Yeol Je Cho · Mohamed Jleli ·
Mohammad Mursaleen ·
Bessem Samet · Calogero Vetro *Editors*

Advances in Metric Fixed Point Theory and Applications

Springer

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Preface

The fixed point theory is a very important branch of mathematics. It has a lot of applications in several areas of mathematics and other sciences. It is a very powerful and important tool in the study of nonlinear phenomena. In particular, fixed point techniques have been useful in investigations connected with the theories of differential equations, integral equations, functional integral equations, and in optimization theory as well as in several diverse fields such as biology, chemistry, economics, engineering, game theory, and physics.

The present book contains a comprehensive treatment of fixed point theory and its applications. The book is addressed to the large audience of mathematical community applying the methods and tools of nonlinear analysis and investigating a lot of topics connected with that important branch of mathematics. This book can also serve as a source of examples, references, and new approaches associated with the fixed point theory and its applications. Apart from Preface, the present book consists of 20 chapters on various topics of fixed point theory and its applications. Each chapter is self-contained and contributed by specialists using their researches.

In Chap. 1 "The Relevance of a Metric Condition on a Pair of Operators in Common Fixed Point Theory", the authors study the following two problems:

I. Which are the metric conditions on f and g which imply all the following conclusions:

(1) $F_{f^n} = F_{g^n} = \{x^*\}$ for each $n \geq 0$;

(2) for each $x_0 \in X$ and $n \geq 1$, the sequence $\{x_n\}$ defined by

$$x_{2n} = (gf)^n(x_0), x_{2n+1} = f(x_{2n})$$

converges to $x^* \in X$;

(3) for each $y_0 \in X$ and $n \geq 1$, the sequence $\{y_n\}$ defined by

$$y_{2n} = (fg)^n(y_0), y_{2n+1} = g(y_{2n})$$

converges to $x^* \in X$;

(4) for each $x_0 \in X$ and $n \geq 1$, the sequence $\{f^n(x_0)\}$ converges to $x^* \in X$;

(5) for each $x_0 \in X$ and $n \geq 1$, the sequence $\{g^n(x_0)\}$ converges to $x^* \in X$.

II. In the above context, they study the data dependence phenomenon for the common fixed point set of two given operators.

In Chap. 2 “Some Convergence Results of the K^* Iteration Process in CAT(0) Spaces”, the authors prove some strong and Δ-convergence theorems of the K^* iteration process for two different classes of generalized nonexpansive mappings in a CAT(0) space. The results presented here extend and improve some recent results announced in the current literature.

In Chap. 3 “Split Variational Inclusion Problem and Fixed Point Problem for Asymptotically Nonexpansive Semigroup with Application to Optimization Problem”, the authors deal with the study of an iterative process to approximate a common solution of the split variational inclusion problem and the fixed point problem for an asymptotically nonexpansive semigroup in real Hilbert spaces. Further, we prove that the sequences generated by the proposed iterative method converge strongly to a common solution of the problems for an asymptotically nonexpansive semigroup. As applications, these results are used to study the split optimization problem and the split variational inequality.

In Chap. 4 “Convergence Theorems and Convergence Rates for the General Inertial Krasnosel’skiĭ–Mann Algorithm”, the authors discuss the convergence analysis of the general inertial Krasnosel’skiĭ–Mann algorithm [13] with the control conditions $\alpha_n \in [0, 1]$, $\beta_n \in (-\infty, 0]$ and $\alpha_n \in [-1, 0]$, $\beta_n \in [0, +\infty)$, respectively. Also, the convergence rate for the general inertial Krasnosel’skiĭ–Mann algorithm under mild conditions on the inertial parameters and some conditions on the relaxation parameters, respectively, are provided. Finally, authors give a numerical experiment that compares the choice of inertial parameters.

In Chap. 5 “Digital Space-Type Fixed Point Theory and Its Applications”, which is a survey paper, the author studies the fixed point property (FPP, for short) and the almost fixed point property (AFPP, for short) for digital spaces whose structures are induced by a digital graph in terms of the Rosenfeld model (or digital metric space), the Khalimsky (K-, for short), or the (extended) Marcus-Wyse (M-, for short) topology. Furthermore, the author investigates various properties of digital isomorphic (or homeomorphic), digital homotopic, retract, and product properties of the FPP and the AFPP.

In Chap. 6 “Existence and Approximations for Order-Preserving Nonexpansive Semigroups over CAT(κ) Spaces”, the author discusses the fixed point property for an infinite family of order-preserving mappings which satisfy the Lipschitzian condition on comparable pairs. The underlying framework of the main results is a metric space of any global upper curvature bound $\kappa \in \mathbb{R}$, i.e., a $CAT(\kappa)$ space. In particular, the existence of a fixed point for a nonexpasive semigroup on comparable pairs is proved. Further, the author proposes and analyzes two algorithms to approximate such a fixed point.

In Chap. 7 “A Solution of the System of Integral Equations in Product Spaces via Concept of Measures of Noncompactness”, the authors deal with the role of measures of noncompactness and related fixed point results to study the existence of solutions for the system of integral equations of the form

$$x_i(t) = a_i(t) + f_i(t, x_1(t), x_2(t), \cdots, x_n(t)) \\ + g_i(t, x_1(t), x_2(t), \cdots, x_n(t)) \int_0^{\alpha(t)} ((k_i(t, s, x_1(s), x_2(s), \cdots, x_n(s)))ds$$

for all $t \in \mathbb{R}_+, x_1, x_2, \cdots, x_n \in E = BC(\mathbb{R}_+)$ and $1 \leq i \leq n$. Moreover, they study a system of fractional integral equations when k_i is defined in a fractal space.

In Chap. 8 "Fixed Points That Are Zeros of a Given Function", the author presents a discussion on (ordered) S-F-contractions in the setting of complete metric spaces, with and without the ordered approach. S-F-contractions are generalizations of (F, φ)-contractions and $\mathscr{Z}$-contractions. These two types of contractions have encountered a great success among the scientific community due to their versatility and usefulness in overcoming different practical situations. A fundamental characteristic of such a kind of contractions is the possibility to be hybridized with other existing conditions to obtain control hypotheses with best performances.

In Chap. 9 "A Survey on Best Proximity Point Theory in Reflexive and Busemann Convex Spaces", the author presents some best proximity point theorems for Kannan cyclic mappings in the setting of Busemann convex spaces which are reflexive. To this end, the authors recall some results obtained in the framework of the fixed point theory for Kannan self-mappings and generalize them to cyclic mappings in order to study the existence of best proximity points. It is done from two different approaches. The first one is based on a geometric property defined on a nonempty and convex pair in a geodesic space called proximal normal structure, and the other one is done by considering some sufficient conditions on the cyclic mappings. Also, the structure of minimal sets for Kannan cyclic nonexpansive mappings is studied in this chapter.

In Chap. 10 "On Monotone Mappings in Modular Function Spaces", the authors deal with the existence and construction of fixed points for monotone nonexpansive mappings acting in modular functions spaces equipped with a partial order or a graph structure.

In Chap. 11 "Contributions to Fixed Point Theory of Fuzzy Contractive Mappings", the author proves some fixed point theorems for fuzzy contractive type mappings in fuzzy metric spaces. The results presented in detail are selected to illustrate the direction research in the field has taken from the last six decades up to the most recent contribution in the subject.

In Chap. 12 "Common Fixed Point Theorems for Four Maps", the authors prove some coincidence and common fixed point theorems for four mappings satisfying Círíc type and Hardy-Rogers type (α, F)-contractions on α-complete metric spaces. They apply the main results to infer several new and old corresponding results in ordered metric spaces and graphic metric spaces. The results also generalize some results obtained previously. An example and an application to support these results are also illustrated.

In Chap. 13 "Measure of Noncompactness in Banach Algebra and Its Application on Integral Equations of Two Variables", the authors introduce a class of measure of noncompactness satisfying certain conditions. The obtained results are applied to establish a few theorems on the existence of solution of integral equations in two variables and also they give some examples to illustrate the results.

In Chap. 14 “Generalization of Darbo-Type Fixed Point Theorem and Applications to Integral Equations”, the authors present a new notion of μ-set contractive mappings for two class of functions involving measure of noncompactness in Banach space and Darbo-type fixed and n-tuple fixed point results. These results include and extend the results of Falset and Latrach [Falset, J. G., Latrach, K.: On Darbo-Sadovskii’s fixed point theorems type for abstract measures of (weak) noncompactness, *Bull. Belg. Math. Soc.* Simon Stevin 22 (2015), 797–812]. The results are also correlated with the classical generalized Banach fixed point theorems. Finally, these results are applied to two different Volterra integral equations in Banach algebras.

In Chap. 15 “Approximating Fixed Points of Suzuki (α, β)-Nonexpansive Mappings in Ordered Hyperbolic Metric Spaces”, the authors discuss the class of monotone (α, β)-nonexpansive mappings and prove that they have an approximate fixed point sequence in partially ordered hyperbolic metric spaces. They also prove Δ and strong convergence of the CR-iteration scheme.

In Chap. 16 “Generalized JS-Contractions in b-Metric Spaces with Application to Urysohn Integral Equations”, the authors deal with the notion of α-G-JS-type contractions for two pairs of self-mappings in b-metric spaces. Coincidence points, common fixed points, their uniqueness as well as periodic points are studied for these mappings under α-compatible and relatively partially α-weakly increasing conditions on α-complete b-metric spaces. The results are verified through an example in order to check their effectiveness and applicability. Also, the results are used to obtain the existence of solutions for a system of Urysohn integral equations.

In Chap. 17 “Unified Multi-tupled Fixed Point Theorems Involving Monotone Property in Ordered Metric Spaces”, the authors introduce a generalized notion of monotone property and prove some results regarding the existence and uniqueness of multi-tupled fixed points for nonlinear contraction mappings satisfying monotone property in ordered complete metric spaces. The main results unify several classical and well-known n-tupled fixed point results existing in literature.

In Chap. 18 “Convergence Analysis of Solution Sets for Minty Vector Quasivariational Inequality Problems in Banach Spaces”, the authors deal with the convergence analysis of the solution sets for vector quasi-variational inequality problems of the Minty type. Based on the nonlinear scalarization function, a key assumption (H_h) by virtue of a sequence of gap functions is obtained. The necessary and sufficient conditions for the Painlevé-Kuratowski lower convergence and Painlevé-Kuratowski convergence are also established.

In Chap. 19 “Common Solutions for a System of Functional Equations in Dynamic Programming Passing Through the $JCLR$-Property in S_b-Metric Spaces”, the authors present a new concept of the joint common limit in the range property in S_b-metric spaces and prove some common fixed point theorems by using the $JCLR$-property in S_b-metric spaces without the completeness of S_b-metric spaces. As applications of these results, they show the existence of common solutions for a system of functional equations in dynamic programming.

In Chap. 20 “A General Approach on Picard Operators”, the authors present the recent investigations concerning the existence and the uniqueness of fixed points for the mappings in the setting of spaces which are not metric with different functions of measuring the distance and in consequence with the various concepts of convergence the sequences. In this way, they obtain the systematized knowledge of fixed point tools which are in some situations more convenient to apply than the known theorems with an underlying usual metric space. The appropriate illustrative examples are also presented.

The editors would like to express their gratitude to the contributors who have submitted chapters to this volume.

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Chapter 1
The Relevance of a Metric Condition on a Pair of Operators in Common Fixed Point Theory

A. Petruşel and I. A. Rus

Abstract Let (X, d) be a complete metric space and $f, g : X \to X$ be two operators satisfying some metric conditions on f and g. We denote by F_f the fixed point set of f. In this paper, we will study the following problems.

I. What are the metric conditions on f and g which imply that **all** the following conclusions hold?

1. $F_{f^n} = F_{g^n} = \{x^*\}$ for each $n \in \mathbb{N}^*$;
2. for each $x_0 \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_{2n} = (gf)^n(x_0), \quad x_{2n+1} = f(x_{2n}), \quad \forall n \geq 0,$$

converges to $x^* \in X$;
3. for each $y_0 \in X$, the sequence $(y_n)_{n\in\mathbb{N}}$ defined by

$$y_{2n} = (fg)^n(y_0), \quad y_{2n+1} = g(y_{2n}), \quad \forall n \geq 0,$$

converges to $x^* \in X$;
4. for each $x_0 \in X$, the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to $x^* \in X$;
5. for each $x_0 \in X$, the sequence $(g^n(x_0))_{n\in\mathbb{N}}$ converges to $x^* \in X$.

II. Under which assumptions does the data dependence phenomenon for the common fixed point problem hold? Other problems, such as well-posedness, Ostrowski property and Ulam-Hyers stability for the common fixed point problem are also considered.

Keywords Common fixed point · Weakly Picard operator · Well-posedness · Ostrowski property · Ulam-Hyers stability · Open problem

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1.1 Introduction and Preliminaries

1.1.1 The Purpose of the Paper

Let X be a nonempty set and $f, g : X \to X$ be two given operators. We consider the following *fixed point equation* for f:

$$x = f(x). \tag{1.1}$$

We denote by F_f the fixed point set of f, i.e., $F_f := \{x \in X \mid f(x) = x\}$.

In the same context, we can consider the *common fixed point equation* for f and g as follows:

$$x = f(x) = g(x). \tag{1.2}$$

We denote by $CFP(f, g)$ the common fixed point set of f and g, i.e.,

$$CFP(f, g) := \{x \in X \mid x = f(x) = g(x)\}.$$

Notice that $CFP(f, g) = F_f \cap F_g$.

Let (X, d) be a complete metric space and $f, g : X \to X$ be two operators satisfying some metric conditions on f and g. We will denote by f^n the nth iterate of f, i.e., $f^n = f \circ f \circ \cdots \circ f$ (n-times). In this paper, we will study the following two problems.

I. What are the metric conditions on f and g which imply that **all** the following conclusions hold?

(1) $F_{f^n} = F_{g^n} = \{x^*\}$, for $n \in \mathbb{N}^*$;

(2) for each $x_0 \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_{2n} = (gf)^n(x_0), \quad x_{2n+1} = f(x_{2n})$$

converges to $x^* \in X$;

(3) for each $y_0 \in X$, the sequence $(y_n)_{n\in\mathbb{N}}$ defined by

$$y_{2n} = (fg)^n(y_0), \quad y_{2n+1} = g(y_{2n})$$

converges to $x^* \in X$;

(4) for each $x_0 \in X$, the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to $x^* \in X$;

(5) for each $x_0 \in X$, the sequence $(g^n(x_0))_{n\in\mathbb{N}}$ converges to $x^* \in X$.

II. Under which assumptions does the data dependence phenomenon for the common fixed point problem hold?

The structure of this paper is as follows:

1. Introduction and preliminaries;
2. A variant of Kannan's common fixed point theorem: a new research direction;
3. Pairs of operators on a set endowed with two metrics;
4. Contraction pairs of operators on ordered metric spaces;
5. Pairs of operators on $\mathbb{R}^m_+$-metric spaces;
6. Data dependence for the common fixed point problem;
7. Other problems and research directions.

1.1.2 Notations

In this paper, we use the usual notations and symbols in Nonlinear Analysis.

Throughout this paper, $\mathbb{N}$ stands for the set of natural numbers, $\mathbb{N}^*$ is the set of natural numbers except 0, while $\mathbb{R}$ is the set of all real numbers. We also use the same symbol $\leq$ on $\mathbb{R}^m$ for the component-wise ordering.

Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^m_+$ is called an $\mathbb{R}^m_+$-*metric* on X if all the classical axioms of the metric are fulfilled, with respect to the above-mentioned partial ordering. A nonempty set X endowed with such a vector-valued metric $d : X \times X \to \mathbb{R}^m_+$ is called a *generalized metric space*.

The notions of convergent sequence, Cauchy sequence, completeness, open and closed balls are defined in a similar way to the case of metric spaces.

We denote by $\mathscr{M}_{m,m}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements, by I_m the identity $m \times m$ matrix and by O_m the null $m \times m$ matrix.

Definition 1.1 A square matrix $A \in \mathscr{M}_{m,m}(\mathbb{R}_+)$ is said to be *convergent to zero* if $A^n \to O_m$ *as* $n \to \infty$.

A classical result in matrix analysis is the following theorem (see [47]).

Theorem 1.1 *Let* $A \in \mathscr{M}_{m,m}(\mathbb{R}_+)$. *The following assertions are equivalent:*

(1) *A is convergent to zero;*
(2) *The spectral radius $\rho(A)$ of A is strictly less than* 1*;*
(3) *The matrix $(I_m - A)$ is non-singular and*

$$(I_m - A)^{-1} = I_m + A + \cdots + A^n + \cdots ;$$

(4) *The matrix $(I_m - A)$ is non-singular and $(I_m - A)^{-1}$ has non-negative elements.*

1.1.3 Picard Operators

It is well known that Banach's contraction principle asserts that any k-contraction $f : X \to X$ on a complete metric space (X, d) has a unique fixed point. If the contraction condition

$$d(f(x), f(y)) \leq kd(x, y), \quad \forall (x, y) \in X \times X$$

holds (for some $k \in]0, 1[$) on the graphic of the operator, i.e.,

$$d(f(x), f^2(x)) \leq kd(x, f(x)), \quad \forall x \in X,$$

then the operator is called a *graphic k-contraction*.

A continuous graphic k-contraction in a complete metric space (X, d) has at least one fixed point.

These two mathematical phenomena gave rise (see [36, 39]) to the following very important abstract concepts:

If (X, d) is a metric space, then, by the definition, f is a *weakly Picard operator* if, for each $x \in X$, the sequence $(f^n(x))_{n \in \mathbb{N}}$ converges in X and its limit x^* (which may depend of x) is a fixed point for f. This definition generates a set retraction given by $f^\infty : X \to F_f$, $f^\infty(x) = \lim_{n \to \infty} f^n(x)$.

For example, a continuous graphic k-contraction on a complete metric space is a weakly Picard operator.

By the notation $F_f = \{x^*\}$, we understand that f has a unique fixed point denoted by x^*. A weakly Picard operator with a unique fixed point is, by the definition, a *Picard operator*. Notice that a k-contraction on a complete metric space is a Picard operator. A Picard operator for which there exists $c > 0$ such that

$$d(x, x^*) \leq cd(x, f(x)), \quad \forall x \in X,$$

is called a *c-Picard operator*.

It is easy to see that a k-contraction on a complete metric space is a $\frac{1}{1-k}$-Picard operator. For some more details on the Picard and weakly Picard operator theories, see also [9, 43] and others.

1.1.4 Graphic Contractions That are Picard Operators

Let (X, d) be a complete metric space and $f : X \to X$ be a graphic k-contraction. If f has a closed graph, then f has at least one fixed point. In general, we cannot say too many things about the fixed point set of a graphic contraction. Actually, for each $Y \subset X$, there exists a graphic contraction $f : X \to X$ such that $F_f = Y$. For a better understanding of the results of this work, the following theorems are very useful.

Theorem 1.2 *Let (X, d) be a metric space and $f : X \to X$ be an operator. We suppose that*

(a) *there exists $k \in]0, 1[$ such that*

$$d(f(x), f^2(x)) \le kd(x, f(x)), \quad \forall x \in X;$$

(b) $F_f = \{x^*\}$;
(c) $f^n(x) \to x^*$ *as* $n \to \infty$, $\forall x \in X$.

Then the following conclusions hold:

(1) $d(x, x^*) \le \frac{1}{1-k} d(x, f(x))$ *for every $x \in X$, i.e., f is a $\frac{1}{1-k}$-Picard operator;*
(2) *if $k < \frac{1}{3}$, then $d(f(x), x^*) \le \frac{k}{1-2k} d(x, x^*)$ for every $x \in X$, i.e., f is a quasi-contraction.*

Proof (1) We have

$$\begin{aligned} d(x, x^*) &\le d(x, f(x)) + d(f(x), f^2(x)) + \cdots + d(f^n(x), x^*) \\ &\le (1 + k + \cdots + k^{n-1}) d(x, f(x)) + d(f^n(x), x^*) \\ &\le \frac{1}{1-k} d(x, f(x)) + d(f^n(x), x^*). \end{aligned}$$

Letting $n \to \infty$, we obtain the first conclusion.

(2) Suppose that $k < \frac{1}{3}$. Then, by (1), we obtain

$$\begin{aligned} d(f(x), x^*) &\le \frac{1}{1-k} d(f(x), f^2(x)) \le \frac{k}{1-k} d(x, f(x)) \\ &\le \frac{k}{1-k} \left(d(x, x^*) + d(f(x), x^*) \right). \end{aligned}$$

Thus we have

$$d(f(x), x^*) \le \frac{k}{1-2k} d(x, x^*), \quad \forall x \in X.$$

This completes the proof.

Theorem 1.3 *Let (X, d) be a metric space and $f : X \to X$ be a c-Picard operator. Then the fixed point problem for f is well-posed, i.e., $F_f = \{x^*\}$ and for any sequence $(u_n)_{n \in \mathbb{N}}$ in X with $d(u_n, f(u_n)) \to 0$, we have that $u_n \to x^*$ as $n \to \infty$.*

Proof Let $(u_n)_{n \in \mathbb{N}}$ be a sequence in X such that

$$d(u_n, f(u_n)) \to 0 \text{ as } n \to \infty.$$

Since f is a c-Picard operator, we have

$$d(u_n, x^*) \leq cd(u_n, f(u_n)), \quad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$, we get that $u_n \to x^*$. Hence the fixed point problem for f is well-posed. This completes the proof.

Theorem 1.4 *Let (X, d) be a metric space and $f : X \to X$ be a k-quasi-contraction such that $F_f = \{x^*\}$. Then f has the Ostrowski property, i.e., $F_f = \{x^*\}$ and for any sequence $(v_n)_{n\in\mathbb{N}}$ in X with $d(v_{n+1}, f(v_n)) \to 0$, we have that $v_n \to x^*$ as $n \to \infty$.*

Proof Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in X such that

$$d(v_{n+1}, f(v_n)) \to 0 \text{ as } n \to \infty.$$

Since f is a k-quasi-contraction, we have

$$\begin{aligned} d(v_{n+1}, x^*) &\leq d(v_{n+1}, f(v_n)) + d(f(v_n), x^*) \\ &\leq d(v_{n+1}, f(v_n)) + kd(v_n, x^*) \\ &\vdots \\ &\leq d(v_{n+1}, f(v_n)) + kd(v_n, x^*) + \cdots + k^n d(v_1, f(v_0)) + k^n d(v_n, x^*). \end{aligned}$$

By the Cauchy-Toeplitz lemma, it follows that $v_n \to x^*$ as $n \to \infty$, which proves that f has the Ostrowski property. This completes the proof.

Remark 1.1 For the well-posedness of the fixed point problems and the Ostrowski property of an operator, see [9, 41–43]. For an extensive study of the fixed point equation with graphic contractions, see [29].

1.2 A Variant of Kannan's Common Fixed Point Theorem: A New Research Direction

In 1969, Kannan [22] proved the following common fixed point result.

Theorem 1.5 *Let (X, d) be a complete metric space and $f, g : X \to X$ be two operators for which there exists $\alpha \in]0, \frac{1}{2}[$ such that*

$$d(f(x), g(y)) \leq \alpha \left[d(x, f(x)) + d(y, g(y))\right], \quad \forall x, y \in X.$$

Then, f and g have a unique common fixed point, i.e., there exists a unique $x^* \in X$ such that $x^* = f(x^*) = g(x^*)$.

There are many variants and generalizations of Kannan's theorem (see [17, 31–33, 38, 43, 45].

Following some ideas from [34, 35], we present now a new variant of this theorem.

Theorem 1.6 *Let (X, d) be a complete metric space and $f, g : X \to X$ be two operators for which there exists $\alpha \in]0, \frac{1}{2}[$ such that*

$$d(f(x), g(y)) \leq \alpha\,[d(x, f(x)) + d(y, g(y))]\,, \quad \forall x, y \in X. \tag{1.3}$$

Then we have the following conclusions:

(a) $F_f = F_g = \{x^*\}$;
(b) *for each $x_0 \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$ defined by*

$$x_{2n} = (g \circ f)^n(x_0), \quad x_{2n+1} = f(x_{2n}), \quad \forall n \in \mathbb{N},$$

converges to x^* as $n \to \infty$;
(c) *for each $y_0 \in X$, the sequence $(y_n)_{n\in\mathbb{N}}$ defined by*

$$y_{2n} = (f \circ g)^n(y_0), \quad y_{2n+1} = g(y_{2n}), \quad \forall n \in \mathbb{N},$$

converges to x^* as $n \to \infty$;
(d) *the operators f and g are graphic contractions;*
(e) *the operators f and g are quasi-contractions;*
(f) *the operators f and g are $\frac{1-\alpha}{1-2\alpha}$-Picard operators;*
(g) *the fixed point problem for f and the fixed point problem for g are well-posed;*
(h) *the operators f and g have the Ostrowski property.*

Proof (a) Note that $F_f = F_g$. Indeed, let, for example, $x^* \in F_f$. Then, by (1.3), we have

$$d(x^*, g(x^*)) \leq \alpha d(x^*, g(x^*)).$$

Thus $x^* \in F_g$. Also, we have $Card(F_f \cap F_g) \leq 1$. Indeed, let $x^*, y^* \in F_f \cap F_g$. Then, by (1.3), we have

$$d(x^*, y^*) = d(f(x^*), g(x^*)) \leq \alpha \left[d(x^*, f(x^*)) + d(y^*, g(y^*))\right] = 0.$$

Hence $x^* = y^*$.

(b) Consider $x_0 \in X$ arbitrarily chosen and the sequence $(x_n)_{n\in\mathbb{N}}$ as in (b). Then, since

$$d(x_1, x_2) = d(f(x_1), g(x_1)) \leq \alpha\,[d(x_0, x_1) + d(x_1, x_2)]\,,$$

we have

$$d(x_1, x_2) \leq \frac{\alpha}{1-\alpha} d(x_0, x_1).$$

By induction, we get

$$d(x_n, x_{n+1}) \le \left(\frac{\alpha}{1-\alpha}\right)^n d(x_0, x_1), \quad \forall n \in \mathbb{N}. \tag{1.4}$$

By the above relation, using a classical approach, it follows that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $x^* \in X$ be its limit. We have

$$\begin{aligned} d(x^*, f(x^*)) &\le d(x^*, x_{2n}) + d(x_{2n}, f(x^*)) \\ &\le d(x^*, x_{2n}) + d(g(x_{2n-1}, f(x_{2n})) \\ &\le d(x^*, x_{2n}) + \alpha\left[d(x^*, f(x^*)) + d(x_{2n-1}, x_{2n})\right]. \end{aligned}$$

Letting $n \to \infty$, we get $d(x^*, f(x^*)) = 0$. Moreover, if we denote $\beta := \frac{\alpha}{1-\alpha} < 1$, by (1.4), we obtain

$$d(x_n, x_{n+p}) \le \frac{\beta^n}{1-\beta} d(x_0, x_1), \quad \forall n \in \mathbb{N},\ p \in \mathbb{N}^*.$$

Letting $p \to \infty$ and taking $n = 0$, we obtain the following retraction-displacement relation:

$$d(x_0, x^*) \le \frac{1}{1-\beta} d(x_0, x_1) = \frac{1-\alpha}{1-2\alpha} d(x_0, f(x_0)). \tag{1.5}$$

By a similar procedure, we can show that the sequence $(y_n)_{n\in\mathbb{N}}$ converges to x^* as $n \to \infty$ and a similar retraction-displacement relation holds.

(d) We prove that f is a graphic contraction. For arbitrary $x \in X$, we have

$$\begin{aligned} d(f^2(x), f(x)) &\le d(f^2(x), g(x^*)) + d(f(x), g(x^*)) \\ &\le \alpha\left[d(f^2(x), f(x)) + d(x^*, g(x^*))\right] \\ &\quad +\alpha\left[d(x, f(x)) + d(x^*, g(x^*))\right] \\ &= \alpha\left[d(f^2(x), f(x)) + d(x, f(x))\right]. \end{aligned}$$

Thus we have

$$d(f^2(x), f(x)) \le \frac{\alpha}{1-\alpha} d(x, f(x)),$$

which proves that f is a graphic contraction.

(e) We have

$$d(f(x), x^*) \le d(f(x), g(x^*)) \le \alpha d(x, f(x)) \le \alpha\left[d(x, x^*) + d(x^*, f(x))\right].$$

Hence

$$d(f(x), x^*) \le \frac{\alpha}{1-\alpha} d(x, x^*), \forall x \in X.$$

This shows that f is a quasi-contraction. Since the condition (1.3) is symmetric with respect to f and g, we get that g is also a quasi-contraction.

(f) We prove now that f is a Picard operator. By (d), using the graphic contraction principle (see [29]), it follows that, for every $x \in X$, the sequence $(f^n(x))_{n\in\mathbb{N}}$ is convergent. Denote by x_f^* its limit. Then we can prove that $x_f^* = x^*$. Indeed, we have

$$\begin{aligned} d(x_f^*, x^*) &\le d(x_f^*, f^n(x)) + d(f^n(x), g(x^*)) \\ &\le d(x_f^*, f^n(x)) + \alpha d(f^{n-1}(x), f^n(x)), \quad \forall n \in \mathbb{N}. \end{aligned}$$

Letting $n \to \infty$, we get that $x_f^* = x^*$.

(g) It follows from (f), while (h) follows from (e). This completes the proof.

By the above theorem, the following open problems arise.

Problem A. There exists, in the literature, a large class of metric conditions on a pair of the operators $f, g : X \to X$. We recall here some of these conditions:

(1) [Chatterjea (1979)] there exists $\alpha \in]0, \frac{1}{2}[$ such that

$$d(f(x), g(y)) \le \alpha\, [d(x, g(y)) + d(y, f(x))]\,,$$

for every $x, y \in X$;

(2) [Rus (1973)] there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $\alpha + 2\beta + 2\gamma < 1$ such that

$$d(f(x), g(y)) \le \alpha d(x, y) + \beta\, [d(x, f(x)) + d(y, g(y))] + \gamma\, [d(x, g(y)) + d(y, f(x))]\,,$$

for every $x, y \in X$;

(3) [Ćirić (1974)] there exists $\alpha \in]0, 1[$ such that

$$d(f(x), g(y)) \le \alpha \max\{d(x, y), d(x, f(x)), d(y, g(y)), \frac{1}{2}\, [d(x, g(y)) + d(y, f(x))]\,,$$

for every $x, y \in X$.

The problem is for which of the above conditions we can get similar conclusions to those in Theorem 1.6.

Problem B. Let (X, d) be a metric space and $f : X \to X$ be an operator. It is known that the following statements are equivalent:

(1) there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $\alpha + 2\beta + 2\gamma < 1$, such that

$$d(f(x), f(y)) \le \alpha d(x, y) + \beta\, [d(x, f(x)) + d(y, f(y))] + \gamma\, [d(x, f(y)) + d(y, f(x))]$$

for every $x, y \in X$.

(2) there exist $\alpha, \beta, \gamma, \delta, \eta, \in \mathbb{R}_+$ with $\alpha + \beta + \gamma + \delta + \eta < 1$ such that

$$d(f(x), f(y)) \le \alpha d(x, y) + \beta d(x, f(x)) + \gamma d(y, f(y)) + \delta d(x, f(y)) + \eta d(y, f(x))$$

for every $x, y \in X$.

In the case of a pair of operators $f, g : X \to X$, the condition

(2') there exist $\alpha, \beta, \gamma, \delta, \eta, \in \mathbb{R}_+$ with $\alpha + \beta + \gamma + \delta + \eta < 1$ such that, for each $x, y \in X$,

$$d(f(x), g(y)) \leq \alpha d(x, y) + \beta d(x, f(x)) + \gamma d(y, g(y)) + \delta d(x, g(y)) + \eta d(y, f(x))$$

is more general than the condition:

(1') there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $\alpha + 2\beta + 2\gamma < 1$ such that, for each $x, y \in X$,

$$d(f(x), g(y)) \leq \alpha d(x, y) + \beta \left[d(x, f(x)) + d(y, g(y))\right] + \gamma \left[d(x, g(y)) + d(y, f(x))\right].$$

Notice also that all the conclusions of Theorem 1.6 can be obtained by the assumption (1'). The problem is in which conditions on $\alpha, \beta, \gamma, \delta, \eta$, the assumption (2') implies the conclusions in Theorem 1.6.

Remark 1.2 For related results and applications in common fixed point theory, see [2, 4, 7, 23, 24, 26, 44, 48, 48, 49] and others.

1.3 Pairs of Operators on a Set with Two Metrics

In this section, we extend Theorem 1.6 to the case of a set endowed with two metrics.

We have the following results.

Theorem 1.7 *Let X be a nonempty set, d, ρ be two metrics on X and $f, g : X \to X$ be two operators. Suppose that*

(a) *there exists $a > 0$ such that $d(x, y) \leq a\rho(x, y)$, for every $x, y \in X$;*
(b) *(X, d) is a complete metric space;*
(c) *the operators $f, g : (X, d) \to (X, d)$ are continuous;*
(d) *there exists $\alpha \in]0, \frac{1}{2}[$ such that*

$$\rho(f(x), g(y)) \leq \alpha \left[\rho(x, f(x)) + \rho(y, g(y))\right], \quad \forall x, y \in X. \tag{1.6}$$

Then we have the following conclusions:

(1) *$F_f = F_g = \{x^*\}$;*
(2) *for each $x_0 \in X$, the sequence $(x_n)_{n \in \mathbb{N}}$ defined by*

$$x_{2n} = (g \circ f)^n(x_0), \quad x_{2n+1} = f(x_{2n}), \quad \forall n \in \mathbb{N},$$

converges to x^* with respect to the metric d as $n \to \infty$;

(3) *for each* $y_0 \in X$ *the sequence* $(y_n)_{n\in\mathbb{N}}$ *defined by*

$$y_{2n} = (f \circ g)^n(y_0), \quad y_{2n+1} = g(y_{2n}), \quad \forall n \in \mathbb{N},$$

converges to x^* *with respect to the metric* d *as* $n \to \infty$;

(4) *the operators* $f, g : (X, \rho) \to (X, \rho)$ *are graphic contractions;*

(5) *the operators* $f, g : (X, \rho) \to (X, \rho)$ *are quasi-contractions;*

(6) *the operators* f *and* g *are Picard operators with respect to* d *and* ρ;

(7) *the fixed point problem for* f *and the fixed point problem for* g *are well-posed with respect to the metric* ρ;

(8) *the operators* f *and* g *have the Ostrowski property with respect to* ρ.

Proof (1)–(3) Similar to the proof of Theorem 1.6, condition (d) implies that $F_f = F_g$ and $Card(F_f \cap F_g) \leq 1$. The assumption (d) also implies that the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ defined in (2) and in (3), respectively, are Cauchy sequences in (X, ρ). By (a) and (b), the sequences are convergent in (X, d). By (c), it follows that their limits are fixed points for f and g. Thus, $F_f = F_g = \{x^*\}$.

(4) and (5) follow in a similar way to the proof of Theorem 1.6.

(6) By the above considerations, it is obvious that f and g are Picard operators in (X, ρ). By (a), f and g are Picard operators in (X, d) too.

(7) The conclusion follows by the fact that f and g are c-Picard operators in (X, ρ).

Finally, (8) follows from (5). This completes the proof.

By the above result, the following problem arises.

Problem C. For which metric conditions in (X, ρ) do we obtain similar conclusions as in Theorem 1.7?

Remark 1.3 For various Maia type theorems for pairs of operators, see [17, 31] and others.

1.4 Contraction Pairs of Operators on Ordered Metric Spaces

Let $(X, \preceq)$ be a partially ordered set. In this framework, we denote

$$X_{\preceq} := \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\},$$

$$[a, b] := \{x \in X : a \preceq x \preceq b\},$$

for $a, b \in X$ with $a \preceq b$.

If $f : X \to X$, then the lower fixed point set and the upper fixed point set are denoted by

$$(LF)_f := \{x \in X : x \preceq f(x)\}, \quad (UF)_f := \{x \in X : f(x) \preceq x\},$$

respectively. If X, Y are two nonempty sets and $f : X \to X$, $g : Y \to Y$ are two mappings, then the Cartesian product of f and g, which is denoted by $f \times g : X \times Y \to X \times Y$, is defined by

$$(f \times g)(x, y) = (f(x), g(y)), \quad \forall (x, y) \in X \times Y.$$

Also, we denote the set of all nonempty invariant subsets of f by

$$I(f) := \{Y \subset X : f(Y) \subset Y\}.$$

We have the following common fixed point theorem for a pair of two operators.

Theorem 1.8 *Let (X, d) be a complete metric space and $\preceq$ be a partial order on X. Let $f, g : X \to X$ be two operators for which the following assumptions take place:*

(a) *there exists $x_0 \in X$ such that $f(x_0) \in (LF)_g \cup (UF)_g$;*
(b) *there exists $\beta \in]0, \frac{1}{2}[$ such that*

$$d(f(x), g(y)) \leq \beta \left[d(x, f(x)) + d(y, g(y))\right], \quad \forall (x, y) \in X_{\preceq}; \tag{1.7}$$

(c) *$X_{\preceq} \in I(f \times g)$ and $X_{\preceq} \in I(g \times f)$;*
(d) *one of the following conditions is satisfied:*

(d1) *f or g has a closed graph;*

or

(d2) *if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $x_n \to x^*$ as $n \to \infty$ and $(x_n, x_{n+1}) \in X_{\preceq}$ for every $n \in \mathbb{N}^*$, then at least one of the subsequences $(x_{2n})_{n\in\mathbb{N}}$ or $(x_{2n+1})_{n\in\mathbb{N}}$ has all the terms comparable to x^*.*

(1) *Then there exists $x^* \in X$ such that $x^* = f(x^*) = g(x^*)$, i.e., $CFP(f, g) \neq \emptyset$.*
(2) *Additionally, if for any $x, y \in CFP(f, g)$, we have that $(x, y) \in X_{\preceq}$, then $CFP(f, g) = \{x^*\}$.*

Proof (1) Let $x_0 \in X$ such that $f(x_0) \preceq g(f(x_0))$. For the reverse inequality, the proof runs in a similar way. We denote $x_1 := f(x_0)$, $x_2 := g(x_1)$, $x_3 := f(x_2)$, $x_4 := g(x_3)$. In general, we have

$$x_{2n+1} := f(x_{2n}), \quad x_{2n+2} := g(x_{2n+1}), \quad \forall n \in \mathbb{N}.$$

Then, by (c), we get that $(x_n, x_{n+1}) \in X_{\preceq}$ for every $n \in \mathbb{N}^*$. We have the following estimations:

I. For every $n \in \mathbb{N}^*$, we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(f(x_{2n}), g(x_{2n+1})) \\ &\le \beta \left[d(x_{2n}, f(x_{2n})) + d(x_{2n+1}, g(x_{2n+1})) \right] \\ &= \beta \left[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) \right]. \end{aligned}$$

Thus we have

$$d(x_{2n+1}, x_{2n+2}) \le \frac{\beta}{1-\beta} d(x_{2n}, x_{2n+1}). \tag{1.8}$$

II. For every $n \in \mathbb{N}^*$, we have

$$\begin{aligned} d(x_{2n+2}, x_{2n+3}) &= d(g(x_{2n+1}), f(x_{2n+2})) \\ &\le \beta \left[d(x_{2n+2}, f(x_{2n+2})) + d(x_{2n+1}, g(x_{2n+1})) \right] \\ &= \beta \left[d(x_{2n+2}, x_{2n+3}) + d(x_{2n+1}, x_{2n+2}) \right]. \end{aligned}$$

Thus we have

$$d(x_{2n+2}, x_{2n+3}) \le \frac{\beta}{1-\beta} d(x_{2n+1}, x_{2n+2}). \tag{1.9}$$

By (1.8) and (1.9), we get

$$d(x_{n+1}, x_{n+2}) \le \frac{\beta}{1-\beta} d(x_n, x_{n+1}), \quad \forall n \in \mathbb{N}. \tag{1.10}$$

Since $\alpha := \frac{\beta}{1-\beta} < 1$, by a classical approach, we obtain that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in X. By the completeness of the metric space (X, d), there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Next, we show that $x^* \in CFP(f, g)$. If f or g has a closed graph, then the conclusion follows by the relations $x_{2n+1} := f(x_{2n})$ and $x_{2n+2} := g(x_{2n+1})$ for every $n \in \mathbb{N}$.

If (d2) takes place, then we first notice that

$$\begin{aligned} d(x^*, f(x^*)) &\le d(x^*, x_{2n+2}) + d(x_{2n+2}, f(x^*)) \\ &\le d(x^*, x_{2n+2}) + d(g(x_{2n+1}), f(x^*)) \\ &\le d(x^*, x_{2n+2}) + \beta \left[d(x^*, f(x^*)) + d(x_{2n+1}, g(x_{2n+1})) \right] \\ &= d(x^*, x_{2n+2}) + \beta \left[d(x^*, f(x^*)) + d(x_{2n+1}, x_{2n+2}) \right] \\ &\vdots \\ &\le d(x^*, x_{2n+2}) + \beta \left[d(x^*, f(x^*)) + \alpha^{2n} d(x_1, x_2) \right]. \end{aligned}$$

Thus we have

$$d(x^*, f(x^*)) \leq \frac{1}{1-\beta}\left[d(x^*, x_{2n+2}) + \beta\alpha^{2n} d(x_1, x_2)\right] \to 0 \ \text{ as } n \to \infty.$$

Hence $x^* \in F_f$. Now, let us observe that

$$d(x^*, g(x^*)) = d(x^*, g(x^*)) \leq \beta\left[d(x^*, f(x^*)) + d(x^*, g(x^*))\right] = \beta d(x^*, g(x^*)).$$

Since $\beta < \frac{1}{2}$, we obtain that $d(x^*, g(x^*)) = 0$. Hence $x^* \in F_g$. As a conclusion, we proved that $x^* \in CFP(f, g)$.

(2) Now, if x^*, y^* are two common fixed points, then $(x^*, y^*) \in X_{\preceq}$ and so we have

$$d(x^*, y^*) = d(f(x^*), g(y^*)) = \beta\left[d(x^*, f(x^*)) + d(y^*, g(y^*))\right] = 0,$$

which proves the uniqueness of the common fixed point. This completes the proof.

Remark 1.4 For the fixed point theory in ordered metric spaces, see [28] and the references therein. See also [46] for a recent survey.

1.5 Pairs of Operators on $\mathbb{R}^m_+$-Metric Spaces

Let (X, d) be a generalized metric space with $d(x, y) \in \mathbb{R}^m_+$.

The following result is an extension of Theorem 1.6 for pairs of operators defined on such generalized metric spaces.

Theorem 1.9 *Let (X, d) be a complete generalized metric space (with $d(x, y) \in \mathbb{R}^m_+$) and $f, g : X \to X$ be two operators, for which there exists $A \in \mathscr{M}_{m,m}(\mathbb{R}_+)$ such that the matrices A and $A(I_m - A)^{-1}$ are convergent to zero and the following property holds:*

$$d(f(x), g(y)) \leq A\left[d(x, f(x)) + d(y, g(y))\right], \quad \forall x, y \in X. \tag{1.11}$$

Then we have the following conclusions:

(1) $F_f = F_g = \{x^*\}$;
(2) *for each $x_0 \in X$ the sequence $(x_n)_{n\in\mathbb{N}}$ defined by*

$$x_{2n} = (g \circ f)^n(x_0), \quad x_{2n+1} = f(x_{2n}), \quad \forall n \in \mathbb{N},$$

converges to x^* as $n \to \infty$;
(3) *for each $y_0 \in X$ the sequence $(y_n)_{n\in\mathbb{N}}$ defined by*

$$y_{2n} = (f \circ g)^n(y_0), \quad y_{2n+1} = g(y_{2n}), \quad \forall n \in \mathbb{N},$$

converges to x^* as $n \to \infty$;

(4) *the operators f and g are graphic contractions;*

(5) *the operators f and g are quasi-contractions;*

(6) *the operators f and g are Picard operators;*

(7) *the fixed point problem for f and the fixed point problem for g are well-posed;*

(8) *the operators f and g have the Ostrowski property.*

Proof (1) Notice first that $F_f = F_g$. Indeed, let, for example, $x^* \in F_f$. Then, by (1.11), we have

$$d(x^*, g(x^*)) = d(f(x^*), g(x^*)) \leq Ad(x^*, g(x^*)) \leq \cdots \leq A^n d(x^*, g(x^*)).$$

Since $A^n \to 0$ as $n \to \infty$, we obtain that $x^* \in F_g$. We also have that $Card(F_f \cap F_g) \leq 1$. Indeed, let $x^*, y^* \in F_f \cap F_g$. Then, by (1.11), we have

$$d(x^*, y^*) = d(f(x^*), g(x^*)) \leq A\left[d(x^*, f(x^*)) + d(y^*, g(y^*))\right] = 0.$$

Hence $x^* = y^*$.

(2) For the second conclusion, consider $x_0 \in X$ arbitrarily chosen and the sequence $(x_n)_{n\in\mathbb{N}}$ constructed as in (2). Notice that the matrices A and $(I - A)^{-1}$ commute. Then, since

$$d(x_1, x_2) = d(f(x_1), g(x_1)) \leq A\left[d(x_0, x_1) + d(x_1, x_2)\right],$$

we have

$$d(x_1, x_2) \leq A(I_m - A)^{-1} d(x_0, x_1).$$

Denote $B := A(I_m - A)^{-1}$. Then B is convergent to zero. By induction, we get

$$d(x_n, x_{n+1}) \leq B^n d(x_0, x_1), \quad \forall n \in \mathbb{N}.$$

By the above relation, using a classical approach, we get that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Let $x^* \in X$ be its limit. We have

$$\begin{aligned} d(x^*, f(x^*)) &\leq d(x^*, x_{2n}) + d(x_{2n}, f(x^*)) \\ &\leq d(x^*, x_{2n}) + d(g(x_{2n-1}), f(x_{2n})) \\ &\leq d(x^*, x_{2n}) + A\left[d(x^*, f(x^*)) + d(x_{2n-1}, x_{2n}).\right]. \end{aligned}$$

Letting $n \to \infty$, we get $d(x^*, f(x^*)) = 0$. By a similar procedure, we can show that the sequence $(y_n)_{n\in\mathbb{N}}$ defined in (3) converges to x^* as $n \to \infty$.

(4) We prove that f is a graphic contraction. For arbitrary $x \in X$, we have

$$\begin{aligned}
d(f^2(x), f(x)) &\leq d(f^2(x), g(x^*)) + d(f(x), g(x^*)) \\
&\leq A\left[d(f^2(x), f(x)) + d(x^*, g(x^*))\right] + A\left[d(x, f(x)) + d(x^*, g(x^*))\right] \\
&= A\left[d(f^2(x), f(x)) + d(x, f(x))\right].
\end{aligned}$$

Thus we have

$$d(f^2(x), f(x)) \leq A(I_m - A)^{-1} d(x, f(x)),$$

which proves that f is a graphic contraction.

(5) We have

$$d(f(x), x^*) = d(f(x), g(x^*)) \leq Ad(x, f(x)) \leq A\left[d(x, x^*) + d(x^*, f(x))\right].$$

Hence we have

$$d(f(x), x^*) \leq A(I_m - A)^{-1} d(x, x^*), \quad \forall x \in X.$$

This shows that f is a quasi-contraction. Since the condition (1.11) is symmetric with respect to f and g, we get that g is also a quasi-contraction.

(6) We prove now that f is a Picard operator. By (4), using the graphic contraction principle (see [27]), it follows that, for every $x \in X$, the sequence $(f^n(x))_{n\in\mathbb{N}}$ is convergent. Denote by x_f^* its limit. Then we can prove that $x_f^* = x^*$. Indeed, we have

$$\begin{aligned}
d(x_f^*, x^*) &\leq d(x_f^*, f^n(x)) + d(f^n(x), g(x^*)) \\
&\leq d(x_f^*, f^n(x)) + Ad(f^{n-1}(x), f^n(x)), \quad \forall n \in \mathbb{N}.
\end{aligned}$$

Letting $n \to \infty$, we get that $x_f^* = x^*$.

(7) follows from (6), while (8) follows by (5). This completes the proof.

In connection to the above results, we present now some open questions.

Problem D. Let (X, d) be a complete generalized metric space ($d(x, y) \in \mathbb{R}^m_+$) and $f, g : X \to X$. We suppose there exist two matrices $A, B \in \mathscr{M}_{m,m}(\mathbb{R}_+)$ such that

$$d(f(x), g(y)) \leq Ad(x, f(x)) + Bd(y, g(y)), \quad \forall x, y \in X.$$

In which conditions on A and B do we have similar conclusions as in Theorem 1.9?

See the papers [35, 43].

Problem E. Another open question is to extend Theorem 1.6 to a generalized metric spaces (X, d) with $d(x, y) \in s(\mathbb{R}_+)$, where $s(\mathbb{R}_+)$ is the space of infinite sequences of real non-negative numbers.

See the papers [15, 43].

Problem F. A more general problem is to extend Theorem 1.6 to cone metric spaces.

See the papers [3, 30, 38, 43].

1.6 Data Dependence for the Common Fixed Point Problem

In this section, we discuss the data dependence phenomenon for the common fixed point problem with a pair of operators. More exactly, if (X, d) is a metric space and $f, g, \tilde{f}, \tilde{g} : X \to X$ are the operators such that

$$d(f(x), \tilde{f}(x)) \leq \eta_1, \quad d(g(x), \tilde{g}(x)) \leq \eta_2, \quad \forall x \in X,$$

the problem is to study the distance between the fixed points of the pairs f, g and $\tilde{f}, \tilde{g}$.

Now, we have the following result.

Theorem 1.10 *Let (X, d) be a complete metric space and $f, g : X \to X$ be two operators for which there exists $\alpha \in]0, \frac{1}{2}[$ such that*

$$d(f(x), g(y)) \leq \alpha \left[d(x, f(x)) + d(y, g(y))\right], \quad \forall x, y \in X.$$

Let $\tilde{f}, \tilde{g} : X \to X$ be two operators such that $F_f \neq \emptyset$, $F_g \neq \emptyset$ and there exists $\eta_1, \eta_2 > 0$ satisfying the relations

$$d(f(x), \tilde{f}(x)) \leq \eta_1, \quad d(g(x), \tilde{g}(x)) \leq \eta_2, \quad \forall x \in X.$$

Then $F_f = F_g = \{x^*\}$ and the following estimations hold:

$$d(x, x^*) \leq (1+\alpha)\eta_1, \quad d(u, x^*) \leq (1+\alpha)\eta_2, \quad \forall x \in F_{\tilde{f}}, \; u \in F_{\tilde{g}}.$$

Proof Since f and g are $\frac{1-\alpha}{1-2\alpha}$-Picard operators, the conclusions follow by Theorem 1.6.

Remark 1.5 Since, in the conditions of Theorem 1.10, f and g are c-Picard operators, we obtain that the fixed point equations $x = f(x)$ and $x = g(x)$ are Ulam-Hyers stable. Recall that, a fixed point problem $x = f(x)$, $x \in X$ (where $f : X \to X$) is said to be Ulam-Hyers stable if there exists $c > 0$ such that, for every $\varepsilon > 0$ and any $z \in X$ with $d(z, f(z)) \leq \varepsilon$, there exists $x^* \in F_f$ with $d(z, x^*) \leq c \cdot \varepsilon$.

1.7 Other Problems

1.7.1 Common Fixed Point Set as a Fixed Point Set

Let X be a normed space, Y be a nonempty, closed and convex subset of X, and let $f, g : Y \to X$ be two operators. In which conditions on the above data does there exist an operator $h : Y \to X$ such that $F_h = F_f \cap F_g$?

Commentaries. (1) An answer to the above problem is the following well-known result:

Theorem 1.11 (Bruck's Theorem) *We suppose that*

(a) *X is a strictly convex Banach space and Y is a closed and convex subset of X;*
(b) *$f, g : Y \to X$ are nonexpansive;*
(c) *$F_f \cap F_g \neq \emptyset$.*

Then, for each $\lambda \in]0, 1[$, we have

$$F_{h_\lambda} = F_f \cap F_g,$$

where $h_\lambda := \lambda f + (1 - \lambda)g$.

(2) Let Y be a nonempty, compact convex subset of X and $f, g : Y \to X$ be two nonexpansive operators such that $F_f \cap F_g \neq \emptyset$. By Schauder's fixed point theorem, we have that $F_{h_\lambda} \neq \emptyset$ for every $\lambda \in]0, 1[$. Thus, in general, F_{h_λ} is nonempty, but this does not imply that $F_f \cap F_g \neq \emptyset$.

(3) Another problem is to consider, instead of the assumption (c) in Bruck's theorem, the following condition:

(c') $f(Y) \subset Y,\ g(Y) \subset Y,\ F_f \neq \emptyset,\ F_g \neq \emptyset,\ f \circ g = g \circ f.$

(4) A more general variant of our problem is the following: Let X be a Banach space, Y be a nonempty and closed subset of X and $G : Y \times Y \to Y$ be such that

(a) $G(x, x) = x$ for every $x \in Y$;
(b) $G(x, y) = x \ \Rightarrow\ y = x$;
(c) $G(x, y) = y \Longrightarrow x = y$.

Let $f, g : Y \to Y$ be two operators. In which conditions do we have that

$$G(f(x), g(x)) = x \Longrightarrow x \in F_f \cap F_g?$$

See the papers [1, 5, 6, 8, 11–14, 21, 25, 33, 36, 40] and others.

1.7.2 Lipschitz Pairs on Compact Convex Subsets

Let X be a Banach space, Y be a nonempty, compact and convex subset of X, and $f, g : Y \to Y$ be two operators with $f \circ g = g \circ f$, for which there exists L_f, $L_g > 0$ such that

$$\|f(x) - f(y)\| \leq L_f\|x - y\|, \ \|g(x) - g(y)\| \leq L_g\|x - y\|, \quad \forall x, y \in Y.$$

In which conditions on L_f, L_g do we have that $F_f \cap F_g \neq \emptyset$?

In this direction, the following result is well known:

Theorem 1.12 (De Marr's Theorem) *Let $I \subset \mathbb{R}$ be a compact interval and $f, g : I \to I$ two commuting mappings having the Lipschitz property with constants L_f and L_g, respectively. If $L_f L_g - L_f - L_g < 1$, then $F_f \cap F_g \neq \emptyset$.*

See the papers [10, 17–20, 33, 37] and others.

1.8 Conclusion

In this paper, we proved existence, uniqueness and approximation theorems for the common fixed point problem with single-valued operators in various metric-type frameworks. Then, some stability results (data dependence of the common fixed point on the operators' perturbation, well-posedness, Ostrowski's property and Ulam-Hyers stability) for the common fixed point problem are presented. Finally, other open questions and new research directions are pointed out.

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Chapter 2
Some Convergence Results of the K^* Iteration Process in CAT(0) Spaces

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Abstract In this paper, we prove some strong and $\triangle$-convergence theorems of the K^* iteration process for two different classes of generalized nonexpansive mappings in CAT(0) spaces.

Keywords CAT(0) space · Iteration processes · $\triangle$-convergence · Strong convergence · Nonexpansive mappings

2.1 Introduction

A CAT(0) *space X* is a metric space which it is geodesically connected and every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane (see [1, p. 159]). The term "CAT" is due to Gromov [2] and it is an acronym for E. Cartan, A. D. Alexanderov, and V. A. Toponogov. The CAT(0) spaces play a fundamental role in various branches of mathematics (see Bridson and Haefliger [1] or Burago et al. [3]). Moreover, there are applications in computer science, biology and graph theory as well (see, e.g., [4–6]). Fixed point theory in CAT(0) spaces was first studied by Kirk [7, 8]. He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0)space always has a fixed point. Since then, the fixed point theory in CAT(0) spaces has been rapidly developed and many papers have appeared (see, e.g., [9–15]).

Let C be a nonempty subset of a metric space (X, d) and T be a self-mapping on C. A point $p \in C$ is called a *fixed point* of T if $Tp = p$ and $F(T)$ denotes the set of all fixed points of T. The mapping T is called *nonexpansive* if

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$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in C,$$

and *quasi-nonexpansive* if

$$d(Tx, p) \leq d(x, p), \quad \forall x \in C, \ p \in F(T).$$

The mapping T is said to be *uniformly* L *-Lipschitzian* if there exists a constant $L \geq 0$ such that

$$d(T^n x, T^n y) \leq L d(x, y), \quad \forall x, y \in C, \ n \geq 1.$$

In 2006, Alber et al. [16] introduced the notion of total asymptotically nonexpansive mappings as follows:

Definition 2.1 (see [16, Definition 1.4]) Let (X, d) be a metric space and C be a subset of X. A mapping $T : C \to C$ is called *total asymptotically nonexpansive* if there are non-negative real sequences $\{k_n^{(1)}\}$ and $\{k_n^{(2)}\}$ for each $n \geq 1$ with $k_n^{(1)}, k_n^{(2)} \to 0$ as $n \to \infty$ and strictly increasing and continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x, y) + k_n^{(1)} \phi(d(x, y)) + k_n^{(2)}, \quad \forall x, y \in C. \tag{2.1}$$

Remark 2.1 (see [16, Remark 1.5]) If $\phi(\lambda) = \lambda$, then the inequality (2.1) takes the form

$$d(T^n x, T^n y) \leq (1 + k_n^{(1)}) d(x, y) + k_n^{(2)}, \quad \forall x, y \in C.$$

In addition, if $k_n^{(2)} = 0$ for all $n \geq 1$, then total asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings defined by Goebel and Kirk [17]. If $k_n^{(1)} = 0$ and $k_n^{(2)} = 0$ for all $n \geq 1$, then we obtain the class of uniformly 1-Lipschitzian mappings from (2.1).

Theorem 2.1 (see [18, Corollary 3.2]) *Let (X, d) be a complete CAT(0) space and C be a nonempty bounded closed convex subset of X. If $T : C \to C$ is a continuous and total asymptotically nonexpansive mapping, then T has a fixed point.*

In 2008, Suzuki [19] introduced a new condition on a mappings, called condition (C), which is weaker than nonexpansiveness. A mapping $T : C \to C$ is said to satisfy the *condition* (C) if, for all $x, y \in C$,

$$\frac{1}{2} d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq d(x, y).$$

The mapping T satisfying the condition (C) is called *Suzuki generalized nonexpansive mapping*.

In 2011, Garcia-Falset et al. [20] introduced a new generalization of nonexpansive mappings which in turn includes Suzuki generalized nonexpansive mappings.

Definition 2.2 (see [20, Definiton 2]) Let T be a mapping defined on a subset C of a metric space (X, d) and $\mu \geq 1$. Then T is said to satisfy the *condition* $\left(E_\mu\right)$ if, for all $x, y \in C$,

$$d(x, Ty) \leq \mu d(x, Tx) + d(x, y).$$

The following example shows that the class of mappings satisfying the condition $\left(E_\mu\right)$ is larger than the class of Suzuki generalized nonexpansive mappings.

Example 2.1 (see [20, Example 1]) In the space $C([0, 1])$, consider the set

$$K := \{x \in C([0, 1]) : 0 = x(0) \leq x(t) \leq x(1) = 1\}.$$

Take a function $g \in K$ and generate a mapping F_g as follows:

$$F_g : K \to K, F_g x(t) := (g \circ x)(t) = g(x(t)).$$

Then the mapping F_g satisfies the condition (E_1), but it fails to be a Suzuki generalized nonexpansive mapping.

Proposition 2.1 (see [20, Proposition 1]) *Let $T : C \to C$ be a mapping satisfying the condition $\left(E_\mu\right)$ on C. If T has some fixed point, then T is quasi-nonexpansive.*

Recently, Ullah and Arshad [21] introduced a new iteration process called K^* iteration process in Banach spaces as follows:

$$\begin{cases} x_1 \in C, \\ z_n = (1 - \beta_n)x_n + \beta_n T x_n, \\ y_n = T((1 - \alpha_n)z_n + \alpha_n T z_n), \\ x_{n+1} = T y_n, \quad \forall n \geq 1. \end{cases}$$

With the help of a numerical example, they showed that this iteration process is faster than the Picard S-iteration [22] and S-iteration [23] for Suzuki generalized nonexpansive mappings.

In this paper, we study the convergence of the K^* iteration process in CAT(0) spaces. This paper contains four sections. In Sect. 2.2, we recollect basic definitions and a detailed overview of the fundamental results. In Sect. 2.3, we prove the strong and $\triangle$-convergence theorems of the K^* iteration process for the class of mappings satisfying the condition $\left(E_\mu\right)$. In Sect. 2.4, we prove the strong and $\triangle$-convergence theorems for total asymptotically nonexpansive mappings by using the K^* iteration process.

