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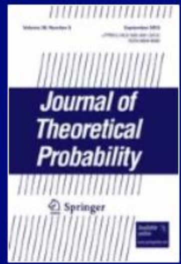

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Chung-Type Strong Laws and Almost Complete Convergence for Arrays of Measurable Operators

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Abstract
The aim of this study is to establish some Chung-type strong laws of large numbers and almost complete convergence for arrays of measurable operators under various conditions. Some related results in the literature are extended to the noncommutative context.

Keywords Strong law of large numbers · Almost complete convergence · Arrays of measurable operators · Noncommutative probability

Mathematics Subject Classification (2020) 46L53 · 60F15 · 60F99

1 Introduction

As is well known, the law of large numbers (LLN) is an essential theory in probability, statistics and related fields. The strong LLN for arrays of random variables was considered by several authors. For example, Hu and Taylor [7] presented some Chung-type strong LLN's for arrays of row-wise independent random variables. Bozorgnia et al. [2] also obtained strong LLN's for Banach spaces under conditions similar to those of Chung [4] and Hu and Taylor [7] with appropriate modifications for the geometric condition type p. Hu et al. [6] gave the complete convergence for arrays of row-wise independent random variables, and Gut [5] extended their results. When considering the complete convergence for arrays of random elements in Banach space, Taylor [15]

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and Hu et al. [8] also derived some interested results without the geometry of the underlying Banach space.

In noncommutative probability theory, there are several versions of LLN, e.g., Batty [1], Jaite [9] and Luczak [12] have proved some weak and strong LLN's for a sequence of successively independent measurable operators. Some strong LLN's for positive measurable operators have been established by Quang et al. [13], and Choi and Ji [3] gave estimates of the rate of convergence for weighted sums of measurable operators. Recently, Quang et al. [14] established several LLN's for the sequence of measurable operators under some kinds of uniform integrability. Other versions of LLN can be found in Chao and YouLiang [16], Klimczak [10], Lindsay and Pata [11] and the references cited therein. However, the strong LLN for arrays of measurable operators has not yet been studied in our knowledge.

The main purpose of this paper is to give some Chung-type strong laws of large numbers and the almost complete convergence for arrays of measurable operators under various conditions. To do this purpose, we establish some new properties and use the “mean type of the Multinomial theorem" for measurable operators. The layout of this paper is as follows: In Sect. 2, we summarize some basic concepts and related properties. Section 3 will establish some Chung-type strong LLN's. Finally, the almost complete convergence for arrays of measurable operators will be considered in Sect. 4.

2 Preliminaries

Let H be a Hilbert space and $\mathcal{B}(H)$ be the algebra of all bounded linear operators on H. A *von Neumann algebra* is a subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$ which is self-adjoint (that is, if $a \in \mathcal{A}$, then $a^* \in \mathcal{A}$), contains $\mathbf{1}$, and is closed in the weak operator topology.

The range $\mathcal{R}(T)$ and the null space $\mathcal{N}(T)$ of a operator $T \in \mathcal{B}(H)$ are subspaces of H; $\mathcal{N}(T)$ is closed, but $\mathcal{R}(T)$ is not necessarily closed. A bounded linear operator U is called a *partial isometry* if for $h \in \mathcal{N}(U)^{\perp}$, $\|U(h)\| = \|h\|$. Let X be a densely defined closed operator in H, $|X| = (X^*X)^{1/2}$, and let U be the partial isometry in the polar decomposition $X = U|X|$ of X. Then, X is said to be *affiliated* to the von Neumann algebra $\mathcal{A}$ if U and all the spectral projections of $|X|$ belong to $\mathcal{A}$. We notate $\widetilde{\mathcal{A}}$ for the set of all operators which are affiliated to the von Neumann algebra $\mathcal{A}$. An element of $\widetilde{\mathcal{A}}$ is called a *measurable operator*. Let τ be a faithful normal tracial state on $\mathcal{A}$. For notational consistency, $\widetilde{\mathcal{A}}$ will be denoted by $L^0(\mathcal{A}, \tau)$. Then, we have natural inclusions:

$$\mathcal{A} \equiv L^{\infty}(\mathcal{A}, \tau) \subset L^q(\mathcal{A}, \tau) \subset L^p(\mathcal{A}, \tau) \subset ... \subset L^0(\mathcal{A}, \tau) = \widetilde{\mathcal{A}}$$

for $1 \leq p \leq q < \infty$, where $L^p(\mathcal{A}, \tau)$ is a Banach space of all elements in $L^0(\mathcal{A}, \tau)$ satisfying

$$||X||_p = [\tau(|X|^p)]^{\frac{1}{p}} < \infty.$$

A densely defined closed operator $X : H \supseteq \mathcal{D}(X) \mapsto \mathcal{R}(X) \subseteq H$ is called *positive*, denoted by $X \geq 0$, if X is self-adjoint and $\langle X(h), h\rangle \geq 0$ for all $h \in \mathcal{D}(X)$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in Hilbert space H. Let S, T be densely defined closed operators. We say that $S \le T$ if $T - S \ge 0$.

For two projection p, q in $\mathcal{A}$, p is called a *subprojection* of q if $p \le q$. Note that if $p \le q$, $X \in \widetilde{\mathcal{A}}$ and $Xp, Xq \in \mathcal{A}$, then $\left\|Xp\right\|_\infty \le \left\|Xq\right\|_\infty$. Indeed, since $p \le q$, we first show that $\mathcal{N}(p) \supset \mathcal{N}(q)$. Let $h \in \mathcal{N}(q)$. Then, $q(h) = 0$. By $\|p(h)\|^2 = \langle p(h), p(h) \rangle = \langle p(h), h \rangle \le \langle q(h), h \rangle = 0$, it follows that $p(h) = 0, h \in \mathcal{N}(p)$, and hence, $\mathcal{N}(p) \supset \mathcal{N}(q)$, which together $H = \mathcal{R}(p) \bigoplus \mathcal{N}(p) = \mathcal{R}(q) \bigoplus \mathcal{N}(q)$, we get $\mathcal{R}(p) \subseteq \mathcal{R}(q)$, and it follows that

$$\begin{aligned}\left\|Xp\right\|_\infty &= \sup\left\{\left\|Xp(h)\right\| : \|h\| \le 1\right\} = \sup\left\{\left\|Xq\left[p(h)\right]\right\| : \|h\| \le 1\right\} \\ &\le \sup\left\{\left\|Xq(k)\right\| : \|k\| \le 1\right\} = \left\|Xq\right\|_\infty.\end{aligned}$$

Two projections p and q are said to be *equivalent*, written $p \sim q$, if there exists a partial isometry U in $\mathcal{A}$ such that $U^*U = p$ and $UU^* = q$. Then, since τ is tracial, we have $\tau(p) = \tau(q)$ when $p \sim q$. For any projections p and q in $\mathcal{A}$, we have $(p \vee q - q) \sim (p - p \wedge q)$, which implies that $\tau(p \vee q) \le \tau(p) + \tau(q)$. We denote $p \prec q$ if p is equivalent to a subprojection of q. If $p \wedge q = 0$, then $p \sim (p \vee q - q) \le \mathbf{1} - q = q^\perp$, i.e., $p \prec q^\perp$, and hence, $\tau(p) \le \tau(q^\perp)$.

