frac{1}{s}\sum_{h=1}^{s}f_{(h-1)k+i,n}(\omega)-x_{ij}\right|\right| + \frac{s}{m}\sum_{i=p+1}^{k}\frac{1}{s}||f_{(s-1)k+i,n}(\omega)|| + \left(\frac{s}{m}-\frac{1}{k}\right)\left|\left|\sum_{i=1}^{k}x_{ij}\right|\right|.$$

For $1\leq i\leq k$, since $\{f_{(s-1)k+i,n} : s\geq 1\}$ is a sequence of 2-exchangeable random elements in $L^2(\mathfrak{X})$, then again by Theorem 2, $||\frac{1}{s}\sum_{h=1}^{s}f_{(h-1)k+i,n}(\omega)-x_{ij}|| \to 0$ a.s. as $s\to\infty$, and, hence,

$$\frac{1}{s}\,||f_{(s-1)k+i,n}(\omega)||$$

$$= \left\| \left(\frac{1}{s} \sum_{h=1}^{s} f_{(h-1)k+i,n}(\omega) - x_{ij} \right) - \frac{s-1}{s} \left(\frac{1}{s-1} \sum_{h=1}^{s-1} f_{(h-1)k+i,n}(\omega) - x_{ij} \right) + \frac{1}{s} x_{ij} \right\|$$

$$\to 0 \quad \text{a.s.} \quad \text{as} \quad s \to \infty.$$

Therefore, $\left\| \frac{1}{m} \sum_{i=1}^{m} f_{in}(\omega) - \frac{1}{k} \sum_{i=1}^{k} x_i \right\| \to 0$ a.s. as $m \to \infty$.

Similarly, we have that for each $m \geq 1$, $\left\| \frac{1}{n} \sum_{j=1}^{n} f_{mj}(\omega) - \frac{1}{k} \sum_{i=1}^{k} x_i \right\| \to 0$ a.s. as $n \to \infty$. Combining the above statements and [4, Proposition 3.5], we get

$$\frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f_{ij}(\omega) \to \frac{1}{k} \sum_{i=1}^{k} x_i \quad \text{a.s. as} \quad \max\{m,n\} \to \infty.$$

Hence, $\frac{1}{k} \sum_{i=1}^{k} x_i \in s-\liminf_{\max\{m,n\}\to\infty} G_{mn}(\omega)$ a.s. Thus, $X \subset s-\liminf_{\max\{m,n\}\to\infty} G_{mn}(\omega)$ a.s.

Since $\mathfrak{X}$ is separable, there is a countable dense subset D of $\mathfrak{X}$. From $X \subset s-\liminf_{\max\{m,n\}\to\infty} G_{mn}(\omega)$ a.s., there is a negligible set $N \in \mathcal{A}$ such that for every $\omega \in \Omega \setminus N$, $X \subset s-\liminf_{\max\{m,n\}\to\infty} G_{mn}(\omega)$. Fix $\omega \in \Omega \setminus N$. Then, for any $a \in D$ and $p \in \mathbf{N}$, there exists $b \in X$ satisfying $||a-b|| \leq d(a,X) + \frac{1}{p}$. Therefore, for each $m,n \geq 1$, we have $g_{mn} \in G_{mn}(\omega)$ such that $g_{mn} \to b$ as $\max\{m,n\} \to \infty$. Thus,

$$\limsup_{\max\{m,n\}\to\infty} d(a, G_{mn}(\omega)) \leq \lim_{\max\{m,n\}\to\infty} ||a - g_{mn}|| = ||a-b|| \leq d(a,X) + \frac{1}{p}.$$