2.2 Preliminaries and Lemmas

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image of c is called a *geodesic* (or *metric*) *segment* joining x and y. If it is unique, this geodesic segment is denoted by $[x, y]$. The space (X, d) is called a *geodesic space* if every two points of X are joined by a geodesic. Furthermore, X is called *uniquely geodesic* if there is exactly one geodesic joining x to y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

In a geodesic metric space (X, d), *a geodesic triangle* $\triangle(x_1, x_2, x_3)$ consists of three points x_1, x_2, x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle} := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane E^2 such that $d_{E^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for each $i, j \in \{1, 2, 3\}$.

Comparison Axiom. Let (X, d) be a geodesic metric space and $\overline{\triangle}$ be a comparison triangle for a geodesic triangle $\triangle$ in X. Then $\triangle$ is said to satisfy the CAT(0) *inequality* if

$$d(x, y) \leq d_{E^2}(\overline{x}, \overline{y})$$

for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$.

A geodesic metric space is called a CAT(0) space [1] if all geodesic triangles of appropriate size satisfy the comparison axiom. A complete CAT(0) space is called *"Hadamard space"*.

If x, y_1, y_2 are points in CAT(0) space and y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d\left(x, y_0\right)^2 \leq \frac{1}{2} d(x, y_1)^2 + \frac{1}{2} d(x, y_2)^2 - \frac{1}{4} d(y_1, y_2)^2. \tag{(CN)}$$

This is *the (CN) inequality* of Bruhat and Tits [24]. In fact, a geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality (see [1, p.163]).

The following lemmas are some elementary facts about CAT(0) spaces:

Lemma 2.1 *Let X be a CAT(0) space. Then we have the following:*

(1) (see [10, Lemma 2.1(iv)]) *For all $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d\left(x, z\right) = t d(x, y) \text{ and } d\left(y, z\right) = (1 - t) d(x, y). \tag{2.2}$$

The notation $(1 - t)x \oplus ty$ is used for the unique point z satisfying (2.2).

(2) (see [10, Lemma 2.4]) *For all $x, y, z \in X$ and $t \in [0, 1]$, one has*

$$d\left((1-t)x \oplus ty, z\right) \leq (1-t)d(x,z) + td(y,z).$$

Lemma 2.2 (see [25, Lemma 3.2]) *Let X be a CAT(0) space, $x \in X$ be a given point and $\{t_n\}$ be a sequence in $[b, c]$ with $b, c \in (0, 1)$ and $0 < b(1-c) \leq \frac{1}{2}$. Let $\{x_n\}$ and $\{y_n\}$ be any sequences in X such that*

$$\limsup_{n\to\infty} d\left(x_n, x\right) \leq r, \;\; \limsup_{n\to\infty} d\left(y_n, x\right) \leq r, \;\; \lim_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r$$

for some $r \geq 0$. Then $\lim_{n\to\infty} d\left(x_n, y_n\right) = 0$.

Now, we give the concept of $\triangle$-convergence and collect some of its basic properties.

Let $\{x_n\}$ be a bounded sequence in CAT(0) space X. For all $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\ \{r(x, \{x_n\}) : x \in X\}.$$

The *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that in a Hadamard space, $A(\{x_n\})$ consists of exactly one point (see [26, Proposition 7]).

Definition 2.3 (see [27, 28]) A sequence $\{x_n\}$ in CAT(0) space X is said to be $\triangle$-*convergent* to a point $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, one can write $\triangle\text{-}\lim_{n\to\infty} x_n = x$ and call x the $\triangle$-*limit* of $\{x_n\}$.

Lemma 2.3 *Let X be a Hadamard space. Then we have the following:*

(1) (see [28, p.3690]) *Every bounded sequence in X has a $\triangle$-convergent subsequence.*
(2) (see [29, Proposition 2.1]) *If K is a closed convex subset of X and $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*
(3) (see [10, Lemma 2.8]) *If $\{x_n\}$ is a bounded sequence in X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.4 (see [30, Lemma 2]) *Let $\{a_n\}$, $\{b_n\}$ and $\{r_n\}$ be three sequences of non-negative real numbers such that $a_{n+1} \leq (1 + r_n)a_n + b_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

2.3 Some Convergence Results for the Class of Mappings Satisfying the Condition (E_μ)

In this section, we prove some strong and $\triangle$-convergence theorems of a sequence generated by the K^* iteration process for the class of mappings satisfying the condition (E_μ) in the setting of CAT(0) spaces.

Theorem 2.2 *Let C be a nonempty closed convex subset of a Hadamard space X and $T : C \to C$ be a mapping satisfying the condition (E_μ) with $F(T) \neq \varnothing$. Let $\{x_n\}$ be an iterative sequence generated by*

$$\begin{cases} x_1 \in C, \\ z_n = (1-\beta_n)x_n \oplus \beta_n T x_n, \\ y_n = T((1-\alpha_n)z_n \oplus \alpha_n T z_n), \\ x_{n+1} = T y_n, \quad \forall n \geq 1, \end{cases} \tag{2.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[a,b]$ for some $a, b \in (0,1)$ with $0 < a(1-b) \leq \frac{1}{2}$. Then the sequence $\{x_n\}$ is $\triangle$-convergent to a fixed point of T.

Proof We divide our proof into three steps.

Step 1. First, we prove that, for each $p \in F(T)$,

$$\lim_{n\to\infty} d(x_n, p) \text{ exists.} \tag{2.4}$$

Step 2. Next, we prove that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{2.5}$$

Step 3. Finally, to show that the sequence $\{x_n\}$ is $\triangle$-convergent to a fixed point of T, we prove that

$$W_\triangle(x_n) = \bigcup_{\{u_n\}\subset\{x_n\}} A(\{u_n\}) \subseteq F(T)$$

and $W_\triangle(x_n)$ consists of exactly one point.

Step 1. By Proposition 2.1 and Lemma 2.1 (2), we have

$$d(x_{n+1}, p) = d(Ty_n, p) \leq d(y_n, p), \tag{2.6}$$

$$\begin{aligned} d(y_n, p) &= d(T((1-\alpha_n)z_n \oplus \alpha_n T z_n), p) \\ &\leq d((1-\alpha_n)z_n \oplus \alpha_n T z_n, p) \\ &\leq (1-\alpha_n)d(z_n, p) + \alpha_n d(Tz_n, p) \\ &\leq (1-\alpha_n)d(z_n, p) + \alpha_n d(z_n, p) \\ &= d(z_n, p) \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} d(z_n, p) &= d((1-\beta_n)x_n \oplus \beta_n T x_n, p) \\ &\le (1-\beta_n)d(x_n, p) + \beta_n d(T x_n, p) \\ &\le (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{2.8}$$

Using (2.6), (2.7) and (2.8), we obtain

$$d(x_{n+1}, p) \le d(x_n, p).$$

This implies that the sequence $\{d(x_n, p)\}$ is non-increasing and bounded below. Hence $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.

Step 2. It follows from (2.4) that $\lim_{n\to\infty} d(x_n, p)$ exists for each given $p \in F(T)$. Let

$$\lim_{n\to\infty} d(x_n, p) = r \ge 0. \tag{2.9}$$

Since

$$d(T x_n, p) \le d(x_n, p),$$

we have

$$\limsup_{n\to\infty} d(T x_n, p) \le r. \tag{2.10}$$

On the other hand, it follows from (2.8) that

$$\limsup_{n\to\infty} d(z_n, p) \le r. \tag{2.11}$$

By using (2.6) and (2.7), we get

$$d(x_{n+1}, p) \le d(z_n, p)$$

which yields that

$$r \le \liminf_{n\to\infty} d(z_n, p). \tag{2.12}$$

Hence, from (2.11) and (2.12), we have that $\lim_{n\to\infty} d(z_n, p) = r$. This implies that

$$\lim_{n\to\infty} d((1-\beta_n)x_n \oplus \beta_n T x_n, p) = r. \tag{2.13}$$

From (2.9), (2.10), (2.13) and Lemma 2.2, we get $\lim_{n\to\infty} d(x_n, T x_n) = 0$.

Step 3. Let $u \in W_\triangle(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.3 (1) and (2), there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle$-$\lim_{n\to\infty} v_n = v \in C$. By (2.5), we have

$$\lim_{n\to\infty} d(v_n, Tv_n) = 0. \tag{2.14}$$

Now, we have to show that v is a fixed point of T. Since T is a mapping satisfying the condition (E_μ), then there exists a $\mu \geq 1$ such that

$$d(v_n, Tv) \leq \mu d(v_n, Tv_n) + d(v_n, v).$$

Taking the limit supremum on both sides of the above estimate and using (2.14), we have

$$\begin{aligned} r(Tv, \{v_n\}) &= \limsup_{n\to\infty} d(Tv, v_n) \\ &\leq \limsup_{n\to\infty} d(v, v_n) = r(v, \{v_n\}). \end{aligned}$$

By the uniqueness of asymptotic center, we get $Tv = v$. Thus $v \in F(T)$. By (2.4), $\lim_{n\to\infty} d(x_n, v)$ exists. Hence, by Lemma 2.3 (3), we have $u = v$. This implies that $W_\triangle(x_n) \subseteq F(T)$.
Finally, we prove that $W_\triangle(x_n)$ consists of exactly one point. In fact, let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in F(T)$. From (2.4), we know that $\{d(x_n, u)\}$ is convergent. In view of Lemma 2.3 (3), we have $x = u \in F(T)$. This shows that $W_\triangle(x_n) = \{x\}$. This completes the proof.

Next, we prove the strong convergence theorem.

Theorem 2.3 *Let X, C, T and $\{x_n\}$ be the same as in Theorem 2.2 and C be a compact subset of X. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof By (2.5), we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since C is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some $p \in C$. Since T satisfies the condition (E_μ), we have

$$d(x_{n_k}, Tp) \leq \mu d(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, p). \tag{2.15}$$

Letting $k \to \infty$, we get $Tp = p$, i.e., $p \in F(T)$. By (2.4), $\lim_{n\to\infty} d(x_n, p)$ exists for every $p \in F(T)$ and so the sequence $\{x_n\}$ converges strongly to p. This completes the proof.

Example 2.2 Let $\mathbb{R}$ be the real line with its usual metric and let $C = [-4, 1]$. Then X is a Hadamard space and C is a compact, closed and convex subset of X. Define a mapping $T : C \to C$ by

$$Tx = \begin{cases} \frac{|x|}{4}, & \text{if } x \in [-4, 1), \\ -\frac{1}{4}, & \text{if } x = 1. \end{cases}$$

In order to see that T satisfies the condition $\left(E_\mu\right)$ on C, we consider the following (non-trivial) cases:

(1) Let $x \in [-4, 0)$ and $y \in [-4, 1]$, then $|x - Tx| = \frac{5}{4}|x|$ and

$$|x - Ty| \le |x| + \frac{1}{4}|y| \le \frac{5}{4}|x| + \frac{1}{4}|x - y| \le 2\ |x - Tx| + |x - y|.$$

(2) Let $x \in [0, 1)$ and $y \in [-4, 1]$, then $|x - Tx| = \frac{3}{4}|x|$ and

$$|x - Ty| \le |x| + \frac{1}{4}|y| \le \frac{3}{2}|x| + \frac{1}{4}|x - y| \le 2|x - Tx| + |x - y|.$$

(3) Let $x = 1$ and $y \in [-4, 1)$, then $|1 - T1| = \frac{5}{4}$ and

$$|1 - Ty| = \frac{3}{4} + \frac{1 - |y|}{4} \le \frac{3}{5}|1 - T1| + \frac{1}{4}|1 - y| \le 2\ |1 - T1| + |1 - y|.$$

In summary, for all $x, y \in C$, we have

$$|x - Ty| \le 2|x - Tx| + |x - y|,$$

that is, T satisfies the condition (E_2) on C. Clearly, $F(T) = \{0\}$. Set

$$\alpha_n = \frac{n}{2n + 1}, \quad \beta_n = \frac{n}{3n + 1}, \quad \forall n \ge 1. \tag{2.16}$$

Therefore, the conditions of Theorem 2.3 are satisfied. So, the sequence $\{x_n\}$ generated by (2.3) is strong and $\triangle$-convergent to 0.

In [31, p. 375], Senter and Dotson introduced the concept of the condition (I) as follows:

A mapping $T : C \to C$ is said to satisfy the *condition* (I) if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$d(x, Tx) \ge f(d(x, F(T))), \quad \forall x \in C, \tag{2.17}$$

where $d(x, F(T)) = \inf\{d(x, p) : p \in F(T)\}$.

By using this definition, we prove the following strong convergence theorem:

Theorem 2.4 *Under the same assumptions of Theorem 2.2, if T satisfies the condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof By (2.5) and (2.17), we have

$$\lim_{n\to\infty} f(d(x_n, F(T))) \le \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

This implies that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since f is a non-decreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r\in(0, \infty)$, we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. The rest of the proof follows the lines of Theorem 3.4 in [32]. This completes the proof.

2.4 Some Convergence Results for Total Asymptotically Nonexpansive Mappings

Now, we give the $\triangle$-convergence theorem of K^* iteration process for total asymptotically nonexpansive mappings in CAT(0) spaces.

Theorem 2.5 *Let C be a nonempty bounded closed convex subset of a Hadamard space X and $T : C \to C$ be a uniformly L-Lipschitzian and total asymptotically nonexpansive mapping. Suppose that the following conditions are satisfied:*

(a) $\sum_{n=1}^{\infty} k_n^{(1)} < \infty$ *and* $\sum_{n=1}^{\infty} k_n^{(2)} < \infty$;
(b) *there exists constants $a, b \in (0, 1)$ with $0 < a(1-b) \le \frac{1}{2}$ such that $\{\alpha_n\}, \{\beta_n\} \subset [a, b]$;*
(c) *there exists a constant $M > 0$ such that $\phi(r) \le Mr$ for all $r \ge 0$.*

Then the sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in C, \\ z_n = (1-\beta_n)x_n \oplus \beta_n T^n x_n, \\ y_n = T^n((1-\alpha_n)z_n \oplus \alpha_n T^n z_n), \\ x_{n+1} = T^n y_n, \quad \forall n \ge 1, \end{cases} \tag{2.18}$$

is $\triangle$-convergent to a fixed point of T.

Proof Since T is uniformly L-Lipschitzian, we conclude that T is continuous. By using Theorem 2.1, we get $F(T) \neq \varnothing$. Let $p \in F(T)$. Since T is a total asymptotically nonexpansive mapping, by the condition (c), then we obtain

$$\begin{aligned} d(z_n, p) &= d((1-\beta_n)x_n \oplus \beta_n T^n x_n, p) \\ &\le (1-\beta_n)d(x_n, p) + \beta_n d(T^n x_n, p) \\ &\le (1-\beta_n)d(x_n, p) + \beta_n[d(x_n, p) + k_n^{(1)}\phi(d(x_n, p)) + k_n^{(2)}] \\ &\le (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) + \beta_n M k_n^{(1)} d(x_n, p) + \beta_n k_n^{(2)} \\ &= (1+\beta_n M k_n^{(1)})d(x_n, p) + \beta_n k_n^{(2)} \end{aligned} \tag{2.19}$$

and

$$\begin{aligned}
d(y_n, p) &= d(T^n((1-\alpha_n)z_n \oplus \alpha_n T^n z_n), p)\\
&\le d((1-\alpha_n)z_n \oplus \alpha_n T^n z_n, p) + k_n^{(1)}\phi(d((1-\alpha_n)z_n \oplus \alpha_n T^n z_n, p)) + k_n^{(2)}\\
&\le (1-\alpha_n)d(z_n, p) + \alpha_n d(T^n z_n, p)\\
&\quad + Mk_n^{(1)}[(1-\alpha_n)d(z_n, p) + \alpha_n d(T^n z_n, p)] + k_n^{(2)}\\
&\le (1-\alpha_n)d(z_n, p) + \alpha_n[d(z_n, p) + k_n^{(1)}\phi(d(z_n, p)) + k_n^{(2)}]\\
&\quad + Mk_n^{(1)}\big[(1-\alpha_n)d(z_n, p) + \alpha_n[d(z_n, p) + k_n^{(1)}\phi(d(z_n, p)) + k_n^{(2)}]\big] + k_n^{(2)}\\
&\le (1-\alpha_n)d(z_n, p) + \alpha_n d(z_n, p) + \alpha_n Mk_n^{(1)}d(z_n, p) + \alpha_n k_n^{(2)}\\
&\quad + (1-\alpha_n)Mk_n^{(1)}d(z_n, p) + \alpha_n Mk_n^{(1)}d(z_n, p) + \alpha_n M^2(k_n^{(1)})^2 d(z_n, p)\\
&\quad + \alpha_n Mk_n^{(1)}k_n^{(2)} + k_n^{(2)}\\
&= \big(1 + (1+\alpha_n)Mk_n^{(1)} + \alpha_n M^2(k_n^{(1)})^2\big)d(z_n, p)\\
&\quad + (1 + \alpha_n + \alpha_n Mk_n^{(1)})k_n^{(2)}.
\end{aligned} \tag{2.20}$$

Substituting (2.19) into (2.20) and simplifying it, we get

$$\begin{aligned}
d(y_n, p) &\le \big(1 + (1+\alpha_n)Mk_n^{(1)} + \alpha_n M^2(k_n^{(1)})^2\big)\big[(1 + \beta_n Mk_n^{(1)})d(x_n, p) + \beta_n k_n^{(2)}\big]\\
&\quad + (1 + \alpha_n + \alpha_n Mk_n^{(1)})k_n^{(2)}\\
&= \Bigg[1 + (1 + \alpha_n + \beta_n)Mk_n^{(1)} + (\alpha_n + \beta_n + \alpha_n\beta_n)M^2(k_n^{(1)})^2\\
&\quad + \alpha_n\beta_n M^3(k_n^{(1)})^3\Bigg]d(x_n, p)\\
&\quad + \big[1 + \alpha_n + \beta_n + (\alpha_n + \beta_n + \alpha_n\beta_n)Mk_n^{(1)} + \alpha_n\beta_n M^2(k_n^{(1)})^2\big]k_n^{(2)}
\end{aligned} \tag{2.21}$$

Also, we have

$$\begin{aligned}
d(x_{n+1}, p) &= d(T^n y_n, p)\\
&\le d(y_n, p) + k_n^{(1)}\phi(d(y_n, p)) + k_n^{(2)}\\
&\le (1 + Mk_n^{(1)})d(y_n, p) + k_n^{(2)}.
\end{aligned} \tag{2.22}$$

Combining (2.21) and (2.22), we conclude that

$$d(x_{n+1}, p) \le (1 + r_n)d(x_n, p) + b_n, \ \forall n \ge 1, \tag{2.23}$$

where

$$\begin{aligned}
r_n &= 1 + (2 + \alpha_n + \beta_n)Mk_n^{(1)} + (1 + 2\alpha_n + 2\beta_n + \alpha_n\beta_n)M^2(k_n^{(1)})^2\\
&\quad + (\alpha_n + \beta_n + 2\alpha_n\beta_n)M^3(k_n^{(1)})^3 + \alpha_n\beta_n M^4(k_n^{(1)})^4
\end{aligned}$$

and

$$
\begin{aligned}
b_n = & [2+\alpha_n+\beta_n+(1+2\alpha_n+2\beta_n+\alpha_n\beta_n)Mk_n^{(1)} \\
& +(\alpha_n+\beta_n+2\alpha_n\beta_n)M^2(k_n^{(1)})^2+\alpha_n\beta_n M^3(k_n^{(1)})^3]k_n^{(2)}.
\end{aligned}
$$

By the condition (a), we have that $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. Hence, by Lemma 2.4, $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$. Without loss of generality, we may assume that

$$
r := \lim_{n\to\infty} d(x_n, p). \tag{2.24}
$$

From (2.19), we conclude that

$$
\limsup_{n\to\infty} d(z_n, p) \le r. \tag{2.25}
$$

Now, using (2.25) and the fact that T is a total asymptotically nonexpansive mapping, we obtain

$$
\begin{aligned}
\limsup_{n\to\infty} d(T^n z_n, p) \le & \limsup_{n\to\infty} \left[d(z_n, p)+k_n^{(1)}\phi(d(z_n, p))+k_n^{(2)}\right] \\
\le & \limsup_{n\to\infty} \left[(1+Mk_n^{(1)})d(z_n, p)+k_n^{(2)}\right] \\
\le & \; r.
\end{aligned} \tag{2.26}
$$

Similarly, we get

$$
\limsup_{n\to\infty} d(T^n x_n, p) \le r. \tag{2.27}
$$

Now, we can write

$$
\begin{aligned}
d(x_{n+1}, p) \le & (1+Mk_n^{(1)})d(y_n, p)+k_n^{(2)} \\
\le & (1+Mk_n^{(1)})\Bigg[\bigg(1+(1+\alpha_n)Mk_n^{(1)}+\alpha_n M^2(k_n^{(1)})^2\big)d(z_n, p) \\
& +(1+\alpha_n+\alpha_n Mk_n^{(1)})k_n^{(2)}\Bigg]+k_n^{(2)} \\
= & \left[1+(2+\alpha_n)Mk_n^{(1)}+(1+2\alpha_n)M^2(k_n^{(1)})^2+\alpha_n M^3(k_n^{(1)})^3\right]d(z_n, p) \\
& +\left[(2+\alpha_n)+(1+2\alpha_n)Mk_n^{(1)}+\alpha_n M^2(k_n^{(1)})^2\right]k_n^{(2)}.
\end{aligned}
$$

Taking limit infimum on both sides in the above inequality, we have

$$
\liminf_{n\to\infty} d(z_n, p) \ge r.
$$

Combining with (2.25), it yields that $\lim_{n\to\infty} d(z_n, p) = r$. This implies that

$$
\lim_{n\to\infty} d((1-\beta_n)x_n \oplus \beta_n T^n x_n, p) = r. \tag{2.28}
$$

By using Lemma 2.2 with (2.24), (2.27) and (2.28), we get

$$\lim_{n\to\infty} d(x_n, T^n x_n) = 0. \tag{2.29}$$

From (2.21) and (2.22), we conclude that

$$\limsup_{n\to\infty} d(y_n, p) \le r, \quad \liminf_{n\to\infty} d(y_n, p) \ge r,$$

respectively. Hence, we have

$$\lim_{n\to\infty} d(y_n, p) = r. \tag{2.30}$$

On the other hand, since

$$\begin{aligned}
&\lim_{n\to\infty} d(y_n, p) \\
&= \lim_{n\to\infty} d(T^n((1-\alpha_n)z_n \oplus \alpha_n T^n z_n), p) \\
&\le \lim_{n\to\infty} \left[d((1-\alpha_n)z_n \oplus \alpha_n T^n z_n, p) + k_n^{(1)} \phi(d((1-\alpha_n)z_n \oplus \alpha_n T^n z_n, p)) + k_n^{(2)} \right] \\
&\le \lim_{n\to\infty} \left[(1 + M k_n^{(1)}) d((1-\alpha_n)z_n \oplus \alpha_n T^n z_n, p) + k_n^{(2)} \right] \\
&= \lim_{n\to\infty} d((1-\alpha_n)z_n \oplus \alpha_n T^n z_n, p) \\
&\le \lim_{n\to\infty} \left[(1-\alpha_n) d(z_n, p) + \alpha_n d(T^n z_n, p) \right] \\
&\le \lim_{n\to\infty} \left[(1-\alpha_n) d(z_n, p) + \alpha_n d(z_n, p) + \alpha_n M k_n^{(1)} d(z_n, p) + \alpha_n k_n^{(2)} \right] \\
&= \lim_{n\to\infty} d(z_n, p),
\end{aligned}$$

we have

$$\lim_{n\to\infty} d((1-\alpha_n)z_n \oplus \alpha_n T^n z_n, p) = r. \tag{2.31}$$

Again, by using Lemma 2.2 with (2.25), (2.26) and (2.31), we get

$$\lim_{n\to\infty} d(z_n, T^n z_n) = 0. \tag{2.32}$$

By (2.29), we obtain

$$\begin{aligned}
d(z_n, T^n x_n) &= d((1-\beta_n)x_n \oplus \beta_n T^n x_n, T^n x_n) \\
&\le (1-\beta_n) d(x_n, T^n x_n) + \beta_n d(T^n x_n, T^n x_n) \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

It follows that

$$d(T^n z_n, T^n x_n) \leq d(T^n z_n, z_n) + d(z_n, T^n x_n) \to 0 \text{ as } n \to \infty. \tag{2.33}$$

By (2.32), we have

$$\begin{aligned} d(y_n, T^n z_n) &= d(T^n((1-\alpha_n)z_n \oplus \alpha_n T^n z_n), T^n z_n) \\ &\leq d((1-\alpha_n)z_n \oplus \alpha_n T^n z_n, z_n) \\ &\quad + k_n^{(1)}\phi(d((1-\alpha_n)z_n \oplus \alpha_n T^n z_n, z_n)) + k_n^{(2)} \\ &\leq (1+Mk_n^{(1)})d((1-\alpha_n)z_n \oplus \alpha_n T^n z_n, z_n) + k_n^{(2)} \\ &\leq (1+Mk_n^{(1)})\left[(1-\alpha_n)d(z_n, z_n) + \alpha_n d(T^n z_n, z_n)\right] + k_n^{(2)} \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.34}$$

From (2.32) and (2.34), we get

$$\begin{aligned} d(T^n y_n, T^n z_n) &\leq d(y_n, z_n) + k_n^{(1)}\phi(d(y_n, z_n)) + k_n^{(2)} \\ &\leq (1+Mk_n^{(1)})d(y_n, z_n) + k_n^{(2)} \\ &\leq (1+Mk_n^{(1)})\left[d(y_n, T^n z_n) + d(T^n z_n, z_n)\right] + k_n^{(2)} \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.35}$$

By using the triangle inequality, (2.33) and (2.35), we conclude that

$$d(T^n x_n, T^n y_n) \leq d(T^n x_n, T^n z_n) + d(T^n z_n, T^n y_n) \to 0 \text{ as } n \to \infty. \tag{2.36}$$

From (2.29) and (2.36), we obtain

$$d(x_{n+1}, x_n) \leq d(T^n y_n, T^n x_n) + d(T^n x_n, x_n) \to 0 \text{ as } n \to \infty. \tag{2.37}$$

Since T is uniformly L-Lipschitzian, therefore we have

$$\begin{aligned} & d(x_n, Tx_n) \\ & \leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + d(T^{n+1}x_{n+1}, T^{n+1}x_n) + d(T^{n+1}x_n, Tx_n) \\ & \leq (1+L)d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + Ld(T^n x_n, x_n). \end{aligned}$$

Hence, (2.29) and (2.37) imply that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{2.38}$$

The rest of the proof follows the pattern of Theorem 1 in [33]. This completes the proof.

Example 2.3 Consider $X = \mathbb{R}$ with its usual metric, so X is a Hadamard space. Let $C = [-1, 1]$ which is a bounded, closed and convex subset of X. Define a mapping $T : C \to C$ by

$$Tx = \begin{cases} 2\sin\frac{x}{2}, & \text{if } x \in [-1, 0), \\ -2\sin\frac{x}{2}, & \text{if } x \in [0, 1]. \end{cases}$$

It was proved in [33] that T is uniformly L-Lipschitzian and total asymptotically nonexpansive mapping with $L = 1$, $k_n^{(1)} = k_n^{(2)} = 0$ for all $n \geq 1$ and $\phi(t) = t$ for all $t \geq 0$. Clearly, $F(T) = \{0\}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be the same as in (2.16). Thus, the conditions of Theorem 2.5 are fullfilled. Therefore the sequence $\{x_n\}$ generated by (2.18) is $\triangle$-convergent to 0.

Next, we give some characterizations of the strong convergence for the sequence $\{x_n\}$ defined by (2.18) in CAT(0) spaces as follows:

Theorem 2.6 *Let $X, C, T, \{\alpha_n\}, \{\beta_n\}$ and $\{x_n\}$ satisfy the hypotheses of Theorem 2.5. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

Proof If $\{x_n\}$ converges strongly to $p \in F(T)$, then $\lim_{n\to\infty} d(x_n, p) = 0$. Since $0 \leq d(x_n, F(T)) \leq d(x_n, p)$, we have $\liminf\limits_{n\to\infty} d(x_n, F(T)) = 0$.
Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. By (2.23), we have

$$d(x_{n+1}, p) \leq (1 + r_n)d(x_n, p) + b_n, \quad \forall p \in F(T).$$

This implies that

$$d(x_{n+1}, F(T)) \leq (1 + r_n)d(x_n, F(T)) + b_n.$$

Since $\sum_{n=1}^{\infty} r_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then, by Lemma 2.4, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Thus, by the hypothesis, we get $\lim_{n\to\infty} d(x_n, F(T)) = 0$. The rest of the proof is similar to the proof of Theorem 2 in [33] and therefore it is omitted. This completes the proof.

Remark 2.2 In Theorem 2.6, the condition $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ may be replaced with $\limsup_{n\to\infty} d(x_n, F(T)) = 0$.

Now, we give the following theorem related to the strong convergence of the sequence $\{x_n\}$ defined by (2.18).

Theorem 2.7 *Let $X, C, T, \{\alpha_n\}, \{\beta_n\}$ and $\{x_n\}$ satisfy the hypotheses of Theorem 2.5 and T be a mapping satisfying the condition (I). Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof As proved in Theorem 2.6, $\lim_{n\to\infty} d(x_n, F(T))$ exists. By (2.38), we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. It follows from the condition (I) that

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0.$$

Hence, by the properties of f, we get $\lim_{n\to\infty} d(x_n, F(T)) = 0$. The conclusion now follows from Theorem 2.6. This completes the proof.

Finally, we give an example of a mapping which satisfies all the assumptions of T in Theorem 2.7, i.e., T is a uniformly L-Lipschitzian and total asymptotically nonexpansive mapping satisfying the condition (I).

Example 2.4 Let $T : [0, 2] \to [0, 2]$ defined as

$$Tx = \begin{cases} 1, & \text{if } x \in [0, 1], \\ \sqrt{\frac{4-x^2}{3}}, & \text{if } x \in (1, 2]. \end{cases}$$

Note that $T^n x = 1$ for all $x \in [0, 2]$ and $n \geq 2$ and $F(T) = \{1\}$. Clearly, T is both uniformly L-Lipschitzian and total asymptotically nonexpansive mapping on $[0, 2]$. Additionally, it was shown in [34, Example 3.7] that T satisfies the condition (I).

2.5 Conclusions

In the above sections, we study the strong and $\triangle$-convergence of the K^* iteration process introduced by Ullah and Arshad [21] for two different classes of generalized nonexpansive mappings in CAT(0) spaces.

Theorems 2.2, 2.3, 2.4 generalize some results of Ullah and Arshad [21] in two ways:

(1) from the class of Suzuki generalized nonexpansive mappings to the class of mappings satisfying the condition (E_μ),

(2) from Banach space to Hadamard space.

Theorems 2.5, 2.6, 2.7 contain the corresponding theorems proved for asymptotically nonexpansive mappings when $k_n^{(2)} = 0$ for all $n \geq 1$ and $\phi(\lambda) = \lambda$ for all $\lambda \geq 0$.

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Chapter 3
Split Variational Inclusion Problem and Fixed Point Problem for Asymptotically Nonexpansive Semigroup with Application to Optimization Problem

Shih-sen Chang, Liangcai Zhao, and Zhaoli Ma

Abstract The purpose of this paper is, by using the shrinking projection method, to introduce and study an iterative process to approximate a common solution of the split variational inclusion problem and the fixed point problem for an asymptotically nonexpansive semigroup in real Hilbert spaces. Further, we prove that the sequences generated by the proposed iterative method converge strongly to a common solution of the problems for an asymptotically nonexpansive semigroup. As applications, we utilize the results to study the split optimization problem and the split variational inequality.

Keywords Split variational inclusion problem · Asymptotically nonexpansive semigroup · Fixed point problem · Nonexpansive semigroup

Mathematics Subject Classification 54E70 · 47H25

3.1 Introduction

Throughout the paper, unless otherwise stated, let $H, \ H_1, \ H_2$ be three real Hilbert spaces, C be a nonempty closed and convex subset of H.

Recall that a mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

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$$||T^n x - T^n y|| \le k_n||x - y||, \quad \forall n \ge 1, \ x, y \in C.$$

A family $\mathfrak{T} := \{T(s) : 0 \le s < \infty\}$ of mappings from C into itself is called an *asymptotically nonexpansive semigroup* on C (resp., *nonexpansive semigroup* on C) if it satisfies the following conditions:

(a) $T(0)x = x$ for all $x \in C$;
(b) $T(s + t) = T(s)T(t)$ for all $s, t \ge 0$;
(c) there exists a sequence $\{k_n\} \subset [1, \infty)$ (resp., $\{k_n = 1\}$) such that $k_n \to 1$ and satisfying the following condition:

$$||T^n(s)x - T^n(s)y|| \le k_n||x - y||, \quad \forall x, y \in C, \ n \ge 1, \ s \ge 0;$$

(d) for all $x \in C$, $s \mapsto T(s)x$ is continuous.

The set of all common fixed points of a semigroup $\mathfrak{T}$ is denoted by $\mathrm{Fix}(\mathfrak{T})$, i.e.,

$$\mathrm{Fix}(\mathfrak{T}) := \{x \in C : T(s)x = x, \ 0 \le s < \infty\} = \bigcap_{0 \le s < \infty} \mathrm{Fix}(T(s)),$$

where $\mathrm{Fix}(T(s))$ is the set of fixed points of $T(s), \ s \ge 0$.

Recall that a mapping $T : H_1 \to H_1$ is said to be

(1) *monotone* if

$$\langle Tx - Ty, x - y\rangle \ge 0, \quad \forall x, y \in H_1.$$

(2) α*-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge \alpha||x - y||^2, \quad \forall x, y \in H_1.$$

(3) *firmly nonexpansive* if

$$||Tx - Ty||^2 \le \langle Tx - Ty, x - y\rangle, \quad \forall x, y \in H_1. \tag{3.1}$$

Remark 1 It is easy to see that the definition of firmly nonexpansive mapping is equivalent to the following:

$(3)'$ A mapping $T : H_1 \to H_1$ is said to be *firmly nonexpansive* if

$$||Tx - Ty||^2 \le ||x - y||^2 - \langle x - y, (x - Tx) - (y - Ty)\rangle, \quad \forall x, y \in C. \tag{3.2}$$

(4) A multi-valued mapping $M : H_1 \to 2^{H_1}$ is said to be *monotone* if, for all $x, y \in H_1$, $u \in Mx$ and $v \in My$ such that

$$\langle x - y, u - v\rangle \geq 0.$$

(5) A monotone mapping $M : H_1 \to 2^{H_1}$ is said to be *maximal* if the Graph(M) is not properly contained in the graph of any other monotone mapping.

It is known that a monotone mapping M is maximal if and only if, for any $(x, u) \in H_1 \times H_1$, $\langle x - y, u - v\rangle \geq 0$ for every $(y, v) \in Graph(M)$ implies that $u \in Mx$.

Let $M : H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping. Then the *resolvent mapping* $J_\lambda^M : H_1 \to H_1$ *associated with* M is defined by

$$J_\lambda^M(x) := (I + \lambda M)^{-1}(x), \quad \forall x \in H_1 \tag{3.3}$$

for some $\lambda > 0$, where I stands the identity operator on H_1.

Note that, for all $\lambda > 0$, the resolvent operator J_λ^M is single-valued, nonexpansive and firmly nonexpansive.

Recently, Moudafi [1] introduced the following *split variational inclusion problem (in short, SVIP)*: Find $x^* \in H_1$ and $y^* = Ax^* \in H_2$ such that

$$0 \in B_1(x^*) \text{ and } 0 \in B_2(y^*), \tag{3.4}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings.

From the definition of resolvent mapping J_λ^M, we have the following technical lemma:

Lemma 1 *The problem SVIP (3.4) is equivalent to the problem: Find $x^* \in H_1$ and $y^* = Ax^* \in H_2$ such that*

$$x^* \in Fix(J_\lambda^{B_1}) \text{ and } y^* \in Fix(J_\lambda^{B_2}) \text{ for some } \lambda > 0. \tag{3.5}$$

In the sequel, we denote the solution set Ω of the problem (3.4) or (3.5) by

$$\begin{aligned}\Omega : &= \{x^* \in H_1 : y^* = Ax^* \in H_2 \text{ such that } x^* \in B_1^{-1}(0),\ Ax^* \in B_2^{-1}(0)\}\\ &= \{x^* \in H_1 : y^* = Ax^* \in H_2 \text{ such that } x^* \in Fix(J_\lambda^{B_1}),\ Ax^* \in Fix(J_\lambda^{B_2})\}.\end{aligned} \tag{3.6}$$

Moudafi [1] also introduced an iterative method for solving the problem SVIP (3.4), which can be seen as an important generalization of an iterative method given by Censor et al. [2] for split variational inequality problem. As Moudafi noted in [1], the problem SVIP (3.4) includes as special cases, the split common fixed point problem, the split variational inequality problem, the split zero problem, and the split feasibility problem (see [1–6]), which have already been studied and used in practice as a model in intensity-modulated radiation therapy treatment planning (see [5, 6]).

This formalism is also at the core of modeling of many inverse problems arising for phase retrieval and other real-world problems, for instance, in sensor networks in computerized tomography and data compression (see, for example, [7, 8]).

In 2012, Byrne et al. [4] studied the weak and strong convergence of the following iterative method for the problem SVIP (3.4): For any $x_0 \in H_1$, compute the iterative sequence $\{x_n\}$ generated by the following scheme:

$$x_{n+1} = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n) \tag{3.7}$$

for some $\lambda > 0$, where $\gamma > 0$ is a constant and A is a linear and bounded operator.

Very recently, Kazmi and Rizvi [9] studied the strong convergence of the following iterative method for split variational inclusion problem and the fixed point problem for a nonexpansive mapping:

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n \end{cases} \tag{3.8}$$

for some $\lambda > 0$, where S is a nonexpansive mapping and f is a contractive mapping.

Motivated by the work of Moudafi [1], Byrne et al. [4], Kazmi and Rizvi [9], Deepho et al. [10] and Sitthithakerngkiet et al. [11], the purpose of this paper is, by using the shrinking projection method, to introduce and study an iterative process to approximate a common solution of split variational inclusion problem and fixed point problem for an asymptotically nonexpansive semigroup in real Hilbert spaces. Further, we prove that the sequences generated by the proposed iterative method converge strongly to a common solution of the split variational inclusion problem and the fixed point problem for an asymptotically nonexpansive semigroup. The results presented in this paper are an extension and generalization of the previously known results to some related topics.

3.2 Preliminaries

In this section, we recall some concepts and lemmas which will be used in proving our main results.

Let C be a nonempty closed and convex subset of H. For each $x \in H$, *the (metric) projection* $P_C : H \to C$ is defined as the unique element $P_C x \in C$ such that

$$||x - P_C x|| = \inf_{y \in C} ||x - y||.$$

It is well known that, for any $x \in H$, $y = P_C(x)$ if and only if

$$\langle y - z,\ x - y\rangle \geq 0, \quad \forall z \in C \tag{3.9}$$

and P_C is a firmly nonexpansive mapping from H onto C, that is,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in C. \tag{3.10}$$

Recall that a mapping $T : C \to H$ is said to be α*-inverse strongly monotone* if there exists $\alpha > 0$ such that

$$\alpha\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle, \quad \forall x, y \in C. \tag{3.11}$$

This implies that each firmly nonexpansive mapping is 1-inverse strongly monotone.

Also, it is easy to prove that the following result holds:

Lemma 2 ([12]) *If $T : C \to H$ is α -inverse strongly monotone, then, for each $\lambda \in (0, 2\alpha]$, $I - \lambda T$ is a nonexpansive mapping of C into H.*

Lemma 3 *Let H be a real Hilbert space, then the following result holds:*

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$$

for all $x, y \in H$ and $t \in [0, 1]$.

Lemma 4 ([13]) *Let H be a real Hilbert space, C be a nonempty closed convex subset of H and $S : C \to C$ be an asymptotically nonexpansive mapping. If the set of fixed points $Fix(S)$ of S is nonempty, then it is closed and convex and the mapping $I - S$ is demiclosed at zero, that is, for any sequence $\{x_n\}$ in C such that, if $\{x_n\}$ converges weakly to $\overline{x}$ and $||x_n - Sx_n|| \to 0$, then $\overline{x} \in Fix(S)$.*

3.3 Main Results

In this section, we prove a strong convergence theorem based on the proposed iterative method for computing a common approximate solution of the problem SVIP (3.4) and a common fixed point of the asymptotically nonexpansive semigroup $\mathfrak{T} = \{T(s) : 0 \leq s < \infty\}$.

Throughout this section we assume the following:

(A1) H_1 and H_2 are two real Hilbert spaces;
(A2) $A : H_1 \to H_2$ is a bounded linear operator, A^* is the adjoint of A and it is strongly positive, i.e., there exists a constant $\gamma > 0$ such that

$$\langle A^* x,\ y\rangle \geq \gamma ||x||||y||, \quad \forall y \in H_1,\ x \in H_2;$$

(A3) $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are two maximal monotone mappings;
(A4) $J_\lambda^{B_1} : H_1 \to H_1$ and $J_\lambda^{B_2} : H_2 \to H_2$ are the resolvent mappings associated with B_1 and B_2 defined by (3.3), respectively,
(A5) $\mathfrak{T} = \{T(s) : 0 \le s < \infty\} : H_1 \to H_1$ is an asymptotically nonexpansive semigroup.

First, we give the following lemma:

Lemma 5 *Let $H_1,\ H_2,\ A,\ A^*, B_1,\ B_2,\ J_\lambda^{B_1},\ J_\lambda^{B_2}$ be the same as above. Let L be the spectral radius of the operator A^*A and $\gamma \in (0, \frac{2}{L})$. Then $(I - \gamma A^*(I - J_\lambda^{B_2})A)$ and $J_\lambda^{B_1}(I - \gamma A^*(I - J_\lambda^{B_2})A)$ both are nonexpansive mappings.*

Proof Since $J_\lambda^{B_2}$ is firmly nonexpansive, $(I - J_\lambda^{B_2})$ is also firmly nonexpansive. Hence it is 1−inverse strongly monotone. So we have

$$\begin{aligned} &\|(I - J_\lambda^{B_2})Ax - (I - J_\lambda^{B_2})Ay\|^2 \\ &= \|Ax - Ay\|^2 - 2\langle Ax - Ay, J_\lambda^{B_2}Ax - J_\lambda^{B_2}Ay\rangle + \|J_\lambda^{B_2}Ax - J_\lambda^{B_2}Ay\|^2 \\ &\le \|Ax - Ay\|^2 - \langle Ax - Ay, J_\lambda^{B_2}Ax - J_\lambda^{B_2}Ay\rangle \\ &= \langle Ax - Ay, (I - J_\lambda^{B_2})Ax - (I - J_\lambda^{B_2})Ay\rangle, \quad \forall x, y \in H_1. \end{aligned} \tag{3.12}$$

It follows from (3.12) that

$$\begin{aligned} &\|A^*(I - J_\lambda^{B_2})Ax - A^*(I - J_\lambda^{B_2})Ay\|^2 \\ &\le L||(I - J_\lambda^{B_2})Ax - (I - J_\lambda^{B_2})Ay\|^2 \\ &\le L\langle Ax - Ay, (I - J_\lambda^{B_2})Ax - (I - J_\lambda^{B_2})Ay\rangle \\ &= L\langle x - y, A^*(I - J_\lambda^{B_2})Ax - A^*(I - J_\lambda^{B_2})Ay\rangle, \quad \forall x, y \in H_1. \end{aligned} \tag{3.13}$$

This implies that $A^*(I - J_\lambda^{B_2})A$ is a $\frac{1}{L}$−inverse strongly monotone mapping. Since $\gamma \in (0, \frac{2}{L})$, by Lemma 2, $I - \gamma A^*(I - J_\lambda^{B_2})A$ is a nonexpansive mapping. So is $J_\lambda^{B_1}(I - \gamma A^*(I - J_\lambda^{B_2})A)$. This completes the proof.

Theorem 3.1 *Let $H_1,\ H_2,\ A,\ A^*, B_1,\ B_2,\ J_\lambda^{B_1},\ J_\lambda^{B_2}$ be the same as in Lemma 5. Let $\mathfrak{T} = \{T(s) : 0 \le s < \infty\}$ be an asymptotically nonexpansive semigroup of mappings from H_1 to itself with the sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Denote by $\Gamma :=Fix(\mathfrak{T})\bigcap \Omega$, where Ω is the solution set of the problem (3.4) defined by (3.6). For any initial point $x_0 \in H_1$, $C_1 = H_1$, $x_1 = P_{C_1}x_0$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} u_n = J_{r_n}^{B_1}(I - \gamma A^*(I - J_{r_n}^{B_2})A)x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T^n(s)u_n ds, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \ge 1, \end{cases} \tag{3.14}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)\sup\{\|x_n - u\|^2 : u \in \Gamma\}$, $\{s_n\}$ *is a sequence of positive numbers,* $0 < a \le \alpha_n < c < 1$ *for all* $n \ge 1$, $0 < b \le r_n < +\infty$ *and* $\gamma \in (0, \frac{2}{L})$, *where* L *is the spectral radius of the operator* A^*A*. If the following conditions are satisfied:*

(a) $\Gamma :=$*Fix(*$\mathfrak{T}$*)*$\bigcap \Omega \ne \emptyset$ *and is bounded;*
(b) $\limsup_{n\to\infty} ||\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds - T(h)(\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds)|| = 0$ *for each* $h > 0$*,*

then the sequence $\{x_n\}$ *generated by (3.14) strongly converges to a point* $x^* \in$ *Fix(*$\mathfrak{T}$*)*$\bigcap \Omega$*.*

Proof We divide the proof of Theorem 3.1 into five steps as follows:

Step 1. We show that C_n is closed and convex for each $n \ge 1$. In fact, since the inequality $||y_n - z||^2 \le ||x_n - z||^2 + \theta_n$ is equivalent to

$$2\langle x_n - y_n, z\rangle \le ||x_n||^2 - ||y_n||^2 + \theta_n, \quad \forall n \ge 1,$$

and $z \mapsto 2\langle x_n - y_n, z\rangle$ is a continuous and convex function. Therefore, for each $n \ge 1$, C_n is a convex and closed subset in H_1.

Step 2. Now, we prove that Fix($\mathfrak{T}$)$\bigcap \Omega \subset C_n$, $\forall n \ge 1$. In fact, let $p \in$ Fix($\mathfrak{T}$)$\bigcap \Omega$, then $p = T(s)p$ for all $s \ge 0$, $J_{r_n}^{B_1} p = p$, $J_{r_n}^{B_2} Ap = Ap$ and so $(I - \gamma A^*(I - T_{r_n}^{B_2})A)p = p$. It is obvious that Fix($\mathfrak{T}$)$\bigcap \Omega \subset C_1$. Let Fix($\mathfrak{T}$)$\bigcap \Omega \subset C_n$ for some $n \ge 2$. Then, by induction, we prove that Fix($\mathfrak{T}$)$\bigcap \Omega \subset C_{n+1}$. In fact, it follows from (3.14) and Lemma 5 that

$$\begin{aligned} \|u_n - p\| &= \|J_{r_n}^{B_1}(I - \gamma A^*(I - J_{r_n}^{B_2})A)x_n - J_{r_n}^{B_1}(I - \gamma A^*(I - J_{r_n}^{B_2})A)p\| \\ &\le \|(I - \gamma A^*(I - J_{r_n}^{B_2})A)x_n - (I - \gamma A^*(I - J_{r_n}^{B_2})A)p\| \\ &= \|x_n - p\|. \end{aligned} \tag{3.15}$$

Also, it follows from (3.14), (3.15) and Lemma 3 that

$$\begin{aligned}
\|y_n - p\|^2 &= \Big\|\alpha_n x_n + (1-\alpha_n)\Big(\frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds\Big) - p\Big\|^2 \\
&= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big\|\frac{1}{s_n}\int_0^{s_n}(T^n(s)u_n - p)ds\Big\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\Big\|x_n - \frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds\Big\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big(\frac{1}{s_n}\int_0^{s_n}||T^n(s)u_n - p||ds\Big)^2 \\
&\quad - \alpha_n(1-\alpha_n)\Big\|x_n - \frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds\Big\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)k_n^2||u_n - p||^2 \\
&\quad - \alpha_n(1-\alpha_n)\Big\|x_n - \frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds\Big\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)k_n^2||x_n - p||^2 \\
&= \|x_n - p\|^2 + (1-\alpha_n)(k_n^2 - 1)||x_n - p||^2 \\
&\le \|x_n - p\|^2 + \theta_n,
\end{aligned} \tag{3.16}$$

where

$$\theta_n = (1-\alpha_n)(k_n^2 - 1)\sup_{u\in\Gamma}\{||x_n - u||^2\}. \tag{3.17}$$

This implies that $p \in C_{n+1}$, so is $\mathrm{Fix}(\mathfrak{T})\bigcap\Omega \subset C_{n+1}$. The conclusion is proved.

Step 3. Now, we prove that $\{x_n\}$ is a Cauchy sequence. In fact, it follows from (3.14) that $x_{n+1} = P_{C_{n+1}}x_0$, $x_n = P_{C_n}x_0$ and $C_{n+1} \subset C_n$. By (3.9), we have

$$\langle x_0 - x_{n+1}, x_{n+1} - y\rangle \ge 0, \forall y \in C_{n+1}.$$

Since $\Gamma = \mathrm{Fix}(\mathfrak{T})\bigcap\Omega \subset C_{n+1}$, we have

$$\langle x_0 - x_{n+1}, x_{n+1} - p\rangle \ge 0, \quad \forall p \in \Gamma.$$

This shows that

$$\begin{aligned}
0 &\le \langle x_0 - x_{n+1}, x_{n+1} - x_0 + x_0 - p\rangle \\
&\le -||x_{n+1} - x_0||^2 + ||x_{n+1} - x_0||||x_0 - p||.
\end{aligned}$$

Simplifying, we have

$$||x_{n+1} - x_0|| \le ||x_0 - p||,$$

i.e., $\{x_n\}$ is bounded and so are $\{u_n\}$ and $\{y_n\}$. Also, since

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \ge 0,$$

we have

$$\begin{aligned}0 &\le \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &\le -||x_n - x_0||^2 + ||x_{n+1} - x_0||||x_0 - x_n||,\end{aligned}$$

i.e., $||x_n - x_0|| \le ||x_{n+1} - x_0||$. Since $\{x_n\}$ is bounded, this implies that the limit $\lim_{n\to\infty} ||x_n - x_0||$ exists. Hence, for any positive integers n, m, it follows from (3.14) that $x_m = P_{C_m} x_0$ and $x_n = P_{C_n} x_0$. By the well-known property of the projection, we have

$$||x_n - x_m||^2 + ||x_m - x_0||^2 \le ||x_n - x_0||^2, \ \forall\, n, m \ge 1.$$

Since the limit $\lim_{n\to\infty} ||x_n - x_0||$ exists, we have

$$||x_n - x_m||^2 \le ||x_n - x_0||^2 - ||x_m - x_0||^2 \to 0$$

as $n,\ m \to \infty$. This implies that $\{x_n\}$ is a Cauchy sequence. Without loss of generality, we can assume that

$$\lim_{n\to\infty} x_n = x^* \in C_n. \tag{3.18}$$

Therefore, since $\{x_n\}$ is bounded and Γ is bounded, it follows from (3.17) that

$$\theta_n \to 0 \tag{3.19}$$

as $n \to \infty$.

Step 4. Next, we prove that

$$\lim_{n\to\infty} \|T(h)x_n - x_n\| = 0, \quad \forall h \ge 0. \tag{3.20}$$

In fact, since $x_{n+1} \in C_{n+1} \subset C_n$, by the construction of C_{n+1}, we have

$$\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n$$

and so

$$\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\| + \sqrt{\theta_n}.$$

This together with (3.18) and (3.19) shows that

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0.$$

Therefore, we have

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \to 0 \tag{3.21}$$

as $n \to \infty$. Since $J_{r_n}^{B_1}$ is firmly nonexpansive, by (3.13), $A^*(I - J_{r_n}^{B_2})A$ is a $\frac{1}{L}$-inverse strongly monotone mapping. If $p \in \Gamma$, then we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|J_{r_n}^{B_1}(x_n - \gamma A^*(I - J_{r_n}^{B_2})Ax_n) - J_{r_n}^{B_1}(p - \gamma A^*(I - J_{r_n}^{B_2})Ap)\|^2 \\
&\le \|(I - \gamma A^*(I - J_{r_n}^{B_2})A)x_n - (I - \gamma A^*(I - J_{r_n}^{B_2})A)p\|^2 \\
&- \|(I - J_{r_n}^{B_1})(I - \gamma A^*(I - J_{r_n}^{B_2})A)x_n - (I - J_{r_n}^{B_1})(I - \gamma A^*(I - J_{r_n}^{B_2})A)p\|^2 \\
&= \|x_n - p - \gamma(A^*(I - J_{r_n}^{B_2})Ax_n - A^*(I - J_{r_n}^{B_2})Ap)\|^2 - \|z_n - J_{r_n}^{B_1}z_n\|^2 \\
&= \|x_n - p\|^2 - 2\gamma\langle x_n - p, A^*(I - J_{r_n}^{B_2})Ax_n - A^*(I - J_{r_n}^{B_2})Ap\rangle \\
&\quad + \gamma^2\|A^*(I - J_{r_n}^{B_2})Ax_n - A^*(I - J_{r_n}^{B_2})Ap\|^2 - \|z_n - J_{r_n}^{B_1}z_n\|^2 \\
&\le \|x_n - p\|^2 + \gamma\Big(\gamma - \frac{2}{L}\Big)\|A^*(I - J_{r_n}^{B_2})Ax_n\|^2 - \|z_n - J_{r_n}^{B_1}z_n\|^2,
\end{aligned} \tag{3.22}$$

where $z_n = (I - \gamma A^*(I - J_{r_n}^{B_2})A)x_n$. This together with (3.16) shows that

$$\begin{aligned}
\|y_n - p\|^2 &= \Big\|\alpha_n x_n + (1 - \alpha_n)\Big(\frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds\Big) - p\Big\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)k_n^2||u_n - p||^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)k_n^2\Big\{\|x_n - p\|^2 \\
&\quad + \gamma\Big(\gamma - \frac{2}{L}\Big)\|A^*(I - J_{r_n}^{B_2})Ax_n\|^2 - \|z_n - J_{r_n}^{B_1}z_n\|^2\Big\}.
\end{aligned}$$

After simplifying and using the condition $0 < a \le \alpha_n < c < 1$, we have

$$\begin{aligned}
&(1 - c)k_n^2\Big[\gamma\Big(\frac{2}{L} - \gamma\Big)\|A^*(I - J_{r_n}^{B_2})Ax_n\|^2 + \|z_n - J_{r_n}^{B_1}z_n\|^2\Big] \\
&\le (1 - \alpha_n)k_n^2\Big[\gamma\Big(\frac{2}{L} - \gamma\Big)\|A^*(I - J_{r_n}^{B_2})Ax_n\|^2 + \|z_n - J_{r_n}^{B_1}z_n\|^2\Big] \\
&\le (\alpha_n + (1 - \alpha_n)k_n^2)\|x_n - p\|^2 - \|y_n - p\|^2 \\
&= \alpha_n\|x_n - p\|^2 - \|y_n - p\|^2 + (1 - \alpha_n)k_n^2\|x_n - p\|^2 \\
&\le (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\| + (1 - \alpha_n)(k_n^2 - 1)\|x_n - p\|^2.
\end{aligned} \tag{3.23}$$

This together with (3.21) shows that

$$\lim_{n\to\infty}\|A^*(I - J_{r_n}^{B_2})Ax_n\| = 0, \quad \lim_{n\to\infty}\|z_n - J_{r_n}^{B_1}z_n\| = 0. \tag{3.24}$$

By the assumption that A^* is a strongly positive linear bounded operator, we can get that

$$\lim_{n\to\infty}\|(I - J_{r_n}^{B_2})Ax_n\| = 0. \tag{3.25}$$

Therefore, it follows from (3.14) and (3.24) that

$$\begin{aligned}\|u_n - x_n\| &= \|J_{r_n}^{B_1} z_n - x_n\| \\ &\le \|J_{r_n}^{B_1} z_n - z_n\| + \|z_n - x_n\| \\ &= \|J_{r_n}^{B_1} z_n - z_n\| + \|(I - \gamma A^*(I - J_{r_n}^{B_2})A)x_n - x_n\| \\ &= \|J_{r_n}^{B_1} z_n - z_n\| + \gamma\|A^*(I - J_{r_n}^{B_2})Ax_n\| \to 0\end{aligned} \tag{3.26}$$

as $n \to \infty$. Now, we prove that

$$\left\|\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds - x_n\right\| \to 0$$

as $n \to \infty$. Indeed, it follows from (3.14) that

$$\begin{aligned}\|y_n - x_n\| &= \left\|\alpha_n x_n + (1-\alpha_n)(\frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds) - x_n\right\| \\ &= (1-\alpha_n)\|\frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds - x_n\|.\end{aligned}$$

Hence, from (3.21), it follows that

$$\left\|\frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds - x_n\right\| = \frac{1}{1-\alpha_n}\|y_n - x_n\| \to 0 \tag{3.27}$$

as $n \to \infty$. This together with (3.26) shows that

$$\begin{aligned}&\left\|\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds - x_n\right\| \\ &\le \left\|\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds - \frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds\right\| + \left\|\frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds - x_n\right\| \\ &\le \frac{1}{s_n}\int_0^{s_n} ||T^n(s)x_n - T^n(s)u_n||ds + \left\|\frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds - x_n\right\| \\ &\le \frac{1}{s_n}k_n\int_0^{s_n} ||x_n - u_n||ds + \left\|\frac{1}{s_n}\int_0^{s_n} T^n(s)u_n ds - x_n\right\| \to 0\end{aligned} \tag{3.28}$$

as $n \to \infty$. By the condition (b) and (3.28), for any $h > 0$, we have

$$
\begin{aligned}
&\limsup_{n\to\infty} ||x_n - T(h)x_n|| \\
&\le \limsup_{n\to\infty} \left\| x_n - \frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds \right\| \\
&\quad + \limsup_{n\to\infty} \left\| \frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds - T(h)(\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds) \right\| \\
&\quad + \limsup_{n\to\infty} \left\| T(h)(\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds) - T(h)x_n \right\| \\
&\le \limsup_{n\to\infty}(1+k_1) \left\| x_n - \frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds \right\| \\
&\quad + \limsup_{n\to\infty} \left\| \frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds - T(h)\Big(\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds\Big) \right\| \\
&= 0.
\end{aligned}
\tag{3.29}
$$

This implies that, for each $h \ge 0$,

$$\lim_{n\to\infty} \|T(h)x_n - x_n\| = 0.$$

Thus the conclusion (3.20) is proved.

Step 5. Finally, we prove that the limit x^* in (3.18) is a solution of the problem SVIP (3.4) and it is also a fixed point of the asymptotically nonexpansive semigroup $\mathfrak{T} = \{T(s) : 0 \le s < \infty\}$, i.e., $x^* \in \mathrm{Fix}(\mathfrak{T})\bigcap \Omega$. In fact, since $x_n \to x^*$ and $||x_n - T(h)x_n|| \to 0$ for each $h \ge 0$, it follows from Lemma 4 that $x^* \in Fix(T(h))$ for each $h \ge 0$, i.e., $x^* \in \mathrm{Fix}(\mathfrak{T})$.