Let $\{X_n, n \ge 1\}$ be a sequence in $L^0(\mathcal{A}, \tau)$ and $X \in L^0(\mathcal{A}, \tau)$. We say that

The sequence $\{X_n, n \ge 1\}$ *converges almost uniformly* to X, denoted by $X_n \xrightarrow{a.u.} X \quad as \quad n \to \infty$ if, for every $\varepsilon > 0$, there exists a projection $p \in \mathcal{A}$ such that $\tau(p^\perp) < \varepsilon$, $(X_n - X)p \in \mathcal{A}$, and

$$\lim_{n\to\infty} \left\|(X_n - X)p\right\|_\infty = 0.$$

The sequence $\{X_n, n \ge 1\}$ *converges almost completely* to X, denoted by $X_n \xrightarrow{a.c.} X \quad as \quad n \to \infty$ if, for every $\varepsilon > 0$, there exists a sequence (q_n) of projections in $\mathcal{A}$ such that $\sum_{n=1}^{\infty} \tau(\mathbf{1} - q_n) < \infty$, $(X_n - X)q_n \in \mathcal{A}$, and $\left\|(X_n - X)q_n\right\|_\infty < \varepsilon$, for $n = 1, 2, \ldots$

Denote $e_B(X)$ by the spectral projection of a self-adjoint operator X corresponding to the Borel subset B of the real line $\mathbb{R}$. Obviously, if $\sum_{n=1}^{\infty} \tau\left[e_{(\varepsilon,\infty)}\left(|X_n - X|\right)\right] < \infty$ for every $\varepsilon > 0$, then $X_n \xrightarrow{a.c.} X \quad as \quad n \to \infty$. Indeed, for every $\varepsilon > 0$, by choosing $\mathbf{1} - q_n = e_{(\varepsilon,\infty)}\left(|X_n - X|\right)$, we easily can get $\sum_{n=1}^{\infty} \tau(\mathbf{1} - q_n) < \infty$, $(X_n - X)q_n \in \mathcal{A}$ and $\left\|(X_n - X)q_n\right\|_\infty < \varepsilon$.

Let $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of measurable operators. The array $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ is said to be *uniformly bounded* by a measurable operator X if for all n, k and for every real number $t > 0$,

$$\tau\left[e_{(t,\infty)}(|X_{nk}|)\right] \le \tau\left[e_{(t,\infty)}(|X|)\right].$$

For convenience, throughout of this paper, the symbol C will denote a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

3 Chung-Type Strong Laws

In this section, Chung-type strong laws of large numbers are obtained for arrays of measurable operators.

At first, we present some lemmas which we will use in the proof of our main results.

The following lemma is a slight generalization of the noncommutative version of Chebyshev' inequality.

Lemma 3.1 (See [13], Lemma 2.3) *Let $X \in L^0(\mathcal{A}, \tau)$, and let $g : (0; \infty) \to (0; \infty)$ be a nondecreasing function such that $\tau(g(|X|)) < \infty$. Then, for each $\varepsilon > 0$, we have*

$$\tau\left(e_{[\varepsilon,\infty)}(|X|)\right) \leq \frac{\tau(g(|X|))}{g(\varepsilon)}.$$

Lemma 3.2 *Let X be a self-adjoint element of $\mathcal{A}$. Then, for each $a \in \mathbb{R}$, we have*

$$|X + a| \leq |X| + |a|.$$

Proof Suppose that the spectral representation of the self-adjoint element X is

$$X = \int_{-\infty}^{+\infty} \lambda e_{d\lambda}(X).$$

Then, for all $h \in H$, we have

$$\begin{aligned}\Big\langle (|X| + |a|)\,(h), h\Big\rangle - \Big\langle |X + a|\,(h), h\Big\rangle &= \int_{-\infty}^{+\infty} (|\lambda| + |a|)\, de_{h,h} - \int_{-\infty}^{+\infty} |\lambda + a|\, de_{h,h} \\ &= \int_{-\infty}^{+\infty} \big[\,(|\lambda| + |a|) - |\lambda + a|\,\big] de_{h,h} \geq 0.\end{aligned}$$

Hence, $\Big\langle |X + a|\,(h), h\Big\rangle \leq \Big\langle (|X| + |a|)\,(h), h\Big\rangle$, for all $h \in H$.
This implies

$$|X + a| \leq |X| + |a|.$$

(where $e_{h,h}$ is the positive measure on $\mathcal{B}(\mathbb{R})$, defined by $e_{h,h}(B) = \Big\langle e(B)(h), h\Big\rangle$ for any Borel subset B of $\mathbb{R}$). ◻

Lemma 3.3 *Suppose that* $r > 0$ *and* X *is a self-adjoint element of* $L^r(\mathcal{A}, \tau)$. *Then, for each* $a \in \mathbb{R}$, *we have*

$$\tau\left(|X+a|^r\right) \le C_r\left[\tau(|X|^r + |a|^r)\right], \text{ where } C_r = \max\{1, 2^{r-1}\}.$$

Proof As above, suppose that the spectral representation of the self-adjoint element X is

$$X = \int_{-\infty}^{+\infty} \lambda e_{d\lambda}(X).$$

Using the elementary inequality $|\alpha+\beta|^r \le C_r\,(|\alpha|^r + |\beta|^r)$, for all $\alpha, \beta \in \mathbb{R}$, we obtain

$$\begin{aligned}\tau\left(|X+a|^r\right) = \int_{-\infty}^{+\infty} |\lambda + a|^r \tau\left(e_{d\lambda}(X)\right) &\le C_r \int_{-\infty}^{+\infty} \left(|\lambda|^r + |a|^r\right)\tau\left(e_{d\lambda}(X)\right) \\ &= C_r\Big[\tau\left(|X|^r\right) + |a|^r\Big].\end{aligned}$$

□

From now until the end of this section, we always suppose that $\{a_n,\ n \ge 1\}$ be a sequence of positive real numbers such that $a_{n+1} > a_n$ and $\lim a_n = +\infty$.

Lemma 3.4 *Let* $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ *be an array of measurable operators such that*

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} \tau\left(e_{(a_n,\infty)}(|X_{ni}|)\right) < \infty, \tag{3.1}$$

Putting $Y_{ni} = X_{ni}e_{[0,a_n]}(|X_{ni}|)$, $\widetilde{S}_n = \sum_{i=1}^{n} Y_{ni}$, $S_n = \sum_{i=1}^{n} X_{ni}$, *we have*

$$\frac{S_n - \widetilde{S}_n}{a_n} \xrightarrow{a.u.} 0 \ \ as \ \ n \to \infty.$$

Proof Let $\varepsilon > 0$ be given.

By (3.1), we can find an integer N such that $\sum_{n=N+1}^{\infty}\sum_{i=1}^{n} \tau\left(e_{(a_n,\infty)}(|X_{ni}|)\right) < \frac{\varepsilon}{3}$.

Since $X_{ki},\ Y_{ki} \in L^0(\mathcal{A}, \tau)$ for all $1 \le k \le N,\ 1 \le i \le k$, there exist projections $p_1,\ p_2$ in $\mathcal{A}$ ($p_1,\ p_2$ depend only on N) with $\tau(p_1^{\perp}) < \frac{\varepsilon}{3}$, $\tau(p_2^{\perp}) < \frac{\varepsilon}{3}$ such that $\sum_{i=1}^{k} X_{ki}p_1 \in \mathcal{A}$, $\sum_{i=1}^{k} X_{ki}p_2 \in \mathcal{A}$.