Letting $p \to \infty$, we get

$$\limsup_{\max\{m,n\}\to\infty} d(a, G_{mn}(\omega)) \leq d(a,X). \tag{3.15}$$

Next, we recall that the function $d(\cdot, A) : \mathfrak{X} \to \mathbf{R}$ $(A \subset \mathfrak{X})$ is 1-Lipschitz, i.e., for every $x, y \in \mathfrak{X}$,

$$|d(x,A) - d(y,A)| \leq d(x,y). \tag{3.16}$$

For any x in $\mathfrak{X}$, there exists a sequence $\{x_k : k \geq 1\} \subset D$ satisfying $\lim_{k\to\infty} x_k = x$. Then, for each $m,n \geq 1$ and $k \geq 1$,

$$d(x, G_{mn}(\omega)) - d(x,X) \leq |d(x, G_{mn}(\omega)) - d(x_k, G_{mn}(\omega))| + \{d(x_k, G_{mn}(\omega)) - d(x_k, X)\}$$
$$+ |d(x_k, X) - d(x,X)| \leq 2d(x,x_k) + \{d(x_k, G_{mn}(\omega)) - d(x_k,X)\} \quad \text{(by (3.16))}.$$

Letting $\max\{m,n\} \to \infty$, we have $\limsup_{\max\{m,n\}\to\infty} \{d(x, G_{mn}(\omega)) - d(x,X)\} \leq 2d(x,x_k)$ (by (3.15)). Then, letting $k \to \infty$, we obtain $\limsup_{\max\{m,n\}\to\infty} d(x, G_{mn}(\omega)) \leq d(x,X)$.

Now define $\left\{x_j^* : j \geq 1\right\}$ as in proof of Theorem 3. Since the conditions (a) and (b) and by applying Lemma 3.1, $\left\{s(x_j^*, F_{mn}) : m \geq 1, n \geq 1\right\}$ is a double array of 2-exchangeable random variables satisfying all conditions of Theorem 2 for real-valued random variables case, and so, by using this theorem, there exists a negligible set $N \in \mathcal{A}$ such that for every $\omega \in \Omega \backslash N$ and $j \geq 1$,

$$s(x_j^*, G_{mn}(\omega)) = \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} s(x_j^*, F_{kl}(\omega)) \to s(x_j^*, X) < \infty \quad \text{as} \quad \max\{m,n\} \to \infty.$$

If $x \in w-\limsup_{\max\{m,n\}\to\infty} G_{mn}(\omega)$ for $\omega \in \Omega \backslash N$, then $x_{rs} \xrightarrow{w} x$ as $\max\{r,s\} \to \infty$ for some $x_{rs} \in G_{m_r n_s}(\omega)$ and hence $\langle x_j^*, x \rangle = \lim_{\max\{r,s\}\to\infty} \langle x_j^*, x_{rs} \rangle \leq \lim_{\max\{r,s\}\to\infty} s(x_j^*, G_{m_r n_s}(\omega)) = s(x_j^*, X)$, $j \geq 1$, which implies $x \in X$. Thus, $w-\limsup_{\max\{m,n\}\to\infty} G_{mn}(\omega) \subset X$ a.s.

Next, the closed unit ball of $\mathfrak{X}^*$ is denoted by B^*. It is known that for any closed convex subset A of $\mathfrak{X}$ and for any $x \in \mathfrak{X}$, we have $d(x, A) = \sup_{x^* \in B^*} \{\langle x^*, x\rangle - s(x^*, A)\}$. By using [13, Lemma 3.1], there exists a countable subset D^* of B^* such that, for any $x \in \mathfrak{X}$, $d(x, X) = \sup_{x^* \in D^*} \{\langle x^*, x\rangle - s(x^*, X)\}$.

Therefore, by virtue of Theorem 2 for each double array $\{s(x^*, F_{mn}) : m \geq 1, n \geq 1\}$, $x^* \in D^*$, we have that for every $x \in \mathfrak{X}$,