Now, we show $x^* \in \Omega$. In fact, by (3.14), we have $u_n = J_{r_n}^{B_1}(I - \gamma A^*(I - J_{r_n}^{B_2})A)x_n$ and so

$$(x_n - \gamma A^*(I - J_{r_n}^{B_2})A)x_n \in (I + r_n B_1)(u_n). \tag{3.30}$$

Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that $u_{n_k} \rightharpoonup w \in H_1$. Since $||x_n - u_n|| \to 0$ and $x_n \to x^*$, this implies that $x^* = w$. Simplifying (3.30), we have

$$\frac{1}{r_{n_k}}(x_{n_k} - u_{n_k} - \gamma A^*(I - J_{r_{n_k}}^{B_2})A)x_{n_k} \in B_1(u_{n_k}). \tag{3.31}$$

By passing to limit $k \to \infty$ in (3.31) and by taking into account (3.24), (3.26) and the fact that the graph of a maximal monotone operator is weakly-strongly closed, we obtain $0 \in B_1(x^*)$, i.e., $x^* \in Fix(J_\lambda^{B_1})$. Furthermore, since $\{x_n\}$ and $\{u_n\}$ have the same asymptotical behavior, $\{Ax_{n_k}\}$ weakly converges to Ax^*. Again, by (3.25), Lemma 4 and the fact that the resolvent $J_\lambda^{B_2}$ is nonexpansive, we obtain that $0 \in B_2(Ax^*)$, i.e., $Ax^* \in Fix(J_\lambda^{B_2})$. Thus $x^* \in \mathrm{Fix}(\mathfrak{T})\bigcap \Omega$, i.e., x^* is not only a solution of the problem SVIP (3.4) but also a fixed point of the asymptotically nonexpansive semigroup $\mathfrak{T} = \{T(s) : 0 \leq s < \infty\}$. This completes the proof.

Example Next, we give an example of asymptotically nonexpansive semigroup which satisfies the condition (b) in Theorem 3.1 (see [14]).

Let H be a real Hilbert space and $L(H)$ be the space of all bounded linear operators on H. For any $\psi \in L(H)$, define $\mathfrak{T} = \{T(s) : 0 \leq s < \infty\}$ of bounded linear operators by using the following exponential expression:

$$T(t) = e^{-t\psi} := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k \psi^k.$$

Then the family $\mathfrak{T} = \{T(s) : 0 \leq s < \infty\}$ satisfies the asymptotically nonexpansive semigroup properties. Moreover, this family forms a one-parameter semigroup of self-mappings of H satisfying the condition (b) in Theorem 3.1.

Now, we consider the cases of nonexpansive semigroup. First, we give the following lemma:

Lemma 6 ([15]) *Let C be a nonempty bounded closed and convex subset of a real Hilbert H and let $\mathfrak{T} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C. Then, for any $h \geq 0$,*

$$\lim_{t \to \infty} \sup_{x \in C} \left\| \frac{1}{t} \int_0^t T(t)x ds - T(h) \left(\frac{1}{t} \int_0^t T(t)x ds \right) \right\| = 0. \tag{3.32}$$

By using Lemma 6, we can obtain the following result:

Theorem 3.2 *Let $H_1,\ H_2,\ A,\ A^*, B_1,\ B_2,\ J_\lambda^{B_1},\ J_\lambda^{B_2}$ be the same as in Lemma 5. Let $\mathfrak{T}_1 = \{T(s) : 0 \leq s < \infty\}$ be an nonexpansive semigroup of mappings from H_1 to itself. Denote by $\Gamma_1 := Fix(\mathfrak{T}_1) \bigcap \Omega$, where Ω is the solution set of the problem (3.4) defined by (3.6). For any initial point $x_0 \in H_1$, $C_1 = H_1$, $x_1 = P_{C_1}x_0$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} u_n = J_{r_n}^{B_1}(I - \gamma A^*(I - J_{r_n}^{B_2})A)x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T(s)u_n ds, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \geq 1, \end{cases} \tag{3.33}$$

where $\{s_n\}$ is a sequence of positive real numbers with $s_n \to \infty$, $0 < a \leq \alpha_n < c < 1$ for all $n \geq 1$, $0 < b \leq r_n < +\infty$ and $\gamma \in (0, \frac{2}{L})$, where L is the spectral radius of the operator A^*A. If $\Gamma_1 \neq \emptyset$, then the sequence $\{x_n\}$ generated by (3.33) strongly converges to a point $x^* \in Fix(\mathfrak{T}_1) \bigcap \Omega$.

Proof In fact, since $\mathfrak{T}_1 = \{T(s) : 0 \le s < \infty\}$ is a nonexpansive semigroup, we have $k_n = 1$. Hence we have

$$\theta_n = (1 - \alpha_n)(k_n^2 - 1) \sup\{\|x_n - u\|^2 : u \in \Gamma_1\} = 0.$$

The condition "Γ_1 being bounded" is not used. On the other hand, it follows from Lemma 6 that

$$\limsup_{n\to\infty} \left\| \frac{1}{s_n} \int_0^{s_n} T(s)x_n ds - T(h)(\frac{1}{s_n} \int_0^{s_n} T(s)x_n ds) \right\| = 0, \quad \forall h \ge 0.$$

By the same way, as given in the proof of Theorem 3.1, we can prove that the conclusion of Theorem 3.2 is true. This completes the proof.

3.4 Applications

3.4.1 Applications to Split Optimization Problems

Let $H_1,\ H_2$ be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded and linear operator. The "so-called" *split optimization problem* (SOP) with respect to the functions $f : H_1 \to \mathbb{R}$ and $g : H_2 \to \mathbb{R}$ is as follows ([16, 17]):

Find points $x^* \in H_1$ and $Ax^* \in H_2$ such that

$$f(x^*) \ge f(x) \ \text{ for all } \ x \in H_1 \ \text{ and } \ g(Ax^*) \ge g(y) \ \text{ for all } \ y \in H_2. \tag{3.34}$$

We denote by Ω_1 the set of solutions of the split optimization problem (3.34).

Let $f : H_1 \to \mathbb{R}$ and $g : H_2 \to R$ be two proper convex and lower semi-continuous functions. Denote by $B_1 = \partial f$ and $B_2 = \partial g$, where ∂f and ∂g are sub-differentials of f and g, respectively. Then $\partial f : H_1 \to H_1$ and $\partial g : H_2 \to H_2$ both are maximal monotone mappings. Denoting by $J_\lambda^{\partial f}$ and $J_\lambda^{\partial g}$ the resolvents associated with ∂f and ∂g defined by (3.3), respectively, then the problem (SOP) (3.34) is equivalent to the following *split variational inclusion problem*:

Find points $x^* \in H_1$ and $y^* = Ax^* \in H_2$ such that

$$0 \in \partial f(x^*) \ \text{ and } \ 0 \in \partial g(Ax^*). \tag{3.35}$$

Therefore, by Theorem 3.1, we have the following:

Theorem 3.3 *Let $H_1,\ H_2,\ A,\ A^*,\ f,\ g, \partial f,\ \partial g,\ J_\lambda^{\partial f},\ J_\lambda^{\partial g}$ be the same as above. Let $\mathfrak{T} = \{T(s) : 0 \le s < \infty\}$ be an asymptotically nonexpansive semigroup of mappings from H_1 to itself with the sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Denote by $\Gamma_2 :=Fix(\mathfrak{T})\bigcap \Omega_2$, where Ω_2 is the solution set of problem (3.35). For any initial* *point $x_0 \in H_1$, $C_1 = H_1$, $x_1 = P_{C_1}x_0$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} u_n = J_{r_n}^{\partial f}(I - \gamma A^*(I - J_{r_n}^{\partial g})A)x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T^n(s)u_n ds, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \ge 1, \end{cases} \tag{3.36}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)\sup\{\|x_n - u\|^2 : u \in \Gamma_2\}$, $\{s_n\}$ is a sequence of positive numbers, $0 < a \le \alpha_n < c < 1$ for all $n \ge 1$, $0 < b \le r_n < +\infty$ and $\gamma \in (0, \frac{2}{L})$, where L is the spectral radius of the operator A^*A. If the following conditions are satisfied:

(a) *$\Gamma_2 \neq \emptyset$ and is bounded;*
(b) $\limsup_{n\to\infty} ||\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds - T(h)(\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds)|| = 0$ *for each $h >$ 0,*

then the sequence $\{x_n\}$ generated by (3.36) strongly converges to a point $x^* \in Fix(\mathfrak{T})\bigcap \Omega_2$.

Theorem 3.4 *Let H_1, H_2, A, A^*, f, g, ∂f, ∂g, $J_\lambda^{\partial f}$, $J_\lambda^{\partial g}$ be the same as in Theorem 3.3. Let $\mathfrak{T}_3 = \{T(s) : 0 \le s < \infty\}$ be an nonexpansive semigroup of mappings from H_1 to itself. Denote by $\Gamma_3 := Fix(\mathfrak{T}_3)\bigcap \Omega_3$, where Ω_3 is the solution set of the problem (3.35). For an initial point $x_0 \in H_1$, $C_1 = H_1$, $x_1 = P_{C_1}x_0$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} u_n = J_{r_n}^{\partial f}(I - \gamma A^*(I - J_{r_n}^{\partial g})A)x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T(s)u_n ds, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \le \|x_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \ge 1, \end{cases} \tag{3.37}$$

where $\{s_n\}$ is a sequence of positive real numbers with $s_n \to \infty$, $0 < a \le \alpha_n < c < 1$ for all $n \ge 1$, $0 < b \le r_n < +\infty$ and $\gamma \in (0, \frac{2}{L})$, where L is the spectral radius of the operator A^*A. If $\Gamma_3 \neq \emptyset$, then the sequence $\{x_n\}$ generated by (3.37) strongly converges to a point $x^* \in Fix(\mathfrak{T}_3)\bigcap \Omega_3$.

3.4.2 Applications to Split Variational Inequality Problems

In [2], Censor et al. proposed the following *split variational inequality problem* (SVIP):

Find a point $x^* \in C$ and $y^* = Ax^* \in Q$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0 \;\; \text{for all} \;\; x \in C \;\; \text{and} \;\; \langle g(y^*), y - y^* \rangle \geq 0 \;\; \text{for all} \;\; y \in Q, \tag{3.38}$$

where $A : C \to Q$ is a bounded linear operator, $f : C \to C$ and $g : Q \to Q$ are α-inverse strongly monotone mappings, where α is a positive constant.

The solution set of split variational inequality problem (3.38) is denoted by Ω_4.

It is obvious that the problem SVIP (3.38) is equivalent to the following *split fixed point problem*:

Find a point $x^* \in C$ and $y^* = Ax^* \in Q$ such that

$$x^* \in Fix(P_C(I - \lambda f)) \;\; \text{and} \;\; Ax^* \in Fix(P_Q(I - \lambda g) \;\; \text{for all} \;\; \lambda \in (0, 2\alpha). \tag{3.39}$$

Next, we prove that $P_C(I - \lambda f)$ and $P_Q(I - \lambda g)$ for all $\lambda \in (0, 2\alpha)$ both are firmly nonexpansive. In fact, since P_C is firmly nonexpansive, by (3.2), we have

$$\begin{aligned} &||P_C(I - \lambda f)x - P_C(I - \lambda f)y||^2 \\ &\qquad \leq ||(I - \lambda f)x - (I - \lambda f)y||^2 \\ &\qquad\quad - ||(I - P_C(I - \lambda f))x - (I - P_C(I - \lambda f))y||^2. \end{aligned} \tag{3.40}$$

Also, since

$$\begin{aligned} &||(I - \lambda f)x - (I - \lambda f)y||^2 \\ &= ||x - y||^2 + \lambda^2 ||fx - fy||^2 - 2\lambda \langle x - y, fx - fy \rangle \\ &\leq ||x - y||^2 + \lambda^2 ||fx - fy||^2 - 2\lambda\alpha ||fx - fy||^2 \\ &= ||x - y||^2 + \lambda(\lambda - 2\alpha) ||fx - fy||^2 \\ &\leq ||x - y||^2 \; (since \; \lambda \in (0, 2\alpha)), \end{aligned} \tag{3.41}$$

substituting (3.41) into (3.40), we have

$$\begin{aligned} &||P_C(I - \lambda f)x - P_C(I - \lambda f)y||^2 \\ &\leq ||x - y||^2 - ||(I - P_C(I - \lambda f))x - (I - P_C(I - \lambda f))y||^2. \end{aligned} \tag{3.42}$$

This shows that $P_C(I - \lambda f), \; \lambda \in (0, 2\alpha)$ is firmly nonexpansive.

Similarly, we can also prove that $P_Q(I - \lambda g)$ for all $\lambda \in (0, 2\alpha)$ is firmly nonexpansive.

These show that the mappings $P_C(I - \lambda f)$ and $P_Q(I - \lambda g)$ in the split variational inequality problem (3.39) have the similar properties as the mappings $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ in the split variational inclusion problem (3.5). Consequently, by Theorem 3.1, we have the following result:

Theorem 3.5 *Let $H_1,\ H_2,\ A,\ A^*, f,\ g$, be the same as above. Let $\mathfrak{T} = \{T(s) : 0 \le s < \infty\}$ be an asymptotically nonexpansive semigroup of mappings from H_1 to itself with the sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Denote by Γ_4: =Fix($\mathfrak{T}$)$\bigcap \Omega_4$, where Ω_4 is the solution set of the split variational inequality problem (3.39). For any initial point $x_0 \in H_1$, $C_1 = H_1$, $x_1 = P_{C_1}x_0$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} u_n = P_C(I - \lambda_n f)(I - \gamma A^*(I - P_Q(I - \lambda_n g))A)x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T^n(s)u_n ds, \\ C_{n+1} = \{z \in C_n :\ \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \ge 1, \end{cases} \tag{3.43}$$

where $\theta_n = (1 - \alpha_n)(k_n^2 - 1)\sup\{\|x_n - u\|^2 : u \in \Gamma_4\}$, $\{s_n\}$ is a sequence of positive numbers, $0 < a \le \alpha_n < c < 1$ for all $n \ge 1$, $\lambda_n \in (0, 2\alpha)$ and $\gamma \in (0, \frac{2}{L})$, where L is the spectral radius of the operator A^*A. If the following conditions are satisfied:

(a) *$\Gamma_4 \ne \emptyset$ and is bounded;*
(b) $\limsup_{n\to\infty} ||\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds - T(h)(\frac{1}{s_n}\int_0^{s_n} T^n(s)x_n ds)|| = 0$ *for each $h > 0$,*

then the sequence $\{x_n\}$ generated by (3.43) strongly converges to a point $x^* \in$ Fix($\mathfrak{T}$)$\bigcap \Omega_4$.

Especially, if $\mathfrak{T} = \{T(s) : 0 \le s < \infty\} : H_1 \to H_1$ is a nonexpansive semigroup, then we have the following:

Theorem 3.6 *Let $H_1,\ H_2,\ A,\ A^*, f,\ g$, be the same as in Theorem 3.5 Let $\mathfrak{T} = \{T(s) : 0 \le s < \infty\}$ be a nonexpansive semigroup of mappings from H_1 to itself. Denote by Γ_5: =Fix($\mathfrak{T}$)$\bigcap \Omega_5$, where Ω_5 is the solution set of the split variational inequality problem (3.39). For any initial point $x_0 \in H_1$, $C_1 = H_1$, $x_1 = P_{C_1}x_0$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} u_n = P_C(I - \lambda_n f)(I - \gamma A^*(I - P_Q(I - \lambda_n g))A)x_n, \\ y_n = \alpha_n x_n + (1 - \alpha_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T(s)u_n ds, \\ C_{n+1} = \{z \in C_n :\ \|y_n - z\|^2 \le \|x_n - z\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \ge 1, \end{cases} \tag{3.44}$$

where $\{s_n\}$ is a sequence of positive numbers with $s_n \to \infty$, $0 < a \le \alpha_n < c < 1$ for all $n \ge 1$, $\lambda_n \in (0, 2\alpha)$ and $\gamma \in (0, \frac{2}{L})$, where L is the spectral radius of the operator A^*A. If $\Gamma_5 \ne \emptyset$, then the sequence $\{x_n\}$ generated by (3.44) strongly converges to a point $x^* \in$ Fix($\mathfrak{T}$)$\bigcap \Omega_5$.

3.5 Conclusions

In this paper, by using the shrinking projection method, an iterative process to approximate a common solution of the split variational inclusion problem and the fixed point problem for asymptotically nonexpansive semigroup in real Hilbert spaces was constructed. We proved that the sequences generated by the proposed iterative process converge strongly to a common solution of the problems for an asymptotically nonexpansive semigroup. Finally, some applications were presented to study the split optimization problem and the split variational inequality problem.

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Chapter 4
Convergence Theorems and Convergence Rates for the General Inertial Krasnosel'skiĭ–Mann Algorithm

Qiao-Li Dong, Shang-Hong Ke, Yeol Je Cho, and Themistocles M. Rassias

Abstract The authors [13] introduced a general inertial Krasnosel'skiĭ–Mann algorithm:

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = x_n + \beta_n(x_n - x_{n-1}), \\ x_{n+1} = (1-\lambda_n)y_n + \lambda_n T(z_n) \end{cases}$$

for each $n \geq 1$ and showed its convergence with the control conditions $\alpha_n, \beta_n \in [0, 1)$. In this paper, we present the convergence analysis of the general inertial Krasnosel'skiĭ–Mann algorithm with the control conditions $\alpha_n \in [0, 1]$, $\beta_n \in (-\infty, 0]$ and $\alpha_n \in [-1, 0]$, $\beta_n \in [0, +\infty)$, respectively. Also, we provide the convergence rate for the general inertial Krasnosel'skiĭ–Mann algorithm under mild conditions on the inertial parameters and some conditions on the relaxation parameters, respectively. Finally, we show that a numerical experiment provided compares the choice of inertial parameters.

Keywords Nonexpansive · Fixed point · Inertial Krasnoselski-Mann algorithm

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4.1 Introduction

Let $\mathscr{H}$ be a Hilbert space and C be a nonempty closed convex subset of $\mathscr{H}$. A mapping $T : C \to C$ is called *nonexpansive* if, for all $x, y \in C$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

Further, let $Fix(T) = \{x \in C : Tx = x\}$ denote the set of all fixed points of T in C.

In this paper, we consider the following *fixed-point problem*:

Problem 1 Suppose that $T : C \to C$ is a nonexpansive mapping with $Fix(T) \neq \emptyset$. Find a point $x^* \in C$ such that

$$T(x^*) = x^*.$$

The fixed-point problems for nonexpansive mappings have a variety of specific applications since many problems can be seen as a fixed point problem of nonexpansive mappings such as convex feasibility problems, monotone variational inequalities (see [4, 5, 31] and references therein).

Recently, Chen et al. [11, 12] showed the convergence of the primal-dual fixed-point algorithms with aid of the fixed-point theory of the nonexpansive mappings. A great deal of literature on the iteration methods for fixed-point problems of nonexpansive mappings has been published (see, for example, [17, 19, 21, 27, 29, 30, 32, 35, 38]).

One of the most used algorithms is the *Krasnosel'skiĭ–Mann algorithm* [22, 25] as follows:

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n)Tx_n \tag{4.1}$$

for each $n \geq 0$. The iterative sequence $\{x_n\}$ converges weakly to a fixed point of T provided that $\{\lambda_n\} \subset [0, 1]$ satisfies

$$\sum_{n=1}^{\infty} \lambda_n(1 - \lambda_n) = +\infty.$$

Some methods for the structured monotone inclusion problems can be casted as the Krasnosel'skiĭ–Mann algorithm, such as the forward–backward method, the Douglas–Rachford method, and the primal-dual method [4, 23].

In general, the convergence rate of the Krasnosel'skiĭ–Mann algorithm is very slow, especially for large-scale problems. To accelerate the convergence of the Krasnosel'skiĭ–Mann algorithm, Iutzeler and Hendrickx [20] recently focused on the two main modification schemes: *relaxation* and *inertia*. He et al. [18] presented the optimal choice of the relaxation parameter λ_n for the Krasnosel'skiĭ–Mann algorithm.

In 1964 and 1987, Polyak [33, 34] first introduced the *inertial extrapolation algorithms* as an acceleration process. The inertial extrapolation algorithm is a two-step iterative method and its main feature is that the next iterate is defined by making use of the previous two iterates.

In 2000, Alvarez [1] proposed an *inertial proximal algorithm* for convex minimization and, in 2001, Attouch and Alvarez [3] extended its maximal monotone operators. In 2003, Moudafi and Oliny [28] introduced the *forward–backward inertial procedure* for solving the problem of finding a zero of the sum of two maximal monotone operators. They also proposed an open question:

"How to investigate, theoretically as well as numerically, which are the best choices for the inertial parameter in order to accelerate the convergence?"

Since the open problem was proposed, there has been a little progress except for some special problems. In 2009, Beck and Teboulle [6] introduced the well-known *fast iterative shrinkage-thresholding algorithm* (shortly, FISTA) to solve the linear inverse problems, which is an inertial version of the *iterative shrinkage-thresholding algorithm* (shortly, ISTA). They proved that the FISTA has the global rate $O(\frac{1}{n^2})$ of the convergence, where n is the iteration number, while the global rate of the convergence of the ISTA is $O(\frac{1}{n})$.

The *inertial parameter* α_k in the FISTA is chosen as follows:

$$\alpha_n = \frac{t_n - 1}{t_{n+1}}$$

for each $n \geq 1$, where $t_1 = 1$, and

$$t_{n+1} = \frac{1 + \sqrt{1 + 4t_n^2}}{2}$$

for each $n \geq 1$. In 2015, Chambolle and Dossal [10] took t_n as follows:

$$t_n = \frac{n + a - 1}{a} \tag{4.2}$$

for each $n \geq 1$, where $a > 2$, and showed that the FISTA has the better property, i.e., the convergence of the iterative sequence when t_n is taken as in (4.2).

The work of Beck and Teboulle [6] revives the study of some inertial-type algorithms. Recently, some researchers constructed many iterative algorithms by using the inertial extrapolation, such as the inertial forward–backward algorithm [3, 8, 24], the inertial extragradient methods [7, 14, 15], and the inertial forward–backward–forward primal-dual splitting method [9].

Recently, in 2017, Stathopoulos and Jones [36] introduced an inertial parallel and asynchronous fixed-point iteration by bringing together the inertial acceleration techniques with asynchronous implementations of a rather wide family of operator splitting schemes.

By using the technique of the inertial extrapolation, 2008, Mainge [26] introduced the classical *inertial Krasnosel'skiĭ–Mann algorithm*:

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ x_{n+1} = (1 - \lambda_n)y_n + \lambda_n T(y_n) \end{cases} \tag{4.3}$$

for each $n \geq 1$. He showed that the sequence $\{x_n\}$ converges weakly to a fixed point of T under the following conditions:

(B1) $\alpha_n \in [0, \alpha)$ for each $n \geq 1$, where $\alpha \in [0, 1)$;
(B2) $\sum_{n=1}^{\infty} \alpha_n \|x_n - x_{n-1}\|^2 < +\infty$;
(B3) $\inf_{n\geq 1} \lambda_n > 0$ and $\sup_{n\geq 1} \lambda_n < 1$.

For satisfying the summability condition (B2) of the sequence $\{x_n\}$, one need to calculate α_n at each step (see [28]).

In 2015, Bot and Csetnek [8] got rid of the condition (B2) and substituted (B1) and (B3) with the following conditions, respectively:

(C1) for each $n \geq 1$, $\{\alpha_n\} \subset [0, \alpha]$ is nondecreasing with $\alpha_1 = 0$ and $0 \leq \alpha < 1$;
(C2) for each $n \geq 1$,

$$\delta > \frac{\alpha^2(1+\alpha) + \alpha\sigma}{1-\alpha^2}, \quad 0 < \lambda \leq \lambda_n \leq \frac{\delta - \alpha[\alpha(1+\alpha) + \alpha\delta + \sigma]}{\delta[1 + \alpha(1+\alpha) + \alpha\delta + \sigma]},$$

where $\lambda, \sigma, \delta > 0$.

Letting $\lambda_n = 1$ in (4.3) and assuming that T is an averaged mapping, in 2018, Iutzeler and Hendrickx [20] presented the online inertial method and the online alternated inertia method, which automatically tune the acceleration coefficients $\{\alpha_n\}$ online.

Very recently, the authors [13] introduced a *general inertial Krasnosel'skiĭ–Mann algorithm* as follows:

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = x_n + \beta_n(x_n - x_{n-1}), \\ x_{n+1} = (1-\lambda_n)y_n + \lambda_n T(z_n) \end{cases} \tag{4.4}$$

for each $n \geq 1$, where $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, 1]$.

Remark 1 In fact, the general inertial Krasnosel'skiĭ–Mann algorithm is the most general Krasnosel'skiĭ–Mann algorithm with inertial effects. It is easy to show that the general inertial Krasnosel'skiĭ–Mann algorithm includes other algorithms as special cases. The relations between the algorithm (4.4) with other works are as follows:

(1) $\alpha_n = \beta_n$, i.e., $y_n = z_n$ for each $n \geq 1$: this is the classical inertial Krasnosel'skiĭ–Mann algorithm (4.3) in [26];
(2) $\beta_n = 0$ for each $n \geq 1$: this becomes the accelerated Krasnosel'skiĭ–Mann algorithm [16]:

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ x_{n+1} = \lambda_n y_n + (1-\lambda_n)Tx_n \end{cases}$$

for each $n \geq 1$;

(3) $\alpha_n = 0$ for each $n \geq 1$: it becomes the reflected Krasnosel'skiĭ–Mann algorithm

$$\begin{cases} z_n = x_n + \beta_n(x_n - x_{n-1}), \\ x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T(z_n) \end{cases}$$

for each $n \geq 1$.

Also, they presented the convergence of the general inertial Krasnosel'skiĭ–Mann algorithm (4.4).

Theorem 1 *Suppose that $T : \mathscr{H} \to \mathscr{H}$ is nonexpansive with $Fix(T) \neq \emptyset$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the conditions:*

(D1) $\{\alpha_n\} \subset [0, \alpha]$ *and* $\{\beta_n\} \subset [0, \beta]$ *are nondecreasing with* $\alpha_1 = \beta_1 = 0$ *and* $\alpha, \beta \in [0, 1)$*;*
(D2) *for any* $\lambda, \sigma, \delta > 0$*,*

$$\delta > \frac{\alpha\xi(1+\xi) + \alpha\sigma}{1 - \alpha^2}, \quad 0 < \lambda \leq \lambda_n \leq \frac{\delta - \alpha[\xi(1+\xi) + \alpha\delta + \sigma]}{\delta[1 + \xi(1+\xi) + \alpha\delta + \sigma]}, \tag{4.5}$$

where $\xi = \max\{\alpha, \beta\}$.

Then the sequence $\{x_n\}$ generated by the general inertial Krasnosel'skiĭ–Mann algorithm (4.4) converges weakly to a point of $Fix(T)$.

Note that, in Theorem 1, the inertial parameters α_n and β_n are nonnegative. In this paper, we relax the choices of α_n and β_n and give further results on the parameters α_n and β_n, which can be taken negative. We mainly consider two cases as follows:

Cases 1: $\alpha_n \in [0, 1]$ and $\beta_n \in (-\infty, 0]$ for each $n \geq 1$;
Cases 2: $\alpha_n \in [-1, 0]$ and $\beta_n \in [0, +\infty)$ for each $n \geq 1$.

Also, we provide the convergence rate for the inertial Krasnosel'skiĭ–Mann algorithm when T is a nonexpansive mapping and $I - T$ is a quasi-strongly monotone mapping. To our knowledge, we have not seen such convergence results in the literature.

The contents of the paper are as follows. In Sect. 4.2, we present some lemmas which will be used in the main results. In Sect. 4.3, we present the convergence of the general inertial Krasnosel'skiĭ–Mann algorithm with negative inertial parameters. In Sect. 4.4, the convergence rate is provided for the inertial Krasnosel'skiĭ–Mann algorithm. Finally, we give a numerical example to compare with different inertial parameters.

4.2 Preliminaries

We use the notations:

(1) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence;
(2) $\omega_w(x^k) = \{x : \exists\, x^{k_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x^k\}$.

The following identity will be used several times in the paper (see Corollary 2.15 of [4]):

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \tag{4.6}$$

for all $\alpha \in \mathbb{R}$ and $(x, y) \in \mathscr{H} \times \mathscr{H}$.

Definition 1 A mapping $T : \mathscr{H} \to \mathscr{H}$ is called an *averaged mapping* if it can be written as the average of the identity I and a nonexpansive mapping, that is,

$$T = (1-\alpha)I + \alpha S, \tag{4.7}$$

where α is a number in $]0, 1[$ and $S : \mathscr{H} \to \mathscr{H}$ is a nonexpansive mapping. More precisely, when (4.7) holds, we say that T is α*-averaged*.

It is obvious that the averaged mapping is nonexpansive.

Definition 2 A mapping $T : \mathscr{H} \to \mathscr{H}$ is called *quasi-μ-strongly monotone*, where $\mu > 0$, if

$$\langle x - y, Tx\rangle \ge \mu\|x-y\|^2$$

for all $x \in \mathscr{H}$ and $y \in zerT := \{y \in \mathscr{H} : Ty = 0\}$. When the inequality holds for $\mu = 0$, T is quasi-monotone.

Lemma 1 ([4, Proposition 4.33]) *The operator $T : \mathscr{H} \to \mathscr{H}$ is nonexpansive if and only if $S = I - T$ is $\frac{1}{2}$-cocoercive (also called $\frac{1}{2}$-inverse strongly monotone), i.e.,*

$$\langle x - y, Sx - Sy\rangle \ge \frac{1}{2}\|Sx - Sy\|^2$$

for all $x, y \in \mathscr{H}$.

Lemma 2 ([2]) *Let $\{\psi_n\}$, $\{\delta_n\}$, and $\{\alpha_n\}$ be the sequences in $[0, +\infty)$ such that*

$$\psi_{n+1} \le \psi_n + \alpha_n(\psi_n - \psi_{n-1}) + \delta_n$$

for each $n \ge 1$, $\sum_{n=1}^{\infty} \delta_n < +\infty$ and there exists a real number α with $0 \le \alpha_n \le \alpha < 1$ for all $n \in \mathbb{N}$. Then the following hold:

(1) $\sum_{n\ge1}[\psi_n - \psi_{n-1}]_+ < +\infty$, *where* $[t]_+ = \max\{t, 0\}$.
(2) *There exists $\psi^* \in [0, +\infty)$ such that* $\lim_{n\to+\infty} \psi_n = \psi^*$.

Lemma 3 ([4]) *Let D be a nonempty closed convex subset of $\mathscr{H}$ and $T : D \to \mathscr{H}$ be a nonexpansive mapping. Let $\{x_n\}$ be a sequence in D and $x \in \mathscr{H}$ such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$ as $n \to +\infty$. Then $x \in Fix(T)$.*

Lemma 4 ([4]) *Let C be a nonempty subset of $\mathscr{H}$ and $\{x_n\}$ be a sequence in $\mathscr{H}$ such that the following two conditions hold:*

(a) *for all $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists;*
(b) *every sequential weak cluster point of $\{x_n\}$ is in C.*

Then the sequence $\{x_n\}$ converges weakly to a point in C.

4.3 The General Inertial Krasnosel'skiĭ–Mann Algorithms with Negative Inertial Parameters

In this section, we present the weak convergence of the general inertial Krasnosel'skiĭ–Mann algorithms with negative inertial parameters.

4.3.1 $\alpha_n \in [0, 1]$ and $\beta_n \in (-\infty, 0]$

Now, we divide $\beta_n \in (-\infty, 0]$ into two cases: $\beta_n \in (-\infty, -1]$ and $\beta_n \in [-1, 0]$ for each $n \geq 1$. Firstly, we establish the convergence of the general inertial Krasnosel'skiĭ–Mann algorithms for $\beta_n \in (-\infty, -1]$ for each $n \geq 1$.

Theorem 2 *Suppose that $T : \mathscr{H} \to \mathscr{H}$ is a nonexpansive mapping with $Fix(T) \neq \emptyset$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$ satisfy the following conditions:*

(D3) $\alpha_1 = \beta_1 = 0$, *$\{\alpha_n\} \subset [\underline{\alpha}, \overline{\alpha}]$ and $\{\beta_n\} \subset [\underline{\beta}, \overline{\beta}]$ are nondecreasing, $\underline{\alpha}, \overline{\alpha} \in [0, 1)$ and $\underline{\beta}, \overline{\beta} \in (-\infty, -1]$;*
(D4) *for any $\underline{\lambda}, \overline{\lambda}, \sigma, \delta > 0$,*

$$\delta > \frac{\overline{\alpha}(\xi + \sigma)}{1 - \overline{\alpha}^2}, \quad 0 < \underline{\lambda} \leq \lambda_n \leq \overline{\lambda}, \tag{4.8}$$

where $\xi = \max\{\overline{\alpha}(1 + \overline{\alpha}), \underline{\beta}(1 + \underline{\beta})\}$ and

$$\overline{\lambda} = \min\left\{\frac{\underline{\alpha}}{\underline{\alpha} - \underline{\beta}}, \frac{\delta - \overline{\alpha}(\xi + \overline{\alpha}\delta + \sigma)}{\delta(\xi + \overline{\alpha}\delta + \sigma + 1)}\right\}. \tag{4.9}$$

Then the sequence $\{x_n\}$ generated by the general inertial Krasnosel'skiĭ–Mann algorithm (4.4) converges weakly to a point of $Fix(T)$.

The techniques of proof of Theorem 2 are similar to those in [13]; however, for completeness reasons, we supply an argument.

Proof Take arbitrarily $p \in Fix(T)$. From (4.6), it follows that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= (1-\lambda_n)\|y_n-p\|^2+\lambda_n\|Tz_n-p\|^2-\lambda_n(1-\lambda_n)\|Tz_n-y_n\|^2\\ &\le (1-\lambda_n)\|y_n-p\|^2+\lambda_n\|z_n-p\|^2-\lambda_n(1-\lambda_n)\|Tz_n-y_n\|^2.\end{aligned} \tag{4.10}$$

Using (4.6) again, we have

$$\begin{aligned}\|y_n-p\|^2 &= \|(1+\alpha_n)(x_n-p)-\alpha_n(x_{n-1}-p)\|^2\\ &= (1+\alpha_n)\|x_n-p\|^2-\alpha_n\|x_{n-1}-p\|^2+\alpha_n(1+\alpha_n)\|x_n-x_{n-1}\|^2.\end{aligned} \tag{4.11}$$

Similarly, we have

$$\|z_n-p\|^2 = (1+\beta_n)\|x_n-p\|^2-\beta_n\|x_{n-1}-p\|^2+\beta_n(1+\beta_n)\|x_n-x_{n-1}\|^2. \tag{4.12}$$

Combining (4.10), (4.11), and (4.12), we have

$$\begin{aligned}&\|x_{n+1}-p\|^2-(1+\theta_n)\|x_n-p\|^2+\theta_n\|x_{n-1}-p\|^2\\ &\qquad\le -\lambda_n(1-\lambda_n)\|Tz_n-y_n\|^2\\ &\qquad\quad+[(1-\lambda_n)\alpha_n(1+\alpha_n)+\lambda_n\beta_n(1+\beta_n)]\|x_n-x_{n-1}\|^2,\end{aligned} \tag{4.13}$$

where

$$\theta_n = \alpha_n(1-\lambda_n)+\beta_n\lambda_n. \tag{4.14}$$

From (4.9) and $\{\lambda_n\} \in [\underline{\lambda}, \overline{\lambda}]$, it follows that $\theta_n \subset [0, \theta] \subset [0, 1)$ is nondecreasing with $\theta_1 = 0$ and $\theta = \overline{\alpha}(1-\underline{\lambda})+\overline{\beta}\underline{\lambda}$. Using (4.4), we have

$$\begin{aligned}\|Tz_n-y_n\| &= \left\|\frac{1}{\lambda_n}(x_{n+1}-x_n)+\frac{\alpha_n}{\lambda_n}(x_{n-1}-x_n)\right\|^2\\ &= \frac{1}{{\lambda_n}^2}\|x_{n+1}-x_n\|^2+\frac{\alpha_n^2}{{\lambda_n}^2}\|x_{n-1}-x_n\|^2\\ &\quad+2\frac{\alpha_n}{{\lambda_n}^2}\langle x_{n+1}-x_n, x_{n-1}-x_n\rangle\\ &\ge \frac{1}{{\lambda_n}^2}\|x_{n+1}-x_n\|^2+\frac{\alpha_n^2}{{\lambda_n}^2}\|x_{n-1}-x_n\|^2\\ &\quad+\frac{\alpha_n}{{\lambda_n}^2}\Big(-\rho_n\|x_{n+1}-x_n\|^2-\frac{1}{\rho_n}\|x_{n-1}-x_n\|^2\Big),\end{aligned} \tag{4.15}$$

where we denote $\rho_n := \frac{1}{\alpha_n+\delta\lambda_n}$. From (4.13) and (4.15), we can derive the inequality

$$\begin{aligned}&\|x_{n+1}-p\|^2-(1+\theta_n)\|x_n-p\|^2+\theta_n\|x_{n-1}-p\|^2\\&\le \frac{(1-\lambda_n)(\alpha_n\rho_n-1)}{\lambda_n}\|x_{n+1}-x_n\|^2+\mu_n\|x_n-x_{n-1}\|^2,\end{aligned}\tag{4.16}$$

where

$$\mu_n=(1-\lambda_n)\alpha_n(1+\alpha_n)+\lambda_n\beta_n(1+\beta_n)+\alpha_n(1-\lambda_n)\frac{1-\rho_n\alpha_n}{\rho_n\lambda_n}.\tag{4.17}$$

From $\beta_n \le -1$, it follows $\beta_n(1+\beta_n)\ge 0$. Due to $\rho_n\alpha_n \le 1$ and $\lambda_n \in (0,1)$, we have $\mu_n \ge 0$. Again, taking into account the choice of ρ_n, we have

$$\delta=\frac{1-\rho_n\alpha_n}{\rho_n\lambda_n}\tag{4.18}$$

and, from (4.17), it follows that

$$\begin{aligned}\mu_n&=(1-\lambda_n)\alpha_n(1+\alpha_n)+\lambda_n\beta_n(1+\beta_n)+\alpha_n\delta\\&\le \xi+\overline{\alpha}\delta\end{aligned}\tag{4.19}$$

for each $n \ge 1$. In the following, we apply some techniques from [3, 8] adapted to our setting. Define the sequences $\{\phi_n\}$ and $\{\Psi_n\}$ by

$$\phi_n := \|x_n-p\|^2,\quad \Psi_n := \phi_n-\theta_n\phi_{n-1}+\mu_n\|x_n-x_{n-1}\|^2$$

for each $n \ge 1$. Using the monotonicity of $\{\theta_n\}$ and the fact that $\phi_n \ge 0$ for all $n \in \mathbb{N}$, we have

$$\Psi_{n+1}-\Psi_n \le \phi_{n+1}-(1+\theta_n)\phi_n+\theta_n\phi_{n-1}+\mu_{n+1}\|x_{n+1}-x_n\|^2-\mu_n\|x_n-x_{n-1}\|^2.$$

By (4.16), we know

$$\Psi_{n+1}-\Psi_n \le \left(\frac{(1-\lambda_n)(\alpha_n\rho_n-1)}{\lambda_n}+\mu_{n+1}\right)\|x_{n+1}-x_n\|^2.\tag{4.20}$$

Now, we claim that

$$\frac{(1-\lambda_n)(\alpha_n\rho_n-1)}{\lambda_n}+\mu_{n+1}\le -\sigma\tag{4.21}$$

for each $n \ge 1$. Indeed, by (4.17), we have

$$
\begin{aligned}
\frac{(1-\lambda_n)(\alpha_n\rho_n-1)}{\lambda_n} &+ \mu_{n+1} \le -\sigma \\
&\iff \lambda_n(\mu_{n+1}+\sigma)+(1-\lambda_n)(\alpha_n\rho_n-1)\le 0 \\
&\iff \lambda_n(\mu_{n+1}+\sigma)-\frac{\delta\lambda_n(1-\lambda_n)}{\alpha_n+\delta\lambda_n}\le 0 \\
&\iff (\alpha_n+\delta\lambda_n)(\mu_{n+1}+\sigma)+\delta\lambda_n\le\delta.
\end{aligned}
\tag{4.22}
$$

Employing (4.19), we have

$$
(\alpha_n+\delta\lambda_n)(\mu_{n+1}+\sigma)+\delta\lambda_n \le (\overline{\alpha}+\delta\lambda_n)(\xi+\overline{\alpha}\delta+\sigma)+\delta\lambda_n\le\delta,
$$

where the last inequality follows by using the upper bound for the sequence $\{\lambda_n\}$ in (4.8). Hence, the claim in (4.21) is true. It follows from (4.20) and (4.21) that

$$
\Psi_{n+1}-\Psi_n \le -\sigma\|x_{n+1}-x_n\|^2 \tag{4.23}
$$

for each $n \ge 1$. The sequence $(\Psi_n)_{n\ge 1}$ is nonincreasing and the boundness for the sequence $\{\theta_n\}$ delivers

$$
-\theta\phi_{n-1}\le\phi_n-\theta\phi_{n-1}\le\Psi_n\le\Psi_1 \tag{4.24}
$$

for each $n \ge 1$. Thus, we obtain

$$
\phi_n \le \theta^n\phi_0+\Psi_1\sum_{k=1}^{n-1}\theta^k \le \theta^n\phi_0+\frac{\Psi_1}{1-\theta} \tag{4.25}
$$

for each $n \ge 1$, where we notice that $\Psi_1=\phi_1\ge 0$ (due to the relation $\theta_1=\alpha_1=\beta_1=0$). Using (4.23)–(4.25), it follows that, for all $n\ge 1$,

$$
\sigma\sum_{k=1}^{n}\|x_{k+1}-x_k\|^2 \le \Psi_1-\Psi_{n+1}\le\Psi_1+\theta\phi_n\le\theta^{n+1}\phi_0+\frac{\Psi_1}{1-\theta},
$$

which means that

$$
\sum_{n=1}^{\infty}\|x_{n+1}-x_n\|^2<+\infty. \tag{4.26}
$$

Thus, we have

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{4.27}
$$

From (4.4), we have

$$
\begin{aligned}
\|y_n-x_{n+1}\| &\le \|x_n-x_{n+1}\|+\alpha_n\|x_n-x_{n-1}\| \\
&\le \|x_n-x_{n+1}\|+\overline{\alpha}\|x_n-x_{n-1}\|,
\end{aligned}
$$

which with (4.27) implies that

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{4.28}$$

Similarly, we obtain

$$\lim_{n\to\infty} \|z_n - x_{n+1}\| = 0. \tag{4.29}$$

For arbitrary $p \in Fix(T)$, by (4.16), (4.19), (4.26) and Lemma 2, we derive that $\lim_{n\to\infty} \|x_n - p\|$ exists (we take into consideration also $\lambda_n \in (0, 1)$ in (4.16)).

On the other hand, let x be a sequential weak cluster point of $\{x_n\}$, that is, there exists a subsequence $\{x_{n_k}\}$ which converge weakly to x. By (4.29), it follows that $z_{n_k} \rightharpoonup x$ as $k \to \infty$. Furthermore, from (4.4), we have

$$\begin{aligned}\|Tz_n - z_n\| &\le \|Tz_n - y_n\| + \|y_n - z_n\| \\ &\le \frac{1}{\lambda_n}\|x_{n+1} - y_n\| + \|y_n - x_{n+1}\| + \|z_n - x_{n+1}\| \\ &\le \left(1 + \frac{1}{\underline{\lambda}}\right)\|x_{n+1} - y_n\| + \|z_n - x_{n+1}\|.\end{aligned}$$

Thus, by (4.28) and (4.29), we obtain $\|Tz_{n_k} - z_{n_k}\| \to 0$ as $k \to \infty$. Applying now Lemma 3 for the sequence $\{z_{n_k}\}$, we conclude that $x \in Fix(T)$. Therefore, from Lemma 4, it follows that $\{x_n\}$ converges weakly to a point in $Fix(T)$. This completes the proof. $\square$

Remark 2 From (4.9), we get $\overline{\lambda} \le \frac{\underline{\alpha}}{\underline{\alpha}-\underline{\beta}}$. Since $\underline{\alpha} \in [0, 1]$ and $\underline{\beta} \in (-\infty, -1]$, we have $\overline{\lambda} \le 0.5$.

Next, we analyze the convergence of the general inertial Krasnosel'skiĭ–Mann algorithm (4.4) for $\beta_n \in [-1, 0]$ for each $n \ge 1$.

Theorem 3 *Suppose that $T : \mathscr{H} \to \mathscr{H}$ is a nonexpansive mapping with $Fix(T) \neq \emptyset$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$ satisfy the following conditions:*

(D5) $\alpha_1 = \beta_1 = 0$, $\{\alpha_n\} \subset [\underline{\alpha}, overline\alpha]$, $\{\beta_n\} \subset [\underline{\beta}, \overline{\beta}]$ *are nondecreasing,* $\underline{\alpha}, \overline{\alpha} \in (0, 1)$ *and* $\underline{\beta}, \overline{\beta} \in [-1, 0)$*;*

(D6) *for any* $\underline{\lambda}, \overline{\lambda}, \sigma, \delta > 0$,

$$\delta > \frac{\overline{\alpha}[\overline{\alpha}(1+\overline{\alpha})+\sigma]}{1-\overline{\alpha}^2}, \quad 0 < \underline{\lambda} \le \lambda_n \le \overline{\lambda}, \tag{4.30}$$

where

$$\overline{\lambda} = \min\left\{\frac{\underline{\alpha}}{\underline{\alpha}-\underline{\beta}}, \frac{\underline{\alpha}(1+\underline{\alpha}+\delta)}{\underline{\alpha}(1+\underline{\alpha}+\delta)-\eta}, \frac{\delta-\overline{\alpha}[\overline{\alpha}(1+\overline{\alpha}+\delta)+\sigma]}{\delta[\overline{\alpha}(1+\overline{\alpha}+\delta)+\sigma+1]}\right\} \tag{4.31}$$

and

$$\eta = \min_{\beta\in[\underline{\beta},\overline{\beta}]}\{\beta(1+\beta)\}.$$

Then the sequence $\{x_n\}$ generated by the general inertial Krasnosel'skiĭ–Mann algorithm (4.4) converges weakly to a point of $Fix(T)$.

Proof Following the proof line of Theorem 2, we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2&-(1+\theta_n)\|x_n-p\|^2+\theta_n\|x_{n-1}-p\|^2\\ &\le \frac{(1-\lambda_n)(\alpha_n\rho_n-1)}{\lambda_n}\|x_{n+1}-x_n\|^2+\mu_n\|x_n-x_{n-1}\|^2,\end{aligned} \tag{4.32}$$

where θ_n is defined by (4.14), $\rho_n := \frac{1}{\alpha_n+\delta\lambda_n}$ and

$$\mu_n = (1-\lambda_n)\alpha_n(1+\alpha_n)+\lambda_n\beta_n(1+\beta_n)+\alpha_n(1-\lambda_n)\delta. \tag{4.33}$$

From (D6), it follows that

$$\mu_n \ge (1-\overline{\lambda})\underline{\alpha}(1+\underline{\alpha})+\overline{\lambda}\eta+\underline{\alpha}(1-\overline{\lambda})\delta \ge 0. \tag{4.34}$$

Using (D5), we have

$$\mu_n \le \overline{\alpha}(1+\overline{\alpha}+\delta). \tag{4.35}$$

Next, we claim that

$$\frac{(1-\lambda_n)(\alpha_n\rho_n-1)}{\lambda_n}+\mu_{n+1}\le -\sigma \tag{4.36}$$

for each $n \ge 1$. Similarly with (4.22), we have

$$\begin{aligned}&\frac{(1-\lambda_n)(\alpha_n\rho_n-1)}{\lambda_n}+\mu_{n+1}\le -\sigma\\ &\qquad\Longleftrightarrow\quad (\alpha_n+\delta\lambda_n)(\mu_{n+1}+\sigma)+\delta\lambda_n\le\delta.\end{aligned}$$

Employing (4.35), we have

$$(\alpha_n+\delta\lambda_n)(\mu_{n+1}+\sigma)+\delta\lambda_n \le (\overline{\alpha}+\delta\lambda_n)[\overline{\alpha}(1+\overline{\alpha}+\delta)+\sigma]+\delta\lambda_n\le\delta,$$

where the last inequality follows by using the upper bound for (λ_n) in (4.31). Hence, the claim in (4.36) is true. The rest proof is similar to the proof of Theorem 2. This completes the proof. □

Remark 3 In the analysis in [20, Sect. 3.2.2 (i)], Iutzeler and Hendrickx concluded that *inertia has a negative effect on the negative side of the spectrum*. The condition based on which the authors got the conclusion is the nonnegativity of the inertial parameters (see Lemma 3.2 in [20]). According to their analysis, it is easy to show that negative inertia satisfying some conditions has a positive effect on the negative side of the spectrum. So, our choices for inertial parameters in Theorems 2 and 3 may have an advantage for the negative side of the spectrum.

4.3.2 $\alpha_n \in [-1, 0]$ *and* $\beta_n \in [0, +\infty)$

In this subsection, we consider $\alpha_n \in [-1, 0]$ and $\beta_n \in [0, +\infty)$ for the inertial Krasnosel'skiĭ–Mann algorithm (4.4).

Theorem 4 *Suppose that* $T : \mathscr{H} \to \mathscr{H}$ *is a nonexpansive mapping with* $Fix(T) \neq \emptyset$. *Assume that* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\lambda_n\}$ *satisfy the following conditions:*

(D7) $\alpha_1 = \beta_1 = 0$, $\{\alpha_n\} \subset [\underline{\alpha}, \overline{\alpha}]$, $\{\beta_n\} \subset [\underline{\beta}, \overline{\beta}]$ *are nondecreasing,* $\underline{\alpha}, \overline{\alpha} \in (-1, 0]$ *and* $\underline{\beta}, \overline{\beta} \in (0, +\infty)$;

(D8) *for any* $\underline{\lambda}, \overline{\lambda}, \sigma, \delta > 0$,

$$\delta > \max\left\{1 + \overline{\alpha}, \frac{\underline{\alpha}[\overline{\beta}(1 + \overline{\beta}) + \sigma]}{\underline{\alpha}^2 - 1}\right\}, \quad 0 < \underline{\lambda} \leq \lambda_n \leq \overline{\lambda}, \tag{4.37}$$

where

$$\underline{\lambda} \geq \frac{\underline{\alpha}}{\underline{\alpha} - \underline{\beta}}, \tag{4.38}$$

and

$$\overline{\lambda} = \min\left\{\frac{1 - \overline{\alpha}}{\overline{\beta} - \overline{\alpha}}, \frac{\delta + \underline{\alpha}[\overline{\beta}(1 + \overline{\beta}) - \underline{\alpha}\delta + \sigma]}{\delta[\overline{\beta}(1 + \overline{\beta}) - \underline{\alpha}\delta + \sigma + 1]}.\right\} \tag{4.39}$$

Then the sequence $\{x_n\}$ *generated by the general inertial Krasnosel'skiĭ–Mann algorithm (4.4) converges weakly to a point of* $Fix(T)$.

Proof Following the proof line of Theorem 2, we have

$$\begin{aligned} &\|x_{n+1} - p\|^2 - (1 + \theta_n)\|x_n - p\|^2 + \theta_n\|x_{n-1} - p\|^2 \\ &\quad \leq -\lambda_n(1 - \lambda_n)\|Tz_n - y_n\|^2 \\ &\qquad + [(1 - \lambda_n)\alpha_n(1 + \alpha_n) + \lambda_n\beta_n(1 + \beta_n)]\|x_n - x_{n-1}\|^2, \end{aligned} \tag{4.40}$$

where

$$\theta_n = \alpha_n(1 - \lambda_n) + \beta_n\lambda_n. \tag{4.41}$$

From (D8), it follows that $\{\theta_n\} \subset [0, \theta) \subset [0, 1)$ is nondecreasing with $\theta_1 = 0$ and $\theta = \overline{\alpha}(1 - \overline{\lambda}) + \overline{\beta}\,\overline{\lambda}$. Similarly, with (4.15), we have

$$\begin{aligned} \|Tz_n - y_n\| \geq \frac{1}{{\lambda_n}^2}\|x_{n+1} - x_n\|^2 + \frac{\alpha_n^2}{{\lambda_n}^2}\|x_{n-1} - x_n\|^2 \\ + \frac{\alpha_n}{{\lambda_n}^2}\Big(\rho_n\|x_{n+1} - x_n\|^2 + \frac{1}{\rho_n}\|x_{n-1} - x_n\|^2\Big), \end{aligned} \tag{4.42}$$

where we denote $\rho_n := \frac{1}{\delta\lambda_n - \alpha_n}$. From (4.40) and (4.42), we can derive the inequality

$$\begin{aligned} &\|x_{n+1} - p\|^2 - (1 + \theta_n)\|x_n - p\|^2 + \theta_n\|x_{n-1} - p\|^2 \\ &\leq -\frac{(1 - \lambda_n)(1 + \alpha_n\rho_n)}{\lambda_n}\|x_{n+1} - x_n\|^2 + \mu_n\|x_n - x_{n-1}\|^2, \end{aligned} \tag{4.43}$$

where

$$\mu_n = (1 - \lambda_n)\alpha_n(1 + \alpha_n) + \lambda_n\beta_n(1 + \beta_n) - \alpha_n(1 - \lambda_n)\frac{1 + \rho_n\alpha_n}{\rho_n\lambda_n}. \tag{4.44}$$

Taking into account the choice of ρ_n, we have

$$\delta = \frac{1 + \rho_n\alpha_n}{\rho_n\lambda_n}. \tag{4.45}$$

From (4.37), it follows that $1 + \alpha_n - \delta \leq 1 + \overline{\alpha} - \delta \leq 0$ and, consequently, we obtain

$$\begin{aligned} \mu_n &= (1 - \lambda_n)\alpha_n(1 + \alpha_n - \delta) + \lambda_n\beta_n(1 + \beta_n) \\ &\geq 0. \end{aligned} \tag{4.46}$$

Using (D7), we have

$$\begin{aligned} \mu_n &= (1 - \lambda_n)\alpha_n(1 + \alpha_n) + \lambda_n\beta_n(1 + \beta_n) - \alpha_n(1 - \lambda_n)\delta \\ &\leq \overline{\beta}(1 + \overline{\beta}) - \underline{\alpha}\delta \end{aligned} \tag{4.47}$$

for each $n \geq 1$.

Next, we claim that

$$-\frac{(1 - \lambda_n)(1 + \alpha_n\rho_n)}{\lambda_n} + \mu_{n+1} \leq -\sigma \tag{4.48}$$

for each $n \geq 1$. By (4.47), we have

$$
\begin{aligned}
-\frac{(1-\lambda_n)(1+\alpha_n\rho_n)}{\lambda_n} & + \mu_{n+1} \leq -\sigma \\
& \Longleftrightarrow \lambda_n(\mu_{n+1}+\sigma) - (1-\lambda_n)(1+\alpha_n\rho_n) \leq 0 \\
& \Longleftrightarrow \lambda_n(\mu_{n+1}+\sigma) - \frac{\delta\lambda_n(1-\lambda_n)}{\delta\lambda_n - \alpha_n} \leq 0 \\
& \Longleftrightarrow (\delta\lambda_n - \alpha_n)(\mu_{n+1}+\sigma) + \delta\lambda_n \leq \delta.
\end{aligned}
$$

Employing (4.45), we have

$$
(\delta\lambda_n - \alpha_n)(\mu_{n+1}+\sigma) + \delta\lambda_n \leq (\delta\lambda_n - \underline{\alpha})[\overline{\beta}(1+\overline{\beta}) - \underline{\alpha}\delta + \sigma] + \delta\lambda_n, \leq \delta
$$

where the last inequality follows by using the upper bound for (λ_n) in (4.39). Hence, the claim in (4.48) is true. The rest proof is similar to the proof of Theorem 2. This completes the proof. □

Remark 4 In Theorem 4, there is an additive restriction on $\underline{\lambda}$ in (4.38). Furthermore, from (4.38) and (4.39), it follows that $\overline{\alpha}$, $\underline{\alpha}$, $\overline{\beta}$, $\underline{\beta}$ satisfy

$$
\frac{\underline{\alpha}}{\underline{\alpha} - \underline{\beta}} \leq \frac{1-\overline{\alpha}}{\overline{\beta} - \overline{\alpha}}.
$$

Remark 5 To use Lemma 2 in the proof of the convergence theorems, we restrict $\theta_n = \alpha_n(1-\lambda_n) + \beta_n\lambda_n \in (0, 1)$. Therefore, at most one of the two inertial parameters α_n and β_n is negative. On the other hand, to guarantee $\delta > 0$, we have to take $\alpha_n \in [-1, 1]$. So, in Theorems 2, 3, and 4, we only discuss two cases on the inertial parameters:

(1) $\alpha_n \in [0, 1]$ and $\beta \in (-\infty, 0]$ for each $n \geq 1$;
(2) $\alpha_n \in [-1, 0]$ and $\beta \in [0, +\infty)$ for each $n \geq 1$.

4.4 Linear Convergence

Let $S = I - T$. In this section, we consider the general inertial Krasnosel'skiǐ–Mann algorithms with the following form:

$$
\begin{cases}
y_n = x_n + \alpha_n(x_n - x_{n-1}), \\
z_n = x_n + \beta_n(x_n - x_{n-1}), \\
x_{n+1} = y_n - \lambda_n S(z_n)
\end{cases}
\tag{4.49}
$$

for each $n \geq 1$. Rearranging the above (4.49) yields

$$
x_{n+1} = (1-\lambda_n)\left(x_n + \frac{\alpha_n - \lambda_n\beta_n}{1-\lambda_n}(x_n - x_{n-1})\right) + \lambda_n T z_n,
$$

which is the general inertial Krasnosel'skiĭ–Mann algorithm (4.4).

In this section, we establish a linear convergence for the general inertial Krasnosel'skiĭ–Mann algorithms (4.49) under the assumption that S is quasi-strongly monotone.

First, we present a key lemma for the main result.

Lemma 5 *Let $T : \mathscr{H} \to \mathscr{H}$ be a nonexpansive mapping. Assume that $\alpha_n \in [-\alpha, \alpha]$ and $\{\beta_n\} \subset [-\beta, \beta]$ with $\alpha_1 = \beta_1 = 0$ and $\beta > 0, \alpha \in \left(0, \frac{3-2\sqrt{2}}{2}\right)$ and the relaxation parameter is fixed, i.e., $\lambda_n = \lambda$, and satisfies*

$$0 < \lambda \leq \overline{\lambda}_1 := \frac{\rho - 1 - 2\alpha\rho(1+\sqrt{\rho})}{4\rho(1+\beta+\beta\sqrt{\rho})} \tag{4.50}$$

for some $\rho \in \left(\frac{1-2\alpha-\sqrt{(1-2\alpha)^2-8\alpha}}{4\alpha}, \frac{1-2\alpha+\sqrt{(1-2\alpha)^2-8\alpha}}{4\alpha}\right)$. Then it follows that, for all $n \geq 1$,

$$\|x_n - x_{n-1}\|^2 \leq \rho\|x_{n+1} - x_n\|^2. \tag{4.51}$$

Furthermore, we have

$$\|x_n - x_{n-1}\|^2 \leq \eta\|x_{n+1} - y_n\|^2 \tag{4.52}$$

for some $\eta \geq \frac{\rho}{1-2\alpha\sqrt{\rho}}$.