Clearly, there exists an integer N_1, such that, for $n \geq N_1$,

$$\frac{1}{a_n}\left\|\sum_{k=1}^{N}\sum_{i=1}^{k} X_{ki}p_1\right\|_{\infty} < \frac{\varepsilon}{4} \text{ and } \frac{1}{a_n}\left\|\sum_{k=1}^{N}\sum_{i=1}^{k} X_{ki}p_2\right\|_{\infty} < \frac{\varepsilon}{4}.$$

Put $p = p_1 \wedge p_2 \wedge \left(\bigwedge_{n>N} e_{[0,a_n]}(|X_{n1}|)\right) \wedge \ldots \wedge \left(\bigwedge_{n>N} e_{[0,a_n]}(|X_{nn}|)\right) = p_1 \wedge p_2 \wedge p_3$,
where $p_3 = \left(\bigwedge_{n>N} e_{[0,a_n]}(|X_{n1}|)\right) \wedge \ldots \wedge \left(\bigwedge_{n>N} e_{[0,a_n]}(|X_{nn}|)\right)$.
We have

$$\begin{aligned}\tau(p^{\perp}) &\leq \tau(p_1^{\perp}) + \tau(p_2^{\perp}) + \tau(p_3^{\perp}) \leq \tau(p_1^{\perp}) + \tau(p_2^{\perp}) \\ &\quad + \sum_{n=N+1}^{\infty}\sum_{i=1}^{n} \tau\left(e_{(a_n,\infty)}(|X_{ni}|)\right) < \varepsilon.\end{aligned}$$

Then, for $n \geq \max\{N, N_1\}$,

$$\begin{aligned}\left\|\frac{S_n - \widetilde{S}_n}{a_n}p\right\|_{\infty} &= \left\|\frac{1}{a_n}\sum_{i=1}^{n}(X_{ni} - Y_{ni})p\right\|_{\infty} \\ &\leq \left\|\frac{1}{a_n}\left[\sum_{k=1}^{n}\sum_{i=1}^{k}(X_{ki} - Y_{ki}) - \sum_{k=1}^{n-1}\sum_{i=1}^{k}(X_{ki} - Y_{ki})\right]p\right\|_{\infty} \\ &\leq \frac{1}{a_n}\left\|\sum_{k=1}^{n}\sum_{i=1}^{k}(X_{ki} - Y_{ki})p\right\|_{\infty} + \frac{1}{a_n}\left\|\sum_{k=1}^{n-1}\sum_{i=1}^{k}(X_{ki} - Y_{ki})p\right\|_{\infty} \\ &\leq \frac{2}{a_n}\left\|\sum_{k=1}^{N}\sum_{i=1}^{k}(X_{ki} - Y_{ki})p\right\|_{\infty} + \frac{1}{a_n}\left\|\sum_{k=N+1}^{n}\sum_{i=1}^{k}(X_{ki} - Y_{ki})p\right\|_{\infty} \\ &\quad + \frac{1}{a_n}\left\|\sum_{k=N+1}^{n-1}\sum_{i=1}^{k}(X_{ki} - Y_{ki})p\right\|_{\infty} \\ &\leq \frac{2}{a_n}\left\|\sum_{k=1}^{N}\sum_{i=1}^{k}X_{ki}p_1\right\|_{\infty} + \frac{2}{a_n}\left\|\sum_{k=1}^{N}\sum_{i=1}^{k}Y_{ki}p_2\right\|_{\infty} \\ &\quad + \frac{1}{a_n}\left\|\sum_{k=N+1}^{n}\sum_{i=1}^{k}(X_{ki} - Y_{ki})p_3\right\|_{\infty} \\ &\quad + \frac{1}{a_n}\left\|\sum_{k=N+1}^{n-1}\sum_{i=1}^{k}(X_{ki} - Y_{ki})p_3\right\|_{\infty} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + 0 + 0 = \varepsilon.\end{aligned}$$

Hence,

$$\frac{S_n - \widetilde{S}_n}{a_n} \xrightarrow{a.u.} 0 \ \ as \ \ n \to \infty.$$

□

Let $\psi(t)$ be a positive, even, continuous function such that $\frac{\psi(|t|)}{|t|^p}$ is a monotonically increasing function of $|t|$ and $\frac{\psi(|t|)}{|t|^{p+1}}$ is a monotonically decreasing function of $|t|$, respectively, that is,

$$\frac{\psi(|t|)}{|t|^p} \uparrow \text{ and } \frac{\psi(|t|)}{|t|^{p+1}} \downarrow \text{ as } |t| \uparrow \tag{3.2}$$

for some nonnegative integer p.
Let $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ be a triangular array of self-adjoint measurable operators. Consider the following conditions:

$$+ \quad \tau(X_{nk}) = 0, \tag{3.3}$$

$$+ \quad \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{\tau\left(\psi(X_{nk})\right)}{\psi(a_n)} < \infty. \tag{3.4}$$

We define $Y_{nk} = X_{nk} e_{[0,a_n]}(|X_{nk}|)$ and $W_{nk} = \frac{Y_{nk} - \tau(Y_{nk})}{a_n}$. For some positive integer s, we have

$$+ \quad \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} \tau\left(\left|\frac{X_{nk}}{a_n}\right|^2\right)\right)^{2s} < \infty, \tag{3.5}$$

and

$$+ \quad \tau\left(\prod_{j=1}^{2ps} W_{nk_j}\right) \le \prod_{i=1}^{t} \tau\left(W_{ns_i}^{\alpha_i}\right), \quad (\alpha_1 + \alpha_2 + ... + \alpha_t = 2ps) \tag{3.6}$$

where W_{ns_i}, $1 \le i \le t$ are the distinguishing operators of the sequence $\{W_{ni}, 1 \le i \le n\}$ and the operator W_{ns_i} appears α_i-times in $\prod_{j=1}^{2ps} W_{nk_j}$.
The conclusion of interest is

$$\frac{1}{a_n}\sum_{k=1}^{n} X_{nk} \xrightarrow{a.u.} 0 \ \ as \ \ n \to \infty. \tag{3.7}$$

(3.4) is a necessary condition for strong law of large numbers (3.7) to hold in some sense (see Chung [4]), even in the case of random variables. Different strong laws of large numbers are obtained for the integers $p \geq 2$, $p = 1$, and $p = 0$. These strong laws of large numbers are explicitly stated in Theorems 3.5, 3.8, and 3.9.

Theorem 3.5 *Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of self-adjoint measurable operators. Let $\psi(t)$ satisfy (3.2) for some integer $p \geq 2$. Then, the conditions (3.3),(3.4),(3.5) and (3.6) imply (3.7).*

Proof For each $n \geq 1$, put

$$Y_{nk}^* = X_{nk} e_{(a_n,\infty)}(|X_{nk}|), \quad S_n = \sum_{k=1}^{n} X_{nk}, \quad \widetilde{S}_n = \sum_{k=1}^{n} Y_{nk}.$$

We prove that, as $n \longrightarrow \infty$,

$$\frac{S_n - \widetilde{S}_n}{a_n} \xrightarrow{a.u.} 0, \tag{3.8}$$

$$\frac{\tau(\widetilde{S}_n)}{a_n} \longrightarrow 0, \tag{3.9}$$

and

$$\frac{\widetilde{S}_n - \tau(\widetilde{S}_n)}{a_n} \xrightarrow{a.u.} 0. \tag{3.10}$$

Since $\psi(|t|)$ is an increasing function of $|t|$, from Lemma 3.1 and condition (3.4), we get

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \tau\left[e_{(a_n,\infty)}(|X_{nk}|)\right] \leq \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{\tau\left[\psi(X_{nk})\right]}{\psi(a_n)} < \infty,$$

which together with Lemma 3.4 yields (3.8).