$$\begin{aligned}
\liminf_{\max\{m,n\}\to\infty} d(x, G_{mn}(\omega)) &\geq \liminf_{\max\{m,n\}\to\infty} d\left(x, \overline{\text{co}}\, G_{mn}(\omega)\right) \\
&= \liminf_{\max\{m,n\}\to\infty} \left(\sup_{x^* \in B^*} \{\langle x^*, x\rangle - s\left(x^*, G_{mn}(\omega)\right)\}\right) \\
&\geq \sup_{x^* \in B^*} \left\{ \liminf_{\max\{m,n\}\to\infty} \left(\langle x^*, x\rangle - \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} s\left(x^*, F_{ij}(\omega)\right)\right)\right\} \\
&\geq \sup_{x^* \in D^*} \left\{ \liminf_{\max\{m,n\}\to\infty} \left(\langle x^*, x\rangle - \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} s\left(x^*, F_{ij}(\omega)\right)\right)\right\} \\
&= \sup_{x^* \in D^*} \{\langle x^*, x\rangle - \mathbf{E}s\left(x^*, F_{11}(\cdot)\right)\} \quad \text{a.s.} \quad (\text{by Theorem 2}) \\
&= \sup_{x^* \in D^*} \{\langle x^*, x\rangle - s(x^*, X)\} = d(x, X) \quad \text{a.s.}
\end{aligned}$$

Since the set having probability one in above statement doesn't depend on x, we obtain that $\liminf_{\max\{m,n\}\to\infty} d(x, G_{mn}(\omega)) \geq d(x, X)$ for all $x \in \mathfrak{X}$, a.s. Hence, the theorem is proved completely. □

Next, we extend the above result to the fuzzy random sets whose level sets may be unbounded.

Theorem 5. *Assume that* $\left\{\widetilde{F}_{ij} : i \geq 1, j \geq 1\right\}$ *is a double array of 2-exchangeable fuzzy random sets in* $F(\mathfrak{X})$ *with the separable dual space* $\mathfrak{X}^*$ *such that* $S^1_{L_1(\widetilde{F}_{11})} \neq \emptyset$ *and* $L_\alpha\left(\overline{\text{co}}\, \mathbf{E}\widetilde{F}_{11}\right) = \text{cl}\left\{L_{\alpha^+}(\overline{\text{co}}\, \mathbf{E}\widetilde{F}_{11})\right\}$ *for every* $\alpha \in [0, 1] \setminus \mathbf{Q}$. *If the following conditions are satisfied:*

(a) for every $x^* \in \mathfrak{X}^*$ *and every* $\alpha \in (0, 1]$, *Cov* $\left(\langle x^*, g(L_\alpha \widetilde{F}_{11})\rangle, \langle x^*, g(L_\alpha \widetilde{F}_{22})\rangle\right) = 0$,

(b) $\mathbf{E}||L_{0^+}\widetilde{F}_{11}||^2 < \infty$,

then

$$\text{M} - \lim_{\max\{m,n\}\to\infty} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \widetilde{F}_{ij}(\omega) = \overline{co}\, \mathbf{E}\widetilde{F}_{11} \quad \text{a.s.},$$

where $g : c(\mathfrak{X}) \to \mathfrak{X}$ *is some measurable function.*

Proof. Let $\widetilde{G}_{mn}(\omega) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\widetilde{F}_{ij}(\omega)$. From the assumptions of theorem, we have that for each $\alpha \in (0, 1]$, the double array of random sets $\left\{L_\alpha \widetilde{F}_{ij} : i \geq 1, j \geq 1\right\}$ satisfies all hypothesis of Theorem 4. Therefore, by using Theorem 4, we obtain that

$$\begin{aligned}
\text{M} - \lim_{\max\{m,n\}\to\infty} L_\alpha \widetilde{G}_{mn}(\omega) &= \text{M} - \lim_{\max\{m,n\}\to\infty} \frac{1}{mn} \text{cl} \sum_{i=1}^{m} \sum_{j=1}^{n} L_\alpha \widetilde{F}_{ij} \\
&= \overline{\text{co}}\, \mathbf{E}\left(L_\alpha \widetilde{F}_{11}\right) = L_\alpha\left(\overline{\text{co}}\, \mathbf{E}\widetilde{F}_{11}\right) \quad \text{a.s.}
\end{aligned}$$

for every fixed $\alpha \in [0, 1]$, in particular, for every $\alpha = r \in \mathbf{Q}$. Since countable set $\mathbf{Q}$ is dense in $[0, 1]$, there exists a negligible subset N of Ω verifying

$$\text{M} - \lim_{\max\{m,n\}\to\infty} L_r \widetilde{G}_{mn}(\omega) = L_r\left(\overline{\text{co}}\, \mathbf{E}\widetilde{F}_{11}\right),$$

$$\text{for every} \quad r \in [0,1] \cap \mathbf{Q} \quad \text{and every} \quad \omega \in \Omega \setminus N. \tag{3.17}$$