Proof Now, we prove (4.51) by induction. Lemma 1 shows that S is $\frac{1}{2}$-cocoercive. Based on the inequality $\|a\|^2 - \|b\|^2 \leq 2\|a\|\|b-a\|$, we observe that, for any $n \geq 1$,

$$\begin{aligned}
&\|x_n - x_{n-1}\|^2 - \|x_{n+1} - x_n\|^2 \\
&\leq 2\|x_n - x_{n-1}\|\|(x_{n+1} - x_n) - (x_n - x_{n-1})\| \\
&\leq 2\|x_n - x_{n-1}\|\|\left[\alpha_n(x_n - x_{n-1}) - \alpha_{n-1}(x_{n-1} - x_{n-2})\right] - \lambda(Sz_n - Sz_{n-1})\| \\
&\leq 2\|x_n - x_{n-1}\|\left[|\alpha_n|\|x_n - x_{n-1}\| + |\alpha_{n-1}|\|x_{n-1} - x_{n-2}\| + \lambda\|Sz_n - Sz_{n-1}\|\right] \\
&\leq 2\|x_n - x_{n-1}\|\left[|\alpha_n|\|x_n - x_{n-1}\| + |\alpha_{n-1}|\|x_{n-1} - x_{n-2}\| + 2\lambda\|z_n - z_{n-1}\|\right],
\end{aligned} \tag{4.53}$$

where the final inequality comes from the cocoercive property of S. Applying the triangle inequality and (4.49) yields

$$\begin{aligned}
\|z_n - z_{n-1}\| &\leq \|z_n - x_n\| + \|x_n - x_{n-1}\| + \|x_{n-1} - z_{n-1}\| \\
&\leq (1+|\beta_n|)\|x_n - x_{n-1}\| + |\beta_{n-1}|\|x_{n-1} - x_{n-2}\|.
\end{aligned} \tag{4.54}$$

Let $n = 1$ in (4.54) and using $\beta_0 = 0$, we have

$$\|z_1 - z_0\| \leq (1+\beta)\|x_1 - x_0\|,$$

which, with (4.53) and $\alpha_0 = 0$, implies

$$\begin{aligned} \|x_1 - x_0\|^2 - \|x_2 - x_1\|^2 &\leq 2[\alpha\|x_1 - x_0\|^2 + 2\lambda\|x_1 - x_0\|\|z_1 - z_0\|] \\ &\leq [2\alpha + 4\lambda(1+\beta)]\|x_1 - x_0\|^2. \end{aligned} \tag{4.55}$$

Rearranging the above inequality yields

$$\begin{aligned} \|x_1 - x_0\|^2 &\leq \frac{1}{1 - [2\alpha + 4\lambda(1+\beta)]}\|x_2 - x_1\|^2 \\ &\leq \rho\|x_2 - x_1\|^2, \end{aligned}$$

where the second inequality comes from (4.50). For the induction step, i.e., $n > 1$, from (4.53) and (4.54), it follows that

$$\begin{aligned} &\|x_n - x_{n-1}\|^2 - \|x_{n+1} - x_n\|^2 \\ &\leq 2\,[\alpha + 2\lambda(1+\beta)]\,\|x_n - x_{n-1}\|^2 + 2(\alpha + 2\lambda\beta]\|x_n - x_{n-1}\|\|x_{n-1} - x_{n-2}\|. \end{aligned} \tag{4.56}$$

By the assumption that (4.51) holds for $1, 2, \ldots, n-1$, we have

$$\|x_n - x_{n-1}\|\|x_{n-1} - x_{n-2}\| \leq \sqrt{\rho}\|x_n - x_{n-1}\|^2. \tag{4.57}$$

Combining (4.56) and (4.57) yields

$$\|x_n - x_{n-1}\|^2 - \|x_{n+1} - x_n\|^2 \leq 2\left[\alpha + 2\lambda(1+\beta) + (\alpha + 2\lambda\beta)\sqrt{\rho}\right]\|x_n - x_{n-1}\|^2. \tag{4.58}$$

Finally, rearranging the above inequality and using (4.50) lead to (4.51). Using the inequality $\|a\|^2 - \|b\|^2 \leq 2\|a\|\|b - a\|$ again, we obtain

$$\begin{aligned} \|x_n - x_{n+1}\|^2 - \|y_n - x_{n+1}\|^2 &\leq 2\|x_n - x_{n+1}\|\|(x_n - x_{n+1}) - (y_n - x_{n+1})\| \\ &\leq 2\alpha\|x_n - x_{n+1}\|\|x_n - x_{n-1}\| \\ &\leq 2\alpha\sqrt{\rho}\|x_n - x_{n+1}\|^2, \end{aligned} \tag{4.59}$$

which yields

$$\|x_n - x_{n+1}\|^2 \leq \frac{1}{1 - 2\alpha\sqrt{\rho}}\|y_n - x_{n+1}\|^2.$$

Combining the above inequality with (4.51) yields

$$\|x_n - x_{n-1}\|^2 \leq \frac{\rho}{1 - 2\alpha\sqrt{\rho}}\|y_n - x_{n+1}\|^2 \tag{4.60}$$

which leads to (4.52). This completes the proof. □

Remark 6 Note that, from (4.50), it follows that

$$\rho - 1 - 2\alpha\rho(1 + \sqrt{\rho}) > 0. \tag{4.61}$$

Due to $\rho > 1$, we have $\rho > \sqrt{\rho}$. So, to obtain (4.61), we may let

$$\rho - 1 - 2\alpha\rho(1 + \rho) > 0. \tag{4.62}$$

After a simple calculation, we have

$$\alpha \leq \frac{3 - 2\sqrt{2}}{2}$$

and

$$\frac{1 - 2\alpha - \sqrt{(1 - 2\alpha)^2 - 8\alpha}}{4\alpha} < \rho < \frac{1 - 2\alpha + \sqrt{(1 - 2\alpha)^2 - 8\alpha}}{4\alpha}. \tag{4.63}$$

Note that $\frac{1-2\alpha-\sqrt{(1-2\alpha)^2-8\alpha}}{4\alpha} > 1$. It is easy to show that $1 - 2\alpha\sqrt{\rho} > 0$ if ρ satisfies (4.63).

Theorem 5 *Assume that $T : \mathscr{H} \to \mathscr{H}$ is a nonexpansive mapping and $S : \mathscr{H} \to \mathscr{H}$ is a quasi-μ-strongly monotone mapping with $\mu > 0$. Let ν satisfy $\mu \in (0, 1)$ and $\theta \in (0, 1)$. Let $\alpha \in \left(0, \min\left\{\frac{3-2\sqrt{2}}{2}, \frac{(1-\nu)(1-\theta)\mu\nu}{\eta}\right\}\right)$ and $\{x_n\}$ be the sequence generated by the algorithm (4.49) with a constant relaxation parameter $\lambda \in (0, \min\{\overline{\lambda}_1, \overline{\lambda}_2\}]$, where $\overline{\lambda}_1$ is given in (4.50) and*

$$\begin{aligned} \overline{\lambda}_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad a = 2(1 - \theta)\beta^2\mu^2\nu^2\eta, \\ b = (1 - \theta)\mu\nu[1 + \alpha\eta + 2(\alpha + \beta)\sqrt{\eta}], \quad c = \alpha\eta - (1 - \nu)(1 - \theta)\mu\nu. \end{aligned} \tag{4.64}$$

Then, we have

$$\|x_n - x^*\|^2 \leq (1 - \theta\mu\nu\lambda)^n \|x_0 - x^*\|^2. \tag{4.65}$$

Proof From (4.49), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|y_n - \lambda S z_n - x^*\|^2 \\ &= \|y_n - x^*\|^2 + \lambda^2 \|S z_n\|^2 + 2\lambda\langle S z_n, x^* - y_n\rangle. \end{aligned} \tag{4.66}$$

Using (4.49), we have

$$
\begin{aligned}
\|y_n - x^*\|^2 &\le (1+\epsilon)\|x_n - x^*\|^2 + \left(1 + \frac{1}{\epsilon}\right)\|x_n - y_n\|^2 \\
&\le (1+\epsilon)\|x_n - x^*\|^2 + \left(1 + \frac{1}{\epsilon}\right)\alpha\|x_n - x_{n-1}\|^2 \\
&\le (1+\epsilon)\|x_n - x^*\|^2 + \left(1 + \frac{1}{\epsilon}\right)\alpha\eta\|x_{n+1} - y_n\|^2.
\end{aligned}
\tag{4.67}
$$

Using (4.49) again, we have

$$
\lambda^2\|Sz_n\|^2 = \|x_{n+1} - y_n\|^2. \tag{4.68}
$$

Note that

$$
\begin{aligned}
\langle Sz_n, x^* - y_n\rangle &= \langle Sz_n, x^* - z_n\rangle + \langle Sz_n, z_n - y_n\rangle \\
&= \langle Sz_n, x^* - z_n\rangle + \frac{1}{\lambda}\langle y_n - x_{n+1}, z_n - y_n\rangle.
\end{aligned}
\tag{4.69}
$$

The cocoercive and quasi-strongly monotone properties of S imply

$$
\begin{aligned}
\langle Sz_n, x^* - z_n\rangle &= \nu\langle Sz_n, x^* - z_n\rangle + (1-\nu)\langle Sz_n - Sx^*, x^* - z_n\rangle \\
&\le -\mu\nu\|z_n - x^*\|^2 - \frac{1-\nu}{2}\|Sz_n\|^2 \\
&\le -\mu\nu\|x_n - x^* + \beta(x_n - x_{n-1})\|^2 - \frac{1-\nu}{2\lambda^2}\|x_{n+1} - y_n\|^2 \\
&\le -\frac{\mu\nu}{2}\|x_n - x^*\|^2 + \beta^2\mu\nu\|x_n - x_{n-1}\|^2 - \frac{1-\nu}{2\lambda^2}\|x_{n+1} - y_n\|^2 \\
&\le -\frac{\mu\nu}{2}\|x_n - x^*\|^2 + \left(\beta^2\mu\nu\eta - \frac{1-\nu}{2\lambda^2}\right)\|x_{n+1} - y_n\|^2,
\end{aligned}
\tag{4.70}
$$

where the third inequality comes from $-\|a+b\|^2 \le -\frac{1}{2}\|a\|^2 + \|b\|^2$ and the last inequality follows from Lemma 5. We also have

$$
\begin{aligned}
\langle y_n - x_{n+1}, z_n - y_n\rangle &\le \frac{1}{2\gamma}\|y_n - x_{n+1}\|^2 + \frac{\gamma}{2}\|z_n - y_n\|^2 \\
&= \frac{1}{2\gamma}\|y_n - x_{n+1}\|^2 + \frac{\gamma(\alpha+\beta)^2}{2}\|x_n - x_{n-1}\|^2 \\
&\le \frac{1}{2}\left(\frac{1}{\gamma} + \gamma(\alpha+\beta)^2\eta\right)\|y_n - x_{n+1}\|^2 \\
&= (\alpha+\beta)\sqrt{\eta}\|y_n - x_{n+1}\|^2,
\end{aligned}
\tag{4.71}
$$

where we have let $\gamma = \frac{1}{(\alpha+\beta)\sqrt{\eta}}$ in the final equality. Combining (4.66)–(4.71) and using Lemma 5, we have

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le (1 - \mu\nu\lambda + \epsilon)\|x_n - x^*\|^2 \\
&\quad + \Big[\Big(1 + \frac{1}{\epsilon}\Big)\alpha\eta + 1 - \frac{1-\nu}{\lambda} + 2\beta^2\mu\nu\eta\lambda + 2(\alpha+\beta)\sqrt{\eta}\Big]\|y_n - x_{n+1}\|^2 \\
&= (1 - \theta\mu\nu\lambda)\|x_n - x^*\|^2 + \Big[\Big(1 + \frac{1}{(1-\theta)\mu\nu\lambda}\Big)\alpha\eta + 1 - \frac{1-\nu}{\lambda} \\
&\quad + 2\beta^2\mu\nu\eta\lambda + 2(\alpha+\beta)\sqrt{\eta}\Big]\|y_n - x_{n+1}\|^2 \\
&\le (1 - \theta\mu\nu\lambda)\|x_n - x^*\|^2,
\end{aligned}
\tag{4.72}
$$

where we let $\epsilon = (1-\theta)\mu\nu\lambda$ in the equality and the last inequality holds because of the choice of λ. Therefore, (4.65) holds. This completes the proof. □

Remark 7 There are some examples for the quasi-strongly monotone mappings, such as the gradient of a restricted strongly convex function. See [31] for more examples.

4.5 Numerical Examples

In this section, we give some numerical examples to compare the numerical results for the general inertial Krasnosel'skiĭ–Mann algorithm with different inertial parameters. The codes are written in Matlab 7.0 and run on personal computer.

Problem 2 (see [14]) Consider the classical variational inequality problem, which is to find a point $x^* \in C$ such that

$$\langle f(x^*), x - x^*\rangle \ge 0 \tag{4.73}$$

for all $x \in C$, where C is a nonempty closed convex subset of a real Hilbert space H and $f : C \to H$ is a mapping. Denote by $VI(C, f)$ the solution set of the variational inequality problem (4.73).

Assume that f be a Lipschitz continuous function with Lipschitz constant L. By using the properties of the metric projection, it is easy to show that the variational inequality problem (4.73) equals to the fixed-point problem, that is,

$$Fix(T) = VI(C, f),$$

where the mapping $T : C \to C$ is defined by

$$T := P_C(I - \gamma f), \tag{4.74}$$

where $0 < \gamma < \frac{2}{L}$.

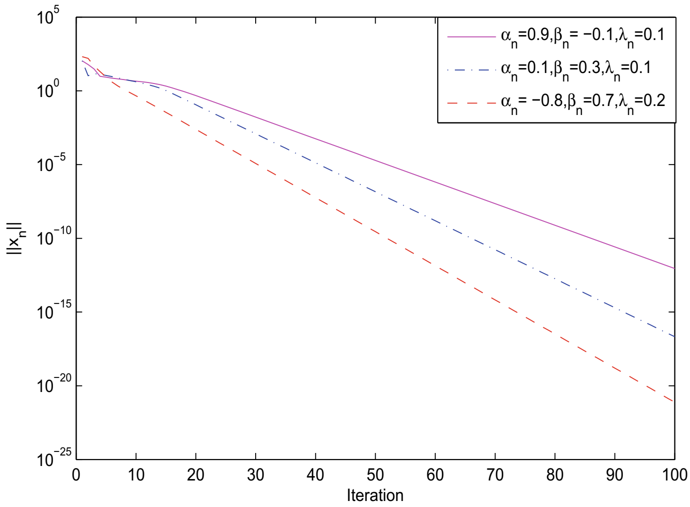

Fig. 4.1 Comparison of different values of the parameters α_n, β_n, and λ_n

Define a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$f(x, y) = (2x + 2y + \sin(x), -2x + 2y + \sin(y))$$

for all $x, y \in \mathbb{R}$. Recently, the authors [14] showed that f is L-Lipschitz continuous with $L = \sqrt{26}$ and strongly monotone. Therefore, the variational inequality (4.73) has a unique solution (see, for example, [39]) and (0, 0) is its solution.

The fact that f is Lipschitz continuous and strongly monotone implies that f is inverse strongly monotone. In [37], Xu showed that T defined in (4.74) is an averaged mapping, that is, T can be written as the average of the identity I and a nonexpansive mapping if f is inverse strongly monotone.

Let $C = \{x \in \mathbb{R}^2 : e_0 \le x \le 10e_1\}$, where $e_0 = (-10, -10)$ and $e_1 = (10, 10)$. Take the initial point $x_0 = (1, 10) \in \mathbb{R}^2$ and $\gamma = \frac{1}{2L}$.

First, we tested three ranges for the parameters $(\alpha_n, \beta_n, \lambda_n)$: $(0, 1) \times (0, 1) \times (0, 1)$, $(-1, 0) \times (0, 1) \times (0, 1)$ and $(0, 1) \times (-1, 0) \times (0, 1)$ to get the optimal values in each range. Then we compared these optimal values in Fig. 4.1, which illustrates $\alpha_n = -0.8$, $\beta_n = 0.7$, and $\lambda_n = 0.2$ have best behavior.

4.6 Conclusions

In this paper, we first showed the weak convergence of the general inertial Krasnosel'skiĭ–Mann algorithm with negative inertial parameters. Also, we provided the convergence rate for the general inertial Krasnosel'skiĭ–Mann algorithm. Finally, we gave an example to compare the choice of inertial parameters.

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Chapter 5
Digital Space-Type Fixed Point Theory and Its Applications

Sang-Eon Han

Abstract The present paper, as a survey paper, studies the fixed point property (*FPP*, for brevity) and the almost fixed point property (*AFPP*, for short) for digital spaces whose structures are induced by a digital graph in terms of the Rosenfeld model (or digital metric space), the Khalimsky (K-, for brevity), or the (extended) Marcus-Wyse (M-, for short) topology. Furthermore, we also investigate various properties of digital isomorphic (or homeomorphic), digital homotopic, retract, and product properties of the *FPP* and the *AFPP* of them. This approach can be used in applied sciences such as some areas of pure and applied topologies, applied analysis, and computer science such as computer graphics, image processing, pattern recognition, mathematical morphology, artificial intelligence, and so forth. All digital spaces are assumed to be connected (or k-connected) unless stated otherwise.

Keywords (Almost) Fixed point property · Marcus Wyse topological space · Digital metric space · Khalimsky topological spaces · digital contractibility

5.1 Introduction

Let $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{Z}^n$ represent the sets of integers, natural numbers, and points in the Euclidean n-dimensional space with integer coordinates, respectively. Digital topology stresses on finding (digital) topological properties of digital spaces in $\mathbb{Z}^n$ for each $n \in \mathbb{N}$ [16, 40, 45, 46], digitized spaces, tiled spaces, and crystalized spaces of subspaces of the n-dimensional Euclidean space and so forth. Thus, it has contributed to the study of some areas of pure and applied topologies, analysis, and computer science such as computer graphics, image processing, pattern recognition, mathematical morphology, artificial intelligence, and so forth. Up to now, several kinds of approaches have been used to study digital spaces (or digital images or digital metric

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spaces) [1, 14, 19, 22, 24, 25, 31, 40, 44, 45, 52]. The present paper deals with three main approaches such as digital graph-based spaces in terms of the Rosenfeld model (or digital metric space), Khalimsky (K-, for brevity), and (extended) Marcus-Wyse (M-, for short) topological spaces [20, 36, 52]. Indeed, digital topology mainly studies these spaces as well as some Alexandroff topological spaces, cellular complexes, combinatorial topological spaces, and so on. The former three spaces are based on the set $\mathbb{Z}^n$ and the latter spaces are related to the Alexandroff topological structure. Furthermore, methods of studying fixed point theory for digital spaces are different from the following typical approaches:

($\star$1) Metric-type fixed point theory [2, 7, 8, 48],
($\star$2) Topology-based fixed point theory [4, 42, 43, 47], and
($\star$3) Order theory-based fixed point theory [51].

Thus, in the present paper, we may add one more approach such as

($\star$4) Digital space-type (or digital topology-based) fixed point theory.

The present paper studies both the fixed point property (*FPP*, for brevity) and the almost fixed point property (*AFPP*, for short) for digital spaces in the corresponding categories. We say that an object X has the *FPP* (*resp. AFPP*) in a category if for any morphism f of the category there is some element $x \in X$ such that $f(x) = x$ (*resp.* $f(x) = x$ or $f(x)$ is adjacent to x depending on the associated adjacency structure) [44]. Since every singleton obviously has the *FPP* in a category, in studying the *FPP* for spaces, all spaces X (*resp.* digital images (X, k)) are assumed to be connected (*resp.* k-connected) and $|X| \geq 2$.

The well-known Lefschetz fixed point theorem [42, 43] as well as algebraic topological tools strongly contributed to the fixed point theory in such a way as to study the *FPP* of a certain topological space X by using homology groups of X [42, 43]. It is also a homotopy invariant. Hence, the theorem implies that a contractible topological space has the *FPP* from the viewpoint of topology-based fixed point theory (see ($\star$2)). However, this approach invokes some difficulties in studying digital spaces (or grid spaces) [37]. To be precise, in digital topology, it turns out that [19, 37] both the ordinary Lefschetz fixed point theorem and its digital version [9] have some limitations of studying the *FPP* of digital spaces, which is not helpful to address the issue of (5.1). Thus, a digital version of the classical Banach contraction principle [2] has been developed [10, 21] to study the *FPP* for digital metric spaces [21, 23, 24, 37]. Indeed, the paper [37] corrected and improved many things in [9]. In this paper, we often use the notation: For all $a, b \in \mathbb{Z}$, we follow the notation $[a, b]_{\mathbb{Z}} := \{x \in \mathbb{Z} : a \leq x \leq b\}$. Let us now recall the notion of digital space defined by Herman [38].

Definition 5.1 ([38]) A *digital space* is a relation set (X, R), where X is a nonempty set and R is a binary symmetric relation on X such that X is R-connected.

In Definition 5.1, we say that the set X is *R-connected* if for any two elements x and y of X there is a finite sequence $(x_i)_{i \in [0,l]_{\mathbb{Z}}}$ of elements in X such that $x = x_0$,

$y = x_l$, and $(x_j, x_{j+1}) \in R$ for $j \in [0, l-1]_{\mathbb{Z}}$. Besides, we should remind that the relation set (X, R) in Definition 5.1 need not be either a preordered set or a partially ordered set. In view of Definition 5.1, we see that not every digital topological space satisfies the T_1-separation axiom [6]. Besides, a relation set without any topological structure can be a digital space [10, 19, 21]. In addition, the digital space of Definition 5.1 can be generalized into a grid space as follows.

Definition 5.2 ([27]) We say that a grid space is a union of some R-connected components with the given relation R instead of just an R-connected component in a digital space.

In digital topology, since fixed point theory deals with only R-connected spaces, in the present paper, we will use the term "digital space" without any distinction from a grid space.

Let (X, R) be a digital space (see Definition 5.1 of the current paper). Then we may pose the following queries:

$$\left\{\begin{array}{l}
\text{(a) Are there some relationships between the contractibility of } X\\
\text{and the existence of the } \mathit{FPP} \text{ of } X\,?\\
\text{(b) Does a finite digital metric plane have the } \mathit{FPP} \text{ or the } \mathit{AFPP}\,?\\
\text{(c) Does a compact Khalimsky topological plane have the } \mathit{FPP}\\
\text{or the } \mathit{AFPP}\,?\\
\text{(d) Are there relationships between the } \mathit{FPP} \text{ of an } MA\text{-space } X\\
\text{and the } MA\text{-contractibility of } X\,?\\
\text{(e) Does a compact Marcus-Wyse topological plane have the } \mathit{FPP}\\
\text{or the } \mathit{AFPP}\,?\\
\text{(f) What about the product properties of the } \mathit{FPP} \text{ and the } \mathit{AFPP}\\
\text{for digital spaces}\,?\\
\text{(g) What about digital topological invariant properties of the } \mathit{FPP}\\
\text{and the } \mathit{AFPP}\,?
\end{array}\right\} \tag{5.1}$$

Rosenfeld [46] first came up with the fixed point theorem for digital images (X, k) in a graph theoretical approach (for more details, see [21, 23, 24]). Indeed, a digital image (X, k) is one of the digital spaces because (X, k) is a kind of relation set (X, R) in the following way: for all $x, y \in X$ with $x \neq y$ we say that $(x, y) \in R$ if and only if x and y are k-adjacent (for more details, see Sect. 5.2).

As for a theorem related to the *FPP* for digital images, we have the following.

Proposition 5.1 ([46], see Theorems 3.3 and 4.1 in [46]) *A (finite) digital plane (or digital metric space or digital image)* (X, k) *in* $\mathbb{Z}^2$ *does not have the FPP, where* X *is* k*-connected and* $|\, X \,| \geq 2$.

Although Rosenfeld [46] investigated the non-*FPP* of a finite digital plane in $\mathbb{Z}^2$ as in Proposition 5.1, we can easily generalize the result into n-dimensional digital cubes or any digital images because a singleton obviously has the *FPP*. This means that only a singleton has the *FPP* in a graph theoretical approach in terms of the Rosenfeld model [46].

Motivated by the *non-FPP* of digital images [46], Rosenfeld [46] further studied the *AFPP* for digital images. Thus, the present paper will also study the *AFPP* for digital spaces and further propose digital topological invariants of the *FPP* and the *AFPP* for digital spaces [25].

The recent papers [20, 22, 49] (*resp.* [23–25, 27, 36, 50]) partially studied the issue of (5.1)(a) from the viewpoint of K- (*resp.* M-) topology. Thus, it turns out that not every MA-space with MA-contractibility has the *FPP* [23]. Furthermore, a compact M-topological plane is proved not to have the *FPP* [24]. Thus, we need to study the questions posed in (5.1) by using only various properties of M-continuous maps.

The rest of the paper proceeds as follows: Sect. 5.2 provides some basic notions on digital topology. Section 5.3 introduces four categories for digital spaces such as *DTC*, *KTC*, and *MTC*. Section 5.4 refers to a digital version of the Banach contraction principle and its utilities. Section 5.5 proposes some relationships between the contractibility of X and the existence of the *FPP* of X in terms of the category of digital MA-spaces, denoted by *MAC*. Section 5.6 investigates the *FPP* and the *AFPP* for digital planes. Section 5.7 studies product properties of the *FPP* and the *AFPP* for digital spaces. Section 5.8 investigates some retract properties and digital topological invariants of the *FPP* and the *AFPP* for digital spaces. Section 5.9 concludes the paper with a summary and suggests further works.

5.2 Preliminaries

To address the issues of (5.1), let us recall basic notions and terminology on digital topology [13, 14, 16, 33, 34, 41, 45, 46]. Since the present paper also studies both the *FPP* and the *AFPP* for digital spaces associated with the K- and the M-topological structure, we will also recall some basic facts and terminology on digital topology such as $\mathbb{Z}^n$ with digital k-connectivity, and K- and M-topological structures. Besides, the well-known T_0-Alexandroff topological structure (i.e., semi-$T_{\frac{1}{2}}$-space [6]) of the K- and the M-topologies [1, 6] will be often used in the present paper.

In this paper, we shall often use the symbol " :=" to introduce new notions without proving the fact. Before studying fixed point theory for digital spaces, first of all, we need to represent the *FPP* for digital spaces as follows.

Remark 5.1 We say that a digital space (X, R) has the *FPP* if every relation preserving self-map f of (X, R) has a point $x \in X$ such that $f(x) = x$, where a self-map f of (X, R) is a relation preserving map if for any $x, y \in X$ with $(x, y) \in R$ and $x \neq y$, $f(x) = f(y)$ or $(f(x), f(y)) \in R$. In case a topological space (X, T) related to a digital space (X, R) such as K- and M-topological spaces as well as Alexandroff spaces, we say that (X, T) has the *FPP* if every continuous self-map f of (X, T) has a point $x \in X$ such that $f(x) = x$, as usual.

In the relation to the study of n-dimensional digital images in a graph theoretical approach, we have often used the k (or $k(t, n)$)-adjacency relations of $\mathbb{Z}^n$ as follows: for a natural number t with $1 \le t \le n$, two distinct points $p = (p_i)_{i\in[1,n]_{\mathbb{Z}}}$ and $q = (q_i)_{i\in[1,n]_{\mathbb{Z}}}$ in $\mathbb{Z}^n$ are called *$k(t, n)$-adjacent* (for short, *k-adjacent*) if

$$\text{at most } t \text{ of their coordinates differs by } \pm 1, \text{ and all others coincide.} \tag{5.2}$$

Indeed, these $k(t, n)$-adjacency relations of $\mathbb{Z}^n$ are determined according to the two numbers $t, n \in \mathbb{N}$ [13] (see also [16, 17]).

Using the above operator, we can obtain the k-adjacency relations of $\mathbb{Z}^n$[13, 16, 17, 29] as follows:

$$\left\{\begin{array}{l} \text{(a) } k := k(t, n) = \displaystyle\sum_{i=n-t}^{n-1} 2^{n-i} C_i^n, \text{ where } C_i^n = \frac{n!}{(n-i)!\, i!} \\ \text{or, equivalently,} \\ \text{(b) } k := k(t, n) = \displaystyle\sum_{i=1}^{t} 2^{i} C_i^n, \text{ where } C_i^n = \frac{n!}{(n-i)!\, i!}. \end{array}\right\} \tag{5.3}$$

Rosenfeld [45] called a set $X \subset \mathbb{Z}^n$ with a k-adjacency a *digital image* denoted by (X, k) for $n \in \{2, 3\}$. The paper [13] generalized this approach into the high-dimensional digital image such as $X \subset \mathbb{Z}^n$ with the k-adjacency of $\mathbb{Z}^n$ for each $n \in \mathbb{N}$. More precisely, using the k-adjacency of $\mathbb{Z}^n$ suggested in (5.3), we say that a digital k-neighborhood of p in $\mathbb{Z}^n$ is the set [45]

$$N_k(p) := \{q \in \mathbb{Z}^n : p \text{ is } k\text{-adjacent to } q\} \cup \{p\}.$$

For a k-adjacency relation of $\mathbb{Z}^n$, a simple k-path with $l + 1$ elements in $\mathbb{Z}^n$ is assumed to be a (injective) finite sequence $(x_i)_{i\in[0,l]_{\mathbb{Z}}} \subset \mathbb{Z}^n$ such that x_i and x_j are k-adjacent if and only if $|i - j| = 1$ [41]. If $x_0 = x$ and $x_l = y$, then the length of the simple k-path, denoted by $l_k(x, y)$, is the number l. A simple closed k-curve with l elements in $\mathbb{Z}^n$, denoted by $SC_k^{n,l}$ [13], is the $(x_i)_{i\in[0,l-1]_{\mathbb{Z}}}$, where x_i and x_j are k-adjacent if and only if $|i - j| = \pm 1 (mod\, l)$ [41].

For a digital image (X, k), as a generalization of $N_k(p)$ [13], a *digital k-neighborhood* of $x_0 \in X$ with radius ε is defined in X as the following subset [14] of X:

$$N_k(x_0, \varepsilon) := \{x \in X : l_k(x_0, x) \le \varepsilon\} \cup \{x_0\},$$

where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x and $\varepsilon \in \mathbb{N}$. Concretely, for any $X \subset \mathbb{Z}^n$, we obtain [17]

$$N_k(x, 1) = N_k(x) \bigcap X. \tag{5.4}$$

To study digital spaces in $\mathbb{Z}^n$ from the viewpoint of fixed point theory, we have often used K- (*resp.* M-) topology on $\mathbb{Z}^n$ (*resp.* $\mathbb{Z}^2$ [52]), denoted by $(\mathbb{Z}^n, \kappa^n)$ (*resp.* $(\mathbb{Z}^2, \gamma)$). We say that a topological space (X, T) is *Alexandroff* if each point $x (\in X)$ has a minimal open neighborhood [1].

Let us now briefly recall some basic facts and terms related to the K-topology. The *Khalimsky line topology* on $\mathbb{Z}$, denoted by $(\mathbb{Z}, \kappa)$, is induced by the set $\{[2n-1, 2n+1]_{\mathbb{Z}} : n \in \mathbb{Z}\}$ as a subbase [1]. Furthermore, the product topology on $\mathbb{Z}^n$ induced by $(\mathbb{Z}, \kappa)$ is called the *Khalimsky product topology* on $\mathbb{Z}^n$ (or *Khalimsky n-dimensional space*) which is denoted by $(\mathbb{Z}^n, \kappa^n)$. Indeed, $(\mathbb{Z}^n, \kappa^n)$ is an Alexandroff space. A point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ is called *pure open* if all coordinates are odd, and *pure closed* if each of the coordinates is even [39]. The other points in $\mathbb{Z}^n$ are called *mixed* [39]. Based on this approach, for a point $p := (p_1, p_2)$ in $(\mathbb{Z}^2, \kappa^2)$, its smallest open neighborhood $SN_K(p)$ is obtained [40] as follows:

$$SN_K(p) := \left\{ \begin{array}{l} \{p\} \text{ if } p \text{ is pure open,} \\ \{(p_1 - 1, p_2), p, (p_1 + 1, p_2)\} \text{ if } p \text{ is closed-open,} \\ \{(p_1, p_2 - 1), p, (p_1, p_2 + 1)\} \text{ if } p \text{ is open-closed,} \\ N_8(p) \text{ if } p \text{ is pure closed,} \end{array} \right\} \tag{5.5}$$

where the point $p := (p_1, p_2)$ is called *closed-open* (*resp. open-closed*) if p_1 is even (*resp. odd*) and p_2 is odd (*resp. even*). In all subspaces of $(\mathbb{Z}^n, \kappa^n)$ of Figs. 5.1 and 5.2, a black jumbo dot (*resp.* a square dot) means a pure open point (*resp.* a pure closed point) and further an ordinary dot means a mixed point.

Hereafter, for a subset $X \subset \mathbb{Z}^n$, we will denote by (X, κ^n_X) for each $n \geq 1$ a subspace induced by $(\mathbb{Z}^n, \kappa^n)$ and it is called a *K-topological space*. For a point x in (X, κ^n_X), we often call $SN_K(x)$ the *smallest open neighborhood* of x in (X, κ^n_X).

Definition 5.3 ([35, 40]) For (X, κ^n_X), we say that two distinct points x and y in X are *K-adjacent* in (X, κ^n_X) if $y \in SN_K(x)$ or $x \in SN_K(y)$, where $SN_K(p)$ is the smallest open set containing the point p in (X, κ^n_X).

According to Definitions 5.1 and 5.3, we obtain the following.

Proposition 5.2 *A K-topological space (X, κ^n_X) is a digital space in terms of the K-adjacency of (X, κ^n_X).*

Let us now recall basic concepts on M-topology. The *M-topology* on $\mathbb{Z}^2$, denoted by $(\mathbb{Z}^2, \gamma)$, is induced by the set $\{U(p) \mid p \in \mathbb{Z}^2\}$ in (5.6) below as a base [52], where, for each point $p = (x, y) \in \mathbb{Z}^2$,

$$U(p) := \left\{ \begin{array}{ll} N_4(p) & \text{if } x + y \text{ is even, and} \\ \{p\} & \text{otherwise.} \end{array} \right\} \tag{5.6}$$

Owing to the property (5.6), the set $U(p)$ is the smallest open neighborhood of the point p in $\mathbb{Z}^2$, denoted by $SN_M(p)$. In relation to the further statement of a point in $\mathbb{Z}^2$, in the paper, we call a point $p = (x_1, x_2)$ *double even* if $x_1 + x_2$ is an even number such that each x_i is even, $i \in \{1, 2\}$; *even* if $x_1 + x_2$ is an even number such that each x_i is odd, $i \in \{1, 2\}$; and *odd* if $x_1 + x_2$ is an odd number [49].

In all subspaces of $(\mathbb{Z}^2, \gamma)$ of Fig. 5.2, the symbols $\Diamond$ and $\bullet$ mean a *double even point* or an *even point*, and an *odd point*, respectively. In view of (5.6), we can obviously obtain the following: under $(\mathbb{Z}^2, \gamma)$ the singleton with either a double even point or an even point is a closed set. In addition, the singleton with an odd point is an open set.

Hereafter, for a subset $X \subset \mathbb{Z}^2$ we will denote by (X, γ_X) a subspace induced by $(\mathbb{Z}^2, \gamma)$, and it is called an *M-topological space*. For a point x in (X, γ_X), we often call $SN_M(x)$ the *smallest open neighborhood* of x in (X, γ_X).

Definition 5.4 ([18]) For (X, γ_X), we say that two distinct points x and y in X are *M-adjacent* in (X, γ_X) if $y \in SN_M(x)$ or $x \in SN_M(y)$, where $SN_M(p)$ is the smallest open set containing the point p in (X, γ_X).

According to Definitions 5.1 and 5.4, we obtain the following.

Proposition 5.3 *An M-topological space (X, γ_X) is a digital space in terms of the M-adjacency of (X, γ_X).*

5.3 Some Categories Associated with the Digital Topological Structures

This section studies several categories for digital spaces associated with Rosenfeld's digital topological structure and the K- and the M-topological structures. To map every k_0-connected subset of a digital image (X, k_0) into a k_1-connected subset of (Y, k_1), the paper [45] established the notion of digital continuity of a map between digital images.

Motivated by this approach, the digital continuity of a map was represented in the following way, which can be substantially used to study digital images X in $\mathbb{Z}^n$.

Proposition 5.4 ([13, 16]) *Let (X_i, k_i) be digital images in $\mathbb{Z}^{n_i}$ with the k_i-adjacency relations of (5.3) for each $i \in \{0, 1\}$. A function $f : (X_0, k_0) \to (X_1, k_1)$ is (k_0, k_1)-continuous if and only if for every $x \in X_0$, $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.*

In Proposition 5.4, in case $k_0 = k_1$, the map f is called a k_1-*continuous map*. By using this concept, we establish the category of digital topological spaces (or digital images), denoted by *DTC*, consisting of the following data [13] (see also [16]):

- The set of (X, k), where $X \subset \mathbb{Z}^n$, as objects of *DTC*;

- For every ordered pair of objects (X_i, k_i) for each $i \in \{0, 1\}$, the set of all (k_0, k_1)-continuous maps between them as morphisms of *DTC*. In *DTC*, in case $k_0 = k_1 := k$, we will particularly use the notation *DTC(k)* [36].

A digital image (X, k) in $\mathbb{Z}^n$ can be assumed to be a set (or graph) X in $\mathbb{Z}^n$ with one of the k-adjacency relations of (5.3). Thus, in classifying digital images, we use the concept of a (k_0, k_1)-isomorphism as in [14] (see also [34]) rather than a (k_0, k_1)-homeomorphism as in [3].

Definition 5.5 ([34], *see also* [14]) For two digital images (X, k_0) in $\mathbb{Z}^{n_0}$ and (Y, k_1) in $\mathbb{Z}^{n_1}$, a map $h : X \to Y$ is called a (k_0, k_1)-isomorphism if h is a (k_0, k_1)-continuous bijection and, further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous.

In Definition 5.5, in case $k_0 = k_1$, we call it a k_0-*isomorphism* [14, 34].

Let us now recall the notion of K-continuity of maps between K-topological spaces. As usual, for two K-topological spaces $(X, \kappa_X^{n_0}) := X$ and $(Y, \kappa_Y^{n_1}) := Y$, a map $f : X \to Y$ is called *continuous* at a point $x \in X$ if, for any open set $O_{f(x)} \subset Y$ containing the point $f(x)$, there is an open set $O_x \subset X$ containing the point x such that $f(O_x) \subset O_{f(x)}$.

Owing to the Alexandroff topological structure of a K-topological space, we can represent the K-continuity of a map at a point x, as follows:

$$f(SN_K(x)) \subset SN_K(f(x))$$

because each point x in a K-topological space X always has the smallest open set $SN_K(x) \subset X$.

By using K-topological spaces $(X, \kappa_X^n) := X$ and K-continuous maps, we have the category of K-topological spaces, denoted by *KTC*, consisting of the following data: [15].

- The set of spaces (X, κ_X^n), where $X \subset \mathbb{Z}^n$, as objects of *KTC* denoted by $Ob(KTC)$.
- For all pairs of elements in $Ob(KTC)$ the set of all K-continuous maps between them as morphisms.

To study K-topological spaces, we need to recall a K-homeomorphism as follows.

Definition 5.6 ([15]) For two spaces $(X, \kappa_X^{n_0}) := X$ and $(Y, \kappa_Y^{n_1}) := Y$, a map $h : X \to Y$ is called a K-*homeomorphism*, denoted by $X \approx_K Y$ if h is a K-continuous bijection, and $h^{-1} : Y \to X$ is K-continuous.

In $(\mathbb{Z}^n, \kappa^n)$, we say that a simple closed K-curve with l elements in $\mathbb{Z}^n$, denoted by $SC_K^{n,l}$, is a *path* $(x_i)_{i\in[0,l-1]_\mathbb{Z}} \subset \mathbb{Z}^n$ for each $l \geq 4$ that is K-homeomorphic to a quotient space of a Khalimsky line interval $[a, b]_\mathbb{Z}$ in terms of the identification of the only two end points a and b [35], where both of the numbers a and b in $[a, b]_\mathbb{Z}$ are even or odd. Namely, $SC_K^{n,l}$ is a finite set $(x_i)_{i\in[0,l-1]_\mathbb{Z}} \subset \mathbb{Z}^n$ such that x_i and x_j are K-adjacent if and only if $|i - j| = \pm 1 (mod\, l)$.

For instance, let us consider the spaces V and W in Fig. 5.1. Then X and Y are kinds of $SC_K^{2,8}$ and $SC_K^{2,4}$, respectively.

Let us now set up the category of M-topological spaces and an M-homeomorphism [31] as follows: owing to the Alexandroff topological structure of M-topology, the M-continuity of a map between M-topological spaces is defined as follows.

Definition 5.7 ([18]) For two M-topological spaces $(X, \gamma_X) := X$ and $(Y, \gamma_Y) := Y$, a function $f : X \to Y$ is said to be *M-continuous* at a point $x \in X$ if $f(SN_M(x)) \subset SN_M(f(x))$. Furthermore, we say that a map $f : X \to Y$ is *M-continuous* if it is M-continuous at every point $x \in X$.

Using M-continuous maps, we establish the category of M-topological spaces, denoted by *MTC*, consisting of the following data [18]:

- The set of spaces (X, γ_X), where $X \subset \mathbb{Z}^2$, denoted by $Ob(MTC)$.
- For every ordered pair of objects (X, γ_X) and (Y, γ_Y), the set of all M-continuous maps between them as morphisms of *MTC*.

Besides, in MTC, for two spaces (X, γ_X) and (Y, γ_Y), we say that a map $f : X \to Y$ is an *M-homeomorphism* [18], denoted by $X \approx_M Y$, if f is an M-continuous bijection and that $f^{-1} : Y \to X$ is M-continuous.

The concepts of both an M-continuous map and an M-homeomorphism play important roles in studying M-topological spaces, as referred to in the paper [18].

Let us now recall the following terminology which has been used to study M-topological spaces.

Definition 5.8 ([18, 31]) Let $(X, \gamma_X) := X$ be an M-topological space. Then we define the following:

(1) Two distinct points $x, y \in X$ are called *M-path connected* if there is a path $(x_i)_{i\in[0,m]_{\mathbb{Z}}}$ on X with $\{x_0 = x, x_1, \ldots, x_m = y\}$ such that $\{x_i, x_{i+1}\}$ is M-connected, $i \in [0, m-1]_{\mathbb{Z}}, m \geq 1$.
(2) A *simple M-path* in X is an M-path $(x_i)_{i\in[0,m]_{\mathbb{Z}}}$ such that the set $\{x_i, x_j\}$ is M-connected if and only if $|i - j| = 1$. Besides, the number m is called the *length* of this simple M-path.
(3) Furthermore, we say that a simple closed M-curve with l elements, denoted by SC_M^l, is a finite set $(x_i)_{i\in[0,l-1]_{\mathbb{Z}}} \subset \mathbb{Z}^2$ if and only if $|i - j| = \pm 1 (mod\, l)$.

For instance, let us consider the spaces X, Y and Z in Fig. 5.1. Then we see that [36] X, Y, and Z are kinds of SC_M^8, SC_M^4, and SC_M^{12}, respectively.

Remark 5.2 ([18]) Let us consider the space $(SC_M^l := (x_i)_{i\in[0,l]_{\mathbb{Z}}}, \gamma_{SC_M^l})$ in Fig. 5.1 such as X, Y, and Z. Consider the self-map f of SC_M^l given by $f(x_i) = x_{i+2m\,(mod\, l)}$, $i \in [0, l]_{\mathbb{Z}}, m \in \mathbb{Z}$. Then we observe that the map f is an M-continuous map. Meanwhile, the map $f(x_i) = x_{i+2m+1\,(mod\, l)}, i \in [0, l]_{\mathbb{Z}}$ for each $m \in \mathbb{Z}$ is not an M-continuous self-map of SC_M^l.

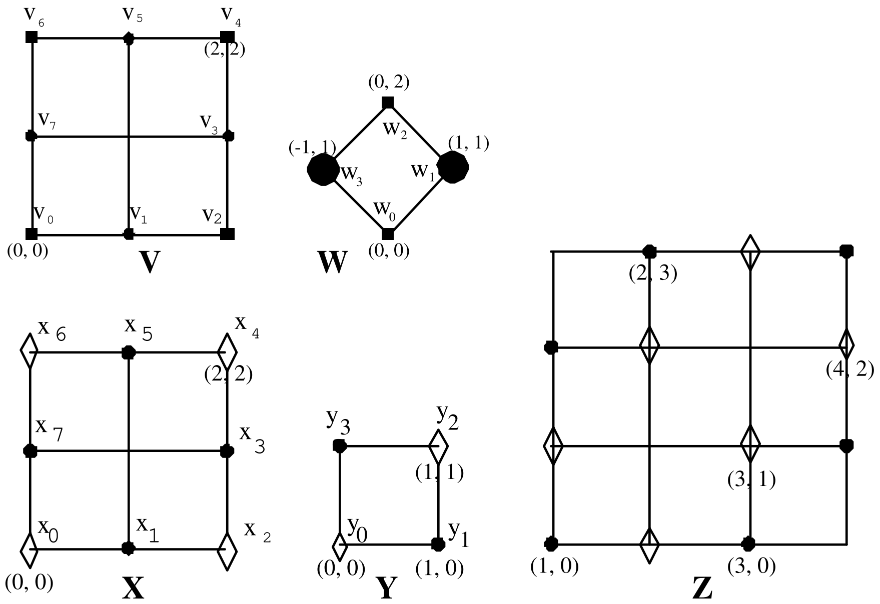

Fig. 5.1 An explanation of $SC_K^{2,l}$ and SC_M^l such as $V := SC_K^{2,8}$, $W := SC_K^{2,4}$ [15], $X := SC_M^8$, $Y := SC_M^4$ [20], and $Z := SC_M^{12}$ [36]

5.4 Digital Versions of the Banach Contraction Principle

As mentioned in Proposition 5.1, any k-connected digital plane (or digital image) (X, k) does not have the *FPP* in *DTC*, where $|X| \geq 2$. Thus, to study the *FPP* of digital images, we need some tools similar to the Banach contraction principle in metric-type fixed point theory. As for the Banach fixed point theorem from the viewpoint of digital topology, while the recent papers [10, 21] studied a digital version of the Banach fixed point theorem, some notions and assertions in [10] were more simplified or improved in [21], as follows.

Definition 5.9 ([10, 21]) We say that $(X, d, k) := (X, k)$ is a *digital metric space* if $X \subset \mathbb{Z}^n$, (X, d) is a metric space inherited from the metric space $(\mathbb{R}^n, d)$ with the standard Euclidean metric d on $\mathbb{R}^n$ and (X, k) is a digital image, $k := k(t, n)$ of (5.3).

Hereafter, we may consider a digital metric space in $\mathbb{Z}^n$ to be a digital image $(X, d, k) := (X, k)$ with the standard Euclidean metric function if there is no danger of ambiguity. In relation to the study of a digital version of the Banach fixed point theorem, first of all, we need to recall basic properties of a digital image (X, k) from the viewpoint of digital space-based fixed point theory, as follows.

Proposition 5.5 ([21]) *For a digital image (X, k) in $\mathbb{Z}^n$, consider two k-adjacent points x_i, x_j in X, where $k := k(t, n)$ of (5.3). Then they have the Euclidean distance*

$d(x_i, x_j)$ *which is greater than or equal to* 1 *and at most* $\sqrt{t}$ *depending on the position of the two points, i.e.,* $d(x_i, x_j) \in \{\sqrt{l} : l \in [1, t]_{\mathbb{Z}}\}$.

According to Proposition 5.5, the notion of "Cauchy sequence" in [10] was improved as follows.

Proposition 5.6 ([21]) *A sequence* $\{x_n\}$ *of points in a digital image* $(X, k := k(t, n))$ *is a Cauchy sequence if and only if there is* $\alpha \in \mathbb{N}$ *such that for all* $n, m \geq \alpha$ *the inequality* $d(x_n, x_m) \lneq 1$ *holds.*

Thus, by Propositions 5.5 and 5.6, we observe that the elements x_n and x_m satisfying $d(x_n, x_m) \lneq 1$ should be equal to each other, as follows.

Theorem 5.1 ([21]) *For a digital image* $(X, k := k(t, n))$*, if a sequence* $\{x_n\} \subset X \subset \mathbb{Z}^n$ *is a Cauchy sequence, then there is* $\alpha \in \mathbb{N}$ *such that for all* $n, m \geq \alpha$*, we have* $x_\alpha = x_n = x_m$.

Owing to Theorem 5.1, the notion of *limit* in [10] was improved, as follows.

Definition 5.10 ([21]) A sequence $\{x_n\}$ of points of a digital image $(X, k := k(t, n))$ converges to a *limit* $L \in X$ if there is $\alpha \in \mathbb{N}$ such that for all $n \geq \alpha$, then $d(x_n, L) \lneq 1$. Finally, we obtain $x_\alpha = x_{\alpha+1} = x_{\alpha+2} = \cdots = L$.

Motivated by the notion of completeness of a sequence in a metric space, its digital version was established [10], as follows.

Definition 5.11 ([10], *for more details, see* [21]) A digital image (X, k) is *complete* if any Cauchy sequence $\{x_n\} \subset X$ converges to a point L of (X, k).

According to Definitions 5.10 and 5.11, we obtain the following.

Theorem 5.2 ([21]) *A digital image* $(X, k) := (X, d, k)$ *is complete, where* $k := k(t, n)$.

To study a digital version of the Banach contraction principle in [2], motivated by Proposition 5.5 and Theorem 5.2, the notion of digital contraction map introduced in [10] was improved, as follows.

Definition 5.12 ([21]) Let $f : (X, k) \to (X, k)$ be a self-map of a digital image (X, k) in $\mathbb{Z}^n$. If there exists $\gamma \in [0, 1)$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) \leq \gamma d(x, y),$$

then we say that f is a *k-DC-self-map*. Besides, we say that f has a *digital version* of the Banach contraction principle (*DBP* for short).

According to the property *DBP*, we obviously obtain the following.

Lemma 5.1 *The composition of two k-DC-self-maps is a k-DC-self-map.*

In Definition 5.12, we need to remind that a k-DC-self-map is certainly associated with both the digital k-connectivity of $\mathbb{Z}^n$ and a typical contraction map on the metric space $(\mathbb{Z}^n, d)$ induced by the Euclidean metric space $(\mathbb{R}^n, d)$. Hence, according to Definition 5.12, we obtain the following.

Proposition 5.7 ([21]) *In the category of k-connected digital images (X, k) in $\mathbb{Z}^n$, a k-DC-self-map of (X, k) implies a k-continuous self-map of (X, k).*

Remark 5.3 The converse of Proposition 5.7 does not hold. For instance, consider the identity map on (X, k). Whereas the identity map is certainly a k-continuous map, it is not a k-DC-self-map.

Although the paper [10] proposed a digital version of the Banach contraction principle (Theorem 3.7 of [10]), owing to Theorem 5.2, it was improved and simplified, as follows.

Theorem 5.3 (Digital version of the Banach contraction principle) [21] (cf. [10]) *Let (X, k) be a digital image and $f : (X, k) \to (X, k)$ be a k-DC-self-map. Then f has a unique fixed point.*

Example 5.1 (1) For a finite digital line $X := [0, l]_{\mathbb{Z}}$ with 2-adjacency, $l \geq 2$. Consider a 2-DC-self-map f of X. Then it is a constant map, which implies that the map f has a fixed point in X.

(2) Let $SC_{2n}^{n,l} := (x_i)_{i \in [0,l-1]_{\mathbb{Z}}}$. Consider a $2n$-DC-self-map f of $SC_{2n}^{n,l}$. Then it is a constant map, which implies that the map f has a fixed point in $SC_{2n}^{n,l}$.

(3) Let $SC_{3^n-1}^{n,l} := (x_i)_{i \in [0,l-1]_{\mathbb{Z}}}$. Consider a $(3^n - 1)$-DC-self-map f of $SC_{3^n-1}^{n,l}$. Whereas it need not be a constant map, the map f has a fixed point in $SC_{3^n-1}^{n,l}$.

(4) Let $X (\subset \mathbb{Z}^n)$ be a finite digital cube with a k-adjacency of (5.3). Every k-DC-self-map f of X has a fixed point.

5.5 Relationships Between the MA-Contractibility of an MA-Space X and the *FPP* and the *AFPP* of X

Let us now deal with spaces $X (\subset \mathbb{Z}^2)$ with M-adjacency whose cardinalities are greater than 1 (see Proposition 5.3). Hereafter, each of these spaces is called an MA-space for short. Indeed, it turns out that an M-topological space (X, γ_X) induces an M-adjacency on the space (see Proposition 5.3). Thus, by (X, γ_X) we may denote an MA-space if there is no ambiguity. Let $(X, \gamma_X) := X$ be an MA-space. Then we say that an *MA-path* in X is the (injective) sequence $(x_i)_{i \in [0,m]_{\mathbb{Z}}}$ such that x_i and x_j are M-adjacent if $|i - j| = 1$ [18]. Besides, an MA-path $(x_i)_{i \in [0,m]_{\mathbb{Z}}}$ is called simple if and only if x_i and x_j are M-adjacent if and only if $|i - j| = 1$ for each $i, j \in [0, m]_{\mathbb{Z}}$. [18]

Motivated by Schauder's theorem, we may have the following query.

$$\text{Does an } MA\text{-contractible space have the } FPP\,? \tag{5.7}$$

To address this question, we need to consider an MA-homotopy (see Definition 5.19). Hence, let MAC be a category (for more details, see the part just below of Remark 5.4 in the present paper) whose objects, denoted by $Ob(MAC)$, are MA-spaces and morphisms, denoted by $Mor(MAC)$, are MA-maps between MA-spaces. This section proves that whereas an MA-space does not have the *FPP* in MAC, a finite simple MA-path has the *AFPP*.

Definition 5.13 ([28]) We say that an MA-space X is *MA-connected* if, for any two distinct points $x, y \in X$, there is an *MA-path* in X connecting these two points.

Under $(\mathbb{Z}^2, \gamma)$ the notions of M-adjacency and M-connectedness are equivalent [18]. Besides, for a point $p \in \mathbb{Z}^2$ and any point $q \in N_4(p)$, we can observe that the subspace $(\{p, q\} := X_1, \gamma_{X_1})$ is both M-connected and MA-connected. Based on this approach, we now define

$$MA(p) := \{q \in \mathbb{Z}^2 : q \text{ is } M\text{-adjacent to } p\}.$$

The paper [18] developed an *MA-map* which can be substantially used to study geometric transformations of MA-spaces.

For a space $(X, \gamma_X) := X$, we now recall an *MA-relation* of a point $p \in X$ as follows.

Definition 5.14 ([18]) For $(X, \gamma_X) := X$ put $MA_X(p) := MA(p) \cap X$. We say that two distinct points $p, q \in X$ are *M-adjacent* to each other if $q \in MA_X(p)$ or $p \in MA_X(q)$.

In Definition 5.14, we say that the two points p, q have an *MA-relation* or p is *MA-related* to q. It is obvious that an MA-relation is an irreflexive and symmetric relation [18].

The following MA-neighborhood of a point $p \in X$ is substantially used to establish an MA-map.

Definition 5.15 ([18]) For a space $(X, \gamma_X) := X$ and a point $p \in X$, we define an *MA-neighborhood* of p in X to be the set $MA_X(p) \bigcup \{p\} := MN_X(p)$.

Hereafter, in (X, γ_X), we use the notation $MN(p)$ for brevity instead of $MN_X(p)$ if there is no danger of ambiguity. In view of (5.4) and Definition 5.15, we conclude that in (X, γ_X)

$$MN(p) = N_4(p, 1). \tag{5.8}$$

By Proposition 5.3, for an M-topological space $(X, \gamma_X) := X$ and each point $x \in X$, owing to the Alexandroff topological structure of (X, γ_X), it is obvious that each point $x \in X$ always has $MN(x) \subset X$ so that the paper [18] established a map sending $MN(x)$ into $MN(f(x))$ as follows.

Definition 5.16 ([18]) For two MA-spaces $(X, \gamma_X) := X$ and $(Y, \gamma_Y) := Y$, we say that a function $f : X \to Y$ is an MA-*map* at a point $x \in X$ if

$$f(MN(x)) \subset MN(f(x)).$$

Furthermore, we say that a map $f : X \to Y$ is an MA-*map* on X if the map f is an MA-map at every point $x \in X$.

Hereafter, we observe the following.

Remark 5.4 (1) An M-continuous map is an MA-map. But the converse does not hold [18].

(2) An MA-map is an M-connectedness preserving map [18].

(3) For a bijective MA-map, its inverse map need not be an MA-map [36].

Using MA-maps, we introduce the category of MA-spaces [18], denoted by *MAC*, consisting of the following data:

- The set of MA-spaces (X, γ_X) with M-adjacency, where $X \subset \mathbb{Z}^2$, as objects of *MAC*.
- For every ordered pair of MA-spaces (X, γ_X) and (Y, γ_Y), the set of all MA-maps $f : (X, \gamma_X) \to (Y, \gamma_Y)$ as morphisms of *MAC*.

As referred to in Remark 5.4 (3), since the inverse of an MA-map (*resp.* M-continuous map) need not be an MA-map (*resp.* M-continuous map), we need to establish the following notion.

Definition 5.17 ([18]) For two MA-spaces $(X, \gamma_X) := X$ and $(Y, \gamma_Y) := Y$, a map $h : X \to Y$ is called an MA-*isomorphism* if h is a bijective MA-map (for short, MA-bijection) and, further, $h^{-1} : Y \to X$ is an MA-map.

In Definition 5.17, we denote by $X \approx_{MA} Y$ an MA-isomorphism from X to Y. In view of Remark 5.4, we obtain the following.

Remark 5.5 Both an MA-map and an MA-isomorphism are generalizations of an M-continuous map and a M-homeomorphism, respectively, [18] so that these maps have strong advantages of studying geometric transformations of M-topological spaces.

Definition 5.18 ([18]) A *simple* MA-*path* in X is the sequence $(x_i)_{i \in [0,m]_{\mathbb{Z}}}$ such that x_i and x_j are M-adjacent to each other if and only if $|i - j| = 1$. Furthermore, we say that a simple closed MA-curve with l elements, denoted by SC^l_{MA}, is the finite set $(x_i)_{i \in [0,l-1]_{\mathbb{Z}}}$ in $\mathbb{Z}^2$ such that x_i and x_j are M-adjacent if and only if $|i - j| = \pm 1 (mod\, l)$.

Indeed, $SC_{MA}^{l_1}$ is MA-isomorphic to $SC_{MA}^{l_2}$ if and only if $l_1 = l_2$ [18]. For SC_{MA}^{l}, the number l is an even number such that $l \in \{2n \mid n \in \mathbb{N} \setminus \{1, 3\}\}$ [36].

For an MA-space X let B be a subset of X. Then (X, B) is called a *MA-space pair*. Furthermore, if B is a singleton set $\{x_0\}$, then (X, x_0) is called a *pointed MA-space*.

By using MA-maps, the paper [36] introduced the notions of MA-homotopy relative to a subset $B \subset X$, MA-contractibility, and an MA-homotopy equivalence, which will be used to study the *FPP* and the *AFPP* for MA-spaces in MAC.

Definition 5.19 ([36]) Let (X, B) and Y be an MA-space pair and an MA-space, respectively. Let $f, g : X \to Y$ be MA-maps. Suppose there exist $m \in \mathbb{N}$ and a function $F : X \times [0, m]_{\mathbb{Z}} \to Y$ such that

(• 1) for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$.
(• 2) for all $x \in X$, the induced function $F_x : [0, m]_{\mathbb{Z}} \to Y$ given by $F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbb{Z}}$ is an MA-map.
(• 3) for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $F_t : X \to Y$ given by $F_t(x) = F(x, t)$ for all $x \in X$ is an MA-map.

Then we say that F is an *MA-homotopy* between f and g.

(• 4) Furthermore, for all $t \in [0, m]_{\mathbb{Z}}$, assume that $F_t(x) = f(x) = g(x)$ for all $x \in B$.

Then we call F an *MA-homotopy* relative to B between f and g, and we say that f and g are *MA-homotopic* relative to B in Y, $f \simeq_{MArel.B} g$ in symbol.

In Definition 5.19, if $B = \{x_0\} \subset X$, then we say that F is a *pointed MA-homotopy* at $\{x_0\}$. When f and g are pointed MA-homotopic in Y, we use the notation $f \simeq_{MA} g$ and $f \in [g]$ which denotes the MA-homotopy class of g. If, for some $x_0 \in X$, 1_X is MA-homotopic to the constant map in the space $\{x_0\}$ relative to $\{x_0\}$, then we say that (X, x_0) is *pointed MA-contractible* (for brevity, *MA-contractible* if there is no danger of ambiguity).

Let us investigate some properties of the MA-contractibility. Motivated by the notion of digital homotopy equivalence [12, 34], we develop the following.

Definition 5.20 ([36]) For two MA-spaces (X, γ_X) and (Y, γ_Y) in MAC, if there are MA-maps $h : X \to Y$ and $l : Y \to X$ such that $l \circ h$ is MA-homotopic to 1_X and $h \circ l$ is MA-homotopic to 1_Y, then the map $h : X \to Y$ is called an *MA-homotopy equivalence* and denote it by $X \simeq_{MA \cdot h \cdot e} Y$.

If a space $X \in Ob(MAC)$ is MA-homotopy equivalent to a singleton in MAC, then it is obvious that the space is MA-contractible.

By using the MA-homotopy $F : SC_{MA}^4 \times [0, 2]_{\mathbb{Z}} \to SC_{MA}^4$ described in Fig. 5.2 as an example, where $SC_{MA}^4 := Y$, we obtain the following.