To prove (3.9), since $\dfrac{\psi(|t|)}{|t|}$ is an increasing function of $|t|$, by conditions (3.3) and (3.4), we get

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{\tau(Y_{nk})}{a_n}\right| &= \left|\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{\tau(Y_{nk}^*)}{a_n}\right| \leq \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{\tau(|Y_{nk}^*|)}{a_n} \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{a_n} \tau\left[|X_{nk}| e_{(a_n,\infty)}(|X_{nk}|)\right] \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{a_n} \int_{a_n}^{+\infty} \lambda \tau\left[e_{d\lambda}(|X_{nk}|)\right] = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \int_{a_n}^{+\infty} \frac{\lambda}{a_n} \tau\left[e_{d\lambda}(|X_{nk}|)\right]
\end{aligned}$$

$$\leq \sum_{n=1}^{\infty} \sum_{k=1}^{n} \int_{a_n}^{+\infty} \frac{\psi(\lambda)}{\psi(a_n)} \tau\left[e_{d\lambda}(|X_{nk}|)\right]$$

$$\text{(because } \frac{\lambda}{a_n} < \frac{\psi(\lambda)}{\psi(a_n)}, \text{ for } \lambda > a_n)$$

$$\leq \sum_{n=1}^{\infty} \sum_{k=1}^{n} \int_{-\infty}^{+\infty} \frac{\psi(\lambda)}{\psi(a_n)} \tau\left[e_{d\lambda}(|X_{nk}|)\right] = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{\tau(\psi(X_{nk}))}{\psi(a_n)} < \infty,$$

which implies that

$$\sum_{k=1}^{n} \frac{\tau(Y_{nk})}{a_n} \longrightarrow 0 \text{ as } n \to \infty,$$

so the condition (3.9) holds.

We now prove (3.10); note that for sufficiently large n, conditions (3.4) and (3.5) provide that

$$\left(\sum_{k=1}^{n} \frac{\tau\big(\psi(X_{nk})\big)}{\psi(a_n)} \right) < 1 \text{ and } \left(\sum_{k=1}^{n} \tau\left(\left| \frac{X_{nk}}{a_n} \right|^2 \right) \right) < 1. \tag{3.11}$$

By Lemma 3.3, for $1 \leq u \leq v$, we obtain

$$\tau(W_{nk})^v \leq \tau(|W_{nk}|^v) = \tau\left(\left| \frac{Y_{nk} - \tau(Y_{nk})}{a_n} \right|^v \right) \leq 2^{v-1} \left[\tau\left(\left| \frac{Y_{nk}}{a_n} \right|^v \right) + \left| \frac{\tau(Y_{nk})}{a_n} \right|^v \right]$$

$$\leq 2^v \tau\left(\left| \frac{Y_{nk}}{a_n} \right|^v \right) = 2^v \tau\left(\frac{|X_{nk}|^v e_{[0,a_n]}(|X_{nk}|)}{a_n^v} \right) = 2^v \int_0^{a_n} \left| \frac{\lambda}{a_n} \right|^v \tau\Big(e_{d\lambda}(|X_{nk}|)\Big)$$

$$\leq 2^v \int_0^{a_n} \left| \frac{\lambda}{a_n} \right|^u \tau\Big(e_{d\lambda}(|X_{nk}|)\Big) = 2^v \tau\left(\left| \frac{Y_{nk}}{a_n} \right|^u \right).$$

It means that

$$\tau(W_{nk})^v \leq 2^v \tau\left(\left| \frac{Y_{nk}}{a_n} \right|^v \right) \leq 2^v \tau\left(\left| \frac{Y_{nk}}{a_n} \right|^u \right). \tag{3.12}$$

Next, similar to the technique in Hu and Taylor [7], we have

$$\tau\left(\sum_{k=1}^{n} W_{nk} \right)^{2ps} = \sum_{k_1,k_2,\ldots,k_{2ps}} \tau\left(\prod_{j=1}^{2ps} W_{nk_j} \right), \tag{3.13}$$

where the sum is extended for all $2ps$-tuples $(k_1, ..., k_{2ps})$ with $k_j = 1, 2, ..., n$ for each j.

It is very interesting that by the noncommutativity of the operators, in noncommutative probability the "Multinomial theorem" is not true. However, using (3.6) and proceeding as in [7], we get the "mean type of the Multinomial theorem" for $\tau\left(\sum_{k=1}^{n} W_{nk}\right)^{2ps}$.

For each $\tau\left(\prod_{j=1}^{2ps} W_{nk_j}\right)$ in the sum on the right side of (3.13), since the condition (3.6) and $\tau(W_{nk}) = 0$, if there is a j with $W_{nk_i} \neq W_{nk_j}$ for all $i \neq j$, then $\tau\left(\prod_{j=1}^{2ps} W_{nk_j}\right) \leq 0$. Therefore, using (3.6), (3.11), (3.12) and following the argument of Hu and Taylor [7], we obtain that

$$\sum_{n=1}^{\infty} \tau\left(\sum_{k=1}^{n} W_{nk}\right)^{2ps} \leq C \sum_{n=1}^{\infty}\left[\left(\sum_{k=1}^{n} \tau\left(\left|\frac{X_{nk}}{a_n}\right|^2\right)\right)^{2s} + \left(\sum_{k=1}^{n} \frac{\tau\big(\psi(X_{nk})\big)}{\psi(a_n)}\right)\right] < \infty.$$

By applying Lemma 3.1, for any $\varepsilon > 0$, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} \tau\left(e_{(\varepsilon,\infty)}\left(\left|\sum_{k=1}^{n} W_{nk}\right|\right)\right) \\
&\leq \frac{1}{\varepsilon^{2ps}} \sum_{n=1}^{\infty} \tau\left(\left|\sum_{k=1}^{n} W_{nk}\right|^{2ps}\right) \\
&= \frac{1}{\varepsilon^{2ps}} \sum_{n=1}^{\infty} \tau\left(\sum_{k=1}^{n} W_{nk}\right)^{2ps} < \infty,
\end{aligned}$$

which implies that

$$\sum_{k=1}^{n} W_{nk} \xrightarrow{a.u.} 0 \quad \text{as} \quad n \to \infty,$$

and hence, (3.10) holds. The proof is completed. □

Remark 3.6 It is obvious that the condition (3.6) is satisfied for an array of row-wise independent real-valued random variables. Moreover, the other conditions of Theorem 3.5 are the same as the ones of Theorem 2.1 of Hu and Taylor [7]. Hence, Theorem 3.5 is an extension of Theorem 2.1 of Hu and Taylor [7] to noncommutative context.

The following example shows a triangular array of self-adjoint measurable operators satisfying the conditions (3.3–3.6), but its elements do not commute

Example 3.7 Let $\mathcal{M}_2$ be the algebra of 2×2 complex matrices, I_2 be the identity of $\mathcal{M}_2$, and τ be the unique tracial state, and let $\psi(t) = t^{p+\frac{1}{2}}$.