Fix $\omega \in \Omega \setminus N$ in the rest of proof. For shortly, we put $u_{mn} = \widetilde{G}_{mn}(\omega)$ $(m, n \geq 1)$ and $u = \overline{\text{co}}\, \mathbf{E}\widetilde{F}_{11}$. It is easy to see that $u_{mn}, m \geq 1, n \geq 1$ are the fuzzy sets and u is the convex fuzzy random set.

Given any $\alpha \in (0,1) \setminus \mathbf{Q}$. Let $\{r_k : k \geq 1\}$ be in $\mathbf{Q} \cap [0,1]$ such that $r_k \nearrow \alpha$ as $k \to \infty$. Then,

$$L_\alpha v = \bigcap_{k=1}^{\infty} L_{r_k} v \quad \text{for any fuzzy set} \quad v. \tag{3.18}$$

Indeed, it is clear that $L_\alpha v \subset L_{r_k} v$ for every k. Consequently, $L_\alpha v \subset \bigcap_{k=1}^{\infty} L_{r_k} v$. For every $x \notin L_\alpha v$, there is an integer k_0 such that $v(x) < r_{k_0} < \alpha$. This implies $x \notin L_{r_{k_0}} v$, so $x \notin \bigcap_{k=1}^{\infty} L_{r_k} v$. Thus, (3.18) is proved.

On the other hand, by (3.17), we obtain

$$w-\limsup_{\max\{m,n\}\to\infty} L_\alpha(u_{mn}) \subset w-\limsup_{\max\{m,n\}\to\infty} L_{r_k}(u_{mn}) \subset L_{r_k} u, \quad \text{for every} \quad k \geq 1.$$

This with (3.18) implies

$$w-\limsup_{\max\{m,n\}\to\infty} L_\alpha(u_{mn}) \subset \bigcap_{k=1}^{\infty} L_{r_k} u = L_\alpha u. \tag{3.19}$$

Next, let $\{s_k : k \geq 1\}$ be in $\mathbf{Q} \cap [0,1]$ such that $s_k \searrow \alpha$ as $k \to \infty$. It is easy to see that $L_{s_k} u \subset L_{\alpha^+} u$ for every k, which yields $\bigcup_{k=1}^{\infty} L_{s_k} u \subset L_{\alpha^+} u$. Also, for each $x \in L_{\alpha^+} u$, there exists an integer k_1 such that $\alpha < s_{k_1} < u(x)$. Then, $x \in L_{s_{k_1}} u$, and so $x \in \bigcup_{k=1}^{\infty} L_{s_k} u$. Therefore, $L_{\alpha^+} u = \bigcup_{k=1}^{\infty} L_{s_k} u$. Hence, $L_\alpha u = \text{cl}\,(\bigcup_{k=1}^{\infty} L_{s_k} u)$.

Further, by virtue of (3.17), we deduce that

$$L_{s_k} u \subset s-\liminf_{\max\{m,n\}\to\infty} L_{s_k}(u_{mn}) \subset s-\liminf_{\max\{m,n\}\to\infty} L_\alpha(u_{mn}),$$

for every $k \geq 1$. Combining this with the closeness of $s-\liminf_{\max\{m,n\}\to\infty} L_\alpha(u_{mn})$, we have that

$$L_\alpha u \subset s-\liminf_{\max\{m,n\}\to\infty} L_\alpha(u_{mn}). \tag{3.20}$$

Hence, $M-\lim_{\max\{m,n\}\to\infty} L_\alpha(u_{mn}) = L_\alpha u$ follows from (3.19) and (3.20). □

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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