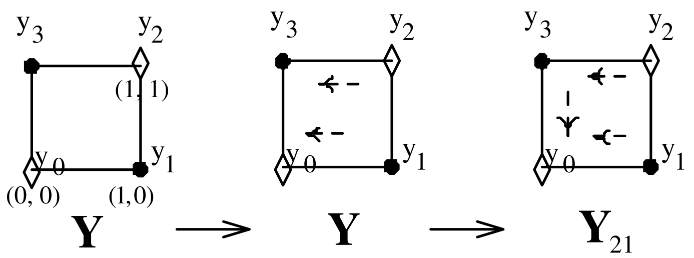

Fig. 5.2 MA-contractibility of SC^4_{MA}

Lemma 5.2 ([36]) *SC^4_{MA} is MA-contractible.*

Indeed, there are several ways including the method described in Fig. 5.2 to proceed with the MA-contractibility of SC^4_{MA}.

Let us now study some relationships between the MA-contractibility of an MA-space X and the existence of the *FPP* of X in the category MAC, which addresses the issue (5.7).

Let us now investigate some properties of MA-spaces from the viewpoint of fixed point theory.

Theorem 5.4 ([23]) *For any MA-connected space $X (\in Ob(MAC))$ such that $|X| \geq 2$, there exists an MA-self-map without a fixed point.*

To support Theorem 5.4, we can take two distinct points x, y in X such that $y \in SN_M(x)$. Then it is obvious that $SN_M(y)$ is the singleton $\{y\}$ and $|SN_M(x)| \geq 2$, where $SN_M(x)$ is the smallest open set of x. Let us consider the self-map f of X given by

$$f(x') = x, \quad x' \in X, \quad x' \neq x \quad \text{and} \quad f(x) = y. \tag{5.9}$$

Then it is obvious that the map f of (5.9) is an MA-map which does not have any fixed point on X. Indeed, the map f of (5.9) does not imply an M-continuous map. To be precise, see the MA-map described by using the arrows in Fig. 5.3. In (5.9), we need to point out that the map f need not be an M-continuous map.

In *MAC*, we say that an *MA-isomorphic invariant* is a property of an MA-space which is invariable under MA-isomorphism.

Proposition 5.8 ([23]) *In MAC, the FPP is an MA-isomorphic invariant.*

Since a digital plane does not have the *FPP* [46], Rosenfeld [46] studied the *AFPP* of a finite digital plane. Before studying this issue, we need to recall the notion of *AFPP* in *DTC*. We say that [46]

$$\text{a digital image } (X, k) \text{ in } \mathbb{Z}^n \text{ has the } AFPP \tag{5.10}$$

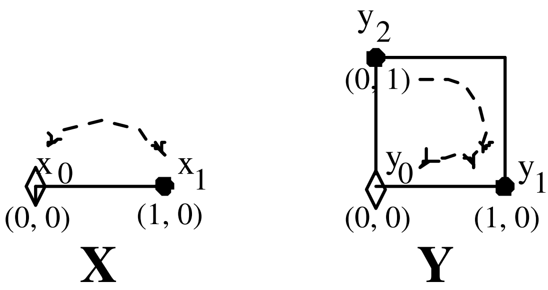

Fig. 5.3 Explanation of the non-*FPP* of the given MA-spaces (X, γ_X) and (Y, γ_Y) in *MAC* [23]

if, for every k-continuous map $f : (X, k) \to (X, k)$, there is a point $x \in X$ such that $f(x) = x$ or $f(x)$ is k-adjacent to x.

Then Rosenfeld [46] studied the *AFPP* of a digital plane (or digital image) (X, k) for k-continuous self-maps f of (X, k). Finally, it turns out that not every digital plane (or digital image) (X, k) for k-continuous self-maps f of (X, k) satisfies the *AFPP* [46].

To study the *AFPP* in *MAC*, we need to introduce the notion of *AFPP* from the viewpoint of *MAC*.

Definition 5.21 ([23]) We say that an MA-space X has the *AFPP* in *MAC* if, for every self-MA-map f of X, there is a point $x \in X$ such that $f(x) = x$ or $f(x)$ is M-adjacent to x.

As referred to in Theorem 5.4, while an MA-space X with $|X| \geq 2$ does not have the *FPP*, we have the *AFPP* of a finite simple MA-path, as follows.

Given an MA-space X, an MA-map preserves an MA-relation on X, we have the following.

Theorem 5.5 ([23]) *In MAC, a finite simple MA-path has the AFPP.*

Motivated by Proposition 5.8, we obtain the following.

Proposition 5.9 ([24]) *In MAC, the AFPP is an MA-isomorphic invariant.*

To study the *AFPP* of X in *MAC*, we often use an MA-retract in [18] (see Definition 5.22).

Definition 5.22 ([18]) In *MAC*, we say that an MA-map $r : (X', \gamma_{X'}) \to (X, \gamma_X)$ is an *MA-retraction* if

(1) (X, γ_X) is a subspace of $(X', \gamma_{X'})$;
(2) $r(a) = a$ for all $a \in (X, \gamma_X)$.

Then we say that (X, γ_X) is an *MA-retract* of $(X', \gamma_{X'})$. Furthermore, we say that the point $a \in X' \setminus X$ is *MA-retractable*.

In view of Definition 5.22, it is clear that an MA-retract holds the reflexivity and the transitivity.

Lemma 5.3 ([23]) *If $X \in Ob(MAC)$ has the AFPP, then its MA-retract $A \in Ob(MAC)$*

In relation to the question (5.7), we have the following.

Proposition 5.10 ([23]) *Not every MA-contractible space in $Ob(MAC)$ has the AFPP.*

To support Proposition 5.10, whereas $SC^4_{MA} := (x_i)_{i \in [0,3]_{\mathbb{Z}}}$ is MA-contractible (see Lemma 5.2), we have an MA-map f of SC^4_{MA} such as $f(x_i) = x_{i+2(mod\,4)}$. Then we see that the map f can have neither the *FPP* nor the *AFPP*.

5.6 *FPP* and *AFPP* for Compact (or Finite) Digital Planes

Let us study both the *FPP* and the *AFPP* for digital planes in *DTC*, *KTC*, or *MTC*, which addresses the issues (b)–(d) posed in Sect. 5.1. As referred to in the part around the property (5.10), the paper [46] studied the *AFPP* of the digital plane with 8-adjacency. Thus, it turns out that not every digital image (X, k) for k-continuous self-maps f of (X, k) satisfies the *AFPP* [46], as follows.

Example 5.2 Consider the map $f : (\mathbb{Z}, 2) \to (\mathbb{Z}, 2)$ given by $f(i) = i + t$ for all $t \in \mathbb{Z}$ with $|t| \geq 2$. Whereas it is a 2-continuous map, $(\mathbb{Z}, 2)$ does not have the *AFPP*.

The recent paper [23] proved that a compact M-topological plane does not have the *FPP* in *MTC*. We may assume a digital plane in $\mathbb{Z}^2$ to be a set $P := \prod_{i \in \{1,2\}} [-l_i, l_i]$ with some $l_i \in \mathbb{N}$. According to the choice of the digital k-connectivity, a K- and an M-topological structure, we can concern the *FPP* and the *AFPP* of digital planes with these structures. For instance, we may consider the compact K-(*resp.* M-) topological plane as the K-(*resp.* M-) space which is homeomorphic to the space (P, κ^2_P) (*resp.* (P, γ_P)). Besides, it is obvious that the infinite M-topological plane $(\mathbb{Z}^2, \gamma)$ does not have the *FPP*.

Definition 5.23 ([22]) We say that a space $(X, \kappa^n_X) := X$ in *KTC* has the *AFPP* if every K-continuous self-map f of X has a point $x \in X$ such that $f(x) = x$ or $f(x)$ is K-adjacent to x.

It is obvious that $SC^{n,l}_k$ does not have the *AFPP* in *DTC* either. Similarly, we have the following.

Corollary 5.1 *Neither $SC^{n,l}_K$ in KTC nor SC^l_M in MTC has the AFPP.*

We say that the *AFPP* is a *K-homeomorphic invariant* in *KTC* if, whenever there exists a K-homeomorphism $h : (X, \kappa^n_X) \to (Y, \kappa^n_Y)$ and (X, κ^n_X) has the *AFPP*, then (Y, κ^n_Y) also has the *AFPP* [22]. Indeed, it turns out that the *AFPP* is a K-homeomorphic invariant, as follows.

Proposition 5.11 ([20]) *The AFPP in KTC is a K-homeomorphic invariant.*

In view of Corollary 5.1, since $SC_K^{n,l}$ does not have the *AFPP*, we obtain the following.

Corollary 5.2 *Not every compact and connected K-topological space has the AFPP.*

In relation to the study of the query (5.1) (c), we have the following.

Theorem 5.6 ([22]) *For every point* $x \in (\mathbb{Z}^n, \kappa^n)$ $SN_K(x)(\subset (\mathbb{Z}^n, \kappa^n))$ *has the FPP.*

Let us study both the *FPP* and the *AFPP* of an M-topological space in *MTC*, which addresses the issue (e) posed in Sect. 5.1.

Definition 5.24 We say that a space $(X, \gamma_X) := X$ in *MTC* has the *AFPP* if every M-continuous self-map f of X has a point $x \in X$ such that $f(x) = x$ or $f(x)$ is M-adjacent to x.

Theorem 5.7 ([25]) *For every point* $x \in (\mathbb{Z}^2, \gamma)$*, every M-connected subspace of* $SN_M(x)$ *has the FPP.*

As a generalization of Theorem 5.7, we obtain the following.

Theorem 5.8 ([25]) *A compact M-topological plane does not have the FPP.*

Proof Unlike the counterexample suggested in [23] (see Fig. 3 of [23]), as another counterexample for Theorem 5.8, let us consider the following self-map f of X in Fig. 5.4 in the present paper as follows (follow the dot arrows in Fig. 5.4):

$$\begin{cases} f(\{x_2, x_4, x_6, x_8\}) = \{x_0\}; \\ f(\{x_5, x_7\}) = \{x_1\}; \\ f(\{x_1, x_3\}) = \{x_7\}; \text{ and} \\ f(\{x_0\}) = \{x_8\}. \end{cases}$$

Then, whereas f is M-continuous, it does not have any fixed point. □

Let us study some properties of K- and M-retracts related to the *FPP* and the *AFPP* for digital spaces in *KTC* or *MTC*.

Definition 5.25 ([20]) In *KTC* we say that a K-continuous map $r : (X', \kappa^n_{X'}) \to (X, \kappa^n_X)$ is a K-retraction if

(1) (X, κ^n_X) is a subspace of $(X', \kappa^n_{X'})$;
(2) $r(a) = a$ for all $a \in X$.

Then we say that (X, κ^n_X) is a *K-retract* of $(X', \kappa^n_{X'})$. Furthermore, we say that the point $a \in X' \setminus X$ is *K-retractable*.

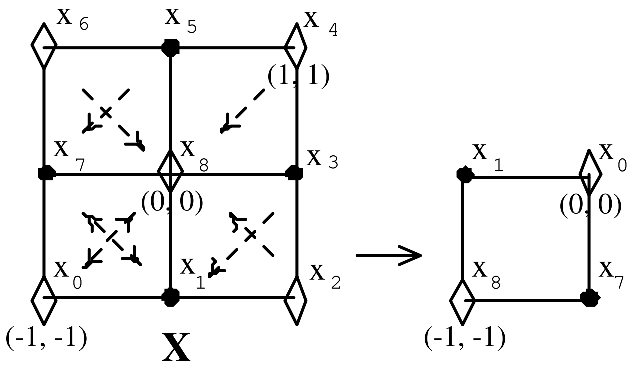

Fig. 5.4 Explanation of the non-*FPP* of a compact *M*-topological plane

Theorem 5.9 ([22]) *In KTC, let* (A, κ_A^n) *be a* K*-retract of* (X, κ_X^n)*. If* (X, κ_X^n) *has the AFPP, then* (A, κ_A^n) *has also the AFPP.*

Definition 5.26 ([18]) In *MTC*, we say that an M-continuous map $r : (X', \gamma_{X'}) \to (X, \gamma_X)$ is an M-retraction if

(1) (X, γ_X) is a subspace of $(X', \gamma_{X'})$;
(2) $r(a) = a$ for all $a \in X$.

Then we say that (X, γ_X) is an M*-retract* of $(X', \gamma_{X'})$. Furthermore, we say that the point $a \in X' \setminus X$ is M*-retractable*.

Theorem 5.10 ([25]) *In MTC, let* (A, γ_A) *be an* M*-retract of* (X, γ_X)*. If* (X, γ_X) *has the AFPP, then* (A, γ_A) *has also the AFPP.*

Theorem 5.11 ([25]) *An* M*-topological plane* (D, γ_D) *does not have the AFPP.*

Proof SC_M^4 does not have the *AFPP*. Furthermore, SC_M^4 is proved an M-retract of any M-topological plane. Owing to the contraposition of Theorem 5.11, the proof is completed. □

5.7 Product Properties of the *FPP* and the *AFPP* for Digital Spaces

To address the queries (5.1)(e), this section proves the following.

Consider M-connected spaces (X_i, γ_{X_i}), where $X_i \subset \mathbb{Z}$ and $2 \leq |X_i| \lneq \infty$ for each $i \in \{1, 2\}$. Then the Cartesian product as an M-topological subspace $(X_1 \times X_2, \gamma_{X_1 \times X_2})$ does not have the *AFPP*. Besides, we compare the *FPP* and the *AFPP* for M-topological spaces and those for digital images in a graph theoretical approach in [46].

Theorem 5.12 ([25]) *Let* (X, γ_X) *and* (Y, γ_Y) *be* M*-topological connected and finite, where* $X, Y \subset \mathbb{Z}$ *and* $|X|, |Y| \geq 2$. *Then* $(X \times Y, \gamma_{X \times Y})$ *does not have the AFPP.*

Remark 5.6 ([25]) Let us compare the *FPP* (*resp. AFPP*) for M-topological spaces (X, γ_X) in *MTC* with the *FPP* (*resp. AFPP*) for digital images (X, k) in *DTC*, $|X| \lneq \infty$ (see [46]). Assume that all M-topological spaces are M-connected and all digital images (X, k) are k-connected.

Let us consider a set $X \subset \mathbb{Z}^2$ with $|X| \geq 2$. Then assume (X, γ_X) in *MTC* and $(X, k), k \in \{4, 8\}$ in *DTC*. Then we can have the following:

(1) While not every (X, γ_X) has the *FPP* (see Theorems 5.7 and 5.8), no digital image (X, k) has the *FPP* (see Proposition 5.1).
Let us consider a finite block $Y \subset \mathbb{Z}^2$, e.g., $Y := [0, m]_{\mathbb{Z}} \times [0, n]_{\mathbb{Z}}, m, n \in \mathbb{N}$. Then we obtain the following:
(2) Every digital image $(Y, 8)$ has the *AFPP* [46].

For instance, consider $Y := [0, 1]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}} = \{(0, 0) := y_0, (1, 0) := y_1, (1, 1) := y_2, (0, 1) := y_3\} \subset \mathbb{Z}^2$. In case we take $(Y, 4)$, it is clear that $(Y, 4)$ cannot have the *AFPP*. To be specific, consider the map given by $f(y_i) = y_{i+2(mod\,4)}$. Then it is clear that f is a 4-continuous map which does not have the *AFPP* [46]. Meanwhile, a finite 4-path in $\mathbb{Z}^2$ has the *AFPP*.

In case we take $(Y, 8)$, it is obvious that $(Y, 8)$ has the *AFPP* [46] because every point $y_i \in Y$ has $N_8(y_i, 1) = Y$. Owing to the 8-continuity of a map, it is clear that any 8-continuous self-map of $(Y, 8)$ has the *AFPP*.

Remark 5.7 Let us consider the finite Cartesian product $X_1 \times X_2$, where $X_i \subset \mathbb{Z}, i \in \{1, 2\}$ and $2 \leq |X_i| \lneq \infty$. Then we have the following:

(1) Neither $(X_1 \times X_2, \gamma_{X_1 \times X_2})$ nor the digital image $(X_1 \times X_2, k), k \in \{4, 8\}$ has the *FPP*.
(2) While $(X_1 \times X_2, \gamma_{X_1 \times X_2})$ does not have the *AFPP* (see Theorem 6.1 of [25]), the digital image $(X_1 \times X_2, 8)$ has the *AFPP* (see Theorem 4.1 of [46]).

Indeed, not every compact and connected K-topological space has the *AFPP*. Thus, we have the following.

Question 5.1 *Does an n-dimensional* K*-topological cube* (D, κ^n_D) *have the FPP or the AFPP ?*

Conjecture 5.1 *A compact* K*-topological plane has the FPP.*

Question 5.2 *Assume that each of digital images* (X_i, k_i) *has the AFPP for each* $i \in \{1, 2\}$. *Under what k-adjacency of* $X_1 \times X_2$ *does it have the AFPP in DTC ?*

5.8 Digital Topological Invariants of the *FPP* and the *AFPP*

Although a digital cube $X(\subset \mathbb{Z}^n)$ with $(3^n - 1)$-connectivity does not have the *FPP*, it has the *AFPP*. This section studies the *FPP* and the *AFPP* of K- (*resp.* M-) topological spaces in *KTC* (*resp. MTC*). The *FPP* in *KTC* has its own feature as follows.

Let us now investigate some properties of K-topological spaces from the viewpoint of fixed point theory.

In *KTC*, we say that a *K-homeomorphic invariant* is a property of a K-topological space which is invariable under K-homeomorphism.

Proposition 5.12 ([22]) *The FPP from the viewpoint of KTC is a K-homeomorphic invariant.*

Proposition 5.13 ([22]) *The AFPP in KTC is a K-homeomorphic invariant.*

Theorem 5.13 ([22]) *In KTC, let (A, κ_A^n) be a K-retract of (X, κ_X^n). If (X, κ_X^n) has the AFPP, then (A, κ_A^n) has also the AFPP.*

The *FPP* in *MTC* also has its own feature as follows.

Theorem 5.14 ([23]) *Not every space $X \in Ob(MTC)$ has the FPP, where X is M-connected and $|X| \geq 2$.*

Example 5.3 ([23]) Consider the map $f : SC_M^l \to SC_M^l$ given by $f(x_i) = x_{i+2(mod\, l)}$. While it is an M-continuous map, SC_M^l does not have the *FPP*, where $SC_M^l := (x_i)_{i\in[0,l-1]_{\mathbb{Z}}}$.

Let us now study some properties of M-topological spaces from the viewpoint of fixed point theory. In *MTC*, we say that an *M-homeomorphic invariant* is a property of an M-topological space which is invariable under M-homeomorphism.

Proposition 5.14 ([24]) *The FPP from the viewpoint of MTC is an M-homeomorphic invariant.*

Proposition 5.15 ([24]) *The AFPP in MTC is an M-homeomorphic invariant.*

Theorem 5.15 ([24]) *In MTC, let (A, γ_A) be an M-retract of (X, γ_X). If (X, γ_X) has the AFPP, then (A, γ_A) has also the AFPP.*

By using Theorem 5.15, we obtain the following.

Corollary 5.3 ([23]) *Not every compact and connected M-topological space has the AFPP.*

5.9 Concluding Remarks and a Further Work

Motivated by the study of the *FPP* and the *AFPP* in *DTC* in [46], we have studied both the *FPP* and the *AFPP* for K-topological spaces in *KTC*, MA-spaces in *MAC*, and further those for M-topological spaces in *MTC*. It turns out that these properties are quite different from those of the typical metric-type fixed point theory.

As a further work, we can study fixed point theory for some other spaces [28, 32] related to digital topology, as follows:

(1) The *FPP* problem of the compactification of the K-(or M-) topological plane.
(2) The *FPP* problem of the compactification of a new topological plane with the Alexandroff topological structure.
(3) The *FPP* of a compact K-topological plane.
(4) The study of some relationships between the contractibility of a given digital space X and the *FPP* and the *AFPP* of X.
(5) Using iterations of a k-DC-self-map of a digital image (X, k), we can estimate the complexity of the *FPP* of a digital image (X, k).
(6) After digitizing a space in $\mathbb{R}^n$ from the viewpoints of K- and M-topologies [5, 11, 26, 31], we can establish some links between the *FPP* of ordinary metric spaces and the *FPP* for their digitized spaces from the viewpoints of K- and M-topologies.
(7) Fixed point theory for digital spaces with multi-valued functions.
(8) After developing an M-homotopy, an M-homotopy equivalence, and M-contractibility, we can use them in fixed point theory.
(9) After generalizing all of a K-homotopy, a K-homotopy equivalence, and K-contractibility, we can use them in fixed point theory.
(10) Development of an Alexandroff topological structure which can be used to study digital spaces.
(11) The study of an alignment of fixed point sets of digital spaces [30] and applications.

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Chapter 6
Existence and Approximations for Order-Preserving Nonexpansive Semigroups over CAT(κ) Spaces

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Abstract In this paper, we discuss the fixed point property for an infinite family of order-preserving mappings which satisfy the Lipschitz condition on comparable pairs. The underlying framework of our main results is a metric space of any global upper curvature bound $\kappa \in \mathbb{R}$, i.e., a CAT(κ) space. In particular, we prove the existence of a fixed point for a nonexpansive semigroup on comparable pairs. Then, we propose and analyze two algorithms to approximate such a fixed point.

Keywords CAT(κ) space · Partially ordered metric space · Nonexpansive semigroup · Fixed point

6.1 Introduction

Metric Fixed Point Theory was assumably started in 1922 by the work of Banach where he introduced the famous Banach Contraction Principle with an application to Cauchy differential equations. This well-known principle applies to every complete metric spaces and has been fruitfully extended to several generalizations of a metric space as well (see [14] for recent results). To appreciate the principle, let us recall that not only the existence and uniqueness of a fixed point is guaranteed, but a simple construction of such fixed point is also provided with a priori error estimates in terms of the contraction constant and the initial data.

As almost a century past, the subjects and objects in Metric Fixed Point Theory grows vastly, but the main theme undeniably roams around the notion of the Lipschitz continuity. Suppose now that (M, d) is a metric space. We say that $T : M \to M$ is *Lipschitz* if there is a constant $L \geq 0$ such that

$$d(Tx, Ty) \leq Ld(x, y) \tag{6.1}$$

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holds for any $x, y \in M$. If (6.1) holds with $L < 1$, we say that T is a *contraction*, while we say that T is *nonexpansive* if (6.1) holds with $L = 1$.

Although many generalizations of the contraction have been carried out, the naive extension to nonexpansive mappings seems not as straightforward as it looks. As a quick glance, we may take $X = \mathbb{R}$ and $Tx = 1 + x$ for each $x \in X$. Then T is nonexpansive with no fixed point. Existence theorems for nonexpansive mappings officially began in 1965, in Hilbert and certain Banach spaces, after the works of Browder, Göhde, and Kirk (see [11] as well as [12, 21, 23]). The results were generalized to a commuting family of nonexpansive mappings by DeMarr [16, 17] and later improved by Lim [27]. Also note some further generalizations.

Assuming the Lipschitz condition (6.1) only on *related* elements has set a new research stream. Ran and Reurings [31] were the first to investigate such situations in the case $L < 1$ and the elements are related with a *partial ordering*, i.e., a relation which is reflexive, antisymmetric, and transitive. Recall that, if $\preceq$ is a partial ordering on a set X, then $x, y \in X$ are said to be $\preceq$-*comparable* if either $x \preceq y$ or $y \preceq x$. The results of Ran and Reurings [31] were later refined and improved by Nieto and Rodríguez-López [28]. These fixed point results were motivated from applications to solve matrix equations and differential equations.

The studies of nonexpansive mappings endowed with a partial ordering in Banach spaces were first considered by Bachar and Khamsi [3] and were complemented with the Mann approximation scheme in [8]. The topic then extended to an order-preserving nonexpansive semigroup [4] under the setting of both Banach and hyperbolic metric spaces. Here, the relationship between approximate fixed point sequences of mappings in the semigroup was thoroughly explained. After that, the full existence result for such semigroups was given in [5, 26] under the framework of Banach spaces and, recently, in [6] for the framework of hyperbolic metric spaces. Finally, an approximation result for this semigroup in Banach spaces was announced by Kozlowski [25] by using the Krasnosel'skiĭ process. It is important to note that Espínola and Wisnicki [20] recently gave a general statement that unifies all the existence results mentioned earlier in Banach spaces. The unification in hyperbolic metric spaces is not known due to an open problem about weak topologies in such spaces (see also [1, 7]).

Let us state the main notions for our study now. Suppose that (X, d) is a metric space endowed with a partial ordering $\preceq$ and $C \subset X$ is nonempty. The family $\Gamma := \{T_t\}_{t \in J}$ of mappings from C into itself, where J is a nontrivial subsemigroup of $[0, \infty)$, is called a $\preceq$-*Lipschitz semigroup on* C if the following conditions are satisfied:

(S1) $T_0 = Id_C$;
(S2) $T_{s+t} = T_s \circ T_t$ for any $s, t \geq 0$;
(S3) For any $x \in C$, the mapping $t \mapsto T_t x$ is continuous;
(S4) For any $t \in J$, T_t preserves $\preceq$ in the sense that $x \preceq y$ implies $T_t x \preceq T_t y$;
(S5) For each $t > 0$, the inequality $d(T_t x, T_t y) \leq Ld(x, y)$ holds whenever $x, y \in C$ are $\preceq$-comparable.

In this paper, we consider a $\preceq$-nonexpansive semigroup for which X is a metric space with curvature bounded above by any $\kappa \in \mathbb{R}$, also known as a CAT(κ) space. Recall that each CAT(κ) space is a hyperbolic metric space if $\kappa \leq 0$. It is, however, unknown to the case $\kappa > 0$.

Our main results can be broke down into three parts. First, we establish an existence result under the assumptions of C being bounded closed and convex. Second, we propose a Krasnosel'skiĭ approximation scheme, similarly to [25], and show its convergence property. Note that the assumptions made in this part are only applicable to discrete (i.e., countable) semigroups. This motivates us to study the final part, where we propose the Browder approximation scheme and show appropriate convergence property. As opposed to the Krasnosel'skiĭ scheme, the Browder scheme is implicit. However, the assumptions for the convergence are less restrictive and apply to any semigroups. The techniques used in this last part are adapted and simplified from [15, 18, 22]. Moreover, to the best of our knowledge, the Browder scheme has not yet been investigated for ordered version of Lipschitz semigroups even in Banach or Hilbert spaces.

The organization of this paper is as follows: The next section collects all the prerequisites of CAT spaces. Sections 6.3–6.5 contain our main materials from Existence Theorems to Explicit and Implicit Approximation Schemes, respectively. The final section then concludes all the results and provides additional remarks and open questions.

6.2 Preliminaries

In this section, we shall recall the prerequisited knowledge for our main results in the next sections. We begin with the notion of geodesic metric spaces and the defining properties of CAT spaces.

Suppose that (X, d) is a metric space. A *geodesic* in X is a curve $c : I \to X$, where $I \subset \mathbb{R}$ is a compact interval, and $d(c(s), c(t)) = |s - t|$ holds for any $s, t \in I$. In other words, a geodesic is curve in X which is isometry to some compact real interval. Without loss of generality, always assume that $I = [0, T]$ for some T. If $c(0) = x$ and $c(T) = y$, we say that c joins x and y. Let $D \in (-\infty, +\infty]$. If any $x, y \in X$ with $d(x, y) < D$ are joined by a geodesic, then X is said to be *D-geodesic*. If such geodesic is unique, then we further say that X is *D-uniquely geodesic*. In the latter case, we write $[\![x, y]\!] := c(I)$ to denote the (unique) geodesic segment. If X is ∞-geodesic (or ∞-uniquely geodesic), we say that X is *geodesic* (or *uniquely geodesic*).

If X is D-uniquely geodesic and $c : [0, T] \to X$ joins x and y, we write $(1 - \lambda)x \oplus \lambda y := c(\lambda T)$ for any $\lambda \in [0, 1]$. If $C \subset X$ and $[\![x, y]\!] \subset C$ for all $x, y \in C$, then C is called *convex*. A function $f : C \to \mathbb{R}$ is called *convex* if C is convex and

$$f((1 - \lambda)x \oplus \lambda y) \leq (1 - \lambda) f(x) + \lambda f(y)$$

for any $x, y \in C$ and $\lambda \in [0, 1]$.

Let $\mathscr{M}_\kappa$ be a simply connected Riemannian 2-surface of constant sectional curvature κ. Denoted by d_κ the intrinsic distance function on $\mathscr{M}_\kappa$, $\measuredangle_q^{(\kappa)}$ the angle at vertex $q \in \mathscr{M}_\kappa$ and by D_κ the diameter of $\mathscr{M}_\kappa$. To be precise, we have $D_\kappa = \infty$ for any $\kappa \leq 0$ and $D_\kappa = \pi/\sqrt{\kappa}$ for any $\kappa > 0$. Note that $\mathscr{M}_0 = \mathbb{R}^2$ and that $\mathscr{M}_1 = \mathbb{S}^2$. To see a more detailed explanation of the subject, refer to [10, 13].

Among other things, the following identity, known as the spherical law of cosines, serves as the main tool for our analysis.

Proposition 6.1 ([10]) *Suppose that Δ is a geodesic triangle in $\mathscr{M}_\kappa$ with $\kappa > 0$. If Δ has side lengths $a, b, c > 0$, and $\gamma > 0$ is the angle opposite to the side with length c. Then*

$$\cos(\sqrt{\kappa}c) = \cos(\sqrt{\kappa}a)\cos(\sqrt{\kappa}b) + \sin(\sqrt{\kappa}a)\sin(\sqrt{\kappa}b)\cos\gamma.$$

Fix $\kappa \in \mathbb{R}$ and let X be D_κ-uniquely geodesic. For any points $p, q, r \in X$, the *geodesic triangle* $\Delta \subset X$ is defined by

$$\Delta(p, q, r) := [\![p, q]\!] \cup [\![q, r]\!] \cup [\![r, p]\!].$$

The geodesic triangle $\overline{\Delta} := \Delta(\overline{p}, \overline{q}, \overline{r})$ with $\overline{p}, \overline{q}, \overline{r} \in \mathscr{M}_\kappa$ is said to be a κ-*comparison triangle* (or, simply, *comparison triangle*) if

$$d_\kappa(\overline{p}, \overline{q}) = d(p, q), \quad d_\kappa(\overline{q}, \overline{r}) = d(q, r), \quad d_\kappa(\overline{r}, \overline{p}) = d(r, p).$$

Note that the triangle inequality of d implies the existence of such comparison triangle. Moreover, the comparison triangle of each geodesic triangle in X is unique up to rigid motions. Suppose that $\Delta(p, q, r) \subset X$ is a geodesic triangle whose comparison triangle is $\Delta(\overline{p}, \overline{q}, \overline{r})$. Given $u \in [\![p, q]\!]$, the point $\overline{u} \in [\![\overline{p}, \overline{q}]\!]$ is said to be a *comparison point* of u if $d_\kappa(\overline{p}, \overline{u}) = d(p, u)$. Comparison points for $u' \in [\![q, r]\!]$ and $u'' \in [\![r, p]\!]$ are defined likewise.

Definition 6.1 Given $\kappa \in \mathbb{R}$. A D_κ-geodesic metric space (X, d) is said to be a $\mathrm{CAT}(\kappa)$ *space* if for each geodesic triangle $\Delta \subset X$ and two points $u, v \in \Delta$, the following $\mathrm{CAT}(\kappa)$ inequality holds:

$$d(u, v) \leq d_\kappa(\overline{u}, \overline{v}),$$

where $\overline{u}, \overline{v} \in \overline{\Delta}$ are the comparison points of u and v, respectively, and $\overline{\Delta} \subset \mathscr{M}_\kappa$ is a κ-comparison triangle of Δ.

Let us give now the following fundamental facts:

Lemma 6.1 ([10, 29]) *Suppose that (X, d) is a complete* $\mathrm{CAT}(\kappa)$ *space. Then, the following are satisfied:*

(1) ([10]) *X is also a* $\mathrm{CAT}(\kappa')$ *space for all* $\kappa' \geq \kappa$.
(2) ([10]) *For each $p \in X$, the function $d(\cdot, p)|_{B(p, D_\kappa/2)}$ is convex.*
(3) ([29]) *If $\kappa > 0$ and $C \subset X$ is nonempty, closed, convex, and bounded with* $\mathrm{diam}(C) < D_\kappa/2$*, then*

$$d^2(p, (1-t)x \oplus ty) \leq (1-t)d^2(p, x) + td^2(p, y) - \frac{k}{2}t(1-t)d^2(x, y)$$

for $p, x, y \in C$ and $t \in [0, 1]$, where $k := 2\,\mathrm{diam}(C)\tan(D_\kappa/2 - \mathrm{diam}(C))$.

Next, let c, c' be two geodesics in a $\mathrm{CAT}(\kappa)$ space X in which $c(0) = c'(0) = p$ and images of both c and c' are not singleton. We define the *Alexandrov angle* between c and c' by

$$\measuredangle_p(c, c') := \limsup_{t, t' \longrightarrow 0^+} \measuredangle_{\overline{p}}^{(\kappa)}(\overline{c(t)}, \overline{c'(t')}),$$

where $\Delta(\overline{p}, \overline{c(t)}, \overline{c'(t')})$ is the κ-comparison triangle of $\Delta(p, c(t), c'(t'))$ for each $t, t' > 0$ near 0. Note that $\measuredangle$ is symmetric and satisfies the triangle inequality whenever all the angles are defined (see [10]).

In 2009, Espínola and Fernández-León [19] studies several results related to fixed point theory and convex analysis. There are two basic results established in this paper. One is the generalization of Δ-convergence to general $\mathrm{CAT}(\kappa)$ spaces (the concept was originally given on $\mathrm{CAT}(0)$ spaces earlier in [24]), and the other is the well-definition of a metric projection. Now, let us recall the notions and properties of the Δ-convergence.

Suppose that (X, d) is a $\mathrm{CAT}(\kappa)$ space and (x^k) a bounded sequence in X. Put $\tau(x; (x^k)) := \limsup_{k \longrightarrow \infty} d(x^k, x)$ for each $x \in X$ and $A(x^k) := \arg\min_X \tau(\cdot; (x^k))$.

Definition 6.2 ([19]) A bounded sequence (x^k) in X is said to be Δ-*convergent* to a point $\overline{x} \in X$ if $A(u^k) = \{\overline{x}\}$ for every subsequence (u^k) of (x^k).

Proposition 6.2 ([19]) *If* $\inf_{x \in X} \tau(x) < D_\kappa/2$*, then the following are satisfied:*
(1) *$A(x^k)$ is singleton.*
(2) *(x^k) contains a Δ-convergent subsequence.*
(3) *If (x^k) is Δ-convergent to $x \in X$, then* $x \in \bigcap_{n \in \mathbb{N}} \mathrm{cl}\,\mathrm{conv}\{x^n, x^{n+1}, \dots\}$.

Finally, let us give the basic results of a metric projection in the following.

Proposition 6.3 ([19]) *Let $C \subset X$ be nonempty, closed, and convex, and $x \in X$ with* $\inf_{c \in C} d(x, c) < D_\kappa/2$*. Then, the following are satisfied:*
(1) *The infimum* $\inf_{c \in C} d(x, c)$ *is uniquely attained. The minimizer is denoted by* $\mathrm{Proj}_C(x)$.
(2) *If $x \notin C$ and $y \in C \setminus \{\mathrm{Proj}_C(x)\}$, then* $\measuredangle_{\mathrm{Proj}_C(x)}(x, y) \geq \pi/2$.

6.3 Existence Theorems

In this section, we establish our first main result—the existence of common fixed points. Before we enter the main part, let us give some notes on the partial ordering first.

6.3.1 Some Introductory Notes

Before we establish the existence of a fixed point of a semigroup $\varGamma$, we need to make an additional assumption on the partial ordering $\preceq$. In particular, we want this partial ordering $\preceq$ to be *compatible* with the CAT structure of X in the following sense:

(A1) For each $u \in X$, the $\preceq$-interval

$$[u, \to) := \{z \in X \mid u \preceq z\}$$

is closed.

(A2) If $a, b, c, d \in X$ satisfy $a \preceq b$ and $c \preceq d$, then $(1-\lambda)a \oplus \lambda c \preceq (1-\lambda)b \oplus \lambda d$ for any $\lambda \in [0, 1]$.

Note that the second assumption implies that $\preceq$-intervals are convex. Moreover, we need the following note in our further investigations.

Lemma 6.2 *If $a \preceq b$ and $0 \leq \lambda \leq \eta \leq 1$, then*

$$(1-\lambda)a \oplus \lambda b \preceq (1-\eta)a \oplus \eta b.$$

Proof Notice that $(1-\lambda)a \oplus \lambda b \preceq b$. The conclusion simply follows from the fact that $(1-\eta)a \oplus \eta b \in [\![(1-\lambda)a \oplus \lambda b, b]\!]$.

If X is a normed linear space, the compatibility with the CAT structure is the same with compatibility with the norm-topology and the linear structure. In particular, suppose that E is a normed linear space. Recall that $K \subset E$ is called a *cone* in E if $\alpha x \in K$ for all $\alpha \geq 0$ whenever $x \in E$. Moreover, a cone K is called pointed if $K \cap (-K) = \{0\}$. When K is a closed convex pointed cone in E, we subsequently have a partial ordering $\sqsubseteq_K$ which is given by

$$a \sqsubseteq_K b \iff b - a \in K,$$

for $a, b \in E$. One can simply notice that $\sqsubseteq_K$ is compatible with the norm-topology and the linear structure in the sense given above.

It seems that the compatibility of $\preceq$ on a general CAT(κ) space is less obvious to be achieved. However, the CAT(κ) spaces which appears practical are often geometrically embedded or isometrically contained in some appropriate linear spaces, which makes the situation less complicated.

As mentioned in the introduction, Espínola and Wiśnicki [20] recently gave a general mechanism for an order-preserving mapping in a Hausdorff topological space to have a fixed point. Their results unify several existence theorems for order-preserving mappings in the literatures assuming similar compatibility including [2, 3, 9]. The results involving a Lipschitz semigroup from [5, 26] are also similarly unified. The key ingredient in such unification is the compactness (in some topology) of the order intervals. In a reflexive normed linear space, every bounded closed convex subset is compact in the weak topology. However, the question of whether or not there is a topology which generates Δ-convergence in CAT spaces is still open (see also [1, 7]). It is therefore safe for now to consider similar existence result in the setting of CAT spaces (or more generally the hyperbolic metric space), as was initiated in [6].

6.3.2 An Existence Theorem

Throughout the rest of this paper, always assume that (X, d) is a complete CAT(κ) space ($\kappa \in \mathbb{R}$) endowed with a partial ordering $\preceq$ which is compatible with the CAT structure. Assume that $C \subset X$ is nonempty, closed, convex, and bounded with $\operatorname{diam}(C) < D_\kappa/2$. Finally, assume that $\Gamma := \{T_t\}_{t\in J}$ is a $\preceq$-nonexpansive semigroup on C.

In view of Lemma 6.1 and the boundedness of C, we can always assume that $\kappa > 0$ so that the κ-spherical law of cosines (Proposition 6.1) is applicable. The following theorem is the main existence result of this section.

Theorem 6.1 *The following statements are true:*

(1) *If there is a point $x^0 \in X$ such that $x^0 \preceq T_t x^0$ for all $t \in J$, then there is $w \in \operatorname{Fix}(\Gamma)$ such that $x^0 \preceq w$.*

(2) *If $z_1, z_2 \in \operatorname{Fix}(\Gamma)$ are $\preceq$-comparable, then $[\![z_1, z_2]\!] \subset \operatorname{Fix}(\Gamma)$.*

Proof (1) The 'only if' part is trivial to see. Let us proof the 'if' part. Suppose that $x^0 \in C$ satisfies $x^0 \preceq T_t x^0$ for all $t \in J$. Set $C_0 := C \cap \left(\bigcap_{t\in J}[T_t x^0, \rightarrow)\right)$. We claim that C_0 is nonempty, closed, and convex. The closedness and convexity of C_0 are obvious since all the sets in the intersection are closed and convex. So we only need to show that C_0 is nonempty. Indeed, suppose that (t_k) is a strictly increasing sequence in J with $t_k \longrightarrow \infty$ as $k \longrightarrow \infty$. It follows that $(T_{t_k} x^0)$ is a sequence in C and is $\preceq$-nondecreasing. By the boundedness of C, there is a subsequence (s_k) of (t_k) in which $(T_{s_k} x^0)$ is Δ-convergent to some point $y \in C$. In view of Proposition 6.2, we have $y \in \bigcap_{n\in\mathbb{N}} \operatorname{cl}\operatorname{conv}\{T_{s_n} x^0, T_{s_{n+1}} x^0, \dots\}$. Since all the $\preceq$-intervals are closed and convex and $(T_{s_n} x^0)$ is $\preceq$-nondecreasing, we further have

$$y \in C \cap \left(\bigcap_{n\in\mathbb{N}} \operatorname{cl}\operatorname{conv}\{T_{s_n} x^0, T_{s_{n+1}} x^0, \dots\}\right) \subset C \cap \left(\bigcap_{n\in\mathbb{N}} [T_{s_n} x^0, \rightarrow)\right).$$

Again, since $(T_t x^0)_{t\in J}$ is $\preceq$-nondecreasing, we have the equality $\bigcap_{n\in\mathbb{N}}[T_{s_n} x^0, \rightarrow) = \bigcap_{t\in J}[T_t x^0, \rightarrow)$ and therefore the set C_0 is nonempty.

Let $p : C_0 \to \mathbb{R}$ be a function defined by

$$p(z) := \limsup_{t \longrightarrow \infty} d^2(T_t x^0, z), \quad \forall z \in C_0.$$

Notice that p is convex and continuous. Since C_0 is bounded, closed, and convex, the function p attains a minimizer $z^* \in C_0$. We may see that $T_t x^0 \preceq z^*$ for all $t \geq 0$. Let $s, t \in J$. By means of Lemma 6.1 and the $\preceq$-nonexpansivity, we have

$$\begin{aligned}
& d^2\left(T_{s+t}x^0, \tfrac{1}{2}z^* \oplus \tfrac{1}{2}T_s z^*\right) \\
& \leq \frac{1}{2}d^2(T_{s+t}x^0, z^*) + \frac{1}{2}d^2(T_{s+t}x^0, T_s z^*) - \frac{k}{8}d^2(z^*, T_s z^*) \\
& \leq \frac{1}{2}d^2(T_{s+t}x^0, z^*) + \frac{1}{2}d^2(T_{r+t}x^0, z^*) - \frac{k}{8}d^2(z^*, T_s z^*),
\end{aligned}$$

where $k := 2\operatorname{diam}(C)\tan(D_\kappa/2 - \operatorname{diam}(C)) \in (0, 2)$. Passing $t \longrightarrow \infty$, since z^* minimizes p, we obtain

$$p(z^*) \leq p\left(\tfrac{1}{2}T_r z^* \oplus \tfrac{1}{2}T_s z^*\right) \leq p(z^*) - \frac{k}{8}d^2(z^*, T_s z^*).$$

This implies $z^* = T_s z^*$ for all $s \in J$. Moreover, since $(T_{s_n}x^0)$ is $\preceq$-nondecreasing and we have $x^0 \preceq z^*$.

(2) Suppose that $z_1, z_2 \in \operatorname{Fix}(\Gamma)$ and that $z_1 \preceq z_2$. We may also assume that $z_1 \neq z_2$, since the conclusion is immediate otherwise. Let $c \in [0, 1]$ and put $z := (1-c)z_1 \oplus cz_2$. By the assumption on $\preceq$, we have $z_1 \preceq z \preceq z_2$. For $t \in J$, we have

$$d(T_t z, z_1) = d(T_t z, T_t z_1) \leq d(z, z_1) = cd(z_1, z_2) \tag{6.2}$$

and also

$$d(T_t z, z_2) = d(T_t z, T_t z_2) \leq d(z, z_2) = (1-c)d(z_1, z_2). \tag{6.3}$$

Using the triangle inequality, we get

$$d(z_1, z_2) \leq d(z_1, T_t z) + d(T_t z, z_2) \leq cd(z_1, z_2) + (1-c)d(z_1, z_2) = d(z_1, z_2).$$

This means $d(z_1, T_t z) + d(T_t z, z_2) = d(z_1, z_2)$, which implies that $T_t z \in [\![z_1, z_2]\!]$. So $T_t z = (1-c')z_1 \oplus c' z_2$ for some $c' \in [0, 1]$. Moreover, we have $d(T_t z, z_1) = c' d(z_1, z_2)$ and $d(T_t z, z_2) = (1-c')d(z_1, z_2)$ for some $c' \in [0, 1]$. Together with (6.2) and (6.3), we obtain $c' \leq c$ and $1 - c' \leq 1 - c$ which yeilds $c' = c$. It follows that $T_t z = z$ for every $t \in J$. Since $c \in [0, 1]$ is taken arbitrarily, we conclude that $[\![z_1, z_2]\!] \subset \operatorname{Fix}(\Gamma)$.

We immediately have the following consequence. Note that this consequence is also new in the setting of a $\mathrm{CAT}(\kappa)$ space.

Corollary 6.1 *Suppose that Γ is a nonexpansive semigroup. Then* $\mathrm{Fix}(\Gamma)$ *is nonempty, closed, and convex.*

6.4 Explicit Approximation Scheme

After we have proved the existence of a fixed point for the semigroup Γ in the previous section, we hereby propose an algorithm to approximate such a solution. The algorithm presented in this section is a modification of the Krasnosel'skiĭ approximation scheme.

Let us now give the formal definition of the Krasnosel'skiĭ approximation scheme associated with Γ as follows: Let $\lambda \in (0, 1)$ and (t_k) be a strictly increasing positive real sequence such that $t \longrightarrow \infty$ as $k \longrightarrow \infty$. Suppose that $x^0 \in X$ has the property $x^0 \preceq T_t x^0$ for all $t \in J$, generate for each $k \in \mathbb{N}$ the successive point

$$x^{k+1} := (1 - \lambda)x^k \oplus \lambda T_{t_k} x^k. \tag{6.4}$$

For this section, always suppose that (x^k) is the sequence given by (6.4) from a point $x^0 \in X$. We shall also refer to this sequence as the *Krasnosel'skiĭ sequence generated from* x^0.

We shall decompose the proof for the convergence of (x^k) into a number of Lemmas as stated in the following.

Lemma 6.3 *The following assertions hold for each* $k \in \mathbb{N}$*:*
(1) $x^k \preceq x^{k+1}$.
(2) $x^k \preceq T_s x^k$ *for* $s \in J$ *with* $s \geq t_k$.
(3) $x^k \preceq T_{t_k} x^k$.

Proof Following from $x^0 \preceq T_{t_0} x^0$, we get

$$x^0 \preceq (1 - \lambda)x^0 \oplus \lambda T_{t_0} x^0 \preceq T_{t_0} x^0 \preceq T_s x^0$$

for all $s \in J$ with $s \geq t_0$. This shows that $x^0 \preceq x^1 \preceq T_s x^0$ for all $s \in J$ with $s \geq t_0$. In particular, we have $x^0 \preceq T_{t_1} x^0$. The conclusion follows by the induction process.

Lemma 6.4 *If* $w \in \mathrm{Fix}(\Gamma)$ *satisfies* $x^0 \preceq w$*, then the limit* $\lim_{k \longrightarrow \infty} d(w, x^k)$ *exists*

Proof Since T_t preserves $\preceq$ and $x^0 \preceq w$, we have $T_t x^0 \preceq T_t w = w$ for each $t \in J$. It follows from Lemma 6.3 that $x^k \preceq T_{t_k} x^0 \preceq w$ for all $k \in \mathbb{N}$. Next, observe that the $\preceq$-nonexpansivity yields

$$\begin{aligned} d(w, x^{k+1}) &= d(w, (1 - \lambda)x^k \oplus \lambda T_{t_k} x^k) \\ &\leq (1 - \lambda)d(w, x^k) + \lambda(w, T_{tk} x^k) \\ &\leq d(w, x^k). \end{aligned}$$

Therefore, the sequence $\big(d(w, x^k)\big)$ is nonincreasing and bounded from below. This shows that the desired limit exists.

Lemma 6.5 *The following limits hold:*
(1) $\lim_{k\longrightarrow\infty} d(x^k, T_{t_k}x^k) = 0.$
(2) $\lim_{k\longrightarrow\infty} d(x^k, x^{k+1}) = 0.$

Proof Let $w \in \mathrm{Fix}(\Gamma)$ satisfies $x^0 \preceq w$.
(1) Observe that, for each $k \in \mathbb{N}$,

$$\begin{aligned} d^2(w, x^{k+1}) &= d^2(w, (1-\lambda)x^k \oplus \lambda T_{t_k}x^k) \\ &\le (1-\lambda)d^2(w, x^k) + \lambda d^2(w, T_{t_k}x^k) - \frac{k_2}{2}\lambda(1-\lambda)d^2(x^k, T_{t_k}x^k) \\ &\le d^2(w, x^k) - \frac{k_2}{2}\lambda(1-\lambda)d^2(x^k, T_{t_k}x^k). \end{aligned}$$

Letting $k \longrightarrow \infty$ and putting $r := \lim_{k\longrightarrow\infty} d(w, x^k)$, we get

$$r^2 \le r^2 - \frac{k_2}{2}\lambda(1-\lambda)\limsup_{k\longrightarrow\infty} d^2(x^k, T_{t_k}x^k).$$

It follows that $\lim_{k\longrightarrow\infty} d(x^k, T_{t_k}x^k) = 0$.
(2) Since $d(x^k, x^{k+1}) = d(x^k, (1-\lambda)x^k \oplus \lambda T_{t_k}x^k) = \lambda d(x^k, T_{t_k}x^k)$, the conclusion follows from (1).

From this point, we need to assume additional conditions on the construction of the sequence (t_k) in relation with the overall structure of the semigroup J. This condition is strong but it allows us to obtain the approximate fixed point sequence.

Lemma 6.6 *Assume that $s \in J$ has the following property:*

There exists a strictly increasing sequence (j_k) of positive integers such that

$$t_{j_{k+1}} = s + t_{j_k} \textit{ for all } k \in \mathbb{N}. \tag{6.5}$$

Then (x^{j_k}) is an approximate fixed point sequence of T_s, i.e.,

$$\lim_{k\longrightarrow\infty} d(x^{j_k}, T_s x^{j_k}) = 0.$$

Proof Suppose that $k \in \mathbb{N}$ is sufficiently large so that $t_{j_k} > s$. In view of (6.3) and the $\preceq$-nonexpansivity of T_s, we have

$$\begin{aligned} d(x^{j_{k+1}}, T_s x^{j_{k+1}}) &\le d(x^{j_{k+1}}, T_{t_{j_{k+1}}} x^{j_{k+1}}) + d(T_{t_{j_{k+1}}} x^{j_{k+1}}, T_s x^{j_{k+1}}) \\ &= d(x^{j_{k+1}}, T_{t_{j_{k+1}}} x^{j_{k+1}}) + d(T_s T_{t_{j_k}} x^{j_{k+1}}, T_s x^{j_{k+1}}) \\ &\le d(x^{j_{k+1}}, T_{t_{j_{k+1}}} x^{j_{k+1}}) + d(T_{t_{j_k}} x^{j_{k+1}}, x^{j_{k+1}}). \end{aligned}$$

Letting $k \longrightarrow \infty$ and applying Lemma 6.5, we get $\lim_{k\longrightarrow\infty} d(x^{j_k}, T_s x^{j_k}) = 0$.

Lemma 6.7 *Suppose that $s \in J$ has the property (6.5) and assume further that*

$$\sup_{k\in\mathbb{N}}(j_k - k) < \infty.$$

Then, the following assertions hold:
(1) $\lim_{k\longrightarrow\infty} d(x^k, x^{j_k}) = 0$.
(2) (x^k) *is an approximate fixed point sequence for* T_s.

Proof (1) Put $P := \sup_{k\in\mathbb{N}}(j_k - k)$. If $P = 0$, then the conclusion is already verified. Hence, assume that $P > 0$. Let $\varepsilon > 0$ be chosen arbitrarily. From Lemma 6.5, we know that $d(x^k, x^{k+1}) < \varepsilon/P$ holds for any k sufficiently large. For such large $k \in \mathbb{N}$, we have

$$d(x^k, x^{j_k}) \leq \sum_{i=k}^{j_k-1} d(x^i, x^{i+1}) < (j_k - k)\frac{\varepsilon}{P} \leq P \cdot \frac{\varepsilon}{P} = \varepsilon.$$

This proves $\lim_{k\longrightarrow\infty} d(x^k, x^{j_k}) = 0$.
(2) By Lemma 6.3 and the $\preceq$-nonexpansivity, we have

$$\begin{aligned} d(x^k, T_s x^k) &\leq d(x^k, x^{j_k}) + d(x^{j_k}, T_s x^{j_k}) + d(T_s x^{j_k}, T_s x^k) \\ &\leq d(x^k, x^{j_k}) + d(x^{j_k}, T_s x^{j_k}) + d(x^{j_k}, x^k). \end{aligned}$$

Letting $k \longrightarrow \infty$ and applying the earlier fact (1) and Lemma 6.6, we have

$$\lim_{k\longrightarrow\infty} d(x^k, T_s x^k) = 0,$$

which is the desired result.

After having gathered all the technical lemmas required for the convergence result, we now state and prove the main theorem of this section.

Theorem 6.2 *Assume that all $s \in J$ has the property (6.5) with $\sup_{k\in\mathbb{N}}(j_k - k) < \infty$. Then, the Krasnosel'skiĭ sequence (x^k) generated from x^0 is Δ-convergent to a point $w \in \mathrm{Fix}(\Gamma)$ with $x^0 \preceq w$.*

Proof First, note that the boundedness of C implies the boundedness of (x^k). So (x^k) contains a Δ-convergent subsequence. Suppose that (y^k) and (z^k) be two Δ-convergent subsequences of (x^k) whose Δ-limits are y^* and z^*, respectively. Suppose that $y^* \neq z^*$.

Since (x^k) is $\preceq$-nondecreasing, we have $y^k \preceq y^*$ and $z^k \preceq z^*$ for all $k \in \mathbb{N}$. Let $s \in J$. From Lemma 6.7, we have

$$\limsup_{k\longrightarrow\infty} d(y^k, T_s y^*) \leq \limsup_{k\longrightarrow\infty} d(y^k, T_s y^k) + \limsup_{k\longrightarrow\infty} d(T_s y^k, T_s y^*)$$
$$\leq \limsup_{k\longrightarrow\infty} d(y^k, y^*).$$

Since y^* is the unique asymptotic center of (y^k), it follows that $y^* = T_s y^*$. With the same arguments, we also have $z^* = T_s z^*$. Since $s \in J$ is arbitrary, we have $y^*, z^* \in \mathrm{Fix}(\Gamma)$ with $x^0 \preceq y^*$ and $x^0 \preceq z^*$. Set $r_1 := \lim_{k\longrightarrow\infty} d(x^k, y^*)$ and $r_2 := \lim_{k\longrightarrow\infty} d(x^k, z^*)$, where the existence of such limits follows from Lemma 6.4. From the fact that (y^k) and (z^k) are subsequences of (x^k) and the uniqueness of the asymptotic center, we have

$$r_1 = \lim_{k\longrightarrow\infty} d(x^k, y^*) = \lim_{k\longrightarrow\infty} d(y^k, y^*) < \limsup_{k\longrightarrow\infty} d(y^k, z^*) = r_2$$

and also

$$r_2 = \lim_{k\longrightarrow\infty} d(x^k, z^*) = \lim_{k\longrightarrow\infty} d(z^k, z^*) < \lim_{k\longrightarrow\infty} d(z^k, y^*) = r_1.$$

This gives a contradiction, and therefore it must be the case that $y^* = z^*$. In other words, (x^*) has only one Δ-accumulation point, denoted with w. Similarly, we have $x^k \preceq w$ for all $k \in \mathbb{N}$. Let $s \in J$. The Lemma 6.7 yields

$$\limsup_{k\longrightarrow\infty} d(x^k, T_s w) \leq \limsup_{k\longrightarrow\infty} d(x^k, T_s x^k) + \limsup_{k\longrightarrow\infty} d(T_s x^k, T_s w)$$
$$\leq \limsup_{k\longrightarrow\infty} d(x^k, w).$$

The uniqueness of the asymptotic center guarantees that $w = T_s w$ and further that $w \in \mathrm{Fix}(\Gamma)$. Additionally, the fact that (x^k) is $\preceq$-nondecreasing yields $x^0 \preceq w$.

6.5 Implicit Approximation Scheme

In the previous section, we deal with the Krasnosel'skiĭ approximation scheme1 where the computation of each iterate can be carried out explicitly by a specific formula. In this section, we present another route to approximate a solution $w \in \mathrm{Fix}(\Gamma)$ by using the Browder approximation schemes which is of different nature to the Krasnosel'skiĭ approximation scheme. In the Browder approximation scheme, there is no specific closed form for each iterate. However, it can be simply computed by the use of Picard's procedure.

Also note again that we have not seen Browder approximation in this setting even when the space is linear. Since a Hilbert space is CAT(0), our next main theorem applies there.

The construction and several properties of the algorithm studied in this section are based on a theorem of Nieto and Rodríguez-López [28].

Theorem 6.3 ([28]) *Let (X, d) be a complete metric space that is endowed with a partial ordering $\preceq$ with the following property:*

$$\text{If a } \preceq -\text{nondecreasing sequence } (x^k) \text{ in } X \text{ converges to } x^*,$$

$$\text{then } x^k \preceq x^* \text{ for each } k \in \mathbb{N}. \tag{6.6}$$

Suppose that $T : X \to X$ is a mapping in which (6.1) holds for each $x, y \in X$ that are $\preceq$-comparable with $L < 1$. If there is a point $x^0 \in X$ such that $x^0 \preceq Tx^0$, then
(1) *T has a fixed point.*
(2) *The orbit $(T^k x^0)$ converges to a fixed point $w \in \mathrm{Fix}(T)$.*
(3) *$T^k x^0 \preceq w$ for all $k \in \mathbb{N}$.*

Recall again that $\Gamma := \{T_t\}_{t \in J}$ is a $\preceq$-nonexpansive semigroup on a bounded closed convex set $C \subset X$, and $x^0 \in C$ is fixed with the property $x^0 \preceq T_t x^0$ for all $t \in J$. At each $t \in J$ and $\lambda \in (0, 1)$, we define $T_t^\lambda := (1 - \lambda)T_t \oplus \lambda x^0$.

Let us now give the following simple facts which are essential in our main construction in this section.

Lemma 6.8 *For each $\lambda \in (0, 1)$ and $t \in J$, the following facts hold:*
(1) *T_t^λ is a $\preceq$-contraction with constant $(1 - \lambda)$.*
(2) *T_t^λ is $\preceq$-nondecreasing.*
(3) *$x^0 \preceq T_t^\lambda x^0 \preceq T_t x^0$.*

Proof (1) Let $x, y \in X$ with $x \preceq y$. We have

$$\begin{aligned} d(T_t^\lambda x, T_t^\lambda y) &= d((1-\lambda)T_t x \oplus \lambda x^0, (1-\lambda)T_t y \oplus \lambda x^0) \\ &\leq (1-\lambda)d(T_t x, T_t y) \leq d(x, y). \end{aligned}$$

This shows the $\preceq$-contractivity of T_t^λ.

(2) Let $x, y \in X$ with $x \preceq y$. Since T_t is $\preceq$-nondecreasing, it is immediate to see that

$$T_t^\lambda x = (1-\lambda)T_t x \oplus \lambda x^0 \preceq (1-\lambda)T_t y \oplus \lambda x^0 = T_t^\lambda y.$$

(3) Since $x^0 \preceq T_t x^0$, we have $x^0 \preceq (1-\lambda)T_t x^0 \oplus x^0 = T_t^\lambda x^0 \preceq T_t x^0$.

The following fact is obvious from the results aforestated. However, we collect it here for convenience and explicitly.

Lemma 6.9 *Let $\lambda \in (0, 1)$ and $t \in J$. Then, $\lim_{n \longrightarrow \infty} (T_t^\lambda)^n = x_t^\lambda \in \mathrm{Fix}(T_t^\lambda)$ with $(T_t^\lambda x^0)^n \preceq x_t^\lambda$ for all $n \in \mathbb{N}$.*

Proof Since all $\preceq$-intervals are closed, the condition (6.6) is satisfied. Apply Lemma 6.8 and Theorem 6.3 to arrive at the conclusion.