Fix

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then,

$$AB = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad BA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and for every positive integer k, we have that

$$\begin{aligned} &A^{2k} = B^{2k} = I_2, \\ &A^{2k+1} = A, \quad B^{2k+1} = B, \\ &(AB)^{4k} = (BA)^{4k} = I_2, \\ &(AB)^{4k+1} = AB, \quad (BA)^{4k+1} = BA, \\ &(AB)^{4k+2} = (BA)^{4k+2} = -I_2, \\ &(AB)^{4k+3} = -AB, \quad (BA)^{4k+3} = -BA. \end{aligned} \tag{3.14}$$

We consider the array of matrices $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ as follows:
For $n = 1$, we choose $X_{11} = A$.
For $n \geq 2$, we put $X_{n1} = A$, $X_{nj} = B$ for $2 \leq j \leq n$.
Then, the condition (3.3) holds.
For each n, let

$$a_n = \max \left\{ \left(n.2^n. \max_{1 \leq k \leq n} \tau\left(\psi(X_{nk})\right)\right)^{\frac{2}{2p+1}}, \left(n.2^n. \max_{1 \leq k \leq n} \tau(|X_{nk}|^2)\right)^{1/2}, n+1 \right\}.$$

Then, the conditions (3.4) and (3.5) hold.
Since

$$|A| = |B| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

with the eigenvalue 1, then $Y_{nk} = X_{nk}$ for all n and k.

Therefore, $W_{11} = \frac{1}{a_1} A$, and for every $n \geq 2$, $W_{n1} = \frac{1}{a_n} A$, $W_{nj} = \frac{1}{a_n} B$ for $2 \leq j \leq n$.

In each $\prod_{j=1}^{2ps} W_{nk_j}$, assume that the operators A, B appear a-times and b-times ($a + b = 2ps$), respectively.

If the numbers a, b are odd, then by (3.14), it implies that $\prod_{j=1}^{2ps} W_{nk_j}$ can be equal to $(AB)^{2k+1}$ or equal to $(BA)^{2k+1}$. In this case, $\tau\left(\prod_{j=1}^{2ps} W_{nk_j}\right) = 0$ and $\tau(A^a).\tau(B^b) = 0$, so the condition (3.6) holds.

If the numbers a, b are even, then by (3.14), it implies that $\prod_{j=1}^{2ps} W_{nk_j}$ can be equal to I_2, or equal to $(AB)^{2k}$, or equal to $(BA)^{2k}$. In this case, $\tau\left(\prod_{j=1}^{2ps} W_{nk_j}\right) = 1$ or $\tau\left(\prod_{j=1}^{2ps} W_{nk_j}\right) = -1$, and also $\tau(A^a).\tau(B^b) = 1$. So, the condition (3.6) holds.

However, if n is odd ($n > 1$), then $\prod_{k=1}^{n} W_{nk} = ABB...B = A$, and

$$W_{nj} \prod_{k=1,k\neq j}^{n} W_{nk} = BABB...B = BAB = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$
$$\neq A, \text{ for every } 2 \leq j \leq n.$$

Also, if n is even, then $\prod_{k=1}^{n} W_{nk} = ABB...B = AB$, and

$$W_{nj} \prod_{k=1,k\neq j}^{n} W_{nk} = BABB...B = BA \neq AB, \text{ for every } 2 \leq j \leq n.$$

Therefore,

$$\prod_{k=1}^{n} W_{nk} \neq W_{nj} \prod_{k=1,k\neq j}^{n} W_{nk} \text{ for every } n \geq 2 \text{ and } 2 \leq j \leq n.$$

Theorem 3.8 *Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of self-adjoint measurable operators. Let $\psi(t)$ satisfy (3.2) for $p = 1$. Then, conditions (3.3), (3.4) and (3.6) imply (3.7).*

Proof As in proof of Theorem 3.5, for each $n \geq 1$, we put

$$Y_{nk} = X_{nk} e_{[0,a_n]}(|X_{nk}|).$$

We can easily verify (3.8) and (3.9). It remains to prove (3.10), and we show that

$$\frac{1}{a_n} \sum_{k=1}^{n} \left[Y_{nk} - \tau(Y_{nk})\right] \xrightarrow{a.u.} 0 \quad \text{as} \quad n \to \infty.$$

Observe that

$$\tau\Big(\psi(Y_{nk})\Big) = \int_{-\infty}^{+\infty} \psi(\lambda)\mathbf{1}_{[0;a_n]}(\lambda)\tau\Big(e_{d\lambda}(X_{nk})\Big) = \int_{0}^{a_n} \psi(\lambda)\tau\Big(e_{d\lambda}(X_{nk})\Big)$$
$$\leq \int_{-\infty}^{+\infty} \psi(\lambda)\tau\Big(e_{d\lambda}(X_{nk})\Big) = \tau\Big(\psi(X_{nk})\Big),$$

(where $\mathbf{1}_B$ is indicator function of a Borel subset B of $\mathbb{R}$),
which implies that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} \tau\left(\frac{\psi(Y_{nk})}{\psi(a_n)}\right) \leq \sum_{n=1}^{\infty}\sum_{k=1}^{n} \tau\left(\frac{\psi(X_{nk})}{\psi(a_n)}\right) < \infty.$$

Thus, condition (3.4) holds for $\{Y_{nk}\}$.
By $p = 1$, condition (3.2) reduces to
$\dfrac{\psi(|t|)}{|t|} \uparrow$ and $\dfrac{\psi(|t|)}{|t|^2} \downarrow$ as $|t| \uparrow$.
Since $\dfrac{\psi(|t|)}{|t|^2}$ is a monotonically decreasing function of $|t|$, we have

$$\tau\left(\left|\frac{Y_{nk}}{a_n}\right|^2\right) = \int_{0}^{a_n} \frac{|\lambda|^2}{a_n^2}\tau\Big(e_{d\lambda}(|X_{nk}|)\Big) \leq \int_{0}^{a_n} \frac{\psi(|\lambda|)}{\psi(a_n)}\tau\Big(e_{d\lambda}(|X_{nk}|)\Big) = \tau\left(\frac{\psi(Y_{nk})}{\psi(a_n)}\right).$$

Therefore, for all positive integer s, we have

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} \tau\left(\left|\frac{Y_{nk}}{a_n}\right|^2\right)\right)^{2s} \leq \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} \tau\left(\frac{\psi(Y_{nk})}{\psi(a_n)}\right)\right)^{2s} \leq \sum_{n=1}^{\infty}\sum_{k=1}^{n} \tau\left(\frac{\psi(Y_{nk})}{\psi(a_n)}\right)$$
$$\leq \sum_{n=1}^{\infty}\sum_{k=1}^{n} \tau\left(\frac{\psi(X_{nk})}{\psi(a_n)}\right) < \infty.$$

This follows that condition (3.5) holds for $\{Y_{nk}\}$, and the proof of Theorem 3.8 follows from the proof of Theorem 3.5. □

When condition (3.2) holds for $p = 0$, conditions (3.3), (3.5) and (3.6) are no longer needed.

Theorem 3.9 *Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of self-adjoint measurable operators. Let $\psi(t)$ satisfy (3.2) for $p = 0$. Then, condition (3.4) implies (3.7).*

Proof Condition (3.8) follows since $\psi(|t|)$ is a monotonically increasing function of $|t|$.