Now, let us define the Browder approximation associated with Γ. Suppose that (λ_k) a strictly decreasing sequence in $(0, 1)$ and (t_k) is a strictly increasing sequence of positive reals. In this situation, we adopt the notions $T^{[k]} := T_{t_k}^{\lambda_k}$ for each $k \in \mathbb{N}$. Next, generate for each $k \in \mathbb{N}$ the successive point

$$x^k := \lim_{n \longrightarrow \infty} (T^{[k]})^n x^0.$$

In this case, the sequence (x^k) is called the *Browder sequence generated from* x^0. One may observe from Lemmas 6.8 and 6.9 that for each $k \in \mathbb{N}$, $x^k \in \mathrm{Fix}(T^{[k]})$ and $(T^{[k]})^n x^0 \preceq x^k$ for all $n \in \mathbb{N}$. Also, we can see that $((T^{[k]})^n x^0)$ is $\preceq$-nondecreasing.

For a technical reason, assume throughout this section that $t^0 \in J \setminus \{0\}$ and $t^{k+1} := 2t^k$ for $k \in \mathbb{N}$.

Lemma 6.10 *The following assertions hold for each* $k \in \mathbb{N}$*:*

(1) $x^k \preceq x^{k+1}$.
(2) $x^k \preceq T_{t_k} x^k$.

Proof (1) Fix $k \in \mathbb{N}$. Since T_{t_k} is $\preceq$-nondecreasing, we apply Lemma 6.8 and obtain

$$T_{t_k} T^{[k]} x^0 \preceq T_{t_k} T_{t_k} x^0 = T_{2t_k} x^0 = T_{t_{k+1}} x^0 \preceq T_{t_{k+1}} T^{[k+1]} x^0.$$

Again, since T_{t_k} is $\preceq$-nondecreasing and (λ_k) is strictly decreasing, we further have

$$\begin{aligned}(T^{[k]})^2 x^0 &= (1 - \lambda_k) T_{t_k} T^{[k]} x^0 \oplus \lambda_k x^0 \preceq (1 - \lambda_k) T_{t_{k+1}} T^{[k+1]} x^0 \oplus \lambda_k x^0 \\ &\preceq (1 - \lambda_{k+1}) T_{t_{k+1}} T^{[k+1]} x^0 \oplus \lambda_{k+1} x^0 = (T^{[k+1]})^2 x^0.\end{aligned}$$

Now, let $n \in \mathbb{N}$ be an integer such that the statement $(T^{[k]})^n x^0 \preceq (T^{[k+1]})^n x^0$ holds true. We may observe using similar facts that

$$\begin{aligned}T_{t_k} (T^{[k]})^n x^0 &\preceq T_{t_k} T_{t_k} (T^{[k]})^{n-1} x^0 = T_{t_{k+1}} (T^{[k]})^{n-1} x^0 \\ &\preceq T_{t_{k+1}} (T^{[k]})^{n-1} T^{[k]} x^0 = T_{t_{k+1}} (T^{[k]})^n x^0 \\ &\preceq T_{t_{k+1}} (T^{[k+1]})^n x^0.\end{aligned}$$

Similarly, using the facts that T_{t_k} is $\preceq$-nondecreasing and (λ_k) is strictly decreasing, we get

$$\begin{aligned}(T^{[k]})^{n+1} x^0 &= (1 - \lambda_k) T_{t_k} (T^{[k]})^n x^0 \oplus \lambda_k x^0 \preceq (1 - \lambda_k) T_{t_{k+1}} (T^{[k+1]})^n x^0 \oplus \lambda_k x^0 \\ &\preceq (1 - \lambda_{k+1}) T_{t_{k+1}} (T^{[k+1]})^n x^0 \oplus \lambda_{k+1} x^0 = (T^{[k+1]})^{n+1} x^0.\end{aligned}$$

Hence, the mathematical induction implies

$$(T^{[k]})^n x^0 \preceq (T^{[k+1]})^n x^0 \tag{6.7}$$

for each $n \in \mathbb{N}$.

Next, recall that Theorem 6.3 gives $\lim_{n\longrightarrow\infty}(T^{[k]})^n x^0 = x^k$ and $(T^{[k]})^n x^0 \preceq x^k$ for all $n, k \in \mathbb{N}$. Taking (6.7) into account, we see now that $((T^{[k]})^n x^0)$ is a sequence in the $\preceq$-interval $(\leftarrow, x^{k+1}]$, which is a closed set. Therefore, the point x^k belongs to $(\leftarrow, x^{k+1}]$ as the limit of $((T^{[k]})^n x^0)$ and we conclude here that $x^k \preceq x^{k+1}$ for any $k \in \mathbb{N}$.

(2) Fix $k \in \mathbb{N}$. Since $x^k \in \mathrm{Fix}(T^{[k]})$, we have

$$x^k = (1-\lambda_k)T_{t_k}x^k \oplus \lambda_k x^0.$$

Recall that $x^0 \preceq x^k$ and $x^0 \preceq T_s x^0$ for all $s \in J$. If $t \in J$ and $t \geq t_k$, we have $x^0 \preceq T_{t_k}x^0 \preceq T_{t_k}x^k$, which further yields

$$x^k = (1-\lambda_k)T_{t_k}x^k \oplus \lambda_k x^0 \preceq (1-\lambda_k)T_{t_k}x^k \oplus \lambda_k T_{t_k}x^k = T_{t_k}x^k.$$

□

Before we go further, let us consider for a while an ordinary metric space (Y, p) and a family $\varXi := \{S_t\}_{t\in J}$ of self-mappings on a bounded subset $K \subset Y$, indexed by a nontrivial subsemigroup J of $[0, \infty)$. The following notions and lemma are variants to the similar definition given by Huang [22] for which J is not necessarily the same as $[0, \infty)$. The proof is carried out in the same way so we leave it to the reader.

Definition 6.3 The family $\varXi$ is called *asymptotically regular* (or briefly, *AR*) if for any $h \in J$ and $y \in K$, the following limit holds:

$$\lim_{\substack{t\in J\\ t\longrightarrow\infty}} d(T_t y, T_h T_t y) = 0.$$

Moreover, it is called *uniformly asymptotically regular* (or brieftly, *UAR*) if for any $h \in J$, the following limit holds:

$$\lim_{\substack{t\in J\\ t\longrightarrow\infty}} \sup_{y\in K} d(T_t y, T_h T_t y) = 0.$$

Lemma 6.11 *If $\varXi$ is AR and $S_t S_{t'} = S_{t+t'}$ for $t, t' \in J$, then* $\mathrm{Fix}(\varXi) = \mathrm{Fix}(S_t)$ *for any $t \in J$.*

Now we get back to our main result.

Theorem 6.4 *Assume that (λ_k) is a strictly decreasing sequence in* $(0, 1)$ *with the limit* $\lim_{k\longrightarrow\infty}\alpha_k = 0$*, and (t_k) is a sequence given by $t_{k+1} = 2t_k$ for $k \in \mathbb{N}$ with $t_0 \in J \setminus \{0\}$. Also suppose that $\varGamma$ is UAR. Then, the Browder sequence converges strongly to $y \in$* $\mathrm{Fix}(\varGamma)$ *with $x^0 \preceq y$. Moreover, if $q \in$* $\mathrm{Fix}(\varGamma)$ *satisfies $v^k \preceq q$ at each $k \in \mathbb{N}$ for some subsequence (v^k) of (x^k), then $d(x^0, y) \leq d(x^0, q)$.*

Proof Note first that if $x^0 \in \mathrm{Fix}(\Gamma)$, then $x^k = x^0$ for all $k \in \mathbb{N}$. Now, consider the case $x^0 \notin \mathrm{Fix}(\Gamma)$. Since C is bounded, (x^k) contains a subsequence (y^k) which is Δ-convergent to some point $y \in C$. Note that $y^k \preceq y$ for any $k \in \mathbb{N}$. Suppose that (β_k) and (s_k) are respective subsequences of (λ_k) and (t_k) for which $y^k = (1-\beta_k)T_{s_k}y^k \oplus \beta_k x^0$ for all $k \in \mathbb{N}$. Fix any $t \in J$. Then, Lemma 6.10 and the convexity of d on C implies

$$\begin{aligned} d(T_t y, y^k) &\le d(T_t y, T_t y^k) + d(T_t y^k, T_t T_{t_k} y^k) + d(T_t T_{t_k} y^k, y^k) \\ &\le d(y, y^k) + d(y^k, T_{t_k} y^k) + \beta_k d(x^0, T_t T_{t_k} y^k) + (1-\beta_k) d(T_{t_k} y^k, T_t T_{t_k} y^k) \\ &= d(y, y^k) + \beta_k d(x^0, T_{t_k} y^k) + \beta_k d(x^0, T_t T_{t_k} y^k) + (1-\beta_k) d(T_{t_k} y^k, T_t T_{t_k} y^k). \end{aligned}$$

Letting $k \longrightarrow \infty$, from $\lim_{k\longrightarrow\infty} \beta_k = 0$ and Γ being UAR, we get

$$\limsup_{k\longrightarrow\infty} d(T_t y, y^k) \le \limsup_{k\longrightarrow\infty} d(y, y^k).$$

By the uniqueness of the asymptotic center, we have $y \in \mathrm{Fix}(T_t)$. Lemma 6.11 implies further that $y \in \mathrm{Fix}(\Gamma)$.

Next, we claim that (y^k) contains a strongly convergent subsequence. Let us suppose to the contrary that $\limsup_{k\longrightarrow\infty} d(y, y^k) = \sigma > 0$. For $k \in \mathbb{N}$, since $y^k \preceq y = T_{s_k} y$, we have

$$\begin{aligned} d(y, y^k) &\le \beta_k d(y, x^0) + (1-\beta_k) d(y, T_{s_k} y^k) \\ &\le \beta_k d(y, x^0) + (1-\beta_k) d(y, y^k). \end{aligned}$$

Passing $k \longrightarrow \infty$, one obtain

$$\limsup_{k\longrightarrow\infty} d(y, T_{s_k} y^k) = \sigma.$$

By passing to a subsequence, we assume, without the loss of generality, that $y^k \ne x^0$ for all $k \in \mathbb{N}$. Recall that

$$d(x^0, y^k) = (1-\beta_k) d(x^0, t_{s_k} y^k) < d(x^0, T_{s_k} y^k).$$

Since $y^k \in [\![x^0, T_{s_k} y^k]\!]$, we have

$$d(y^k, T_{s_k} y^k) = d(x^0, T_{s_k} y^k) - d(x^0, y^k) > 0.$$

Note that $x^0 \ne y$ since $x^0 \notin \mathrm{Fix}(\Gamma)$. The uniqueness of the asymptotic center yields

$$\limsup_{k\longrightarrow\infty} d(x^0, y^k) > \limsup_{k\longrightarrow\infty} d(y, y^k) = \sigma > 0. \tag{6.8}$$

Again, by passing to a subsequence, we may assume that (y^k) has the following property for all $k \in \mathbb{N}$:

$$d(x^0, y^k) > 0, \quad d(y, y^k) > 0, \quad d(y, T_{s_k} y^k) > 0. \tag{6.9}$$

For each $k \in \mathbb{N}$, let $\Delta(\overline{x^0}, \overline{y}, \overline{T_{s_k} y^k})$ be the κ-comparison triangle of $\Delta(x^0, y, T_{s_k} y^k)$ that share the common side $[\![\overline{x^0}, \overline{y}]\!]$. In view of (6.8) and (6.9), the κ-angles $\measuredangle^{(\kappa)}_{\overline{y_k}}(\overline{x^0}, \overline{y})$, $\measuredangle^{(\kappa)}_{\overline{y_k}}(\overline{x^0}, \overline{T_{s_k} y^k})$, and $\measuredangle^{(\kappa)}_{\overline{y_k}}(\overline{y}, \overline{T_{s_k} y^k})$ exist, where $\overline{y^k}$ is the corresponding comparison point for y^k. We claim that $\measuredangle^{(\kappa)}_{\overline{y_k}}(\overline{x^0}, \overline{y}) \geq \pi/2$. Let us assume to the contrary that $\measuredangle^{(\kappa)}_{\overline{y_k}}(\overline{x^0}, \overline{y}) < \pi/2$. Since $\measuredangle^{(\kappa)}_{\overline{y_k}}(\overline{x^0}, \overline{T_{s_k} y^k}) = \pi$, this also implies that $\measuredangle^{(\kappa)}_{\overline{y_k}}(\overline{y}, \overline{T_{s_k} y^k}) \geq \pi/2$. On one hand, we have

$$\begin{aligned} \cos\sqrt{\kappa} d_\kappa(\overline{y}, \overline{T_{s_k} y^k}) &= \cos\sqrt{\kappa} d_\kappa(\overline{y^k}, \overline{T_{s_k} y^k}) \cos\sqrt{\kappa} d_\kappa(\overline{y}, \overline{y^k}) \\ &\quad + \sin\sqrt{\kappa} d_\kappa(\overline{y^k}, \overline{T_{s_k} y^k}) \sin\sqrt{\kappa} d_\kappa(\overline{y}, \overline{y^k}) \cos\measuredangle^{(\kappa)}_{\overline{y_k}}(\overline{x^0}, \overline{y}) \\ &< \cos\sqrt{\kappa} d_\kappa(\overline{y^k}, \overline{T_{s_k} y^k}), \end{aligned}$$

which means $d_\kappa(\overline{y^k}, \overline{T_{s_k} y^k}) < d_\kappa(\overline{y}, \overline{T_{s_k} y^k})$. On the other hand, the fact that $y^k \preceq y$ gives

$$\begin{aligned} d_\kappa(\overline{y}, \overline{T_{s_k} y^k}) &= d(y, T_{s_k} y^k) = d(T_{s_k} y, T_{s_k} y^k) \\ &\leq d(y, y^k) \leq d_\kappa(\overline{y}, \overline{y^k}), \end{aligned}$$

which contradicts the earlier inequality. Therefore, it must be the case that $\measuredangle^{(\kappa)}_{\overline{y_k}}(\overline{x^0}, \overline{y}) \geq \pi/2$.

Again, by the κ-spherical law of cosines (Proposition 6.1), we have

$$\begin{aligned} \cos\sqrt{\kappa} d_\kappa(\overline{x^0}, \overline{y}) &= \cos\sqrt{\kappa} d_\kappa(\overline{x^0}, \overline{y^k}) \cos\sqrt{\kappa} d_\kappa(\overline{y^k}, \overline{y}) \\ &\quad + \sin\sqrt{\kappa} d_\kappa(\overline{x^0}, \overline{y^k}) \sin\sqrt{\kappa} d_\kappa(\overline{y^k}, \overline{y}) \cos\measuredangle^{(\kappa)}_{\overline{y_k}}(\overline{x^0}, \overline{y}) \\ &\leq \cos\sqrt{\kappa} d_\kappa(\overline{x^0}, \overline{y^k}) \cos\sqrt{\kappa} d_\kappa(\overline{y^k}, \overline{y}). \end{aligned} \tag{6.10}$$

Since $0 < d(x^0, y^k) \leq d_\kappa(\overline{x^0}, \overline{y^k})$, (6.10) further yields

$$d_\kappa(\overline{y^k}, \overline{y}) < d_\kappa(\overline{x^0}, \overline{y}). \tag{6.11}$$

By the diameter assumption on C, the point

$$u^k := \mathrm{Proj}_{[\![x^0, y]\!]}\, y^k$$

is well defined for each $k \in \mathbb{N}$. Thus, (u^k) is a sequence in $[\![x^0, y]\!]$. Since every geodesic interval is isometry to a compact interval in $\mathbb{R}$, we pass again to a subsequece and assume that (u^k) is strongly convergent to a point $u \in [\![x^0, y]\!]$. Using the definitions of an asymptotic center and a projection, we obtain

$$\begin{aligned}\sigma &= \limsup_{k\longrightarrow\infty} d(y, y^k) \le \limsup_{k\longrightarrow\infty} d(u, y^k)\\ &\le \limsup_{k\longrightarrow\infty} d(u, u^k) + \limsup_{k\longrightarrow\infty} d(u^k, y^k)\\ &= \limsup_{k\longrightarrow\infty} d(u^k, y^k)\\ &\le \limsup_{k\longrightarrow\infty} d(y, y^k) = \sigma.\end{aligned}$$

This shows $u = y$. Passing again to a subsequence, we may assume that $d(u^k, y^k) > \sigma/2$ for all $k \in \mathbb{N}$. For $k \in \mathbb{N}$, let $\overline{u^k}$ and $\overline{u}$ b comparison points for u^k and u, respectively, in the comparison triangle $\Delta(\overline{x^0}, \overline{y}, \overline{y^k})$ of $\Delta(x^0, y, y^k)$. Note that $u^k \neq x^0$ for all $k \in \mathbb{N}$. Otherwise, the Proposition 6.3 gives

$$\measuredangle^{(\kappa)}_{\overline{x^0}}(\overline{y}, \overline{y^k}) = \measuredangle^{(\kappa)}_{\overline{u^k}}(\overline{y}, \overline{y^k}) \ge \pi/2.$$

Note that the angles above are defined in view of facts we derived earlier. By the κ-spherical law of cosines (Proposition 6.1), we subsequently get

$$\begin{aligned}\cos\sqrt{\kappa}d_\kappa(\overline{y}, \overline{y^k}) &= \cos\sqrt{\kappa}d_\kappa(\overline{x^0}, \overline{y})\cos\sqrt{\kappa}d_\kappa(\overline{x^0}, \overline{y^k})\\ &\qquad + \sin\sqrt{\kappa}d_\kappa(\overline{x^0}, \overline{y})\sin\sqrt{\kappa}d_\kappa(\overline{x^0}, \overline{y^k})\cos\measuredangle^{(\kappa)}_{\overline{x^0}}(\overline{y}, \overline{y^k})\\ &\le \cos\sqrt{\kappa}d_\kappa(\overline{x^0}, \overline{y}).\end{aligned}$$

This means $d_\kappa(\overline{x^0}, \overline{y}) \le d_\kappa(\overline{y}, \overline{y^k})$, which contradicts with (6.11). Thus $u^k \neq x^0$ for all $k \in \mathbb{N}$. This shows that the angle $\gamma_k := \measuredangle^{(\kappa)}_{\overline{u^k}}(\overline{x^0}, \overline{y^k})$ is well defined and the Proposition 6.3 implies that $\gamma_k \ge \pi/2$ for all $k \in \mathbb{N}$. Apart from this, we also define for each $k \in \mathbb{N}$ the following quantities:

$$a_k := d_\kappa(\overline{x^0}, \overline{u^k}), \quad b_k := d_\kappa(\overline{u^k}, \overline{y^k}), \quad c_k := d_\kappa(\overline{x^0}, \overline{y^k}).$$

We may see now that

$$\sigma/2 < b_k \le c_k < d_\kappa(\overline{x^0}, \overline{y})$$

at each $k \in \mathbb{N}$. By the κ-spherical law of cosines (Proposition 6.1), we obtain

$$\begin{aligned}\cos\sqrt{\kappa}c_k &= \cos\sqrt{\kappa}a_k\cos\sqrt{\kappa}b_k + \sin\sqrt{\kappa}a_k\sin\sqrt{\kappa}b_k\cos\gamma_k\\ &\le \cos\sqrt{\kappa}a_k\cos\sqrt{\kappa}b_k.\end{aligned}$$

The two inequalities above implies

$$\cos\sqrt{\kappa}a_k \geq \frac{\cos\sqrt{\kappa}c_k}{\cos\sqrt{\kappa}b_k} > \frac{\cos\sqrt{\kappa}d_\kappa(\overline{x^0},\overline{y})}{\cos\sqrt{\kappa}(\sigma/2)} > \cos\sqrt{\kappa}d_\kappa(\overline{x^0},\overline{y}).$$

For convenience, we put

$$\delta := \frac{1}{\sqrt{\kappa}}\arccos\left(\frac{\cos\sqrt{\kappa}d_\kappa(\overline{x^0},\overline{y})}{\cos\sqrt{\kappa}(\sigma/2)}\right).$$

Note that δ is independent of $k \in \mathbb{N}$. Hence, we get $a_k < \delta < d_\kappa(\overline{x^0},\overline{y})$ and then

$$\begin{aligned} d(y,u^k) = d_\kappa(\overline{y},\overline{u^k}) &= d_\kappa(\overline{x^0},\overline{y}) - d_\kappa(\overline{x^0},\overline{u^k}) \\ &> d_\kappa(\overline{x^0},\overline{y}) - \delta > 0. \end{aligned}$$

This shows that $(d(y,u^k))$ is bounded away from 0, which together implies that $u \neq y$. This is a contradiction. Therefore, the sequence (y^k) is convergent to y. Since all subsequence of (x^k) contains a subsequent convergent to y, we conclude that (x^k) converges to $y \in \mathrm{Fix}(\Gamma)$. Since (y^k) is $\preceq$-nondecreasing, we have $x^0 \preceq y$.

Next, we show the second conclusion. Suppose that $q \in \mathrm{Fix}(\Gamma)$ satisfies $v^k \preceq q$ at each $k \in \mathbb{N}$, for some subsequence (v^k) of (x^k). Let (β_k) and (s_k) be the subsequences of (λ_k) and (t_k), respectively, in which $v^k = (1-\beta_k)T_{s_k}v^k \oplus \beta_k x^0$ for $k \in \mathbb{N}$. We may also assume that $v^k \neq x^0$ at all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $\Delta(\overline{q},\overline{x^0},\overline{T_{s_k}v^k})$ be the comparison triangle of $\Delta(q,x^0,T_{s_k}v^k)$ that share the common side $[\![\overline{q},\overline{x^0}]\!]$. Observe that we have $d(T_{s_k}v^k,q) \leq d(v^k,q)$. If $\measuredangle^{(\kappa)}_{\overline{v^k}}(\overline{q},\overline{T_{s_k}v^k}) > \pi/2$, then we further have

$$\begin{aligned} \cos\sqrt{\kappa}d_\kappa(\overline{q},\overline{T_{s_k}v^k}) &= \cos\sqrt{\kappa}d_\kappa(\overline{q},\overline{v^k})\cos\sqrt{\kappa}d_\kappa(\overline{v^k},\overline{T_{s_k}v^k}) \\ &\quad + \sin\sqrt{\kappa}d_\kappa(\overline{q},\overline{v^k})\sin\sqrt{\kappa}d_\kappa(\overline{v^k},\overline{T_{s_k}v^k})\cos\measuredangle^{(\kappa)}_{\overline{v^k}}(\overline{q},\overline{T_{s_k}v^k}) \\ &< \cos\sqrt{\kappa}d_\kappa(\overline{q},\overline{v^k}) \leq \cos\sqrt{\kappa}d_\kappa(\overline{q},\overline{T_{s_k}v^k}), \end{aligned}$$

which is absurd. Hence, it must be the case that $\measuredangle^{(\kappa)}_{\overline{v^k}}(\overline{q},\overline{T_{s_k}v^k}) \leq \pi/2$. If follows that $\measuredangle^{(\kappa)}_{\overline{v^k}}(\overline{x^0},\overline{q}) > \pi/2$. Again, from the κ-spherical law of cosines (Proposition 6.1), we have

$$\begin{aligned} \cos\sqrt{\kappa}d_\kappa(\overline{x^0},\overline{q}) &= \cos\sqrt{\kappa}d_\kappa(\overline{q},\overline{v^k})\cos\sqrt{\kappa}d_\kappa(\overline{x^0},\overline{v^k}) \\ &\quad + \sin\sqrt{\kappa}d_\kappa(\overline{q},\overline{v^k})\sin\sqrt{\kappa}d_\kappa(\overline{x^0},\overline{v^k})\cos\measuredangle^{(\kappa)}_{\overline{v^k}}(\overline{x^0},\overline{q}) \\ &\leq \cos\sqrt{\kappa}d_\kappa(\overline{x^0},\overline{v^k}). \end{aligned}$$

Subsequently, we may see that

$$d(x^0,v^k) = d_\kappa(\overline{x^0},\overline{v^k}) \leq d_\kappa(\overline{x^0},\overline{q}) = d(x^0,q).$$

The final conclusion follows by letting $k \longrightarrow \infty$.

6.6 An Example

In this section, we give a validating example to confirm our main results. This example is a sample application of our explicit algorithm presented in Sect. 6.4.

Example 6.1 Let us consider the model space of constant curvature $= 1$ which is represented by the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ endowed with the spherical metric $d(x, y) := \arccos\langle x, y\rangle$ for $x, y \in \mathbb{S}^2$. Put $p := (1, 0, 0) \in \mathbb{S}^2$ and take $C := \overline{B}(p, r)$, the closed ball around p of radius r, where $0 < r < \pi/4$. Obviously, C is a complete CAT(1) space with respect the restriction of d. Moreover, we have $\operatorname{diam}(C) < \pi/2 = D_1/2$. Define the partial order $\preceq$ on C by

$$x \preceq y \iff x_2 \leq y_2 \text{ and } x_3 \leq y_3$$

for all $x, y \in C$, where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. It is not difficult to see that $\preceq$ is compatible with the CAT structure.

Consider the semigroup $J := \mathbb{N} \cup \{0\}$. Define for each $t \in J$ a mapping $T_t : C \to C$ given by

$$T_t(x) := \left(1 - \frac{1}{2^t}\right) p \oplus \frac{1}{2^t} x$$

for all $x \in C$. Then the family $\Gamma := \{T_t\}_{t\in J}$ is a $\preceq$-nonexpansive semigroup (see [30, Lemma 3.3]) and $\operatorname{Fix}(\Gamma) = \{p\}$.

Next, we shall verify all the assumptions of Theorem 6.2 and validate the theorem. Firstly, the semigroup J is seen to satisfy all the requirements in Theorem 6.2 with $s = 1$. On the other hand, a point $x^0 \in C$ has the property $x^0 \preceq T_t x^0$ for all $t \in J$ if and only if the $\preceq$-interval $[x^0, \rightarrow)$ contains the point p. Equivalently, if and only if $x_2^0 \leq 0$ and $x_3^0 \leq 0$ where $x^0 = (x_1^0, x_2^0, x_3^0) \in C$. We may see that the iterates $x^{k+1} := (1 - \lambda)x^k \oplus \lambda T_{t_k} x^k$ converges to $p \in \operatorname{Fix}(\Gamma)$ for any $x^0 \in C$. However, to obtain the domination property $x^0 \preceq p$, we need to start from x^0 for which $p \in [x^0, \rightarrow)$. The properties discussed above are all in accordance with the Theorem 6.2. In addition, we obtain from the construction of (x^k) the following rate of convergence:

$$\begin{aligned} d(x^k, \operatorname{Fix}(\Gamma)) = d(x^k, p) &= \left(1 - \lambda + \frac{\lambda}{2^{t_{k-1}}}\right) d(x^{k-1}, p) \\ &\leq d(x^0, p) \prod_{i=1}^{k-1} \left(1 - \lambda + \frac{\lambda}{2^{t_i}}\right) \\ &\leq (1 - \lambda/2)^{k-1} d(x^0, p). \end{aligned}$$

6.7 Conclusions and Remarks

As a quick summary, we have established an existence theorem for the class of $\preceq$-nonexpansive semigroups. Then, we proposed two approximation schemes, the Krasnosel'skiĭ's and the Browder's. The first is explicit but works only with discrete semigroup while the second is implicit but works in any semigroups. However, there are still limitations in terms of generality. In the non-ordered case (over both linear and nonlinear spaces), the choices of parameter sequences (λ_k) and (t_k) are more freely available. Based on these inspections, we shall pose here the following open questions:

(Q1) How to generalize parameter conditions on (t_k) of the Krasnosel'skiĭ approximation to any semigroups not necessarily discrete?
(Q2) How to generalize the parameter conditions on (λ_k) and (t_k) in the Browder approximation?

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Chapter 7
A Solution of the System of Integral Equations in Product Spaces via Concept of Measures of Noncompactness

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Abstract In this chapter, we present the role of measures of noncompactness and related fixed point results to study the existence of solutions for the system of integral equations of the form

$$x_i(t) = a_i(t) + f_i(t, x_1(t), x_2(t), \ldots, x_n(t)) + g_i(t, x_1(t), x_2(t), \ldots, x_n(t)) \int_0^{\alpha(t)} k_i(t, s, x_1(s), x_2(s), \ldots, x_n(s)))ds,$$

for all $t \in \mathbb{R}_+,\ x_1, x_2, \ldots, x_n \in E = BC(\mathbb{R}_+)$ and $1 \leq i \leq n$. We mainly focus on introducing new notion of $\mu - (F, \varphi, \psi)-$set contractive operator and establishing some new generalization of Darbo fixed point theorem and Krasnoselskii fixed point result associated with measures of noncompactness. Moreover, we deal with a system of fractional integral equations when k_i is defined in a fractal space.

Keywords Measures of noncompactness · Set contractive map · Integral equations · Darbo fixed point · Krasnoselskii fixed point

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7.1 Introduction

The integral equation creates a very important and significant part of the mathematical analysis and has various applications into real-world problems. On the other hand, measures of noncompactness are very useful tools in the wide area of functional analysis such as the metric fixed point theory and the theory of operator equations in Banach spaces. These are also used in the studies of functional equations, ordinary and partial differential equations, fractional partial differential equations, integral and integro-differential equations, optimal control theory, and others (see [1–7, 12, 19–23]). In our investigations, we apply the technique of measures of noncompactness in order to generalize the Darbo fixed point theorem [14], and we also study the existence of solutions for the following system of integral equations:

$$\begin{aligned} x_i(t) &= a_i(t) + f_i(t, x_1(t), x_2(t), \ldots, x_n(t)) \\ &\quad + g_i(t, x_1(t), x_2(t), \ldots, x_n(t)) \int_0^{\alpha(t)} k_i(t, s, x_1(s), x_2(s), \ldots, x_n(s)))ds, \end{aligned} \tag{7.1}$$

for all $t \in \mathbb{R}_+,\ x_1, x_2, \ldots, x_n \in E = BC(\mathbb{R}_+)$ and $1 \leq i \leq n$.

The present work generalizes the existing related work available in the literature. Further, we generalize our system into fractal integral equations when k_i is defined in a fractal space.

7.2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this work. Denote $\mathbb{R}$, the set of real numbers and $\mathbb{R}_+ = [0, +\infty)$. Let $(E, \|\cdot\|)$ be a real Banach space with zero element 0. Let $\overline{B}(x, r)$ denote the closed ball centered at x with radius r. The symbol $\overline{B}_r$ stands for the ball $\overline{B}(0, r)$. For X, a nonempty subset of E, we denote by $\overline{X}$ and $ConvX$, the closure and the closed convex hull of X, respectively. Moreover, let us denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

In 1930, Kuratowski suggested a new direction for the researchers with the introduction of measure of noncompactness (MNC, for short) [18]. The MNC joins some algebraic arguments, studies the mathematical formulations and solves the existence of solutions for some nonlinear problems involving certain conditions [18].

We use the following definition of the measure of noncompactness given in [14].

Definition 7.1 A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a *measure of noncompactness* in E if it satisfies the following conditions:

(1^0) The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{N}_E$;

(2^0) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$;

(3^0) $\mu(\overline{X}) = \mu(X)$;
(4^0) $\mu(ConvX) = \mu(X)$;
(5^0) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for all $\lambda \in [0, 1]$;
(6^0) If (X_n) is a sequence of closed sets from m_E such that $X_{n+1} \subset X_n$ $(n = 1, 2, \ldots)$ and, if $\lim\limits_{n\longrightarrow\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $ker\mu$ defined in axiom (1^0) is called the kernel of the measure of noncompactness μ.

One of the properties of the measure of noncompactness is $X_\infty \in ker\mu$. Indeed, from the inequality $\mu(X_\infty) \leq \mu(X_n)$ for $n = 1, 2, 3, \ldots,$ we infer that $\mu(X_\infty) = 0$. Further facts concerning measures of noncompactness and their properties may be found in [13, 14].

The *Kuratowski measure of noncompactness* is the map $\alpha : \mathfrak{M}_E \to \mathbb{R}^+$ with

$$\alpha(\mathcal{Q}) = \inf\left\{\varepsilon > 0 : \mathcal{Q} \subset \bigcup_{k=1}^{n} S_k,\, S_k \subset E,\, diam(S_k) < \varepsilon\ (k \in \mathbb{N})\right\}. \tag{7.2}$$

For all $\mathcal{Q} \in \mathfrak{M}_E$, the *Hausdorff measure of noncompactness*

$$\chi(\mathcal{Q}) = \inf\{\varepsilon > 0 : \mathcal{Q} \text{ can be covered by a finite number of balls of radii } \varepsilon\}. \tag{7.3}$$

A continuous mapping $T : X \to X$ is called a *densifying map* if, for any bounded set $\mathcal{Q}$ with $\mu(\mathcal{Q}) > 0$, we have $\mu(T(\mathcal{Q})) < \mu(\mathcal{Q})$. If $\mu(T(\mathcal{Q}) \leq k\mu(\mathcal{Q}), 0 < k < 1$, then T is a $k-$set contraction.

If $\mu(T(\mathcal{Q}) \leq \mu(\mathcal{Q})$, then T is said to be $1-$ *set contraction*.

A nonexpansive map is an example of $1-$set contraction. A contraction map is densifying and so is a compact mapping. In the history, there are results in fixed point theory that dealt with combination of two maps, for instance, $T_1 + T_2$, where T_1 is a contraction map and T_2 is a compact map.

If both T_1 and T_2 are continuous functions, then $T_1 + T_2$ is also a continuous map and the fixed point theorem for continuous map is applicable for $T_1 + T_2$. However, if T_1 is a contraction map, then Banach fixed point theorem is applied, and if T_2 is a compact map, then Schauder fixed point theorem is applicable. If T_1 is densifying and T_2 is densifying, then $T_1 + T_2$ is also densifying.

Darbo [17] used the notion of MNC for the first time and defined some classes of operators. He proved the generalized Schauder's and Banach's fixed point theorems. Krasnoselskii combined Schauder's and Banach's fixed point theorems together in one result (see [9–11, 15, 16]).

Theorem 7.1 ([1]) *Let C be a closed, convex subset of a Banach space E. Then every compact, continuous map $T : C \to C$ has at least one fixed point.*

In the following, we state a fixed point theorem of Darbo type proved by Banaś and Goebel [14]:

Theorem 7.2 *Let C be a nonempty, closed, bounded, and convex subset of the Banach space E and $T : C \to C$ be a continuous mapping. Assume that there exist a constant $k \in [0, 1)$ such that $\mu(TX) \leq k\mu(X)$ for any nonempty subset X of C. Then T has a fixed point in the set C.*

The following concept of $\mathscr{O}(f; .)$ and its examples was given by Altun and Turkoglu [8]:

Let $\mathscr{F}([0, \infty))$ be class of all function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and let Θ be class of all operators

$$\mathscr{O}(\bullet; .) : \mathscr{F}(\mathbb{R}_+) \longrightarrow \mathscr{F}(\mathbb{R}_+), \quad f \to \mathscr{O}(f; .)$$

satisfying the following conditions:

(a) $\mathscr{O}(f; t) > 0$ for $t > 0$ and $\mathscr{O}(f; 0) = 0$;
(b) $\mathscr{O}(f; t) \leq \mathscr{O}(f; s)$ for $t \leq s$;
(c) $\lim_{n\to\infty} \mathscr{O}(f; t_n) = \mathscr{O}(f; \lim_{n\to\infty} t_n)$;
(d) $\mathscr{O}(f; max\{t, s\}) = max\{\mathscr{O}(f; t), \mathscr{O}(f; s)\}$ for some $f \in \mathscr{F}(\mathbb{R}_+)$.

Example 7.1 If $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a Lebesgue-integrable mapping which is finite integral on each compact subset of $\mathbb{R}_+$, non-negative and such that for each $t > 0$, $\int_0^t f(s)ds > 0$, then the operator defined by

$$\mathscr{O}(f; t) = \int_0^t f(s)ds$$

satisfies the above conditions.

7.3 Main Results

In this section, we introduce a new notion of a μ-set contraction and establish new results for said notion.

In the sequel, we fix the set of functions by $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\psi, \varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

(a) F is nondecreasing, continuous and $F(0, 0) = 0 < F(t, s)$ for every $t, s > 0$;
(b) φ is continuous;
(c) ψ is a nondecreasing function such that $\lim\limits_{n\longrightarrow\infty} \psi^n(t) = 0$ for each $t \geq 0$.

Define $\mathbb{F} = \{F : F$ satisfies (a)$\}$, $\Phi = \{\varphi : \varphi$ satisfies (b)$\}$ and $\Psi = \{\psi : \psi$ satisfies (c)$\}$.

As a result, we state the existence of a fixed point for a continuous (but not necessarily compact) operator satisfying a $\mu(F, \varphi, \psi)$-set contractive condition.

Theorem 7.3 *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E, and $T : C \to C$ be a continuous and $\mu(F, \varphi, \psi)$-set contractive operator, that is,*

$$\mathscr{O}(f; F(\mu(T(X)), \varphi(\mu(TX)))) \leq \psi[\mathscr{O}(f; F(\mu(X), \varphi(\mu(X))))] \tag{7.4}$$

for any nonempty subset X of C, where μ is an arbitrary measure of noncompactness, $F \in \mathbb{F}$, $\varphi \in \Phi$, $\mathscr{O}(\bullet; .) \in \Theta$ and $\psi \in \Psi$. Then T has at least one fixed point in C.

Proof Let $C_0 = C$, we construct a sequence $\{C_n\}$ such that $C_{n+1} = Conv(TC_n)$, for $n \geq 0$. Then $TC_0 = TC \subseteq C = C_0$, $C_1 = Conv(TC_0) \subseteq C = C_0$. Continuing this process we have

$$C_0 \supseteq C_1 \supseteq \cdots \supseteq C_n \supseteq C_{n+1} \supseteq \cdots.$$

If there exists a natural number N such that $\mu(C_N) = 0$, then C_N is compact. In this case, Theorem 7.1 implies that T has a fixed point. So we assume that $\mu(C_n) \neq 0$ for $n = 0, 1, 2, \ldots$. Also, by (7.4), we have

$$\begin{aligned} \mathscr{O}(f; F(\mu(C_{n+1}), \varphi(\mu(C_{n+1})))) &= \mathscr{O}(f; F(\mu(Conv(TC_n)), \varphi(\mu(Conv(TC_n))))) \\ &= \mathscr{O}(f; F(\mu(TC_n), \varphi(\mu(TC_n)))) \\ &\leq \psi[\mathscr{O}(f; F(\mu(C_n), \varphi(\mu(C_n))))] \\ &\leq \psi^2[\mathscr{O}(f; F(\mu(C_{n-1}), \varphi(\mu(C_{n-1}))))] \\ &\ldots \\ &\leq \psi^n[\mathscr{O}(f; F(\mu(C_0), \varphi(\mu(C_0))))] \\ &= \psi^n[\mathscr{O}(f; F(\mu(C), \varphi(\mu(C))))]. \end{aligned} \tag{7.5}$$

Taking the limit $n \longrightarrow \infty$ in (7.5), we have

$$\lim_{n\to\infty} \mathscr{O}(f; F(\mu(C_{n+1}), \varphi(\mu(C_{n+1})))) = 0$$

and so

$$\lim_{n\to\infty} F(\mu(C_{n+1}), \varphi(\mu(C_{n+1}))) = 0 \Longrightarrow \lim_{n\to\infty} \mu(C_{n+1}) = 0.$$

Since $C_n \supseteq C_{n+1}$ and $TC_n \subseteq C_n$ for all $n = 1, 2, \ldots$, then, from (6^0), $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is a nonempty convex closed set, invariant under T and belongs to $Ker\mu$. Therefore, Theorem 7.1 completes the proof.

An immediate consequence of Theorem 7.3 is the following:

Theorem 7.4 *Let C be a nonempty bounded closed and convex subset of a Banach space E, $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and $T : C \to C$ be continuous functions. Suppose that there exists a constant $0 < \lambda < 1$ such that, for all $\varnothing \neq X \subseteq C$,*

$$\mathscr{O}(f; F(\mu(T(X)), \varphi(\mu(TX)))) \leq \lambda[\mathscr{O}(f; F(\mu(X), \varphi(\mu(X)))],$$

where μ is an arbitrary measure of noncompactness $F \in \mathbb{F}$, $\varphi \in \Phi$ and $\mathscr{O}(\bullet;.) \in \Theta$. Then T has at least one fixed point in C.

Taking $F(t,s) = t + s$ in Theorem 7.3, we obtain the following:

Theorem 7.5 *Let C be a nonempty bounded closed and convex subset of a Banach space E, $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and $T : C \to C$ be continuous functions such that, for all $\varnothing \neq X \subseteq C$,*

$$\mathscr{O}(f; \mu(T(X)) + \varphi(\mu(TX)))) \leq \psi[\mathscr{O}(f; \mu(X) + \varphi(\mu(X)))],$$

where μ is an arbitrary measure of noncompactness $\psi \in \Psi$, $\varphi \in \Phi$ and $\mathscr{O}(\bullet;.) \in \Theta$. Then T has at least one fixed point in C.

Theorem 7.6 *Let C be a nonempty bounded closed and convex subset of a Banach space E, $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and $T : C \to C$ be continuous functions such that, for all $\varnothing \neq X \subseteq C$,*

$$F(\mu(T(X)), \varphi(\mu(TX)))) \leq \psi[F(\mu(X), \varphi(\mu(X)))],$$

where μ is an arbitrary measure of noncompactness, $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathbb{F}$. Then T has at least one fixed point in C.

The following corollary gives us a fixed point theorem with a contractive condition of integral type:

Corollary 7.1 *Let C be a nonempty bounded closed and convex subset of a Banach space E and $T : C \to C$ be a continuous operator such that, for any $\varnothing \neq X \subseteq C$,*

$$\int_0^{F(\mu(T(X)),\varphi(\mu(TX))))} f(s)\, ds \leq \psi\Big(\int_0^{F(\mu(X),\varphi(\mu(X)))} f(s)\, ds\Big),$$

where μ is an arbitrary measure of noncompactness and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of $\mathbb{R}_+$, non-negative and such that, for each $\varepsilon > 0$, $\int_0^{\varepsilon} f(s)\, ds > 0$, $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathbb{F}$. Then T has at least one fixed point in C.

Next we present a Krasnoselskii type fixed point result.

Theorem 7.7 *Let $(E, \|\cdot\|)$ be a Banach space and X be a closed convex subsets of E. Let $T_1, T_2 : X \to X$ be two operators satisfying the following conditions:*
(a) *$(T_1 + T_2)(X) \subseteq X$;*
(b) *there exist $F \in \mathbb{F}$, $\varphi \in \Phi$, $\mathscr{O}(\bullet;.) \in \Theta$ and $\psi \in \Psi$ such that, for all $u, v \in X$,*

$$\mathscr{O}(f; F(\|T_1u - T_1v\|), \varphi(\|T_1u - T_1v\|))) \leq \psi[\mathscr{O}(f; F(\|u - v\|, \varphi(\|u - v\|)))]; \quad (7.6)$$

(c) *T_2 is a continuous and compact operator.*
Then $\mathscr{J} := T_1 + T_2 : X \to X$ has a fixed point $\hat{u} \in X$.

Proof Suppose that M is a subset of X with $\mu(M) > 0$. By the notion of Kuratowski MNC, for each $n \in \mathbb{N}$, there exist bounded subsets $\mathscr{C}_1, \ldots, \mathscr{C}_{m(n)}$ such that $M \subseteq \bigcup_{i=1}^{m(n)} \mathscr{C}_i$ and $diam(\mathscr{C}_i) \leq \mu(M) + \frac{1}{n}$. Suppose that $\mu(T_1(M)) > 0$. Since $T_1(M) \subseteq \bigcup_{i=1}^{m(n)} T_1(\mathscr{C}_i)$, there exists $i_0 \in \{1, 2, \ldots, m(n)\}$ such that $\mu(T_1(M)) \leq diam(T_1(\mathscr{C}_{i_0}))$. Using (7.6) condition of T_1 with discussed arguments, we have

$$\begin{aligned} &\mathscr{O}(f; F(\mu(T_1(M)), \varphi(\mu(T_1 M)))) \\ &\leq \mathscr{O}(f; F(diam(T_1(\mathscr{C}_{i_0})), \varphi(diam(T_1(\mathscr{C}_{i_0}))))) \\ &\leq \psi[\mathscr{O}(f; F(diam(\mathscr{C}_{i_0}), \varphi(diam(\mathscr{C}_{i_0}))))] \\ &\leq \psi\left[\mathscr{O}\left(f; F\left(\mu(M) + \frac{1}{n}, \varphi\left(\mu(M) + \frac{1}{n}\right)\right)\right)\right]. \end{aligned} \tag{7.7}$$

Passing to the limit $n \to \infty$ in (7.7), we get

$$\mathscr{O}(f; F(\mu(T_1(M)), \varphi(\mu(T_1 M)))) \leq \psi[\mathscr{O}(f; F(\mu(M), \varphi(\mu(M))))].$$

Using the condition (c), we have, by the notion of μ, that

$$\begin{aligned} &\mathscr{O}(f; F(\mu(\mathscr{J}(M)), \varphi(\mu(\mathscr{J}(M))))) \\ &= \mathscr{O}(f; F(\mu(T_1(M) + T_2(M)), \varphi(\mu(T_1(M) + T_2(M))))) \\ &\leq \mathscr{O}(f; F(\mu(T_1(M)) + \mu(\mathscr{T}_2(M)), \varphi(\mu(T_1(M)) + \mu(T_2(M))))) \\ &= \mathscr{O}(f; F(\mu(T_1(M)), \varphi(\mu(T_1(M))))) \\ &\leq \psi[\mathscr{O}(f; F(\mu(M), \varphi(\mu(M))))]. \end{aligned}$$

Thus, by Theorem 7.3, $\mathscr{J}$ has a fixed point $\hat{u} \in X$. This completes the proof.

Here, we recall a useful theorem concerning the construction of a measure of noncompactness on a finite product space.

Theorem 7.8 ([14]) *Suppose that $\mu_1, \mu_2, \ldots, \mu_n$ are the measures in $E_1, E_2, \ldots, E_n$ respectively. Moreover, assume that the function $F : {\mathbb{R}_+}^n \to \mathbb{R}_+$ is convex and $F(x_1, x_2, \ldots, x_n) = 0$ if and only if $x_i = 0$ for each $i = 1, 2, \ldots, n$. Then*

$$\mu(X) = F(\mu_1(X_1), \mu_2(X_2), \ldots, \mu_n(X_n))$$

defines a measure of noncompactness in $E_1 \times E_2 \times \cdots \times E_n$, where X_i denotes the natural projection of X into E_i for $i = 1, 2, \ldots, n$.

Example 7.2 Let $\mu_i (1 \leq i \leq n)$ be the measures of noncompactness in Banach spaces $E_1, E_2, \ldots, E_n$, respectively. Considering $F_1(x_1, \ldots, x_n) = k \max\{x_1, x_2, \ldots, x_n\}$ and $F_2(x_1, \ldots, x_n) = k(x_1 + \cdots + x_n), k \in \mathbb{R}_+$ for any $(x_1, \ldots, x_n) \in \mathbb{R}^n_+$, then all the conditions of Theorem 7.8 are satisfied. Therefore, $\widetilde{\mu}_1 := k \max\{\mu(X_1), \mu(X_2), \ldots, \mu(X_n)\}$ and $\widetilde{\mu}_2 := k(\mu(X_1) + \cdots + \mu(X_n))$ define measures of noncompactness in the space $E_1 \times E_2 \times \cdots \times E_n$, where X_i, $i = 1, 2, \ldots, n$, denote the natural projections of X into E_i.

Our next main result is the n-tuple Darbo fixed point result in product spaces.

Theorem 7.9 *Let C_i be a nonempty bounded convex and closed subset of a Banach space E_i $(i = 1, 2, \ldots, n)$ and $T_i : C_1 \times C_2 \times \cdots \times C_n \longrightarrow C_i$ $(i = 1, 2, \ldots, n)$ be a continuous operator such that*

$$\mathscr{O}(f; F(\mu(T_i(X_1 \times X_2 \times \cdots \times X_n)), \varphi(\mu(T_i(X_1 \times X_2 \times \cdots \times X_n)))) \leq \psi[\mathscr{O}(f; F(\max_j \mu(X_j), \varphi(\max_j \mu(X_j))))] \tag{7.8}$$

for any subset $\varnothing \neq X_i$ of C_i, where μ_i is an arbitrary measure of noncompactness on E_i $(i = 1, 2, \ldots, n)$, $F \in \mathbb{F}$, $\psi \in \Psi$, $\mathscr{O}(\bullet; .) \in \Theta$, $\varphi \in \Phi$ and nondecreasing. Then there exist $(x_1^*, x_2^*, \ldots, x_n^*) \in C_1 \times C_2 \times \cdots \times C_n$ such that, for all $1 \leq i \leq n$,

$$T_i(x_1^*, x_2^*, \ldots, x_n^*) = x_i^*. \tag{7.9}$$

Proof First, note that, from Example 7.2, $\widetilde{\mu}$ defined by

$$\widetilde{\mu}(X) = \max\{\mu(X_1), \mu(X_2), \ldots, \mu(X_n)\},$$

for any bounded subset $X \subset E_1 \times E_2 \times \cdots \times E_n$, where X_i $(i = 1, 2, \ldots, n)$ denote the natural projections of X into E_i defines a measure of noncompactness on $E_1 \times E_2 \times \cdots \times E_n$. Also, we define a mapping $\widetilde{T} : C_1 \times C_2 \times \cdots \times C_n \longrightarrow C_1 \times C_2 \times \cdots \times C_n$ as follows:

$$\widetilde{T}(x_1, x_2, \ldots, x_n) = (T_1(x_1, x_2, \ldots, x_n), T_2(x_1, x_2, \ldots, x_n), \ldots, T_n(x_1, x_2, \ldots, x_n)).$$

It is obvious that $\widetilde{T}$ is continuous. Now, we claim that $\widetilde{T}$ satisfies all the conditions of Theorem 7.3. To prove this, let X be any nonempty and bounded subset of $C_1 \times C_2 \times \cdots \times C_n$. Then by (2°), (7.8) and the fact that $\varphi(max\{a, b\}) = max\{\varphi(a), \varphi(b)\}$ and $F(max\{a\}, max\{b\}) = max F(a, b)$ for $a, b \in [0, +\infty)$, we obtain

$$\begin{aligned}
&\mathscr{O}(f; F(\widetilde{\mu}(\widetilde{T}(X)), \varphi(\widetilde{\mu}(\widetilde{T}(X)))))\\
&\leq \mathscr{O}(f; F(\widetilde{\mu}(T_1(X_1 \times X_2 \times \cdots \times X_n) \times \cdots \times T_n(X_1 \times X_2 \times \cdots \times X_n)),\\
&\quad \varphi(\widetilde{\mu}(T_1(X_1 \times X_2 \times \cdots \times X_n) \times \cdots \times T_n(X_1 \times X_2 \times \cdots \times X_n))))\\
&= \mathscr{O}(f; F(\max_{1 \leq k \leq n} \mu(T_k(X_1 \times X_2 \times \cdots \times X_n)), \varphi(\max_{1 \leq k \leq n} \mu(T_k(X_1 \times X_2 \times \cdots \times X_n))))\\
&\leq \max_{1 \leq k \leq n} [\mathscr{O}(f; F(\mu(T_k(X_1 \times X_2 \times \cdots \times X_n))), \varphi(\mu(T_k(X_1 \times X_2 \times \cdots \times X_n)))))]\\
&\leq \max_{1 \leq k \leq n} \psi[\mathscr{O}(f; F(\max_{1 \leq i \leq n} \mu(X_i), \varphi(\max_{1 \leq i \leq n} \mu(X_i))))]\\
&= \psi[\mathscr{O}(f; F(\widetilde{\mu}(X), \varphi(\widetilde{\mu}(X))))].
\end{aligned}$$

Hence, from Theorem 7.3, $\widetilde{T}$ has at least one fixed point, i.e., there exists $(x_1^*, x_2^*, \ldots, x_n^*) \in C_1 \times C_2 \times \cdots \times C_n$ such that

$$\begin{aligned}(x_1^*, x_2^*, \ldots, x_n^*) &= \widetilde{T}(x_1^*, x_2^*, \ldots, x_n^*)\\ &= (T_1(x_1^*, x_2^*, \ldots, x_n^*), T_2(x_1^*, x_2^*, \ldots, x_n^*), \ldots, T_n(x_1^*, x_2^*, \ldots, x_n^*))\end{aligned}$$

which gives (7.9) and the proof is complete.

Taking $F(t, s) = t + s$ and $\mathscr{O}(f, t) = t$ in Theorem 7.9, we obtain the following:

Corollary 7.2 *Let C_i be a nonempty bounded convex and closed subset of a Banach space E_i $(i = 1, 2, \ldots, n)$ and $T_i : C_1 \times C_2 \times \cdots \times C_n \longrightarrow C_i$ $(i = 1, 2, \ldots, n)$ be a continuous operator such that*

$$\begin{aligned}\mu(T_i(X_1 \times X_2 \times \cdots \times X_n)) + \varphi(\mu(T_i(X_1 \times X_2 \times \cdots \times X_n)))\\ \leq \psi[\max_j \mu(X_j) + \varphi(\max_j \mu(X_j))]\end{aligned}$$

for any nonempty subset X_i of C_i where μ_i is an arbitrary measure of noncompactness on E_i $(i = 1, 2, \ldots, n)$, $\psi \in \Psi$, $\varphi \in \Phi$ and nondecreasing. Then there exist $(x_1^*, x_2^*, \ldots, x_n^*) \in C_1 \times C_2 \times \cdots \times C_n$ such that for all $1 \leq i \leq n$

$$T_i(x_1^*, x_2^*, \ldots, x_n^*) = x_i^*.$$

The following result is a generalization of similar results in [5, 24]:

Corollary 7.3 *Let C_i be a nonempty bounded convex and closed subset of a Banach space E_i $(i = 1, 2, \ldots, n)$ and $T_i : C_1 \times C_2 \times \cdots \times C_n \longrightarrow C_i$ $(i = 1, 2, \ldots, n)$ be a continuous operator such that*

$$\mu(T_i(X_1 \times X_2 \times \cdots \times X_n)) \leq k \max_j \mu(X_j)$$

for any nonempty subset X_i of C_i, where μ_i is an arbitrary measure of noncompactness on E_i and $k \in [0, 1)$. Then there exist $(x_1^*, x_2^*, \ldots, x_n^*) \in C_1 \times C_2 \times \cdots \times C_n$ such that for all $1 \leq i \leq n$

$$F_i(x_1^*, x_2^*, \ldots, x_n^*) = x_i^*.$$

Proof Take $\varphi \equiv 0$ and $\psi(t) = kt$ in Corollary 7.2.

Corollary 7.4 *Let C_i be a nonempty bounded convex and closed subset of a Banach space E_i $(i = 1, 2, \ldots, n)$ and $T_i : C_1 \times C_2 \times \cdots \times C_n \longrightarrow C_i$ $(i = 1, 2, \ldots, n)$ be a continuous operator such that*

$$\mu(T_i(X_1 \times X_2 \times \cdots \times X_n)) \leq \psi[\max_j \mu(X_j))]$$

for any subset X_i of C_i where μ_i is an arbitrary measure of noncompactness on E_i $(i = 1, 2, \dots, n)$ and $\psi \in \Psi$. Then there exist $(x_1^*, x_2^*, \dots, x_n^*) \in C_1 \times C_2 \times \cdots \times C_n$ such that for all $1 \le i \le n$

$$T_i(x_1^*, x_2^*, \dots, x_n^*) = x_i^*.$$

7.4 Systems of Ordinary and Fractal Integral Equations

In what follows, we will work in the classical Banach space $BC(\mathbb{R}_+)$ consisting of all real functions defined, bounded and continuous on $\mathbb{R}_+$ equipped with the standard norm

$$||x|| = \sup\{|x(t)| : t, s \ge 0\}.$$

Now, we present the definition of a special measure of noncompactness in $BC(\mathbb{R}_+)$ which was introduced and studied in [14].

To do this, let X be fixed as a nonempty and bounded subset of $BC(\mathbb{R}_+)$ and also fixed a positive number N. For $x \in X$ and $\varepsilon > 0$, denote by $\omega^N(x, \varepsilon)$ the modulus of the continuity of function x on the interval $[0, N]$, i.e.,

$$\omega^N(x, \varepsilon) = sup\{|x(t) - x(s)| : t, s \in [0, N], |t - s| \le \varepsilon\}.$$

Further, let us put

$$\omega^N(X, \varepsilon) = sup\{\omega^N(x, \varepsilon) : x \in X\},$$

$$\omega_0^N(X) = \lim_{\varepsilon \to 0} \omega^N(X, \varepsilon)$$

and

$$\omega_0(X) = \lim_{N \to \infty} \omega_0^N(X).$$

Moreover, for a fixed number $t \in \mathbb{R}_+$ let us the define the function μ on the family $\mathfrak{M}_{BC(\mathbb{R}_+)}$ by the following formula:

$$\mu(X) = \omega_0(X) + \alpha(X),$$

where

$$\alpha(X) = \limsup_{t \to \infty} diam X(t), \ \ X(t) = \{x(t) : x \in X\}$$

and

$$diam X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}.$$

In this section, as an application of our results we are going to study the existence of solutions for the system of integral equations (7.1). Consider the following assumptions:

(a_1) $a_i : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and bounded with $a = \sup\{a_i(t) : t \in \mathbb{R}_+, 1 \le i \le n\}$;

(a_2) $k_i : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ are continuous and there exists a positive constant M such that

$$M = \sup\left\{ \int_0^{\alpha(t)} |k_i(t, s, x_1(s), x_2(s), \dots, x_n(s))| ds : t \in \mathbb{R}_+, x_i \in E,\ 1 \le i \le n \right\}. \tag{7.10}$$

Moreover,

$$\lim_{t\to\infty} \left| \int_0^{\alpha(t)} [k_i(t, s, x_1(s), x_2(s), \dots, x_n(s)) - k_i(t, s, y_1(s), y_2(s), \dots, y_n(s))] ds \right| = 0 \tag{7.11}$$

uniformly respect to $x_i, y_i \in E$;

(a_3) $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, nondecreasing and $\lim_{t\to\infty} \alpha(t) = \infty$;

(a_4) the functions $f_i, g_i : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ are continuous and there exists an upper semicontinuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}$ with $\lim_{n\longrightarrow\infty} \psi^n(t) = 0$ for each $t \ge 0$. Also there exist bounded functions $b_i, c_i : \mathbb{R}_+ \to \mathbb{R}$ with bound

$$K = \max\{ \sup_{t\in\mathbb{R}_+} b_1(t),\ \sup_{t\in\mathbb{R}_+} b_2(t), \dots,\ \sup_{t\in\mathbb{R}_+} b_n(t),\ \sup_{t\in\mathbb{R}_+} c_1(t),\ \sup_{t\in\mathbb{R}_+} c_2(t), \dots,\ \sup_{t\in\mathbb{R}_+} c_n(t)\}$$

and a positive constant D such that

$$|f_i(t, x_1(t), x_2(t), \dots, x_n(t)) - f_i(t, y_1(t), y_2(t), \dots, y_n(t))| \le \frac{b_i(t)\psi(\max_{1\le j\le n} |x_j - y_j|)}{D + \psi(\max_{1\le j\le n} |x_j - y_j|)}$$

and

$$|g_i(t, x_1(t), x_2(t), \dots, x_n(t)) - g_i(t, y_1(t), y_2(t), \dots, y_n(t))| \le \frac{c_i(t)\psi(\max_{1\le j\le n} |x_j - y_j|)}{D + \psi(\max_{1\le j\le n} |x_j - y_j|)}$$

for all $t \in \mathbb{R}_+$ and $x_i, y_i \in \mathbb{R}$. Additionally, we assume that ψ is superadditive, i.e., $\psi(t) + \psi(s) \le \psi(t + s)$ for all $t, s \in \mathbb{R}_+$. Moreover, we assume that $K(1 + M) \le D$;

(a_5) the functions $H_1, H_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by $H_1(t) = |f(t, 0, 0, \dots, 0)|$ and $H_2(t) = |g(t, 0, 0, \dots, 0)|$ are bounded on $\mathbb{R}_+$ with

$$H_0 = max\{\sup_{t\in\mathbb{R}_+} H_1(t),\ \sup_{t\in\mathbb{R}_+} H_2(t)\}.$$

Theorem 7.10 *If the assumptions* $(a_1)-(a_5)$ *are satisfied, then the system of equation (7.1) has at least one solution* $(x_1, x_2, \ldots, x_n) \in E \times E \times \cdots \times E$.