If the condition (3.2) holds for $p = 0$, then $\dfrac{\psi(|t|)}{|t|}$ is a monotonically decreasing function of $|t|$.
Therefore,

$$\tau\left(\frac{|Y_{nk}|}{a_n}\right) = \int_0^{a_n} \frac{|\lambda|}{a_n} \tau\Big(e_{d\lambda}(|X_{nk}|)\Big) \le \int_0^{a_n} \frac{\psi(|\lambda|)}{\psi(a_n)} \tau\Big(e_{d\lambda}(|X_{nk}|)\Big) = \tau\left(\frac{\psi(Y_{nk})}{\psi(a_n)}\right),$$

and hence, by the condition (3.4), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\frac{1}{a_n}\sum_{k=1}^{n}\tau(Y_{nk})\right| &\le \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\tau(|Y_{nk}|)}{a_n} \le \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\tau(\psi(Y_{nk}))}{\psi(a_n)}\\ &\le \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\tau(\psi(X_{nk}))}{\psi(a_n)} < \infty,\end{aligned}$$

which implies that $\sum_{n=1}^{\infty}\frac{1}{a_n}\sum_{k=1}^{n}\tau(Y_{nk})$ converges. Thus, by the Kronecker lemma, we get

$$\frac{1}{a_n}\sum_{k=1}^{n}\tau(Y_{nk}) \to 0 \quad \text{as} \quad n \to \infty,$$

so the condition (3.9) holds.

In addition, using Lemma 3.1 and condition (3.4), we obtain, for arbitrary $\varepsilon > 0$,

$$\sum_{n=1}^{\infty}\tau\left[e_{(\varepsilon,\infty)}\left(\left|\frac{1}{a_n}\sum_{k=1}^{n}Y_{nk}\right|\right)\right] \le \frac{1}{\varepsilon}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\tau\left(\left|\frac{Y_{nk}}{a_n}\right|\right) \le \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\tau(\psi(X_{nk}))}{\psi(a_n)} < \infty.$$

It follows that (3.10) holds, and the proof is completed. □

4 Almost Complete Convergence for Arrays of Measurable Operators

We need some more lemmas

Lemma 4.1 *Let $\{X_n, n \ge 1\}$, $\{Y_n, n \ge 1\}$ be two sequences of measurable operators such that $X_n - Y_n \xrightarrow{a.c.} 0$ and $Y_n \xrightarrow{a.c.} 0$. Then, we have $X_n \xrightarrow{a.c.} 0$.*

Proof Let $\varepsilon > 0$ be given.

Since $X_n - Y_n \xrightarrow{a.c.} 0$, there exists a sequence (p_n) of projections in $\mathcal{A}$ such that $\sum_{n=1}^{\infty}\tau(\mathbf{1} - p_n) < \infty$, $(X_n - Y_n)p_n \in \mathcal{A}$, and $\left\|(X_n - Y_n)p_n\right\|_{\infty} < \frac{\varepsilon}{2}$.

Since $Y_n \xrightarrow{a.c.} 0$, there exists a sequence (q_n) of projections in $\mathcal{A}$ such that $\sum_{n=1}^{\infty} \tau(\mathbf{1} - q_n) < \infty$, $Y_n q_n \in \mathcal{A}$, and $\left\| Y_n q_n \right\|_\infty < \frac{\varepsilon}{2}$.

Put $r_n = p_n \wedge q_n$, we have $\sum_{n=1}^{\infty} \tau(\mathbf{1} - r_n) \leq \sum_{n=1}^{\infty} \tau(\mathbf{1} - p_n) + \sum_{n=1}^{\infty} \tau(\mathbf{1} - q_n) < \infty$, $X_n r_n \in \mathcal{A}$, and
$\left\| X_n r_n \right\|_\infty \leq \left\| (X_n - Y_n) r_n \right\|_\infty + \left\| Y_n r_n \right\|_\infty \leq \left\| (X_n - Y_n) p_n \right\|_\infty + \left\| Y_n q_n \right\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
It follows that $X_n \xrightarrow{a.c.} 0$. □

Remark 4.2 Lemma 4.1 yields that if $\{X_n, n \geq 1\}$ is a sequence of measurable operators and $\{a_n, n \geq 1\}$ is a numerical sequence such that $X_n \xrightarrow{a.c.} 0$ and $a_n \to 0$, then $X_n + a_n \xrightarrow{a.c.} 0$.

Lemma 4.3 *For any $r \geq 1$, let X be a self-adjoint measurable operators such that $\tau\left(|X|^r\right) < \infty$. Then, we have*

$$\frac{r}{2^r} \sum_{n=1}^{\infty} n^{r-1} \tau\left(e_{(n,\infty)}(|X|)\right) \leq \tau\left(|X|^r\right).$$

Proof By direct computation using Fubini's theorem, we have

$$\begin{aligned}
\tau\left(|X|^r\right) &= \int_0^\infty |\lambda|^r \tau\big(e_{d\lambda}(|X|)\big) \\
&= r \int_0^\infty \left(\int_0^\lambda t^{r-1} dt \right) \tau\big(e_{d\lambda}(|X|)\big) = r \int_0^\infty t^{r-1} \left(\int_t^\infty \tau\big(e_{d\lambda}(|X|)\big) \right) dt \\
&\geq r \sum_{n=0}^{\infty} \int_n^{n+1} t^{r-1} \left(\int_t^\infty \tau\big(e_{d\lambda}(|X|)\big) \right) dt \geq r \sum_{n=0}^{\infty} \int_n^{n+1} t^{r-1} \left(\int_{n+1}^\infty \tau\left(e_{d\lambda}(|X|)\right) \right) dt \\
&= r \left\{ \int_0^1 t^{r-1} \left(\int_1^\infty \tau\left(e_{d\lambda}(|X|)\right) \right) dt + \sum_{n=1}^{\infty} \int_n^{n+1} t^{r-1} \left(\int_{n+1}^\infty \tau\left(e_{d\lambda}(|X|)\right) \right) dt \right\} \\
&\geq r \left\{ \tau\left(e_{(1,\infty)}(|X|)\right) + \sum_{n=1}^{\infty} n^{r-1} \tau\left(e_{(n+1,\infty)}(|X|)\right) \right\} \\
&\geq r \left\{ \tau\left(e_{(1,\infty)}(|X|)\right) + \sum_{n=2}^{\infty} (n-1)^{r-1} \tau\left(e_{(n,\infty)}(|X|)\right) \right\}
\end{aligned}$$

$$\geq \frac{r}{2^r} \sum_{n=1}^{\infty} n^{r-1} \tau \left(e_{(n,\infty)}(|X|) \right).$$

□

In this section, we will consider an array $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ of self-adjoint measurable operators such that for all n and k

$$\tau(X_{nk}) = 0, \tag{4.1}$$

and such that $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ is uniformly bounded by a measurable operator X with

$$\tau(|X|^{2p}) < \infty \text{ for some } \ 1 \leq p < 2. \tag{4.2}$$

Let $Y_{nk} = X_{nk} e_{[0,n^{1/p}]}(|X_{nk}|)$ and $Z_{nk} = Y_{nk} - \tau(Y_{nk})$. In addition, for all positive integer v

$$+ \quad \tau \left(\prod_{j=1}^{2v} Z_{nk_j} \right) \leq \prod_{i=1}^{t} \tau \left(Z_{ns_i}^{\alpha_i} \right), \quad (\alpha_1 + \alpha_2 + ... + \alpha_t = 2v) \tag{4.3}$$

where Z_{ns_i}, $1 \leq i \leq t$ are the distinguishing operators of the sequence $\{Z_{ni}, 1 \leq i \leq n\}$ and the operator Z_{ns_i} appears α_i-times in $\prod_{j=1}^{2v} Z_{nk_j}$.