Proof Define the operator $T_i : E \times E \times E \cdots \times E \to E$ associated with the integral equation (7.1) by

$$T_i(x_1, x_2, \ldots, x_n)(t) = a_i(t) + f_i(t, x_1(t), x_2(t), \ldots, x_n(t)) + g_i(t, x_1(t), x_2(t), \ldots, x_n(t))F_i(x_1, x_2, \ldots, x_n)(t), \tag{7.12}$$

where

$$F_i(x_1, x_2, \ldots, x_n)(t) = \int_0^{\alpha(t)} k_i(t, s, x_1(s), x_2(s), \ldots, x_n(s))ds. \tag{7.13}$$

Solving Eq. (7.1) is equivalent to finding a point $(x_1, x_2, \ldots, x_n)$ of the operator T_i defined on the space $E \times E \times \cdots \times E$ such that $T_i(x_1, x_2, \ldots, x_n) = x_i$. For better readability, we break the proof into a sequence of steps.

Step 1. T_i transforms the space $E \times E \times \cdots \times E$ into E.
By considering conditions of theorem we infer that $T_i(x_1, x_2, \ldots, x_n)$ are continuous on $\mathbb{R}_+ \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+$. Now we prove that $T_i(x_1, x_2, \ldots, x_n) \in E$ for any $(x_1, x_2, \ldots, x_n) \in E \times E \times \cdots \times E$ and $1 \le i \le n$. For arbitrarily fixed $t \in \mathbb{R}_+$ we have

$$\begin{aligned} T_i(x_1, x_2, \ldots, x_n)(t) &\le |a_i(t)| + |f_i(t, x_1(t), x_2(t), \ldots, x_n(t))| \\ &\quad + |g_i(t, x_1(t), x_2(t), \ldots, x_n(t))||F_i(x_1, x_2, \ldots, x_n)(t)| \\ &\le |a_i(t)| + \frac{K\psi(\max\limits_{1\le j\le n}|x_j|)}{D+\psi(\max\limits_{1\le j\le n}|x_j|)} + H_0 \\ &\quad + \Big[\frac{K\psi(\max\limits_{1\le j\le n}|x_j|)}{D+\psi(\max\limits_{1\le j\le n}|x_j|)} + H_0\Big]M. \end{aligned} \tag{7.14}$$

Indeed, we have

$$\begin{aligned} &|f_i(t, x_1(t), x_2(t), \ldots, x_n(t))| \\ &\le |f_i(t, x_1(t), x_2(t), \ldots, x_n(t)) - f_i(t, 0, \ldots, 0)| + |f_i(t, 0, \ldots, 0)| \\ &\le \frac{b_i(t)\psi(\max\limits_{1\le j\le n}|x_j|)}{D+\psi(\max\limits_{1\le j\le n}|x_j|)} + H_1(t) \le \frac{K\psi(\max\limits_{1\le j\le n}|x_j|)}{D+\psi(\max\limits_{1\le j\le n}|x_j|)} + H_0, \\ &|g_i(t, x_1(t), x_2(t), \ldots, x_n(t))| \\ &\le |g_i(t, x_1(t), x_2(t), \ldots, x_n(t)) - g_i(t, 0, \ldots, 0)| + |g_i(t, 0, \ldots, 0)| \\ &\le \frac{c_i(t)\psi(\max\limits_{1\le j\le n}|x_j|)}{D+\psi(\max\limits_{1\le j\le n}|x_j|)} + H_2(t) \le \frac{K\psi(\max\limits_{1\le j\le n}|x_j|)}{D+\psi(\max\limits_{1\le j\le n}|x_j|)} + H_0, \end{aligned}$$

$$
\begin{aligned}
|F_i(x_1, x_2, \ldots, x_n)(t)| &= \Big| \int_0^{\alpha(t)} k_i(t, s, x_1(s), x_2(s), \ldots, x_n(s)))ds \Big| \\
&\leq \int_0^{\alpha(t)} |k_i(t, s, x_1(s), x_2(s), \ldots, x_n(s)))|ds \leq M.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
||T_i(x_1, x_2, \ldots, x_n)|| &\leq ||a|| + [\frac{K\psi(\max_i ||x_i||)}{D + \psi(\max_i ||x_i||)} + H_0](1 + M) \\
&\leq ||a|| + (K + H_0)(1 + M).
\end{aligned}
\tag{7.15}
$$

Therefore, T_i maps the space $E \times E \times \cdots \times E$ into E. More precisely, from (7.15), we obtain that $T_i(\overline{B}_r \times \overline{B}_r \times \cdots \times \overline{B}_r) \subseteq \overline{B}_r$, where $r = ||a|| + (K + H_0)(1 + M)$.

Step 2. We show that map $T : \overline{B}_r \times \overline{B}_r \times \cdots \times \overline{B}_r \to \overline{B}_r$ is continuous. For this, let us fix arbitrarily $\varepsilon > 0$ and take $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \overline{B}_r \times \overline{B}_r \times \cdots \times \overline{B}_r$ such that $||(x_1, x_2, \ldots, x_n) - (y_1, y_2, \ldots, y_n)|| \leq \varepsilon$. Then we have

$$
\begin{aligned}
&|T_i(x_1, x_2, \ldots, x_n)(t) - T_i(y_1, y_2, \ldots, y_n)(t)| \\
&= |f_i(t, x_1(t), x_2(t), \ldots, x_n(t)) + g_i(t, x_1(t), x_2(t), \ldots, x_n(t))[F_i(x_1, x_2, \ldots, x_n)(t)] \\
&\quad - f_i(t, y_1(t), y_2(t), \ldots, y_n(t)) - g_i(t, y_1(t), y_2(t), \ldots, y_n(t))[F_i(y_1, y_2, \ldots, y_n)(t)]| \\
&\leq |f_i(t, x_1(t), x_2(t), \ldots, x_n(t)) - f_i(t, y_1(t), y_2(t), \ldots, y_n(t))| \\
&\quad + |g_i(t, x_1(t), x_2(t), \ldots, x_n(t))||F_i(x_1, x_2, \ldots, x_n)(t) - F_i(y_1, y_2, \ldots, y_n)(t)| \\
&\quad + |g_i(t, x_1(t), x_2(t), \ldots, x_n(t)) - g_i(t, y_1(t), y_2(t), \ldots, y_n(t))||F_i(y_1, y_2, \ldots, y_n)(t)| \\
&\leq \frac{b_i(t)\psi(\max_{1\leq j\leq n} |x_j(t) - y_j(t)|)}{D + \psi(\max_{1\leq j\leq n} |x_j(t) - y_j(t)|)} \\
&\quad + \Big[\frac{K\psi(\max_{1\leq j\leq n} |x_j(t)|)}{D + \psi(\max_{1\leq j\leq n} |x_j(t)|)} + H_0\Big]|F_i(x_1, x_2, \ldots, x_n)(t) - F_i(y_1, y_2, \ldots, y_n)(t)| \\
&\quad + \Big[\frac{c_i(t)\psi(\max_{1\leq j\leq n} |x_j(t) - y_j(t)|)}{D + \psi(\max_{1\leq j\leq n} |x_j(t) - y_j(t)|)}\Big]M \\
&\leq \frac{K(1 + M)\psi(\max_j ||x_j - y_j||)}{D + \psi(\max_j ||x_j - y_j||)} \\
&\quad + \Big[\frac{K\psi(\max_j ||x_j||)}{D + \psi(\max_j ||x_j||)} + H_0\Big]|F_i(x_1, x_2, \ldots, x_n)(t) - F_i(y_1, y_2, \ldots, y_n)(t)|,
\end{aligned}
\tag{7.16}
$$

Furthermore, with due attention to the condition (a_2) there exists $N > 0$ such that for $t > N$ we have

$$\begin{aligned}
&|F_i(x_1, x_2, \ldots, x_n)(t) - F_i(y_1, y_2, \ldots, y_n)(t)| \\
&= \left| \int_0^{\alpha(t)} [k_i(t, s, x_1(s), x_2(s), \ldots, x_n(s)) - k_i(t, s, y_1(s), y_2(s), \ldots, y_n(s))] ds \right| < \varepsilon.
\end{aligned} \tag{7.17}$$

Suppose that $t, s > N$. It follows (7.16) and (7.17) that

$$|T_i(x_1, x_2, \ldots, x_n)(t) - T_i(y_1, y_2, \ldots, y_n)(t)| < \varepsilon. \tag{7.18}$$

If $t, s \in [0, N]$, then we obtain

$$|F_i(x_1, x_2, \ldots, x_n)(t) - F_i(y_1, y_2, \ldots, y_n)(t)| \leq \alpha_N \omega_1(k, \varepsilon), \tag{7.19}$$

where we denote

$$\alpha_N = sup\{\alpha(t) : t \in [0, N]\}$$

and

$$\begin{aligned}
\omega_1(k_i, \varepsilon) =& sup\{|k_i(t, s, x_1, x_2, \ldots, x_n) - k_i(t, s, y_1, y_2, \ldots, y_n)| : t \in [0, N], s \in [0, \alpha_N] \\
& x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in [-r, r], ||(x_1, x_2, \ldots, x_n) - (y_2, \ldots, y_n)|| \leq \varepsilon\}.
\end{aligned}$$

By using the continuity of k on $[0, N] \times [0, \alpha_N] \times [-r, r] \times \cdots \times [-r, r]$, we have $\omega_1(k, \varepsilon) \to 0$ as $\varepsilon \to 0$. Now, linking the inequalities (7.16) and (7.19) we deduce that

$$|T_i(x_1, x_2, \ldots, x_n)(t) - T_i(y_1, y_2, \ldots, y_n)(t)| \leq \varepsilon + [K + H_0]\alpha_N \omega_1(k, \varepsilon). \tag{7.20}$$

This conclude that T_i is continuous on $\overline{B}_r \times \overline{B}_r \times \cdots \times \overline{B}_r$.

Step 3. In the sequel, we show that for any nonempty set $X_1, X_2, \ldots, X_n \subseteq \overline{B}_r$,

$$\mu(T_i(X_1 \times X_2 \times \cdots \times X_n)) \leq \psi(\max_j \mu(X_j)).$$

Indeed, from the assumptions $(a_1) - (a_5)$, we conclude that, for any $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n) \in X_1 \times X_2 \times \cdots \times X_n$ and $t \in \mathbb{R}_+$,

$$\begin{aligned}
&|T_i(x_1, x_2, \ldots, x_n)(t) - T_i(y_1, y_2, \ldots, y_n)(t)| \\
&\leq \frac{K(1 + M)\psi(\max\limits_{1 \leq j \leq n} |x_j(t) - y_j(t)|)}{D + \psi(\max\limits_{1 \leq j \leq n} |x_j(t) - y_j(t)|)} + \Big[\frac{K\psi(\max\limits_{1 \leq j \leq n} |x_j(t)|)}{D + \psi(\max\limits_{1 \leq j \leq n} |x_j(t)|)} + H_0\Big]\beta(t) \\
&\leq \psi(\max_{1 \leq j \leq n} |x_j(t) - y_j(t)|) + \Big[\frac{K\psi(\max\limits_{1 \leq j \leq n} |x_j(t)|)}{D + \psi(\max\limits_{1 \leq j \leq n} |x_j(t)|)} + H_0\Big]\beta(t),
\end{aligned}$$

where

$$\beta(t) = \sup\Big\{\Big|\int_0^{\alpha(t)} [k_i(t,s,x_1(s),x_2(s),\dots,x_n(s)) \\ - k_i(t,s,y_1(s),y_2(s),\dots,y_n(s))]ds\Big| : x_i, y_i \in E\Big\}.$$

This estimate allows us to derive the following one:

$$diam(T_i(X_1 \times X_2 \times \cdots \times X_n))(t) \le \psi(\max_{1\le j\le n}(diam X_j(t)) \\ + \Big[\frac{K\psi(\max_{1\le j\le n}|x_j(t)|)}{D+\psi(\max_{1\le j\le n}|x_j(t)|)} + H_0\Big]\beta(t). \tag{7.21}$$

Consequently, from (7.21) and the assumption (7.11) that

$$\limsup_{t\longrightarrow\infty} diam(T_i(X_1 \times X_2 \times \cdots \times X_n))(t) \le \psi(\max_{1\le j\le n}(\limsup_{t\longrightarrow\infty} diam X_j(t))). \tag{7.22}$$

Next, fix arbitrarily $N > 0$ and $\varepsilon > 0$. Let us choose $t, s \in [0, N]$ with $|t - s| \le \varepsilon$. Without loss of generality, we may assume that $s \le t$. Then, for any $(x_1, x_2, \dots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$ we get

$$\begin{aligned}
&|f_i(t,x_1(t),x_2(t),\dots,x_n(t)) - f_i(s,x_1(s),x_2(s),\dots,x_n(s))| \\
&\le |f_i(t,x_1(t),x_2(t),\dots,x_n(t)) - f_i(t,x_1(s),x_2(s),\dots,x_n(s))| \\
&\quad + |f_i(t,x_1(s),x_2(s),\dots,x_n(s)) - f_i(s,x_1(s),x_2(s),\dots,x_n(s))| \\
&\le \frac{K\psi(\max_{1\le j\le n}|x_j(t)-x_j(s)|)}{D+\psi(\max_{1\le j\le n}|x_j(t)-x_j(s)|)} \\
&\quad + |f_i(t,x_1(s),x_2(s),\dots,x_n(s)) - f_i(s,x_1(s),x_2(s),\dots,x_n(s))| \\
&\le \frac{1}{(1+M)}\psi(\max_{1\le j\le n}\omega^N(x_j,\varepsilon)) + \omega^N(f_i,\varepsilon),
\end{aligned}$$

$$\begin{aligned}
&|F_i(x_1,x_2,\dots,x_n)(t) - F_i(x_1,x_2,\dots,x_n)(s)| \\
&= \Big|\int_0^{\alpha(t)} k_i(t,u,x_1(u),x_2(u),\dots,x_n(u))du - \int_0^{\alpha(s)} k_i(s,u,x_1(u),x_2(u),\dots,x_n(u))du\Big| \\
&\le \int_0^{\alpha(t)} |k_i(t,u,x_1(u),x_2(u),\dots,x_n(u)) - k_i(s,u,x_1(u),x_2(u),\dots,x_n(u))|du \\
&\quad + \int_{\alpha(s)}^{\alpha(t)} |k_i(s,u,x_1(u),x_2(u),\dots,x_n(u))|du \le \int_0^{\alpha(t)} \omega^N(k_i,\varepsilon)du + \int_{\alpha(s)}^{\alpha(t)} K^N du \\
&\le \alpha_N\,\omega^N(k_i,\varepsilon) + \omega^N(\alpha,\varepsilon)\,K^N
\end{aligned}$$

and

$$
\begin{aligned}
&|g_i(t, x_1(t), x_2(t), \dots, x_n(t)) F_i(x_1, x_2, \dots, x_n)(t) \\
&\quad - g_i(s, x_1(s), x_2(s), \dots, x_n(s)) F_i(x_1, x_2, \dots, x_n)(s)| \\
&\le |g_i(t, x_1(t), x_2(t), \dots, x_n(t)) F_i(x_1, x_2, \dots, x_n)(t) \\
&\quad - g_i(s, x_1(s), x_2(s), \dots, x_n(s)) F_i(x_1, x_2, \dots, x_n)(t)| \\
&\quad + |g_i(s, x_1(s), x_2(s), \dots, x_n(s)) F_i(x_1, x_2, \dots, x_n)(t) \\
&\quad - g_i(s, x_1(s), x_2(s), \dots, x_n(s)) F_i(x_1, x_2, \dots, x_n)(s)| \\
&\le \frac{K\psi(\max\limits_{1\le j\le n} |x_j(t) - x_j(s)|)}{D + \psi(\max\limits_{1\le j\le n} |x_j(t) - x_j(s)|)} |F_i(x_1, x_2, \dots, x_n)(t)| \\
&\quad + \Big[\frac{K\psi(\max\limits_{1\le j\le n} |x_j(s)|)}{D + \psi(\max\limits_{1\le j\le n} |x_j(s)|)} + H_0\Big] |F_i(x_1, x_2, \dots, x_n)(t) - F_i(x_1, x_2, \dots, x_n)(s)| \\
&\le \frac{M}{1+M} \psi(\max_{1\le j\le n} \omega^N(x_j, \varepsilon)) + (K + H_0)[\alpha_N\, \omega^N(k_i, \varepsilon) + \omega^N(\alpha, \varepsilon)\, K^N].
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&|T_i(x_1, x_2, \dots, x_n)(t) - T_i(x_1, x_2, \dots, x_n)(s)| \\
&\le |a_i(t) - a_i(s)| + |f_i(t, x_1(t), x_2(t), \dots, x_n(t)) - f_i(s, x_1(s), x_2(s), \dots, x_n(s))| \\
&\quad + |g_i(t, x_1(t), x_2(t), \dots, x_n(t)) F_i(x_1, x_2, \dots, x_n)(t) \\
&\quad - g_i(s, x_1(s), x_2(s), \dots, x_n(s)) F_i(x_1, x_2, \dots, x_n)(s)| \\
&\le \omega^N(a_i, \varepsilon) + \frac{1}{(1+M)} \psi(\max_{1\le j\le n} \omega^N(x_j, \varepsilon)) + \omega^N(f_i, \varepsilon) \\
&\quad + \frac{M}{1+M} \psi(\max_{1\le j\le n} \omega^N(x_j, \varepsilon)) + (K + H_0)[\alpha_N\, \omega^N(k_i, \varepsilon) + \omega^N(\alpha, \varepsilon)\, K^N],
\end{aligned} \tag{7.23}
$$

where we define

$$
\begin{aligned}
\omega^N(f_i, \varepsilon) = \sup\{&|f_i(t, x_1, x_2, \dots, x_n) - f_i(s, x_1, x_2, \dots, x_n)| : \\
&t, s \in [0, N], |t - s| \le \varepsilon, x_i \in [-r, r]\},
\end{aligned}
$$

$$
\omega^N(x_j, \varepsilon) = \sup\{|x_j(t) - x_j(s)| : t, s \in [0, N], |t - s| \le \varepsilon\},\ 1 \le j \le n,
$$

$$
\begin{aligned}
\omega^N(k_i, \varepsilon) = \sup\{&|k_i(t, u, x_1, x_2, \dots, x_n) - k_i(s, u, x_1, x_2, \dots, x_n)| : \\
&t, s \in [0, N], u \in [0, \alpha_N], |t - s| \le \varepsilon, x_j \in [-r, r], 1 \le j \le n\},
\end{aligned}
$$

$$
K^N = \sup\{|k_i(t, u, x_1, x_2, \dots, x_n)| : t \in [0, N], u \in [0, \alpha_N], x_j \in [-r, r], 1 \le j \le n\},
$$

$$
\omega^N(a_i, \varepsilon) = \sup\{|a_i(t) - a_i(s)| : t, s \in [0, N], |t - s| \le \varepsilon\},
$$

$$\omega^N(\alpha, \varepsilon) = sup\{|\alpha(t) - \alpha(s)| : t, s \in [0, N], |t - s| \leq \varepsilon\}.$$

Since $(x_1, x_2, \ldots, x_n)$ was an arbitrary element of $X_1 \times X_2 \times \cdots \times X_n$, the inequality (7.23) implies that

$$\begin{aligned}
&\omega^N(T_i(X_1 \times X_2 \times \cdots \times X_n)), \varepsilon) \\
&\leq \omega^N(a_i, \varepsilon) + \frac{1}{(1+M)}\psi(\max_{1\leq j\leq n} \omega^N(X_j, \varepsilon)) + \omega^N(f_i, \varepsilon) \\
&\quad + \frac{M}{1+M}\psi(\max_{1\leq j\leq n} \omega^N(X_j, \varepsilon)) + (K + H_0)[\alpha_N\, \omega^N(k_i, \varepsilon) + \omega^N(\alpha, \varepsilon)\; K^N] \\
&= \omega^N(a_i, \varepsilon) + \psi(\max_{1\leq j\leq n} \omega^N(X_j, \varepsilon)) + (K + H_0)[\alpha_N\, \omega^N(k_i, \varepsilon) + \omega^N(\alpha, \varepsilon)\; K^N].
\end{aligned} \tag{7.24}$$

In view of the uniform continuity of the functions a_i, f_i and k_i on $[0, N]$ and $[0, N] \times [-r, r] \times \cdots \times [-r, r]$ and $[0, N] \times [0, \alpha_N] \times [-r, r] \times \cdots \times [-r, r]$, respectively, we have that $\omega^N(a_i, \varepsilon) \to 0, \omega^N(f_i, \varepsilon) \to 0$ and $\omega^N(k_i, \varepsilon) \to 0$. Moreover, it is obvious that the constant K^N is finite and $\omega^T(\alpha, \varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, linking the established facts with the estimate (7.24), we get

$$\omega_o(T_i(X_1 \times X_2 \times \cdots \times X_n)) \leq \varphi(\max_{1\leq j\leq n} \omega_o(X_j)). \tag{7.25}$$

Finally, from (7.22),(7.25) and the definition of the measure of noncompactness μ, we obtain

$$\begin{aligned}
&\mu(T_i(X_1 \times X_2 \times \cdots \times X_n)) \\
&= \omega_0(T_i(X_1 \times X_2 \times \cdots \times X_n)) + \limsup_{t\to\infty} diam(T_i(X_1 \times X_2 \times \cdots \times X_n))(t) \\
&\leq \psi(\max_{1\leq j\leq n} \omega_o(X_j)) + \psi(\max_{1\leq j\leq n} (\limsup_{t\longrightarrow\infty} diam X_j(t))) \\
&\leq \psi(\max_{1\leq j\leq n} \omega_o(X_j) + \max_{1\leq j\leq n} (\limsup_{t\longrightarrow\infty} diam X_j(t))) \\
&= \psi(\max_{1\leq j\leq n} \mu(X_j)).
\end{aligned} \tag{7.26}$$

Finally, applying Corollary 7.4, we obtain the desired result. This completes the proof.

In another application, we study the existence of the integral system with fractal order taking the form:

$$\begin{aligned}
x_i(t) &= a_i(t) + f_i(t, x_1(t), x_2(t), \ldots, x_n(t)) \\
&\quad + g_i(t, x_1(t), x_2(t), \ldots, x_n(t))\frac{1}{\Gamma(\wp+1)}\int_0^{\alpha(t)} k_i(t, s, x_1(s), x_2(s), \ldots, x_n(s)))(ds)^{\wp} \\
&= a_i(t) + f_i(t, x_1(t), x_2(t), \ldots, x_n(t)) + g_i(t, x_1(t), x_2(t), \ldots, x_n(t))I^{\wp}k_i
\end{aligned} \tag{7.27}$$

for all $t \in \mathbb{R}_+,\ x_1, x_2, \ldots, x_n \in E = BC(\mathbb{R}_+)$ and $k_i \in C_\wp[0, b],\ \alpha(t) < b,\ 1 \leq i \leq n$, where $C_\wp$ is the local fractional continuous space satisfying $|k_i(\chi) - k_i(\Upsilon)| < \varepsilon^\wp$ when $|\chi - \Upsilon| < \delta,\ \varepsilon > 0, \delta > 0$. The integral $I^\wp_{[a,b]}$ is called the *fractal integral operator* satisfying the following property [25]:

Property 7.1 (a) $I^\wp_{[a,b]} 1 = (b - a)^\wp / \Gamma(\wp + 1)$;
(b) $|I^\wp_{[a,b]} f(x)| \leq I^\wp_{[a,b]} |f|$;
(c) $I^\wp_{[a,b]}[f(x) + g(x)] = I^\wp_{[a,b]} f(x) + I^\wp_{[a,b]} g(x)$.

Consider the function $a_i, \alpha,\ f_i$ and g_i satisfying the assumptions $(a_1), (a_3) - (a_5)$, respectively. In addition, we consider the following assumption:

$(\hat{a}_2)$ $k_i \in C_\wp[0, b]$ which are continuous and there exists a positive constant $\hat{M}_\wp$ such that

$$\hat{M}_\wp = \sup \left\{ \frac{1}{\Gamma(\wp + 1)} \int_0^{\alpha(t)} |k_i(t, s, x_1(s), x_2(s), \ldots, x_n(s))| (ds)^\wp : r \right. \\ \left. t \in \mathbb{R}_+, x_i \in E, 1 \leq i \leq n \right\}.$$

Moreover,

$$\lim_{t \to \infty} \left| \int_0^{\alpha(t)} [k_i(t, s, x_1(s), x_2(s), \ldots, x_n(s)) - k_i(t, s, y_1(s), y_2(s), \ldots, y_n(s))] (ds)^\wp \right| = 0 \tag{7.28}$$

uniformly respect to $x_i, y_i \in E$. We have the following result:

Theorem 7.11 *If the assumptions* $(a_1), (\hat{a}_2), (a_3) - (a_5)$ *are satisfied, then the system of the equation* (7.27) *has at least one solution* $(x_1, x_2, \ldots, x_n) \in E \times E \times \cdots \times E$.

Proof Define the operator $\Theta_i : E \times E \times E \cdots \times E \to E$ by

$$\Theta_i(x_1, x_2, \ldots, x_n)(t) = a_i(t) + f_i(t, x_1(t), x_2(t), \ldots, x_n(t)) \\ + g_i(t, x_1(t), x_2(t), \ldots, x_n(t)) \hat{F}_i(x_1, x_2, \ldots, x_n)(t), \tag{7.29}$$

where

$$\hat{F}_i(x_1, x_2, \ldots, x_n)(t) = I^\wp k_i(x_1, x_2, \ldots, x_n)(t). \tag{7.30}$$

Our aim is to apply Corollary 7.4. To show that $\Theta_i \in E$. By using the assumptions of our theorem, we have $\Theta_i(x_1, x_2, \ldots, x_n)$ are continuous on $\mathbb{R}_+ \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+$. We proceed to show that $\Theta_i(x_1, x_2, \ldots, x_n) \in E$ for any $(x_1, x_2, \ldots, x_n) \in E \times E \times \cdots \times E$ and $1 \leq i \leq n$. For any $t \in \mathbb{R}_+$ we have

$$
\begin{aligned}
\Theta_i(x_1, x_2, \ldots, x_n)(t) &\leq |a_i(t)| + |f_i(t, x_1(t), x_2(t), \ldots, x_n(t))| \\
&\quad + |g_i(t, x_1(t), x_2(t), \ldots, x_n(t))||\hat{F}_i(x_1, x_2, \ldots, x_n)(t)| \\
&\leq |a_i(t)| + \frac{K\psi(\max\limits_{1\leq j\leq n}|x_j|)}{D+\psi(\max\limits_{1\leq j\leq n}|x_j|)} + H_0 \\
&\quad + [\frac{K\psi(\max\limits_{1\leq j\leq n}|x_j|)}{D+\psi(\max\limits_{1\leq j\leq n}|x_j|)} + H_0]\hat{M}_{\wp}.
\end{aligned} \tag{7.31}
$$

In the same manner of Theorem 7.10, we conclude that

$$
\begin{aligned}
||\Theta_i(x_1, x_2, \ldots, x_n)|| &\leq ||a|| + \Big[\frac{K\psi(\max\limits_i ||x_i||)}{D+\psi(\max\limits_i ||x_i||)} + H_0\Big](1+\hat{M}_{\wp}) \\
&\leq ||a|| + (K + H_0)(1+\hat{M}_{\wp}).
\end{aligned} \tag{7.32}
$$

Therefore, Θ_i maps the space $E \times E \times \cdots \times E$ into E. That is $\Theta_i(\overline{B}_r \times \overline{B}_r \times \cdots \times \overline{B}_r) \subseteq \overline{B}_r$,where $r = ||a|| + (K + H_0)(1+\hat{M}_{\wp})$. Moreover, we have

$$
|\Theta_i(x_1, x_2, \ldots, x_n)(t) - \Theta_i(y_1, y_2, \ldots, y_n)(t)| \leq \varepsilon + [K + H_0]\alpha_N \omega_1(k, \varepsilon), \tag{7.33}
$$

where all the above parameters are in Theorem 7.10. We obtain that Θ_i is continuous on $\overline{B}_r \times \overline{B}_r \times \cdots \times \overline{B}_r$.

Lastly, we show that, for any nonempty set $X_1, X_2, \ldots, X_n \subseteq \overline{B}_r$,

$$
\mu(\Theta_i(X_1 \times X_2 \times \cdots \times X_n)) \leq \psi(\max_j \mu(X_j)).
$$

In view of the assumptions (a_1) $(a_3) - (a_5)$, we have

$$
\begin{aligned}
&|\Theta_i(x_1, x_2, \ldots, x_n)(t) - \Theta_i(y_1, y_2, \ldots, y_n)(t)| \\
&\leq \frac{K(1+M)\psi(\max\limits_{1\leq j\leq n}|x_j(t)-y_j(t)|)}{D+\psi(\max\limits_{1\leq j\leq n}|x_j(t)-y_j(t)|)} + \Big[\frac{K\psi(\max\limits_{1\leq j\leq n}|x_j(t)|)}{D+\psi(\max\limits_{1\leq j\leq n}|x_j(t)|)} + H_0\Big]\rho(t) \\
&\leq \psi(\max_{1\leq j\leq n}|x_j(t)-y_j(t)|) + \Big[\frac{K\psi(\max\limits_{1\leq j\leq n}|x_j(t)|)}{D+\psi(\max\limits_{1\leq j\leq n}|x_j(t)|)} + H_0\Big]\rho(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\rho(t) = \sup\{\frac{1}{\Gamma(\wp+1)}\Big|\int_0^{\alpha(t)} &[k_i(t, s, x_1(s), x_2(s), \ldots, x_n(s)) \\
&- k_i(t, s, y_1(s), y_2(s), \ldots, y_n(s))](ds)^{\wp}\Big| : x_i, y_i \in E\}.
\end{aligned}
$$

Consequently, we obtain

$$diam(\Theta_i(X_1 \times X_2 \times \cdots \times X_n))(t) \le \psi(\max_{1\le j\le n}(diam X_j(t)) + \Big[\frac{K\psi(\max_{1\le j\le n}|x_j(t)|)}{D+\psi(\max_{1\le j\le n}|x_j(t)|)} + H_0\Big]\rho(t). \tag{7.34}$$

Combining (7.34) and the assumption (7.28), we attain

$$\limsup_{t\longrightarrow\infty} diam(\Theta_i(X_1 \times X_2 \times \cdots \times X_n))(t) \le \psi(\max_{1\le j\le n}(\limsup_{t\longrightarrow\infty} diam X_j(t))). \tag{7.35}$$

Next, fix arbitrarily $N > 0$ and $\varepsilon > 0$. Let us choose $t, s \in [0, N]$, with $|t - s| \le \varepsilon$. Without loss of generality, we may assume that $s \le t$. A calculation implies

$$\begin{aligned}
&|f_i(t, x_1(t), x_2(t), \ldots, x_n(t)) - f_i(s, x_1(s), x_2(s), \ldots, x_n(s))| \\
&\le |f_i(t, x_1(t), x_2(t), \ldots, x_n(t)) - f_i(t, x_1(s), x_2(s), \ldots, x_n(s))| \\
&\quad + |f_i(t, x_1(s), x_2(s), \ldots, x_n(s)) - f_i(s, x_1(s), x_2(s), \ldots, x_n(s))| \\
&\le \frac{K\psi(\max_{1\le j\le n}|x_j(t)-x_j(s)|)}{D+\psi(\max_{1\le j\le n}|x_j(t)-x_j(s)|)} \\
&\quad + |f_i(t, x_1(s), x_2(s), \ldots, x_n(s)) - f_i(s, x_1(s), x_2(s), \ldots, x_n(s))| \\
&\le \frac{1}{(1+\hat{M}_\wp)}\psi(\max_{1\le j\le n}\omega^N(x_j, \varepsilon)) + \omega^N(f_i, \varepsilon),
\end{aligned}$$

and, using Proposition 7.1, we have

$$\begin{aligned}
&|\hat{F}_i(x_1, x_2, \ldots, x_n)(t) - \hat{F}_i(x_1, x_2, \ldots, x_n)(s)| \\
&= |I^\wp k_i(x_1, x_2, \ldots, x_n)(t) - I^\wp k_i(x_1, x_2, \ldots, x_n)(s)| \\
&\le \frac{\alpha(t)^\wp}{\Gamma(\wp+1)}\|k_i\|_{C_\wp}, \quad \wp \in (0, 1] \\
&\le \alpha_N\, \omega^N(k_i, \varepsilon) + \omega^N(\alpha, \varepsilon)\, K^N \\
&:= \rho^N.
\end{aligned}$$

Thus we obtain

$$
\begin{aligned}
&|g_i(t, x_1(t), x_2(t), \ldots, x_n(t))\hat{F}_i(x_1, x_2, \ldots, x_n)(t) \\
&\quad - g_i(s, x_1(s), x_2(s), \ldots, x_n(s))\hat{F}_i(x_1, x_2, \ldots, x_n)(s)| \\
&\leq \frac{K\psi(\max\limits_{1\leq j\leq n} |x_j(t) - x_j(s)|)}{D + \psi(\max\limits_{1\leq j\leq n} |x_j(t) - x_j(s)|)} |\hat{F}_i(x_1, x_2, \ldots, x_n)(t)| \\
&\quad + \Big[\frac{K\psi(\max\limits_{1\leq j\leq n} |x_j(s)|)}{D + \psi(\max\limits_{1\leq j\leq n} |x_j(s)|)} + H_0\Big] |\hat{F}_i(x_1, x_2, \ldots, x_n)(t) - \hat{F}_i(x_1, x_2, \ldots, x_n)(s)| \\
&\leq \frac{\hat{M}_\wp}{1 + \hat{M}_\wp} \psi(\max_{1\leq j\leq n} \omega^N(x_j, \varepsilon)) + (K + H_0)\rho^N.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&|\Theta_i(x_1, x_2, \ldots, x_n)(t) - \Theta_i(x_1, x_2, \ldots, x_n)(s)| \\
&\leq |a_i(t) - a_i(s)| + |f_i(t, x_1(t), x_2(t), \ldots, x_n(t)) - f_i(s, x_1(s), x_2(s), \ldots, x_n(s))| \\
&\quad + |g_i(t, x_1(t), x_2(t), \ldots, x_n(t))\hat{F}_i(x_1, x_2, \ldots, x_n)(t) \\
&\quad - g_i(s, x_1(s), x_2(s), \ldots, x_n(s))\hat{F}_i(x_1, x_2, \ldots, x_n)(s)| \\
&\leq \omega^N(a_i, \varepsilon) + \frac{1}{(1 + \hat{M}_\wp)} \psi(\max_{1\leq j\leq n} \omega^N(x_j, \varepsilon)) + \omega^N(f_i, \varepsilon) \\
&\quad + \frac{\hat{M}_\wp}{1 + \hat{M}_\wp} \psi(\max_{1\leq j\leq n} \omega^N(x_j, \varepsilon)) + (K + H_0)\rho^N,
\end{aligned}
\tag{7.36}
$$

which implies that

$$
\begin{aligned}
&\omega^N(\Theta_i(X_1 \times X_2 \times \cdots \times X_n)), \varepsilon) \\
&\leq \omega^N(a_i, \varepsilon) + \frac{1}{(1 + \hat{M}_\wp)} \psi(\max_{1\leq j\leq n} \omega^N(X_j, \varepsilon)) + \omega^N(f_i, \varepsilon) \\
&+ \frac{\hat{M}_\wp}{1 + \hat{M}_\wp} \psi(\max_{1\leq j\leq n} \omega^N(X_j, \varepsilon)) + (K + H_0)\rho^N \\
&= \omega^N(a_i, \varepsilon) + \psi(\max_{1\leq j\leq n} \omega^N(X_j, \varepsilon)) + (K + H_0)\rho^N.
\end{aligned}
\tag{7.37}
$$

In view of the uniform continuity of the functions a_i, f_i and k_i on $[0, N]$ and $[0, N] \times [-r, r] \times \cdots \times [-r, r]$ and $[0, N] \times [0, \alpha_N] \times [-r, r] \times \cdots \times [-r, r]$, respectively, we conclude that $\omega^N(a_i, \varepsilon) \to 0, \omega^N(f_i, \varepsilon) \to 0$ and $\omega^N(k_i, \varepsilon) \to 0\,(\rho^N \to 0)$. Clearly, the constant K^N is finite and $\omega^T(\alpha, \varepsilon) \to 0$ as $\varepsilon \to 0$. This leads to

$$
\omega_o(\Theta_i(X_1 \times X_2 \times \cdots \times X_n)) \leq \varphi(\max_{1\leq j\leq n} \omega_o(X_j)). \tag{7.38}
$$

Thus, from (7.35), (7.38) and the definition of the measure of noncompactness μ, we obtain

$$\begin{aligned}
&\mu(\Theta_i(X_1 \times X_2 \times \cdots \times X_n)) \\
&= \omega_0(\Theta_i(X_1 \times X_2 \times \cdots \times X_n)) + \limsup_{t\to\infty} diam(\Theta_i(X_1 \times X_2 \times \cdots \times X_n))(t) \\
&\leq \psi(\max_{1\leq j\leq n} \omega_o(X_j)) + \psi(\max_{1\leq j\leq n} (\limsup_{t\longrightarrow\infty} diam X_j(t))) \\
&\leq \psi(\max_{1\leq j\leq n} \omega_o(X_j) + \max_{1\leq j\leq n} (\limsup_{t\longrightarrow\infty} diam X_j(t))) \\
&= \psi(\max_{1\leq j\leq n} \mu(X_j)).
\end{aligned} \tag{7.39}$$

Finally, applying Corollary 7.4, we complete the proof.

7.5 Conclusion

The notion of measures of noncompactness (MNC) has been widely used in functional analysis such as the metric fixed point theory and the theory of operator equations in Banach spaces. Due to its importance, in this work, we have used MNC concept to obtain the existence of solutions for the system of integral equations. To achieve the solution, we have introduced a new notion of $\mu - (F, \varphi, \psi)-$set contractive operator, and based on Darbo fixed point theorem and Krasnoselskii fixed point result in generalized sense. We have also discussed the solution of a system of fractional integral equations when k_i is defined in a fractal space.

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Chapter 8
Fixed Points That Are Zeros of a Given Function

Francesca Vetro

Abstract We present a discussion on (ordered) S-F-contractions in the setting of complete metric spaces, with and without the ordered approach. S-F-contractions are generalizations of (F, φ)-contractions and $\mathscr{Z}$-contractions. These two types of contractions have encountered a great success among the scientific community due to their versatility and usefulness in overcoming different practical situations. A fundamental characteristic of such a kind of contractions is the possibility to be hybridized with other existing conditions to obtain control hypotheses with best performances.

8.1 Introduction

This chapter is devoted to the study of sufficient and necessary conditions to establishing the existence and uniqueness of fixed points for self-mappings, defined in a metric space or in an ordered metric space. In particular, these fixed points have to be zeros of a given function. This theory is very interesting in its own right, due to the fact that fixed point results have constructive proofs, and hence they have nice applications in industrial fields such as image processing, engineering, physics, computer science, economics, and telecommunication. Indeed, this recognized success is due to the fact that the basic fixed point problem $x = Tx$, where $T : X \to X$ is a self-mapping of a space X, is a model representative of many practical situations arising in theoretical and applied sciences. For instance, the solutions of differential problems can be obtained in terms of fixed points of integro-differential operators. Also, working with suitable operators, it is possible to approach the solution of equilibrium problems by searching the fixed points of such operators. These remarks give

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sufficient motivations in order to understand the interest of researchers to establishing extensions and generalizations of the Banach fixed point theorem [4], which is the fundamental result of metric fixed point theory. Thus the fixed point theory is a vivid research field for researchers in mathematics and other disciplines. Of course, the literature is reach in extensions of the pioneering Banach's result and so an infinitely long list could be provided; see, for example [5, 22, 23, 26, 28, 29, 33–35, 38, 43, 45].

Note that, in metric fixed point theory, we study problems which involve concepts of an essentially metric nature. Successive approximations played a major role in the study of the metric fixed point theory to establishing, for example, the existence and uniqueness of solutions. In fact, successive approximations find their roots in the papers of Cauchy, Fredholm, Liouville, Lipschitz, Peano, and Picard.

As said above, metric fixed point theory furnishes useful methods and notions for dealing with various problems. In particular, we refer to the existence of solutions of mathematical problems reducible to equivalent fixed point problems.

In Samet et al. [37], and in Vetro and Vetro [40], discussed fixed point results in metric spaces by using a contractive condition where is present an additional semi-continuous function φ. So, they obtained results of existence and uniqueness of fixed points that are zeros of φ. These results generalize and improve many existing fixed point theorems in the literature. As an application of the presented results, the authors gave some theorems in the setting of partial metric spaces.

Recently, two new notions of contractions have been introduced by Jleli et al. and Khojasteh et al. Precisely, in 2014, Jleli et al. [12] introduced the notion of (F, φ)-contraction and, in 2015, Khojasteh et al. [16] introduced the notion of $\mathscr{Z}$-contraction. (F, φ)-contractions were used to establish results of existence of φ-fixed points, that is, fixed points that are zeros of a suitable function φ. We remark that the notion of (F, φ)-contraction is associated with a family $\mathscr{F}$ of functions that have some properties. The concept of $\mathscr{Z}$-contraction was used to prove existence and uniqueness of fixed points. $\mathscr{Z}$-contractions are a new type of nonlinear contractions defined by using a specific function called simulation function. We point out that the advantage of this new approach is in providing a unique point of view for several fixed point problems. For results connected to these two new types of contractions, the reader can see recent results in [1–3, 7–11, 13–15, 17–19, 24, 31, 32, 36, 39, 44].

These two new types of contractions have encountered a great success among the scientific community due to their versatility and usefulness in overcoming a wide range of situations. A fundamental characteristic of such kinds of contractions is the possibility to be hybridized with different other existing contractive conditions to get new conditions with major performances. Thus, we propose to the reader a review of (ordered) S-F-contractions in the setting of complete metric spaces with and without the ordered approach. Clearly, S-F-contractions are generalization of (F, φ)-contractions and $\mathscr{Z}$-contractions. Furthermore, we have that the fixed points belong to the zero-set of a given function.

8.2 (F, φ)-Contractions

In this section, we recall some definitions and results regarding (F, φ)-contractions.

In order to obtain a generalization of the "Banach contraction principle", in [12], Jleli et al. consider the family $\mathscr{F}$ of functions $F : [0, +\infty[\times[0, +\infty[\times[0, +\infty[\to [0, +\infty[$ satisfying the following conditions:

(F_1) $\max\{a, b\} \leq F(a, b, c)$ for all $a, b, c \in [0, +\infty[$;

(F_2) $F(0, 0, 0) = 0$;

(F_3) F is continuous.

They use the family $\mathscr{F}$ to introduce the following notion of (F, φ)-contraction.

Definition 8.1 Let (X, d) be a metric space, $\varphi : X \to [0, +\infty[$ be a given function and $F \in \mathscr{F}$. We say that a mapping $T : X \to X$ is a (F, φ)-*contraction* with respect to the metric d if there exists a constant $k \in]0, 1[$ such that

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq kF(d(x, y), \varphi(x), \varphi(y)) \quad \forall\, x, y \in X.$$

Let (X, d) be a metric space, $\varphi : X \to [0, +\infty[$ be a given function and $T : X \to X$ be a mapping. Here, we denote by Z_φ the set $\{x \in X : \varphi(x) = 0\}$ and by F_T the set $\{x \in X : Tx = x\}$. Let $x_0 \in X$ and $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is called the *sequence of Picard* of initial point at x_0. We say that the mapping T is a φ-*Picard mapping* if $F_T \cap Z_\varphi = \{z\}$ and $x_n \to z$ as $n \to +\infty$, whenever $\{x_n\} \subset X$ is a Picard sequence starting at a point $x_0 \in X$.

By using the notion of (F, φ)-contraction, Jleli et al. give the following generalization of the Banach fixed point theorem [4].

Theorem 8.1 ([12], Theorem 2.1) *Let (X, d) be a complete metric space, $\varphi : X \to [0, +\infty[$ be a given function, and $F \in \mathscr{F}$. Suppose that the following conditions hold*

(H1) *φ is lower semi-continuous;*

(H2) *$T : X \to X$ is a (F, φ)-contraction with respect to the metric d.*

Then we have the following:

(1) *$F_T \subset Z_\varphi$.*

(2) *T is a φ-Picard mapping.*

(3) *for all $x \in X$ and $n \in \mathbb{N}$, we have*

$$d(T^n x, z) \leq \frac{k^n}{1-k} F(d(Tx, x), \varphi(Tx), \varphi(x)),$$

where $\{z\} = F_T \cap Z_\varphi = F_T$.

It is easy to see that the function $F : [0, +\infty[\times[0, +\infty[\times[0, +\infty[\to [0, +\infty[$ defined by one of the following rules:

(a) $F(a, b, c) = a + b + c$ for all $a, b, c \in [0, +\infty[$;

(b) $F(a, b, c) = \max\{a, b\} + c$ for all $a, b, c \in [0, +\infty[$

belong to $\mathscr{F}$.

Remark 8.1 Every Banach contraction is a concrete example of (F, φ)-contraction. It satisfies Definition 8.1 by taking $F(a, b, c) = a + b + c$, for all $a, b, c \in [0, +\infty[$ and $\varphi(x) = 0$, for all $x \in X$.

In [19], Kumrod and Sintunavarat use the family $\mathscr{F}$ to introduce the concepts of (F, φ, θ)-contraction mapping ([19], Definition 2.4) and (F, φ, θ)-weak contraction mapping ([19], Definition 2.8) in the setting of metric spaces. We notice that θ is a function with some properties. Furthermore, thanks to the family $\mathscr{F}$, they establish φ-fixed point results for such mappings ([19], Theorems 2.5 and 2.9).

Finally, we remark that in [11] Işık et al. use the family $\mathscr{F}$ for establishing the existence and uniqueness of φ-best proximity point (see [11], Theorems 7, 10 and 12) for non-self-mappings satisfying (F, φ)-proximal and (F, φ)-weak proximal contraction conditions (see [11], Definition 6) in the context of complete metric spaces.

As applications of the obtained results, Işık et al. give some new best proximity point results in partial metric spaces and discuss sufficient conditions to ensure the existence of a unique solution for a variational inequality problem (see [11]).

8.3 $\mathscr{Z}$-Contractions

In this section, we consider some basic definitions and results on simulation functions obtained by Khojasteh et al. [16] and Argoubi et al. [2]. In Khojasteh et al. [16], give the following definition of simulation function.

Definition 8.2 A simulation function is a function $\zeta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(t, s) < s - t$ for all $t, s > 0$;

(ζ_2) if $\{t_n\}, \{s_n\}$ are sequences in $]0, +\infty[$ such that $\lim_{n\to+\infty} t_n = \lim_{n\to+\infty} s_n = \ell \in]0, +\infty[$, then $\limsup_{n\to+\infty} \zeta(t_n, s_n) < 0$;

(ζ_3) $\zeta(0, 0) = 0$.

Further, in [16], they introduce, by using the simulation functions, the class of $\mathscr{Z}$-contractions, as follows.

Definition 8.3 Let (X, d) be a metric space. A mapping $T : X \to X$ is a $\mathscr{Z}$-contraction if there exists a simulation function ζ such that

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0 \quad \text{for all } x, y \in X.$$

For the class of $\mathscr{Z}$-contractions they give the next result.

Theorem 8.2 ([16], Theorem 2.8) *Let (X, d) be a complete metric space and $T : X \to X$ be a $\mathscr{Z}$-contraction with respect to a certain simulation function ζ, that is,*

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0 \quad \forall\, x, y \in X.$$

Then T has a unique fixed point. Moreover, for every $x_0 \in X$, the sequence of Picard $\{x_n\}$ starting at x_0 converges to this fixed point.

Remark 8.2 Every Banach contraction is again a concrete example of $\mathscr{Z}$-contraction. It satisfies the Definition 8.3 by taking $\zeta(t, s) = ks - t$ for all $t, s \in [0, +\infty[$ and $k \in]0, 1[$.

Remark 8.3 Every $\mathscr{Z}$-contraction $T : X \to X$ is a contractive mapping and hence it is continuous. In fact, if, for some $x, y \in X$ with $x \neq y$, we have $d(Tx, Ty) \geq d(x, y) > 0$, then, by the property (ζ_1) of the function ζ, it follows that

$$\zeta(d(Tx, Ty), d(x, y)) < d(x, y) - d(Tx, Ty) \leq 0,$$

which is a contradiction. This leads us to the conclusion that T is a contractive mapping, that is, $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$.

In Argoubi et al. [2], note that the condition (ζ_3) was not used for the proof of Theorem 8.2. Taking into account this, Argoubi et al. revised the previous definition slightly. More precisely, they withdraw the condition (ζ_3) and hence they give the following definition.

Definition 8.4 A *simulation function* is a function $\zeta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ satisfying the conditions (ζ_1) and (ζ_2).

In Roldán-López-de-Hierro et al. [31], observed that the condition (ζ_2) is symmetric in both arguments of the function ζ, but in the proof of Theorem 8.2, this property is not necessary. In fact, the arguments of the function ζ have different significance and so have a different role. Thus, Roldán-López-de-Hierro et al. modify the Definition 8.2 in order to put in evidence the different role of the two arguments of ζ.

Definition 8.5 ([31], Definition 3.2) A function $\zeta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ is a *simulation function* if it satisfies the following conditions:

(ζ_1) $\zeta(t, s) < s - t$ for all $t, s > 0$;

(ζ_2') if $\{t_n\}, \{s_n\}$ are sequences in $]0, +\infty[$ such that $\lim_{n\to+\infty} t_n = \lim_{n\to+\infty} s_n = \ell \in]0, +\infty[$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\limsup_{n\to+\infty} \zeta(t_n, s_n) < 0$;

(ζ_3) $\zeta(0, 0) = 0$.

Example 8.1 Let $\zeta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ be defined by

$$\zeta(t, s) = s - \frac{f(t, s)}{g(t, s)}t \quad \text{for all } t, s \in [0, +\infty[,$$

where $f, g : [0, +\infty[\times[0, +\infty[\to]0, +\infty[$ are two continuous functions with respect to each variable such that $f(t, s) > g(t, s)$ for all $t, s > 0$. Then ζ is a simulation function.

Note that every simulation function in (original) Khojasteh et al. sense (Definition 8.2) is a simulation function in Argoubi et al. sense (Definition 8.4) and in Roldán-López-de-Hierro et al. sense (Definition 8.5), but the converse is not true (see Example 2.4 of [2] and Example 3.3 of [31]).

The following example shows that there exists a function $\zeta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ that satisfies the conditions (ζ_1) and (ζ_2'), but not the conditions (ζ_2) and (ζ_3).

Example 8.2 Let $k \in [0, 1[$ and let $\zeta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ be the function defined by

$$\zeta(t, s) = \begin{cases} 3(s - t), & \text{if } s < t, \\ 1, & \text{if } s = t = 0, \\ ks - t, & \text{otherwise.} \end{cases}$$

Clearly, the function ζ verifies the condition (ζ_1). If we choose $t_n = 2$ and $s_n = \dfrac{2n-1}{n}$ for all $n \in \mathbb{N}$, then $\lim_{n\to+\infty} t_n = \lim_{n\to+\infty} s_n = 2$ and

$$\limsup_{n\to+\infty} \zeta(t_n, s_n) = \limsup_{n\to+\infty} 3\left(\frac{2n-1}{n} - 2\right) = 0.$$

Consequently, ζ has not the property (ζ_2), but has the property (ζ_2'). Moreover, it has not the property (ζ_3) since $\zeta(0, 0) = 1$.

Later on, we consider the family $\mathcal{S}$ of functions $\zeta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ that have the properties (ζ_1) and (ζ_2) and the family $\mathcal{S}'$ of functions $\zeta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ satisfying the conditions (ζ_1) and (ζ_2'). Clearly, $\mathcal{S} \subset \mathcal{S}'$.

We point out that, in Tchier et al. [39], use the family $\mathcal{S}$ to introduce the notions of $\mathcal{Z}$-proximal contraction of the first kind and second kind (see [39], Definitions 3.1 and 3.2) and to establish existence and uniqueness of g-best proximity points (see [39], Theorems 3.1 and 3.2). Also, in Abbas et al. [1], use the family $\mathcal{S}$ to introduce the notions of proximal simulative contraction of the first kind and second kind (see [1], Definitions 11 and 12). For these classes of proximal contractions, they establish existence and uniqueness of best proximity points (see [1], Theorems 1 and 3).

Finally, we remark that Roldán-López-de-Hierro et al. in [31] study the existence of coincidence points. They explore the existence and uniqueness of coincidence points of two given mappings defined on a complete metric space (see [31], Theorem 4.8) by introducing the notion of $(\mathcal{Z}, g)$-contraction (see [31], Definition 4.1) that employs the simulation function given in Defintion 8.5.

8.4 Fixed Points for S-F-Contractions

In this section, we start pointing out that the classes of functions $\mathcal{F}$ and $\mathcal{S}'$ introduced in Sects. 8.2 and 8.3 are needed to define implicitly the notion of the S-F-contraction. This is clearly shown by the next definition (see [41]).

Definition 8.6 Let (X, d) be a metric space. A mapping $T : X \to X$ is a *S-F-contraction* if there exist a function $\zeta \in \mathscr{S}'$, a function $F \in \mathscr{F}$ and a function $\varphi : X \to [0, +\infty[$ such that

$$\zeta\,(F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)), F(d(x, y), \varphi(x), \varphi(y))) \geq 0 \quad \text{for all } x, y \in X. \tag{8.1}$$

Remark 8.4 Every (F, φ)-contraction is a concrete example of S-F-contraction. It satisfies Definition 8.6 by taking $\zeta(t, s) = ks - t$ for all $t, s \in [0, +\infty[$ and $k \in]0, 1[$. Also, each $\mathscr{Z}$-contraction is an example of S-F-contraction. It satisfies the Definition 8.6 by taking $F(a, b, c) = a + b + c$ for all $a, b, c \in [0, +\infty[$ and $\varphi(x) = 0$ for all $x \in X$.

The following auxiliary result takes a leading role in the development of the chapter.

Lemma 8.1 (see [41], Lemma 3.1) *Let (X, d) be a metric space and $T : X \to X$ be a S-F-contraction with respect to $\zeta \in \mathscr{S}'$, $F \in \mathscr{F}$ and a function $\varphi : X \to [0, +\infty[$. If $\{x_n\}$ is a sequence of Picard starting at $x_0 \in X$ such that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

Proof Let x_0 be an arbitrary point in X and let $\{x_n\}$ be a sequence of Picard starting at $x_0 \in X$. Assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. First, we prove that

$$\lim_{n\to+\infty} d(x_{n-1}, x_n) = 0 \quad \text{and} \quad \lim_{n\to+\infty} \varphi(x_n) = 0. \tag{8.2}$$

From $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$ it follows that $d(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$. Then the property (F_1) of the function F ensures that

$$F(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n)) \geq d(x_{n-1}, x_n) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Now, put $d_{n-1} = d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. Using (8.1) and the property (ζ_1) of the function ζ, with $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned} 0 &\leq \zeta(F(d_n, \varphi(x_n), \varphi(x_{n+1})), F(d_{n-1}, \varphi(x_{n-1}), \varphi(x_n))) \\ &< F(d_{n-1}, \varphi(x_{n-1}), \varphi(x_n)) - F(d_n, \varphi(x_n), \varphi(x_{n+1})) \end{aligned}$$

for all $n \in \mathbb{N}$. The above inequality shows that

$$F(d_n, \varphi(x_n), \varphi(x_{n+1})) < F(d_{n-1}, \varphi(x_{n-1}), \varphi(x_n)) \quad \text{for all } n \in \mathbb{N}.$$

Consequently, $\{F(d_{n-1}, \varphi(x_{n-1}), \varphi(x_n))\}$ is a decreasing sequence of positive real numbers. Thus there is some $\ell \geq 0$ such that

$$\lim_{n\to+\infty} F(d_{n-1}, \varphi(x_{n-1}), \varphi(x_n)) = \ell.$$

Suppose $\ell > 0$. We notice that from the condition (ζ_2'), with $t_n = F(d_n, \varphi(x_n), \varphi(x_{n+1}))$ and $s_n = F(d_{n-1}, \varphi(x_{n-1}), \varphi(x_n))$, it follows that

$$0 \leq \limsup_{n \to +\infty} \zeta\,(F(d_n, \varphi(x_n), \varphi(x_{n+1})), F(d_{n-1}, \varphi(x_{n-1}), \varphi(x_n))) < 0,$$

which is a contradiction. Hence we conclude that $\ell = 0$. Now, using the property (F_1) of the function F, we deduce that

$$\max\{d_{n-1}, \varphi(x_{n-1})\} \leq F(d_{n-1}, \varphi(x_{n-1}), \varphi(x_n)), \quad \text{for all } n \in \mathbb{N},$$

and hence

$$\lim_{n \to +\infty} d(x_{n-1}, x_n) = 0 \quad \text{and} \quad \lim_{n \to +\infty} \varphi(x_{n-1}) = 0.$$

Next, we prove that the sequence $\{x_n\}$ is a Cauchy sequence. Let us assume that $\{x_n\}$ is not a Cauchy sequence. Then there exist a positive real number ε and two sequences $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k \geq k$, $d(x_{m_k}, x_{n_k}) \geq \varepsilon > d(x_{m_k}, x_{n_k-1})$ for all $k \in \mathbb{N}$. Hence, by using the first condition of (8.2), we obtain

$$\lim_{k \to +\infty} d(x_{m_k}, x_{n_k}) = \lim_{k \to +\infty} d(x_{m_k-1}, x_{n_k-1}) = \varepsilon. \tag{8.3}$$

Taking into account that the function F is continuous, we further get

$$\begin{aligned} &\lim_{k \to +\infty} F(d(x_{m_k-1}, x_{n_k-1}), \varphi(x_{m_k-1}), \varphi(x_{n_k-1})) \\ &= \lim_{k \to +\infty} F(d(x_{m_k}, x_{n_k}), \varphi(x_{m_k}), \varphi(x_{n_k})) \\ &= F(\varepsilon, 0, 0) \geq \varepsilon > 0. \end{aligned}$$

We notice that, thanks to (8.3), we can assume $d(x_{m_k-1}, x_{n_k-1}) > 0$ for all $k \in \mathbb{N}$. Then, using the property (F_1) of the function F, we have

$$t_k = F(d(x_{m_k}, x_{n_k}), \varphi(x_{m_k}), \varphi(x_{n_k})) \geq d(x_{m_k}, x_{n_k}) > 0 \quad \text{for all } k \in \mathbb{N}$$

and

$$s_k = F(d(x_{m_k-1}, x_{n_k-1}), \varphi(x_{m_k-1}), \varphi(x_{n_k-1})) \geq d(x_{m_k-1}, x_{n_k-1}) > 0 \quad \text{for all } k \in \mathbb{N}.$$

Using (8.1) and the property (ζ_1) of the function ζ, with $x = x_{m_k-1}$ and $y = x_{n_k-1}$, we infer

$$\begin{aligned} 0 &\leq \zeta(F(d(x_{m_k}, x_{n_k}), \varphi(x_{m_k}), \varphi(x_{n_k})), F(d(x_{m_k-1}, x_{n_k-1}), \varphi(x_{m_k-1}), \varphi(x_{n_k-1}))) \\ &< F(d(x_{m_k-1}, x_{n_k-1}), \varphi(x_{m_k-1}), \varphi(x_{n_k-1})) - F(d(x_{m_k}, x_{n_k}), \varphi(x_{m_k}), \varphi(x_{n_k})) \end{aligned}$$

for all $k \in \mathbb{N}$. This proves that $t_k < s_k$ for all $k \in \mathbb{N}$. Then, by using the property (ζ_2') of the function ζ, we obtain

$$\begin{aligned} 0 &\leq \limsup_{k\to+\infty} \zeta(F(d(x_{m_k}, x_{n_k}), \varphi(x_{m_k}), \varphi(x_{n_k})), \\ &\quad F(d(x_{m_k-1}, x_{n_k-1}), \varphi(x_{m_k-1}), \varphi(x_{n_k-1}))) \\ &< 0, \end{aligned}$$

which is a contradiction. Hence the sequence $\{x_n\}$ is a Cauchy sequence. This completes the proof.