Theorem 4.4 *Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be an array of self-adjoint measurable operators such that (4.1), (4.3) are satisfied and $\{X_{nk}\}$ are uniformly bounded by a measurable operator X satisfying (4.2). Then,*

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{nk} \xrightarrow{a.c.} 0 \ \ as \ \ n \to \infty.$$

Proof Let

$$S_n = \frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{nk}, \quad \widetilde{S}_n = \frac{1}{n^{1/p}} \sum_{k=1}^{n} Y_{nk}.$$

By Lemma 4.1, it suffices to prove that as $n \to \infty$,

$$S_n - \widetilde{S}_n \xrightarrow{a.c.} 0, \tag{4.4}$$

and

$$\widetilde{S}_n \xrightarrow{a.c.} 0. \tag{4.5}$$

We first show (4.4). For any arbitrary $\varepsilon > 0$, we have

$$q \equiv e_{(\varepsilon,\infty)}\left(|S_n - \widetilde{S}_n|\right) \wedge \left(\bigwedge_{k=1}^{n} e_{[0,n^{1/p}]}(|X_{nk}|)\right) = 0$$

Indeed, if there exists h of norm one, $h \in q(H)$, then $h \in e_{[0,n^{1/p}]}(|X_{nk}|)$, for all $1 \le k \le n$, and consequently $X_{nk}(h) = Y_{nk}(h)$, which yields that $S_n(h) = \widetilde{S}_n(h)$.

Thus, from the elementary properties of the spectral decomposition, we obtain

$$\varepsilon = \varepsilon\|h\| \le \left\|\left|S_n - \widetilde{S}_n\right| e_{(\varepsilon,\infty)}\left(\left|S_n - \widetilde{S}_n\right|\right)(h)\right\|_\infty = \left\|\left(S_n - \widetilde{S}_n\right)(h)\right\|_\infty = 0,$$

which is impossible, so $q = 0$ and this implies that

$$e_{(\varepsilon,\infty)}\left(\left|S_n - \widetilde{S}_n\right|\right) \prec \bigvee_{k=1}^{n} e_{(n^{1/p},\infty)}(|X_{nk}|).$$

It follows that $\tau\left(e_{(\varepsilon,\infty)}\left(\left|S_n - \widetilde{S}_n\right|\right)\right) \le \sum_{k=1}^{n} \tau\left(e_{(n^{1/p},\infty)}(|X_{nk}|)\right)$, which together with Lemma 4.3 (with $r = 2$) yields

$$\begin{aligned}\sum_{n=1}^{\infty} \tau\left(e_{(\varepsilon,\infty)}\left(\left|S_n - \widetilde{S}_n\right|\right)\right) &\le \sum_{n=1}^{\infty}\sum_{k=1}^{n} \tau\left(e_{(n^{1/p},\infty)}(|X_{nk}|)\right) \le \sum_{n=1}^{\infty} n\tau\left(e_{(n^{1/p},\infty)}(|X|)\right)\\ &= \sum_{n=1}^{\infty} n\tau\left(e_{(n,\infty)}(|X|^p)\right)\\ &\le 2\tau(|X|^{2p}) < \infty,\end{aligned}$$

so (4.4) holds.

To prove (4.5), by Remark 4.2, we will show that

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} \tau\left(Y_{nk}\right) \to 0 \quad \text{as} \quad n \to \infty, \tag{4.6}$$

and

$$\frac{1}{n^{1/p}} \sum_{k=1}^{n} Z_{nk} \xrightarrow{a.c.} 0 \quad \text{as} \quad n \to \infty. \tag{4.7}$$

Since

$$Y_{nk} = X_{nk} e_{[0,n^{1/p}]}\left(|X_{nk}|\right) = X_{nk} - X_{nk} e_{(n^{1/p},\infty)}\left(|X_{nk}|\right),$$

and $\tau\left(X_{nk}\right)=0$, we have

$$\left|\tau\left(Y_{nk}\right)\right| \leq \tau\left(\left|X_{nk}\right| e_{\left(n^{1/p},\infty\right)}\left(\left|X_{nk}\right|\right)\right).$$

Thus,

$$\begin{aligned}
\left|\sum_{n=1}^{\infty}\frac{1}{n^{1/p}}\sum_{k=1}^{n}\tau\left(Y_{nk}\right)\right| &\leq \sum_{n=1}^{\infty}\frac{1}{n^{1/p}}\sum_{k=1}^{n}\tau\left(\left|X_{nk}\right| e_{\left(n^{1/p},\infty\right)}\left(\left|X_{nk}\right|\right)\right) \\
&= \sum_{n=1}^{\infty}\frac{1}{n^{1/p}}\sum_{k=1}^{n}\left\{n^{1/p}\tau\left(e_{\left(n^{1/p},\infty\right)}\left(\left|X_{nk}\right|\right)\right)+\int_{n^{1/p}}^{\infty}\tau\left(e_{(t,\infty)}\left(\left|X_{nk}\right|\right)\right)dt\right\} \\
&\leq \sum_{n=1}^{\infty}\left\{n\tau\left(e_{\left(n^{1/p},\infty\right)}\left(|X|\right)\right)+\frac{n}{n^{1/p}}\int_{n^{1/p}}^{\infty}\tau\left(e_{(t,\infty)}\left(|X|\right)\right)dt\right\} := I_1.
\end{aligned}$$

Putting $t=n^{1/p}s$ and applying Lemma 4.3 (with $r=2$), we get

$$\begin{aligned}
I_1 &= \sum_{n=1}^{\infty} n\tau\left(e_{(n,\infty)}\left(|X|^{p}\right)\right)+\sum_{n=1}^{\infty} n\int_{1}^{\infty}\tau\left(e_{\left(n^{1/p}s,\infty\right)}\left(|X|\right)\right)ds \\
&\leq 2\tau\left(|X|^{2p}\right)+\int_{1}^{\infty}\sum_{n=1}^{\infty} n\tau\left(e_{(n,\infty)}\left(\left|s^{-1}X\right|^{p}\right)\right)ds \\
&\leq 2\tau\left(|X|^{2p}\right)+2\int_{1}^{\infty}s^{-2p}\tau\left(|X|^{2p}\right)ds \\
&= \frac{4p}{2p-1}\tau\left(|X|^{2p}\right)<\infty.
\end{aligned}$$

It follows that

$$\left|\sum_{n=1}^{\infty}\frac{1}{n^{1/p}}\sum_{k=1}^{n}\tau\left(Y_{nk}\right)\right|<\infty,$$

which together with Kronecker lemma yields (4.6).

We now prove (4.7). By Lemma 3.3, for $1\leq q\leq 2p$, we have

$$\tau\left(\left|Z_{nk}\right|^{q}\right)=\tau\left(\left|Y_{nk}-\tau(Y_{nk})\right|^{q}\right)\leq 2^{q}\tau\left(\left|Y_{nk}\right|^{q}\right),$$

which implies that

$$\left[\tau\left(\left|Z_{nk}\right|^{q}\right)\right]^{1/q}\leq 2\left[\tau\left(\left|Y_{nk}\right|^{q}\right)\right]^{1/q}\leq 2\left[\tau\left(\left|Y_{nk}\right|^{2p}\right)\right]^{1/2p}\leq 2\left[\tau\left(|X|^{2p}\right)\right]^{1/2p},$$

so that, by (4.2),

$$\tau\left(|Z_{nk}|^{q}\right) \le 2^{q}\left[\tau\left(|X|^{2p}\right)\right]^{q/2p} < \infty. \tag{4.8}$$

In addition, by Lemma 3.2,

$$|Z_{nk}| \le |Z_{nk}| + |\tau(Z_{nk})| \le 2n^{1/p} \tag{4.9}$$

Let v denote the least integer such that

$$\frac{2v}{3}\left(\frac{2}{p}-1\right) > 1. \tag{4.10}$$

It is easy to see that $v > \frac{3}{2}p$.

Following the arguments in proof of Theorem 3.5, we have

$$\tau\left(\sum_{k=1}^{n} Z_{nk}\right)^{2v} = \sum_{k_1,k_2,\ldots,k_{2v}} \tau\left(\prod_{j=1}^{2v} Z_{nk_j}\right). \tag{4.11}$$

By using (4.3), (4.8), (4.9) and techniques as in Hu et al. [6], we get

$$\tau\left(\sum_{k=1}^{n} Z_{nk}\right)^{2v} \le \sum_{k=1}^{n}\tau\left(Z_{nk}^{2v}\right) + \sum_{t=2}^{v}\ \sum_{q_1,\ldots,q_m;\ r_1,\ldots,r_l}{}^{(t)} n^{\frac{2v}{p}-t-m\left(\frac{2}{p}-2\right)}, \tag{4.12}$$

where $\sum^{(t)}$ means that the sum is extended over all m-tuples $(q_1, \ldots, q_m)$ and l-tuples $(r_1, \ldots, r_l)$ such that $m + l = t$ with conditions

$$2 \le q_i \le 2p,\ r_j > 2p,\ \text{and}\ \sum_{i=1}^{m} q_i + \sum_{j=1}^{l} r_j = 2v.$$

Using Lemma 3.1 and (4.12), we obtain, for any $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\tau\left[e_{(\varepsilon,\infty)}\left(\left|\frac{1}{n^{1/p}}\sum_{k=1}^{n} Z_{nk}\right|\right)\right] \le \sum_{n=1}^{\infty}\frac{1}{(\varepsilon n^{1/p})^{2v}}\tau\left(\sum_{k=1}^{n} Z_{nk}\right)^{2v}\\
&\le C\left\{\sum_{n=1}^{\infty}\frac{1}{n^{2v/p}}\sum_{k=1}^{n}\tau\left(Z_{nk}^{2v}\right) + \sum_{n=1}^{\infty}\frac{1}{n^{2v/p}}\sum_{t=2}^{v}\ \sum_{q_1,\ldots,q_m;\ r_1,\ldots,r_l}{}^{(t)} n^{\frac{2v}{p}-t-m\left(\frac{2}{p}-2\right)}\right\}\\
&= C\left\{\sum_{n=1}^{\infty}\frac{1}{n^{2v/p}}\sum_{k=1}^{n}\tau\left(Z_{nk}^{2v}\right) + \sum_{t=2}^{v}\ \sum_{q_1,\ldots,q_m;\ r_1,\ldots,r_l}{}^{(t)} \sum_{n=1}^{\infty} n^{-t-m\left(\frac{2}{p}-2\right)}\right\}\\
&:= C(I_2 + I_3).
\end{aligned} \tag{4.13}$$

For I_2, by direct computation using Fubini's theorem, we obtain that

$$
\begin{aligned}
I_2 &\leq 2^{2v} \sum_{n=1}^{\infty} \frac{1}{n^{2v/p}} \sum_{k=1}^{n} \tau \left(|Y_{nk}|^{2v} \right) \\
&= 2^{2v} \sum_{n=1}^{\infty} \frac{1}{n^{2v/p}} \sum_{k=1}^{n} \tau \left(\int_0^{\infty} \lambda^{2v} \mathbf{1}_{[0,n^{1/p}]}(\lambda) e_{d\lambda} \left(|X_{nk}| \right) \right) \\
&= 2^{2v} \sum_{n=1}^{\infty} \frac{1}{n^{2v/p}} \sum_{k=1}^{n} 2v.\tau \left(\int_0^{\infty} \left(\int_0^{\lambda} t^{2v-1} dt \right) \mathbf{1}_{[0,n^{1/p}]}(\lambda) e_{d\lambda} \left(|X_{nk}| \right) \right) \\
&\leq 2^{2v} \sum_{n=1}^{\infty} \frac{1}{n^{2v/p}} 2v \sum_{k=1}^{n} \left(\int_0^{n^{1/p}} t^{2v-1} \tau \left[e_{(t,\infty)} \left(|X_{nk}| \right) \right] dt \right) \\
&\leq 2^{2v} \sum_{n=1}^{\infty} \frac{1}{n^{2v/p}} 2vn \left(\int_0^{n^{1/p}} t^{2v-1} \tau \left[e_{(t,\infty)} \left(|X| \right) \right] dt \right).
\end{aligned}
\tag{4.14}
$$

Putting $t = n^{1/p} s^{1/2v}$, by applying Lemma 4.1 (with $r = 2$), we have

$$
\begin{aligned}
& 2^{2v} \sum_{n=1}^{\infty} \frac{2vn}{n^{2v/p}} \left(\int_0^{n^{1/p}} t^{2v-1} \tau \left[e_{(t,\infty)} \left(|X| \right) \right] dt \right) \\
&= 2^{2v} \sum_{n=1}^{\infty} n \int_0^1 \tau \left[e_{(n^{1/p} s^{1/2v},\infty)} \left(|X| \right) \right] ds \\
&= 2^{2v} \int_0^1 \sum_{n=1}^{\infty} n \tau \left[e_{(n,\infty)} \left(\left| s^{-1/2v} X \right|^p \right) \right] ds \\
&\leq 2^{2v+1} \int_0^1 s^{-p/v} \tau \left(|X|^{2p} \right) \\
&= 2^{2v+1} \frac{v}{v-p} \tau \left(|X|^{2p} \right) < \infty.
\end{aligned}
\tag{4.15}
$$

For I_3, by noting that $t + m \left(\frac{2}{p} - 2 \right) > 1$, we have the exponent of n which is less than -1. Since the number of terms in each of $\sum^{(t)}$ is finite, we get

$$
I_3 < \infty. \tag{4.16}
$$

Combining (4.13), (4.14), (4.15), and (4.16), for every $\varepsilon > 0$, we obtain

$$\sum_{n=1}^{\infty} \tau \left[e_{(\varepsilon,\infty)} \left(\left| \frac{1}{n^{1/p}} \sum_{k=1}^{n} Z_{nk} \right| \right) \right] < \infty.$$

It follows that (4.7) holds, and thereby completing the proof of Theorem 4.4. □

Remark 4.5 As in the Remark 3.6, the condition (4.3) is obvious for an array of row-wise independent real-valued random variables. Hence, Theorem 4.4 is a non-commutative version of Theorem 2 of Hu, Móricz and Taylor [6].

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