Lemma 8.2 *Let (X, d) be a metric space and let $T : X \to X$ be a S-F-contraction with respect to $\zeta \in \mathscr{S}'$, $F \in \mathscr{F}$ and a function $\varphi : X \to [0, +\infty[$. If $\{x_n\}$ is a sequence of Picard starting at $x_0 \in X$ such that $x_k = x_{k+1}$ for some $k \in \mathbb{N}$, then $x_n \to x_k$ and x_k is a fixed point of T such that $\varphi(x_k) = 0$.*

Proof Let x_0 be a point of X and let $\{x_n\}$ be a sequence of Picard starting at x_0. Furthermore, we assume that $x_k = x_{k+1}$ for some $k \in \mathbb{N}$. This assures that $x_n \to x_k$ and that x_k is a fixed point of T, in fact, $x_k = x_{k+1} = Tx_k$. We claim that $\varphi(x_k) = 0$. We assume, by contradiction, that $\varphi(x_k) > 0$. Using the property (F_1) of the function F, we get

$$0 < \varphi(x_k) \leq F(d(x_k, x_{k+1}), \varphi(x_k), \varphi(x_{k+1})).$$

Since $x_n = x_k$ for all $n \geq k$, $n \in \mathbb{N}$, using (8.1) with $x = x_k$ and $y = x_{k+1}$ and the property (ζ_1) of the function ζ, we deduce that

$$\begin{aligned} 0 &\leq \zeta(F(d(x_{k+1}, x_{k+2}), \varphi(x_{k+1}), \varphi(x_{k+2})), F(d(x_k, x_{k+1}), \varphi(x_k), \varphi(x_k + 1))) \\ &< F(d(x_k, x_{k+1}), \varphi(x_k), \varphi(x_{k+1})) - F(d(x_{k+1}, x_{k+2}), \varphi(x_k + 1), \varphi(x_{k+2})) \\ &= F(0, \varphi(x_k), \varphi(x_k)) - F(0, \varphi(x_k), \varphi(x_k)) = 0. \end{aligned}$$

Clearly, this is a contradiction, and hence we can affirm that $\varphi(x_k) = 0$. Taking into account of this, we conclude that if $x_k = x_{k+1}$ for some $k \in \mathbb{N}$, then x_k is a fixed point of T such that $\varphi(x_k) = 0$. This completes the proof.

For the class of S-F-contractions, we have the following result of existence and uniqueness of a fixed point.

Theorem 8.3 (see [41], Theorem 3.2) *Let (X, d) be a complete metric space and $T : X \to X$ be a S-F-contraction with respect to $\zeta \in \mathscr{S}'$, $F \in \mathscr{F}$ and a lower semi-continuous function $\varphi : X \to [0, +\infty[$. Then T has a unique fixed point z such that $\varphi(z) = 0$. Moreover, for every $x_0 \in X$, the sequence of Picard $\{x_n\}$ starting at x_0 converges to z.*

Proof Let $x_0 \in X$ and $\{x_n\}$ be a sequence of Picard starting at x_0. We observe that if $x_k = x_{k+1}$ for some $k \in \mathbb{N}$, then, by Lemma 8.2, it follows that $z := x_k$ is a fixed point of T such that $\varphi(z) = 0$ and that $\{x_n\}$ converges to z. Therefore, we can suppose that $x_n \neq x_{n+1}$ for every $n \in \mathbb{N}$.

Now, by Lemma 8.1 we deduce that the sequence $\{x_n\}$ is a Cauchy sequence and, since (X, d) is complete, there exists some $z \in X$ such that

$$\lim_{n\to+\infty} x_n = z. \tag{8.4}$$

We notice that the second statement of (8.2) and the lower semi-continuity of the function φ give

$$0 \leq \varphi(z) \leq \liminf_{n\to+\infty} \varphi(x_n) = 0,$$

that is, $\varphi(z) = 0$.

Now, we claim that z is a fixed point of T. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = z$ or $Tx_{n_k} = Tz$ for all $k \in \mathbb{N}$, then z is a fixed point of T. If this does not occur, then we can assume that $x_n \neq z$ and $Tx_n \neq Tz$ for all $n \in \mathbb{N}$. Using (8.1) and the condition (ζ_1) with $x = x_n$ and $y = z$, we deduce that

$$\begin{aligned}0 &\leq \zeta(F(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz)), F(d(x_n, z), \varphi(x_n), \varphi(z)))\\ &< F(d(x_n, z), \varphi(x_n), \varphi(z)) - F(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz)).\end{aligned}$$

This implies

$$F(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz)) < F(d(x_n, z), \varphi(x_n), \varphi(z)) \ \ \forall\, n \in \mathbb{N}$$

and, consequently,

$$\begin{aligned}d(z, Tz) &\leq d(z, x_{n+1}) + d(Tx_n, Tz)\\ &\leq d(z, x_{n+1}) + F(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz))\\ &< d(z, x_{n+1}) + F(d(x_n, z), \varphi(x_n), \varphi(z))\end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to +\infty$ in the above inequality, since F is continuous in $(0, 0, 0)$, we obtain that $d(z, Tz) \leq F(0, 0, 0) = 0$, that is, $z = Tz$.

Now, we establish uniqueness of the fixed point. Suppose that there exists $w \in X$ such that $w = Tw$ and $z \neq w$. The property (F_1) of the function F ensures that $F(d(w, z), \varphi(w), \varphi(z)) \geq d(w, z) > 0$. Using (8.1) and the property (ζ_1) of the function ζ, with $x = w$ and $y = z$, we get

$$\begin{aligned}0 &\leq \zeta(F(d(Tw, Tz), \varphi(Tw), \varphi(Tz)), F(d(w, z), \varphi(w), \varphi(z)))\\ &= \zeta(F(d(w, z), \varphi(w), \varphi(z)), F(d(w, z), \varphi(w), \varphi(z)))\\ &< F(d(w, z), \varphi(w), \varphi(z)) - F(d(w, z), \varphi(w), \varphi(z)) = 0,\end{aligned}$$

which is a contradiction and hence $w = z$. This completes the proof of Theorem 8.3 since, by (8.4), the sequence $\{x_n\}$ of Picard starting at x_0 converges to z.

We remark that, if, in Theorem 8.3, we choose $F \in \mathscr{F}$ defined by $F(a, b, c) = a + b + c$ for all $a, b, c \in [0, +\infty[$, then we have the following result.

Theorem 8.4 ([20], Theorem 3.2) *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. Suppose that there exist a simulation function ζ and a lower semi-continuous function $\varphi : X \to [0, +\infty[$ such that*

$$\zeta\left(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty), d(x, y) + \varphi(x) + \varphi(y)\right) \geq 0 \quad \forall\, x, y \in X.$$

Then T has a unique fixed point z such that $\varphi(z) = 0$.

Further, we notice that if we take in Theorem 8.3 the function F defined by $F(a, b, c) = a + b + c$ for all $a, b, c \in [0, +\infty[$ and $\varphi(x) = 0$, for all $x \in X$, then we obtain Theorem 8.2, that is, Theorem 2.8 of [16].

The following example shows that Theorem 8.3 is a proper generalization, in the setting of metric spaces, of the Theorem 2.8 of [16] and hence of the Banach contraction principle.

Example 8.3 ([40], Example 4) Let $X = [0, 1]$ endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Obviously, (X, d) is a complete metric space. Fix $k \in [0, 1[$ and define a mapping $T : X \to X$ by

$$Tx = \begin{cases} 0, & \text{if } x = 0, \\ \dfrac{k}{2n} - k\dfrac{2n-1}{2n}(2nx - 1), & \text{if } \dfrac{1}{2n} \leq x \leq \dfrac{1}{2n-1}, \\ \dfrac{k}{2n} + k\dfrac{2n+1}{2n}(2nx - 1), & \text{if } \dfrac{1}{2n+1} \leq x \leq \dfrac{1}{2n}. \end{cases}$$

Firstly, we note that if we choose k appropriately, then T is not a contractive mapping. In fact, if, for odd $n > 1$, we choose $x = \dfrac{1}{2n-1}$ and $y = \dfrac{1}{n-1}$, then we have

$$d(Tx, Ty) = \frac{k}{n-1} \quad \text{and} \quad d(x, y) = \frac{n}{(n-1)(2n-1)} \leq \frac{3}{5(n-1)}.$$

The previous inequalities ensure that $d(Tx, Ty) \geq d(x, y)$ whenever $k \geq 3/5$. So, T is not a contractive mapping. By Remark 8.3, the mapping T is not a $\mathscr{Z}$-contraction. So, Theorem 2.8 of [16], that is, Theorem 8.2 cannot be applied to estabishing that T has a fixed point if $k \geq 3/5$. This implies that also the Banach contraction principle cannot be applied to estabishing that T has a fixed point if $k \geq 3/5$.

On the other hand, if we consider the function $\varphi : X \to [0, +\infty[$ defined by $\varphi(x) = x$ for all $x \in X$ and the function $F \in \mathscr{F}$ defined by $F(a, b, c) = a + b + c$ for all $a, b, c \in [0, +\infty[$, then we obtain

$$\begin{aligned}F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\\ &= 2\max\{Tx,Ty\} \le 2k\max\{x,y\}\\ &= k[d(x,y)+\varphi(x)+\varphi(y)]\\ &= kF(d(x,y),\varphi(x),\varphi(y))\end{aligned}$$

for all $x, y \in X$. Thus T is a S-F-contraction with respect to $\zeta \in \mathscr{S}'$ defined by $\zeta(t,s) = ks - t$. Therefore, taking into account that φ is a lower semi-continuous function, we can apply Theorem 8.3 in order to deduce that T has a unique fixed point $z = 0$ in X. Obiviously, $\varphi(0) = 0$.

8.5 Fixed Points for Ordered *S*-*F*-Contractions

In this section, we collect some fixed point results involving an ordered S-F-contraction defined in a complete ordered metric space. Again, we work with the families of functions $\mathscr{F}$ and $\mathscr{S}'$ introduced in Sects. 8.2 and 8.3.

We start fixing the notation. If (X,d) is a metric space and $(X,\preceq)$ is a partially ordered set, then we say that $(X,d,\preceq)$ is an *ordered metric space*. Two elements $x, y \in X$ are said to be *comparable* if $x \preceq y$ or $y \preceq x$ holds. The mapping $T : (X,\preceq) \to (X,\preceq)$ is called *nondecreasing* if $Tx \preceq Ty$ whenever $x \preceq y$. A sequence $\{x_n\}$ is *nondecreasing* if $x_{n-1} \preceq x_n$ for all $n \in \mathbb{N}$.

Later on, we will use the following properties of an ordered metric space:

(R) An ordered metric space $(X,d,\preceq)$ is *regular* if, for every nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \to z \in X$, we have $x_{n-1} \preceq z$ for all $n \in \mathbb{N}$.

(U) An ordered metric space $(X,d,\preceq)$ has the *property* (U) if, for each pair of not comparable elements $x, y \in X$, there exists $u \in X$ such that $x \preceq u$ and $y \preceq u$.

Ran and Reurings in [25] investigate a similar conclusion to the Banach contraction principle in metric sets endowed with an order. Following the Ran and Reurings' work many mathematicians got interested into the investigation of the metric fixed point problem for monotone mappings defined in an ordered metric space. We stress that the main fixed point result of [25] was discovered investigating the solutions to some special matrix equations.

Theorem 8.5 (see [25]) *Let $(X,d,\preceq)$ be a complete ordered metric space. Let $T : X \to X$ be a continuous monotone contraction mapping. Assume that there exists $x_0 \in X$ such that x_0 and Tx_0 are comparable. Then the sequence $\{x_n\}$ of Picard starting at x_0 converges to a fixed point z of T. Moreover, if $u_0 \in X$ is comparable to x_0, then we have $\lim_{n\to+\infty} u_n = z$, where $\{u_n\}$ is the sequence of Picard starting at u_0. In addition, if every pair $x, y \in X$ has an upper bound and a lower bound in X, then T has a unique fixed point z and $\lim_{n\to+\infty} u_n = z$ for any $u_0 \in X$.*

In [21], Nieto and Rodríguez-López observe that the continuity assumption in Theorem 8.5 may be relaxed. Thus they formulate the following result.

Theorem 8.6 (see [21], Theorems 2.2 and 2.3) *Let $(X, d, \preceq)$ be a complete ordered metric space. Let $T : X \to X$ be a nondecreasing contraction mapping. Assume that X is regular and that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. Then T has a fixed point. In addition, if every pair $x, y \in X$ has an upper bound or a lower bound in X, then T has a unique fixed point.*

Now, we introduce the notion of ordered S-F-contraction in order to obtain a generalization of the previous result in the setting of ordered metric spaces.

Definition 8.7 Let $(X, d, \preceq)$ be an ordered metric space. A mapping $T : X \to X$ is called an *ordered S-F-contraction* if there exist a function $\zeta \in \mathscr{S}'$, a function $F \in \mathscr{F}$ and a function $\varphi : X \to [0, +\infty[$ such that

$$\zeta\,(F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)), F(d(x, y), \varphi(x), \varphi(y))) \geq 0 \tag{8.5}$$

for all $x, y \in X$ with $x \preceq y$.

Firstly, we give some useful remarks to get our main results.

Remark 8.5 Let $(X, d, \preceq)$ be an ordered metric space and $T : X \to X$ be an ordered S-F-contraction with respect to $\zeta \in \mathscr{S}'$, $F \in \mathscr{F}$ and $\varphi : X \to [0, +\infty[$. If $z \in X$ is a fixed point of T, then $\varphi(z) = 0$.

In fact, if we suppose $\varphi(z) > 0$, by the property (F_1) of the function F, we obtain

$$F(d(z, z), \varphi(z), \varphi(z)) \geq \varphi(z) > 0.$$

Using (8.5) with $x = y = z$ and the property (ζ_1) of the function ζ, we get

$$\begin{aligned}
0 &\leq \zeta(F(d(Tz, Tz), \varphi(Tz), \varphi(Tz), F(d(z, z), \varphi(z), \varphi(z))) \\
&= \zeta(F(d(z, z), \varphi(z), \varphi(z)), F(d(z, z), \varphi(z), \varphi(z))) \\
&< F(d(z, z), \varphi(z), \varphi(z)) - F(d(z, z), \varphi(z), \varphi(z)) = 0.
\end{aligned}$$

Clearly, this is not possible and so $\varphi(z) = 0$.

Remark 8.6 Let $(X, d, \preceq)$ be an ordered metric space and let $T : X \to X$ be an ordered S-F-contraction with respect to $\zeta \in \mathscr{S}'$, $F \in \mathscr{F}$ and $\varphi : X \to [0, +\infty[$. If $z, w \in X$ are two fixed points of T, then $z \neq w$ if and only if z and w are not comparable. In fact, if z and w are comparable and $z \neq w$, then using the property (F_1) of the function F, we obtain

$$F(d(z, w), \varphi(z), \varphi(w)) \geq d(z, w) > 0.$$

Using (8.5) with $x = z$ and $y = w$ and the property (ζ_1) of the function ζ, we obtain

$$\begin{aligned}
0 &\leq \zeta(F(d(Tz, Tw), \varphi(Tz), \varphi(Tw)), F(d(z, w), \varphi(z), \varphi(w))) \\
&< F(d(z, w), \varphi(z), \varphi(w)) - F(d(Tz, Tw), \varphi(Tz), \varphi(Tw)) \\
&= F(d(z, w), \varphi(z), \varphi(w)) - F(d(z, w), \varphi(z), \varphi(w)) = 0,
\end{aligned}$$

which is a contradiction.

Now, we establish some auxiliary results which will be used in the sequel.

Lemma 8.3 *Let $(X, d, \preceq)$ be an ordered metric space and $T : X \to X$ be a nondecreasing ordered S-F-contraction with respect to $\zeta \in \mathscr{S}'$, $F \in \mathscr{F}$ and a function $\varphi : X \to [0, +\infty[$. If $x_0, u_0 \in X$ are such that $x_0 \preceq u_0$ and $\{x_n\}$ and $\{u_n\}$ are the sequences of Picard starting at x_0 and u_0, respectively, then $d(x_n, u_n) \to 0$ as $n \to +\infty$.*

Proof Assume that x_0, u_0 are points of X such that $x_0 \preceq u_0$. The hypothesis that T is nondecreasing implies $x_n \preceq u_n$ for all $n \in \mathbb{N}$. If $x_k = u_k$ for some $k \in \mathbb{N}$, then the conclusion is obvious. So, we assume that $x_{n-1} \neq u_{n-1}$, that is, $x_{n-1} \prec u_{n-1}$ for all $n \in \mathbb{N}$. Consequently, by the property (F_1) of the function F, we have

$$F(d(x_{n-1}, u_{n-1}), \varphi(x_{n-1}), \varphi(u_{n-1})) \geq d(x_{n-1}, u_{n-1}) > 0 \quad \forall\, n \in \mathbb{N}.$$

Now, using (8.5) and the property (ζ_1) of the function ζ, with $x = x_{n-1}$ and $y = u_{n-1}$, we get

$$\begin{aligned} 0 \leq\, &\zeta(F(d(Tx_{n-1}, Tu_{n-1}), \varphi(Tx_{n-1}), \varphi(Tu_{n-1})),\\ &\qquad\qquad F(d(x_{n-1}, u_{n-1}), \varphi(x_{n-1}), \varphi(u_{n-1})))\\ <\, &F(d(x_{n-1}, u_{n-1}), \varphi(x_{n-1}), \varphi(u_{n-1})) - F(d(x_n, u_n), \varphi(x_n), \varphi(u_n)). \end{aligned}$$

The previous inequality ensures that the sequence

$$\{F(d(x_{n-1}, u_{n-1}), \varphi(x_{n-1}), \varphi(u_{n-1}))\}$$

of positive real numbers is decreasing and so there exists a nonnegative real number ℓ such that

$$\lim_{n \to +\infty} F(d(x_{n-1}, u_{n-1}), \varphi(x_{n-1}), \varphi(u_{n-1})) = \ell.$$

If $\ell > 0$, using (8.5) and the property (ζ_2') of the function ζ with $t_n = F(d(x_n, u_n), \varphi(x_n), \varphi(u_n))$ and $s_n = F(d(x_{n-1}, u_{n-1}), \varphi(x_{n-1}), \varphi(u_{n-1}))$, taking into account that $t_n < s_n$ for all $n \in \mathbb{N}$, we infer

$$\begin{aligned} 0 \leq \limsup_{n \to +\infty} &\zeta(F(d(Tx_{n-1}, Tu_{n-1}), \varphi(Tx_{n-1}), \varphi(Tu_{n-1})),\\ &\qquad\qquad F(d(x_{n-1}, u_{n-1}), \varphi(x_{n-1}), \varphi(u_{n-1}))) < 0, \end{aligned}$$

which is a contradiction. Hence $\ell = 0$. Now, we notice that the property (F_1) of the function F implies

$$0 \leq \lim_{n \to +\infty} d(x_{n-1}, u_{n-1}) \leq \lim_{n \to +\infty} F(d(x_{n-1}, u_{n-1}), \varphi(x_{n-1}), \varphi(u_{n-1})) = 0$$

and so the lemma is proved. This completes the proof.

Lemma 8.4 *Let $(X, d, \preceq)$ be an ordered metric space and let $T : X \to X$ be a nondecreasing ordered S-F-contraction with respect to $\zeta \in \mathscr{S}'$, $F \in \mathscr{F}$ and a function $\varphi : X \to [0, +\infty[$. If X has the property (U), then T admits at most one fixed point.*

Proof Assume that $z, w \in X$ are two distinct fixed points of T. We notice that Remark 8.6 assures that z and w are not comparable. Hence, by the property (U) of the space X, there exists $u \in X$ such that $z \preceq u$ and $w \preceq u$. Let $\{u_n\}$, $\{z_n\}$ and $\{w_n\}$ be the sequence of Picard starting at u, z and w, respectively. Since $z_n = z$ and $w_n = w$ for all $n \in \mathbb{N}$, using Lemma 8.3, we deduce

$$\lim_{n\to+\infty} d(z, u_{n-1}) = 0 \quad \text{and} \quad \lim_{n\to+\infty} d(w, u_{n-1}) = 0.$$

From

$$d(z, w) \le d(z, u_{n-1}) + d(u_{n-1}, w),$$

letting $n \to +\infty$, we get $d(z, w) = 0$, that is, $z = w$. This completes the proof.

Lemma 8.5 *Let $(X, d, \preceq)$ be an ordered metric space and $T : X \to X$ be a nondecreasing ordered S-F-contraction with respect to $\zeta \in \mathscr{S}'$, $F \in \mathscr{F}$ and a function $\varphi : X \to [0, +\infty[$. If there exists a point $x_0 \in X$ such that $x_0 \preceq Tx_0$, then the Picard sequence $\{x_n\}$ starting at x_0 is Cauchy.*

Proof Let $x_0 \in X$ be such that $x_0 \preceq Tx_0$ and $\{x_n\}$ be the sequence of Picard starting at x_0. If $x_k = x_{k+1}$ for some $k \in \mathbb{N}$, then $x_n = x_k$ for all $n \ge k$ and hence $\{x_n\}$ is a Cauchy sequence. Therefore, we can assume that $x_{n-1} \ne x_n$ for all $n \in \mathbb{N}$. First, we prove that

$$\lim_{n\to+\infty} d(x_{n-1}, x_n) = 0 \quad \text{and} \quad \lim_{n\to+\infty} \varphi(x_n) = 0. \tag{8.6}$$

We notice that from $x_{n-1} \ne x_n$ for all $n \in \mathbb{N}$, it follows that $d(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$. Then, the property (F_1) of the function F ensures that

$$F(d(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n)) \ge d(x_{n-1}, x_n) > 0 \quad \forall\, n \in \mathbb{N}.$$

Furthermore, the hypothesis that T is nondecreasing ensures that $x_{n-1} \preceq x_n$, for all $n \in \mathbb{N}$. Now, put $d_{n-1} = d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. Using (8.5) and the property (ζ_1) of the function ζ, with $x = x_{n-1}$ and $y = x_n$, we infer that

$$\begin{aligned}
0 &\le \zeta(F(d_n, \varphi(x_n), \varphi(x_{n+1})), F(d_{n-1}, \varphi(x_{n-1}), \varphi(x_n))) \\
&< F(d_{n-1}, \varphi(x_{n-1}), \varphi(x_n)) - F(d_n, \varphi(x_n), \varphi(x_{n+1}))
\end{aligned}$$

for all $n \in \mathbb{N}$. The above inequality shows that

$$F(d_n, \varphi(x_n), \varphi(x_{n+1})) < F(d_{n-1}, \varphi(x_{n-1}), \varphi(x_n)) \ \text{ for all } n \in \mathbb{N}.$$

Consequently, $\{F(d_{n-1}, \varphi(x_{n-1}), \varphi(x_n))\}$ is a decreasing sequence of positive real numbers. Thus, there is some $\ell \ge 0$ such that

$$\lim_{n\to+\infty} F(d_{n-1}, \varphi(x_{n-1}), \varphi(x_n)) = \ell.$$

We remark that if $\ell > 0$, by using condition (ζ_2') with $t_n = F(d_n, \varphi(x_n), \varphi(x_{n+1}))$ and $s_n = F(d_{n-1}, \varphi(x_{n-1}), \varphi(x_n))$, since $t_n < s_n$ for all $n \in \mathbb{N}$, we get

$$0 \le \limsup_{n\to+\infty} \zeta\,(F(d_n, \varphi(x_n), \varphi(x_{n+1})), F(d_{n-1}, \varphi(x_{n-1}), \varphi(x_n))) < 0,$$

which is a contradiction. Hence we conclude that $\ell = 0$. Now, thanks to the property (F_1) of the function F, we deduce that

$$\max\{d_{n-1}, \varphi(x_{n-1})\} \le F(d_{n-1}, \varphi(x_{n-1}), \varphi(x_n)) \;\forall\, n \in \mathbb{N}$$

and hence

$$\lim_{n\to+\infty} d(x_{n-1}, x_n) = 0 \quad \text{and} \quad \lim_{n\to+\infty} \varphi(x_{n-1}) = 0.$$

Next, we show that the sequence $\{x_n\}$ is a Cauchy sequence. We assume, by way of contradiction, that $\{x_n\}$ is not a Cauchy sequence. Then there exist a positive real number ε and two sequences $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k \ge k$, $d(x_{m_k}, x_{n_k}) \ge \varepsilon > d(x_{m_k}, x_{n_k-1})$ for all $k \in \mathbb{N}$. Using the first condition of (8.6), we obtain

$$\lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) = \lim_{k\to+\infty} d(x_{m_k-1}, x_{n_k-1}) = \varepsilon. \tag{8.7}$$

Using the continuity of the function F, we get

$$\begin{aligned} &\lim_{k\to+\infty} F(d(x_{m_k-1}, x_{n_k-1}), \varphi(x_{m_k-1}), \varphi(x_{n_k-1})) \\ &= \lim_{k\to+\infty} F(d(x_{m_k}, x_{n_k}), \varphi(x_{m_k}), \varphi(x_{n_k})) \\ &= F(\varepsilon, 0, 0) \ge \varepsilon > 0. \end{aligned}$$

By (8.7), we can assume $d(x_{m_k-1}, x_{n_k-1}) > 0$ for all $k \in \mathbb{N}$. Now, thanks to the property (F_1) of the function F, we have

$$t_k = F(d(x_{m_k}, x_{n_k}), \varphi(x_{m_k}), \varphi(x_{n_k})) \ge d(x_{m_k}, x_{n_k}) > 0 \;\; \forall\, k \in \mathbb{N}$$

and

$$s_k = F(d(x_{m_k-1}, x_{n_k-1}), \varphi(x_{m_k-1}), \varphi(x_{n_k-1})) \ge d(x_{m_k-1}, x_{n_k-1}) > 0 \;\forall\, k \in \mathbb{N}.$$

We remark that the hypothesis that T is nondecreasing ensures that $x_{m_k-1} \preceq x_{n_k-1}$ for all $k \in \mathbb{N}$. This permits to apply (8.5) with $x = x_{m_k-1}$ and $y = x_{n_k-1}$ and so we obtain

$$0 \le \zeta(F(d(x_{m_k}, x_{n_k}), \varphi(x_{m_k}), \varphi(x_{n_k})), F(d(x_{m_k-1}, x_{n_k-1}), \varphi(x_{m_k-1}), \varphi(x_{n_k-1}))) \tag{8.8}$$

for all $k \in \mathbb{N}$. By the property (ζ_1) of the function ζ, we get

$$F(d(x_{m_k}, x_{n_k}), \varphi(x_{m_k}), \varphi(x_{n_k})) < F(d(x_{m_k-1}, x_{n_k-1}), \varphi(x_{m_k-1}), \varphi(x_{n_k-1}))$$

for all $k \in \mathbb{N}$. This proves that $t_k < s_k$ for all $k \in \mathbb{N}$. Then, by the property (ζ_2') of the function ζ, we deduce that

$$\limsup_{k\to+\infty} \zeta(F(d(x_{m_k}, x_{n_k}), \varphi(x_{m_k}), \varphi(x_{n_k})),\\ F(d(x_{m_k-1}, x_{n_k-1}), \varphi(x_{m_k-1}), \varphi(x_{n_k-1}))) < 0,$$

which is in contradiction with (8.8). Consequently, the sequence $\{x_n\}$ is Cauchy. This completes the proof.

For the class of ordered S-F-contractions, we have the following result of existence of a fixed point.

Theorem 8.7 (see [42], Theorem 1) *Let $(X, d, \preceq)$ be a complete ordered metric space and $T : X \to X$ be a nondecreasing ordered S-F-contraction with respect to $\zeta \in \mathscr{S}'$, $F \in \mathscr{F}$ and a function $\varphi : X \to [0, +\infty[$. If there exists a point $x_0 \in X$ such that $x_0 \preceq Tx_0$ and T is continuous, then T has a fixed point z such that $\varphi(z) = 0$. Moreover, if $u_0 \in X$ is comparable to x_0 then we have $\lim_{n\to+\infty} u_n = z$, where $\{u_n\}$ is the Picard sequence starting at u_0.*

Proof Let $x_0 \in X$ be such that $x_0 \preceq Tx_0$ and $\{x_n\}$ be the sequence of Picard starting at x_0. We stress that, if $x_k = x_{k+1}$ for some $k \in \mathbb{N}$, then $x_k = x_{k+1} = Tx_k$, that is, $z := x_k$ is a fixed point of T. Thus, by Remark 8.5, we have $\varphi(z) = 0$ and the proof of existence of a fixed point is complete. Therefore, it is not restrictive to suppose that $x_n \neq x_{n-1}$ for each $n \in \mathbb{N}$. Thanks to Lemma 8.5, we can affirm that the sequence $\{x_n\}$ is a Cauchy sequence. Further, the completeness of $(X, d, \preceq)$ ensures that there exists some $z \in X$ such that

$$\lim_{n\to+\infty} x_n = z.$$

Now, in order to complete the proof, we notice that the continuity of the mapping T ensures that z is a fixed point of T and, further, by Remark 8.5, we have $\varphi(z) = 0$. Finally, if $x_0, u_0 \in X$ are comparable, thanks to Lemma 8.3, we have $d(x_n, u_n) \to 0$ and hence

$$0 \leq d(u_n, z) \leq d(u_n, x_n) + d(x_n, z) \to 0,$$

that is, $\lim_{n\to+\infty} u_n = z$. This completes the proof.

Theorem 8.8 (see [42], Theorem 2) *Let $(X, d, \preceq)$ be a complete ordered metric space and $T : X \to X$ be a nondecreasing ordered S-F-contraction with respect to $\zeta \in \mathscr{S}'$, $F \in \mathscr{F}$ and a lower semi-continuous function $\varphi : X \to [0, +\infty[$. If there exists a point $x_0 \in X$ such that $x_0 \preceq Tx_0$ and X is regular, then T has a fixed point z such that $\varphi(z) = 0$. Moreover, if $u_0 \in X$ is comparable to x_0, then we have $\lim_{n\to+\infty} u_n = z$, where $\{u_n\}$ is the Picard sequence starting at u_0.*

Proof Let x_0 be a point of X such that $x_0 \preceq Tx_0$ and let $\{x_n\}$ be the sequence of Picard starting at x_0. Following the proof of Theorem 8.7, we say that it is not restrictive to suppose that $x_n \neq x_{n-1}$ for each $n \in \mathbb{N}$. Furthermore, thanks to Lemma 8.5, we say that $\{x_n\}$ is a Cauchy sequence. Now, we notice that the completeness of $(X, d, \preceq)$ ensures that there exists some $z \in X$ such that

$$\lim_{n\to+\infty} x_n = z.$$

Further, the lower semi-continuity of the function φ and the second statement of (8.6) assure that

$$0 \le \varphi(z) \le \liminf_{n\to+\infty} \varphi(x_n) = 0,$$

that is, $\varphi(z) = 0$.

Next, we claim that z is a fixed point of T. Obviously, z is a fixed point of the mapping T if there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = z$ or $Tx_{n_k} = Tz$, for all $k \in \mathbb{N}$. If such a subsequence does not exist, then we can assume that $x_n \neq z$ and $Tx_n \neq Tz$ for all $n \in \mathbb{N}$. Consequently, we have

$$F(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz)),\ F(d(x_n, z), \varphi(x_n), \varphi(z)) \in]0, +\infty[\quad \forall\, n \in \mathbb{N}.$$

Now, the hypothesis that T is nondecreasing together with the condition $x_0 \preceq Tx_0$ ensure that the sequence $\{x_n\}$ is nondecreasing. So, the hypothesis that X is regular implies that $x_{n-1} \preceq z$ for all $n \in \mathbb{N}$. Using (8.5) with $x = x_n$ and $y = z$ and the property (ζ_1) of the function ζ, we get

$$\begin{aligned} 0 &\le \zeta(F(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz)), F(d(x_n, z), \varphi(x_n), \varphi(z))) \\ &< F(d(x_n, z), \varphi(x_n), \varphi(z)) - F(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz)). \end{aligned}$$

From the previous inequality, we get

$$F(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz)) < F(d(x_n, z), \varphi(x_n), \varphi(z)) \;\forall\, n \in \mathbb{N}$$

and so

$$\begin{aligned} d(z, Tz) &\le d(z, x_{n+1}) + d(Tx_n, Tz) \\ &\le d(z, x_{n+1}) + F(d(Tx_n, Tz), \varphi(Tx_n), \varphi(Tz)) \\ &< d(z, x_{n+1}) + F(d(x_n, z), \varphi(x_n), \varphi(z)) \end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to +\infty$ in the above inequality, taking into account that F is continuous in $(0, 0, 0)$, we deduce that $d(z, Tz) \le F(0, 0, 0) = 0$, that is, $z = Tz$. Finally, if $x_0, u_0 \in X$ are comparable, then, by Lemma 8.3, we have $d(x_n, u_n) \to 0$ and hence

$$0 \le d(u_n, z) \le d(u_n, x_n) + d(x_n, z) \to 0,$$

that is, $\lim_{n\to+\infty} u_n = z$. This completes the proof.

From Theorem 8.7 and Lemma 8.4 we deduce the following result.

Theorem 8.9 *Let $(X, d, \preceq)$ be a complete ordered metric space and let $T : X \to X$ be a nondecreasing ordered S-F-contraction with respect to $\zeta \in \mathscr{S}'$, $F \in \mathscr{F}$ and a function $\varphi : X \to [0, +\infty[$. If there exists a point $x_0 \in X$ such that $x_0 \preceq Tx_0$ and T is continuous, then T has a fixed point z such that $\varphi(z) = 0$. Moreover if X has the property (U), then T has a unique fixed point z such that $\varphi(z) = 0$ and, further, for all $u_0 \in X$, we have $\lim_{n\to+\infty} u_n = z$, where $\{u_n\}$ is the Picard sequence starting at u_0.*

Proof The existence of a unique fixed point z such that $\varphi(z) = 0$ is consequence of Theorem 8.7 and Lemma 8.4. Also, we stress that if $\lim_{n\to+\infty} d(u_n, x_n) = 0$, where $\{x_n\}$ is the Picard sequence starting at x_0 such that $x_n \to z$ (see Theorem 8.7), from

$$0 \leq d(u_n, z) \leq d(u_n, x_n) + d(x_n, z) \to 0,$$

we infer the claim $\lim_{n\to+\infty} u_n = z$. Clearly, if x_0 and u_0 are comparable, it holds thanks to Lemma 8.3. If x_0 and u_0 are not comparable, the property (U) of X ensures that there exists $w_0 \in X$ such that x_0 and u_0 are comparable with w_0. If $\{w_n\}$ is the Picard sequence starting at w_0, thanks to Lemma 8.3, we have

$$\lim_{n\to+\infty} d(x_n, w_n) = \lim_{n\to+\infty} d(u_n, w_n) = 0.$$

Then $\lim_{n\to+\infty} d(u_n, x_n) = 0$ and hence $\lim_{n\to+\infty} u_n = z$. This completes the proof.

From Theorem 8.8 and Lemma 8.4 we deduce the following result.

Theorem 8.10 *Let $(X, d, \preceq)$ be a complete ordered metric space and $T : X \to X$ be a nondecreasing ordered S-F-contraction with respect to $\zeta \in \mathscr{S}'$, $F \in \mathscr{F}$ and a lower semi-continuous function $\varphi : X \to [0, +\infty[$. If there exists a point $x_0 \in X$ such that $x_0 \preceq Tx_0$ and X is regular, then T has a fixed point z such that $\varphi(z) = 0$. Moreover if X has the property (U), then T has a unique fixed point z such that $\varphi(z) = 0$ and, further, for all $u_0 \in X$, we have $\lim_{n\to+\infty} u_n = z$, where $\{u_n\}$ is the Picard sequence starting at u_0.*

Proof We stress that the existence of a unique fixed point is a consequence of Theorem 8.8 and Lemma 8.4. Further, if if $\lim_{n\to+\infty} d(u_n, x_n) = 0$, where $\{x_n\}$ is the Picard sequence starting at x_0 such that $x_n \to z$ (see Theorem 8.8), from

$$0 \leq d(u_n, z) \leq d(u_n, x_n) + d(x_n, z) \to 0,$$

it follows the claim $\lim_{n\to+\infty} u_n = z$. Clearly, if x_0 and u_0 are comparable, then this is a consequence of Lemma 8.3. If x_0 and u_0 are not comparable, taking into account that X has the property (U), there exists $w_0 \in X$ such that x_0 and u_0 are comparable with w_0. If $\{w_n\}$ is the Picard sequence starting at w_0, thanks to Lemma 8.3 gives

$$\lim_{n\to+\infty} d(x_n, w_n) = \lim_{n\to+\infty} d(u_n, w_n) = 0.$$

Then $\lim_{n\to+\infty} d(x_n, u_n) = 0$ and thus $\lim_{n\to+\infty} u_n = z$. This completes the proof.

Example 8.4 Let $X = [0, 2]$ endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Also, X can be equipped with a partial order $\preceq$ given by

$$x, y \in X,\ x \preceq y \ \text{ if } \ x = y, \quad \left(x \leq y,\ x, y \in \left[0, \frac{15}{8}\right]\right) \ \text{ or } \ (x \in [0, 2[\ \text{ and } \ y = 2).$$

Obviously, $(X, d, \preceq)$ is an ordered complete metric space that is regular and has the property (U). Consider the nondecreasing function $T : X \to X$ given by

$$Tx = \begin{cases} \frac{x}{2}, & \text{if } x \in \left[0, \frac{15}{8}\right], \\ \frac{3}{2}, & \text{if } x \in \left]\frac{15}{8}, 2\right]. \end{cases}$$

The function T satisfies condition (8.5) with respect to the function $\zeta \in \mathscr{S}'$ defined by

$$\zeta(t, s) = s - \frac{t+2}{t+1}t \ \text{ for all } t, s \in [0, +\infty[,$$

the function $F \in \mathscr{F}$ defined by $F(a, b, c) = a + b + c$ for all $a, b, c \in [0, +\infty[$ and the lower semi-continuous function $\varphi : X \to [0, +\infty[$ defined by $\varphi(x) = x$ for all $x \in X$. Indeed, if $x \preceq y$ and $x, y \in \left[0, \frac{15}{8}\right]$, then we have

$$\begin{aligned}
&\zeta(F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)), F(d(x, y), \varphi(x), \varphi(y))) \\
&= \zeta(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty), d(x, y) + \varphi(x) + \varphi(y)) \\
&= \zeta(y, 2y) = 2y - \frac{y+2}{y+1}y \\
&= \frac{y^2}{y+1} \geq 0.
\end{aligned}$$

If $x \preceq y$ with $x \in [0, 2]$ and $y = 2$, then we have

$$\begin{aligned}
&\zeta(F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)), F(d(x, y), \varphi(x), \varphi(y))) \\
&= \zeta(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty), d(x, y) + \varphi(x) + \varphi(y)) \\
&= \zeta(3, 4) = 4 - \frac{5}{4}3 \\
&= \frac{16 - 15}{4} \geq 0.
\end{aligned}$$

If $x = y \in]\frac{15}{8}, 2[$, then we have

$$\begin{aligned}
&\zeta(F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)), F(d(x, y), \varphi(x), \varphi(y))) \\
&= \zeta(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty), d(x, y) + \varphi(x) + \varphi(y)) \\
&= \zeta(3, 2x) = 2x - \frac{5}{4}3 \\
&= \frac{8x - 15}{4} \geq 0.
\end{aligned}$$

Since all the conditions of Theorem 8.10 are satisfied, T has a unique fixed point $z = 0$ in X, obviously $\varphi(0) = 0$. We stress that Theorem 8.9 cannot be used to deduce that T has a unique fixed point since T is not continuous. Moreover, from

$$d\left(T\frac{15}{8}, T2\right) = \frac{3}{2} - \frac{15}{16} = \frac{9}{16} > \frac{1}{8} = d\left(\frac{15}{8}, 2\right),$$

we infer that we cannot use Theorem 2.2 of [21] (see Theorem 8.6) in order to affirm that T has a fixed point. This completes the proof.

For completeness, we remark that, in Argoubi et al. [2], consider a pair of nonlinear operators satisfying a nonlinear contraction involving a simulation function (in the sense of Definition 8.4) in a metric space endowed with a partial order. For this kind of contractions, they establish coincidence and common fixed point results. Furthermore, Argoubi et al. introduce the notion of right-monotone simulation function. A function $\zeta \in \mathscr{S}$ is called a *right-monotone simulation function* if, for $t \geq 0$, we have $\zeta(t, s_1) \leq \zeta(t, s_2)$ whenever $s_1 \leq s_2$.

We stress that the followimg result can be deduced from Corollary 4.3 of [2].

Theorem 8.11 *Let $(X, d, \preceq)$ be a complete ordered metric space and $T : X \to X$ be a nondecreasing mapping. Assume that X is regular and there exists a right-monotone simulation function $\zeta \in \mathscr{S}$ such that*

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0 \quad \forall\, x, y \in X,\ x \preceq y.$$

Then T has a fixed point.

Obviously, the above result holds for any simulation function $\zeta \in \mathscr{S}$ that is not necessarily right-monotone as specified by Argoubi et al. in [2]. We remark that Theorem 8.11 follows from Theorem 8.8 if we choose the function $F \in \mathscr{F}$ defined by $F(a, b, c) = a + b + c$ for all $a, b, c \in [0, +\infty[$ and the lower semi-continuous function $\varphi : X \to [0, +\infty[$ defined by $\varphi(x) = 0$ for all $x \in X$. Again, choosing F and φ as above and $\zeta(t, s) = ks - t$ for some $k \in [0, 1[$, we infer Theorem 8.6 from Theorem 8.10.

8.6 Consequences

In this section, we point out that particularizing the function $\zeta \in \mathcal{S}'$ in Theorems 8.3 and 8.9–8.10 we get in the setting of metric spaces and ordered metric spaces several special results known in the literature. For instance, if we choose $\zeta \in \mathcal{S}'$ defined by $\zeta(t,s) = k\,s - t$ with $k \in [0,1[$, thanks to Theorem 8.3, we obtain the following corollary.

Corollary 8.1 (see [12], Theorem 2.1) *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping. Suppose that there exist $k \in [0,1[$, a function $F \in \mathcal{F}$ and a lower semi-continuous function $\varphi : X \to [0,+\infty[$ such that*

$$F(d(Tx,Ty), \varphi(Tx), \varphi(Ty)) \leq k\,F(d(x,y), \varphi(x), \varphi(y)) \quad \forall\, x, y \in X.$$

Then T has a unique fixed point z such that $\varphi(z) = 0$. Moreover, for all $u_0 \in X$, the Picard sequence $\{u_n\}$ starting at u_0 converges to z.

Again, choosing the function ζ as above, we obtain the following result in the setting of ordered metric spaces.

Corollary 8.2 *Let $(X,d,\preceq)$ be a complete ordered metric space and $T : X \to X$ be a nondecreasing mapping. Suppose that there exist $k \in [0,1[$, a function $F \in \mathcal{F}$ and a lower semi-continuous function $\varphi : X \to [0,+\infty[$ such that*

$$F(d(Tx,Ty), \varphi(Tx), \varphi(Ty)) \leq k\,F(d(x,y), \varphi(x), \varphi(y)) \quad \forall\, x, y \in X,\ x \preceq y.$$

If there exists a point $x_0 \in X$ such that $x_0 \preceq Tx_0$ and one of the following conditions:
(a) T is continuous;
(b) X is regular,
then T has a fixed point z such that $\varphi(z) = 0$. Moreover, if X has the property (U), then T has a unique fixed point z such that $\varphi(z) = 0$ and, for all $u_0 \in X$, we have also $\lim_{n\to+\infty} u_n = z$, where $\{u_n\}$ is the Picard sequence starting at u_0.

We remark that, if we put $F(a,b,c) = a+b+c$ for all $a,b,c \in [0,+\infty[$ and $\varphi(x) = 0$ for all $x \in X$, then, from Corollary 8.1, we obtain the Banach contraction principle. Further, from Corollary 8.2, we obtain Theorem 8.6 that gives the results of Nieto et al. [21].

Let $\zeta : [0,+\infty[\times[0,+\infty[\to \mathbb{R}$ be defined by $\zeta(t,s) = s - \psi(s) - t$ for all $t, s \in [0,+\infty[$, where $\psi : [0,+\infty[\to [0,+\infty[$ is a lower semi-continuous function such that $\psi(t) = 0$ if and only if $t = 0$. We notice that such a function ζ belongs to $\mathcal{S}'$.

In correspondence of this choice of ζ, in the setting of metric spaces, we obtain the following result of Rhoades type [30].

Corollary 8.3 *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping. Suppose that there exist a function $F \in \mathcal{F}$ and two lower semi-continuous functions $\psi : [0,+\infty[\to [0,+\infty[$ with $\psi^{-1}(0) = \{0\}$ and $\varphi : X \to [0,+\infty[$ such that*

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq F(d(x, y), \varphi(x), \varphi(y)) - \psi(F(d(x, y), \varphi(x), \varphi(y)))$$

$\forall\, x, y \in X$. *Then T has a unique fixed point z such that $\varphi(z) = 0$. Moreover, for all $u_0 \in X$, the Picard sequence $\{u_n\}$ starting at u_0 converges to z.*

In the setting of ordered metric spaces, we can formulate the following result.

Corollary 8.4 *Let $(X, d, \preceq)$ be a complete ordered metric space and $T : X \to X$ be a nondecreasing mapping. Suppose that there exist a function $F \in \mathscr{F}$ and two lower semi-continuous functions $\psi : [0, +\infty[\to [0, +\infty[$ with $\psi^{-1}(0) = \{0\}$ and $\varphi : X \to [0, +\infty[$ such that*

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq F(d(x, y), \varphi(x), \varphi(y)) - \psi(F(d(x, y), \varphi(x), \varphi(y)))$$

for all $x, y \in X$ with $x \preceq y$. If there exists a point $x_0 \in X$ such that $x_0 \preceq Tx_0$ and one of the following conditions:

(a) T is continuous;
(b) X is regular,

then T has a fixed point z such that $\varphi(z) = 0$. Moreover, if X has the property (U), then T has a unique fixed point z such that $\varphi(z) = 0$ and for all $u_0 \in X$, we have $\lim_{n\to+\infty} u_n = z$, where $\{u_n\}$ is the Picard sequence starting at u_0.

Let $\zeta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ be defined by $\zeta(t, s) = s\,\psi(s) - t$ for all $t, s \in [0, +\infty[$, where $\psi : [0, +\infty[\to [0, 1[$ is a function such that $\limsup_{t\to r^+} \psi(t) < 1$ for all $r > 0$. Again, we have that $\zeta \in \mathscr{S}'$. So, if we choose such a function ζ, we get in the setting of metric spaces the following result (see [27]).

Corollary 8.5 *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. Suppose that there exist a function $F \in \mathscr{F}$, a function $\psi : [0, +\infty[\to [0, 1[$ with $\limsup_{t\to r^+} \psi(t) < 1$ for all $r > 0$ and a lower semi-continuous function $\varphi : X \to [0, +\infty[$ such that*

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq \psi(F(d(x, y), \varphi(x), \varphi(y)))\, F(d(x, y), \varphi(x), \varphi(y))$$

for all $x, y \in X$. Then T has a unique fixed point z such that $\varphi(z) = 0$. Moreover, for all $u_0 \in X$, the Picard sequence $\{u_n\}$ starting at u_0 converges to z.

In the setting of ordered metric spaces, we have the following result.

Corollary 8.6 *Let $(X, d, \preceq)$ be a complete ordered metric space and $T : X \to X$ be a nondecreasing mapping. Suppose that there exist a function $F \in \mathscr{F}$, a function $\psi : [0, +\infty[\to [0, 1[$ with $\limsup_{t\to r^+} \psi(t) < 1$ for all $r > 0$ and a lower semi-continuous function $\varphi : X \to [0, +\infty[$ such that*

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq \psi(F(d(x, y), \varphi(x), \varphi(y)))\, F(d(x, y), \varphi(x), \varphi(y))$$

for all $x, y \in X$ with $x \preceq y$. If there exists a point $x_0 \in X$ such that $x_0 \preceq Tx_0$ and one of the following conditions:

(a) *T is continuous;*

(b) *X is regular,*

then T has a fixed point z such that $\varphi(z) = 0$. Moreover, if X has the property (U), then T has a unique fixed point z such that $\varphi(z) = 0$ and, for all $u_0 \in X$, we have $\lim_{n\to+\infty} u_n = z$, *where $\{u_n\}$ is the Picard sequence starting at u_0.*

Let $\zeta : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$ be defined by $\zeta(t, s) = \psi(s) - t$ for all $t, s \in [0, +\infty[$, where $\psi : [0, +\infty[\to [0, +\infty[$ is an upper semi-continuous function such that $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$, then $\zeta \in \mathscr{S}'$.

In correspondence of this choice of ζ, in the setting of metric spaces, we obtain the following result of Boyd-Wong type [6].

Corollary 8.7 *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. Suppose that there exist a function $F \in \mathscr{F}$, an upper semi-continuous function ψ : $[0, +\infty[\to [0, +\infty[$ with $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$ and a lower semi-continuous function $\varphi : X \to [0, +\infty[$ such that*

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq \psi(F(d(x, y), \varphi(x), \varphi(y))) \quad \forall\, x, y \in X.$$

Then T has a unique fixed point z such that $\varphi(z) = 0$. Moreover, for all $u_0 \in X$, the Picard sequence $\{u_n\}$ starting at u_0 converges to z.

Finally, in the setting of ordered metric spaces, we state the following result.

Corollary 8.8 *Let $(X, d, \preceq)$ be a complete ordered metric space and $T : X \to X$ be a nondecreasing mapping. Suppose that there exist a function $F \in \mathscr{F}$, an upper semi-continuous function $\psi : [0, +\infty[\to [0, +\infty[$ with $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$ and a lower semi-continuous function $\varphi : X \to [0, +\infty[$ such that*

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq \psi(F(d(x, y), \varphi(x), \varphi(y)))$$

for all $x, y \in X$ with $x \preceq y$. If there exists a point $x_0 \in X$ such that $x_0 \preceq Tx_0$ and one of the following conditions:

(a) *T is continuous;*

(b) *X is regular,*

then T has a fixed point z such that $\varphi(z) = 0$. Moreover if X has the property (U), then T has a unique fixed point z such that $\varphi(z) = 0$ and, for all $u_0 \in X$, we have $\lim_{n\to+\infty} u_n = z$, *where $\{u_n\}$ is the Picard sequence starting at u_0.*

We notice that we obtain the Boyd-Wong result from Corollary 8.7 if we assume $F(a, b, c) = a + b + c$ for all $a, b, c \in [0, +\infty[$ and $\varphi(x) = 0$ for all $x \in X$.

8.7 Conclusions

We gave a short survey of S-F-contractions in the setting of complete metric spaces, by using also the ordered approach. The presented generalized contractions had a significant impact over the development of fixed point theory and its applications. Indeed, they are useful in providing hybrid versions of already known results. This opens the road to possibilities to get more interesting applications, by covering a large amount of practical situations.

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Chapter 9
A Survey on Best Proximity Point Theory in Reflexive and Busemann Convex Spaces

Moosa Gabeleh

Abstract In this chapter, we present some best proximity point theorems for Kannan cyclic mappings in the setting of Busemann convex spaces which are reflexive. To this end, we recall some results obtained in the framework of the fixed point theory for Kannan self mappings and generalize them to cyclic mappings in order to study the existence of best proximity points. We do it from two different approaches. The first one is based on a geometric property defined on a nonempty and convex pair in a geodesic space, called proximal normal structure, and the other one will be done by considering some sufficient conditions on the cyclic mappings. We also study the structure of minimal sets for Kannan cyclic nonexpansive mappings.

Keywords Best proximity point · Kannan cyclic mapping · Busemann convex space · Proximal quasi-normal structure

9.1 Introduction

9.1.1 Kannan Contractions

A mapping T defined on a metric space (X, d) is called a *Kannan contraction* [26] if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)], \quad \forall x, y \in X. \tag{9.1}$$

In 1968, Kannan established the following fixed point theorem which is independent of the Banach contraction principle [4].

This work is dedicated to Professor Ali Abkar.

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Theorem 9.1 (see also [25] for more new details) *Let (X, d) be a complete metric space and $T : X \to X$ be a Kannan contraction mapping. Then T has a unique fixed point $p \in X$ and, for any $x \in X$, the sequence iterates $\{T^n x\}$ converges to p and*

$$d(T^{n+1}x, p) \leq \alpha\Big(\frac{\alpha}{1-\alpha}\Big)^n d(x, Tx), \quad \forall n \in \mathbb{N} \cup \{0\}.$$

An interesting observation about the Kannan's fixed point problem is that it characterizes the metric completeness, that is, a metric space X is complete if and only if every Kannan self-mapping defined on X has a fixed point [37].

9.1.2 Kannan Nonexpansive Mappings

Let (X, d) be a metric space and let $T : X \to X$ be a self-mapping on X. Then T is called *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in X. \tag{9.2}$$

Also, T is called *Kannan nonexpansive* provided that

$$d(Tx, Ty) \leq \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}, \quad \forall x, y \in X. \tag{9.3}$$

Clearly, the class of Kannan nonexpansive mappings contains the class of Kannan contractions as a subclass. Moreover, there exists a nonexpansive mapping which is not Kannan nonexpansive and a Kannan nonexpansive mapping which is not nonexpansive. So, we cannot compare both conditions directly.

Example 9.1 Consider $X = \mathbb{R}$ with the usual metric and let $A = [0, 1]$. Define the self-mapping $T : A \to A$ with

$$Tx = \begin{cases} 1 - x, & \text{if} \quad x \in \mathbb{Q}^c \cap [0, 1], \\ \frac{1+x}{3}, & \text{if} \quad x \in \mathbb{Q} \cap [0, 1]. \end{cases}$$

It is easy to check that T is a Kannan nonexpansive mapping which is not continuous and so is not nonexpansive. Besides, if we consider $B = [0, \infty)$ and define $S : B \to B$ with $Sx = x + 1$, then clearly T is nonexpansive but for all $x, y \in B$ with $|x - y| > 1$ so that we have

$$\frac{1}{2}\{d(x, Sx) + d(y, Sy)\} = 1 < |x - y| = |Sx - Sy|,$$

which implies that S is not a Kannan nonexpansive mapping.

It is well known that, if K is a nonempty, compact and convex subset of a Banach space X, then any nonexpansive and Kannan nonexpansive mapping of K into K has a fixed point [28]. It is remarkable to note that if K is a weakly compact and convex subset of a Banach space X, then the existence of fixed points for nonexpansive and Kannan nonexpansive mappings cannot be concluded. Indeed, it was shown by Alspach that there exists a weakly compact and convex subset K of $L^1[0, 1]$ and a nonexpansive mapping $T : K \to K$ which is fixed point free [3].

In 1948, a useful geometric property was introduced by Brodskii and Milman as follows.

Definition 9.1 ([5]) A nonempty and convex subset A of a Banach space X is said to have the *normal structure* if for each bounded, closed and convex subset K of A which contains more than one point, there exists a point $x^* \in K$ such that $\sup\{\|x^* - y\| : y \in K\} < \mathrm{diam}(K)$, where $\mathrm{diam}(K)$ denotes the diameter of K.

It is well-known that every nonempty, compact and convex subset of a Banach space has the normal structure. Furthermore, every nonempty, bounded, closed and convex subset of a uniformly convex Banach space has the normal structure too [28].

Using this geometric notion, the following famous fixed point theorem due to Kirk, was proved.

Theorem 9.2 ([29]) *Let K be a nonempty, weakly compact and convex subset of a Banach space X and $T : K \to K$ be a nonexpansive mapping. If K has the normal structure, then T has a fixed point.*

A counterpart result of Theorem 9.2 was established for Kannan nonexpansive mappings by Soardi in [36]. In a separate paper, Kannan used the notion of normal structure dependent on the considered self-mapping and proved the following theorem.

Theorem 9.3 ([27]) *Let K be a nonempty and convex subset of a reflexive Banach space X and $T : K \to K$ be a Kannan nonexpansive mapping. If, for any bounded closed convex and T-invariant subset H of K with more than one point, we have*

$$\sup\{\|y - Ty\| : y \in H\} < \mathrm{diam}(H),$$

then T has a fixed point.

A weaker notion of normal structure was introduced by Wong as follows.

Definition 9.2 ([39]) A convex subset A of a Banach space X is said to have the *quasi-normal structure* if, for any bounded closed and convex subset K of A with $\mathrm{diam}(K) > 0$, there exists $p \in K$ such that

$$\|x - p\| < \mathrm{diam}(K), \quad \forall x \in K.$$

Finally, Wong proved the next fixed point result for Kannan nonexpansive self-mappings by using the notion of quasi-normal structure.

Theorem 9.4 ([40]) *Let K be a nonempt, weakly compact and convex subset of a Banach space X and $T : K \to K$ be a Kannan nonexpansive mapping. If K has the quasi-normal structure, then T has a fixed point.*

9.2 Geodesic Metric Spaces

A metric space (X, d) is said to be a *geodesic space* if every two points x and y of X are joined by a geodesic, i.e. a map $c : [0, l] \subseteq \mathbb{R} \to X$ such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. A metric space (X, d) is called *uniquely geodesic* if there exists exactly one geodesic joining x and y for each $x, y \in X$. If there is only one geodesic between two points x and y, the image of this geodesic which is called *geodesic segment*, is denoted by $[x, y]$.

For instance, any Banach space is a geodesic space with usual segments as geodesic segment. Other interesting examples are the Hilbert ball [24] and the hyperbolic spaces [34].

For a geodesic segment $[x, y]$, we set $]x, y[:= [x, y] - \{x, y\}$. If X is a uniquely geodesic metric space, then, for each $x, y \in X$ and $t \in (0, 1)$, we set $c(t0 + (1 - t)l) := tx \oplus (1 - t)y$. A subset A of a uniquely geodesic metric space (X, d) is said to be *convex* if the geodesic segment joining each pair of points x and y of A is contained in A.

For more details about geodesic metric spaces, one can refer to [6, 7, 32]. The notion of *strictly convexity* in metric spaces was introduced in [2] as follows.

Definition 9.3 A geodesic metric space (X, d) is said to be *strictly convex* provided that, for every $r > 0$ and $a, x, y \in X$ with $d(x, a) \leq r$, $d(y, a) \leq r$ and $x \neq y$, we have $d(a, p) < r$, where $p \in]x, y[$.

Note that every strictly convex metric space is uniquely geodesic. The reader can see [32] for more information. Unless explicitly stated otherwise, from now on, we will just use geodesic metric space to refer to a uniquely geodesic space.

Here, we recall another geometric notion on geodesic spaces which will be used in the sequel.

Definition 9.4 ([24]) A geodesic metric space (X, d) is said to be *uniformly convex* if, for any $r > 0$ and $\varepsilon \in (0, 2]$ there exists $\eta \in (0, 1]$ such that, for all $a, x, y \in X$ with $d(x, a) \leq r$, $d(y, a) \leq r$ and $d(x, y) \geq \varepsilon r$, we have

$$d(m, a) \leq (1 - \eta)r,$$

where m is a midpoint of x and y.

Obviously, every uniformly convex geodesic space is strictly convex, but as we know the inverse implication does not hold in Banach spaces as a subclass of geodesic spaces.

A geodesic metric space (X, d) is said to be *reflexive* if any descending chain of nonempty, bounded, closed and convex subsets of X has a nonempty intersection. Every reflexive Banach space is a reflexive metric space too. Moreover, any uniformly convex geodesic space is reflexive (see [31] for more details).

Let (X, d) be a uniquely geodesic space. A metric $d\colon X \times X \to \mathbb{R}$ is said to be *convex* if, for any $x, y, z \in X$, one has

$$d(x, (1-t)y \oplus tz) \le (1-t)d(x, y) + td(x, z), \quad \forall t \in [0, 1].$$

Definition 9.5 ([9]) A geodesic space (X, d) is called *convex in the sense of Busemann* if, given any pair of geodesics $c_1 : [0, l_1] \to X$ and $c_2 : [0, l_2] \to X$, one has

$$d(c_1(tl_1), c_2(tl_2)) \le (1-t)d(c_1(0), c_2(0)) + td(c_1(l_1), c_2(l_2)), \quad \forall t \in [0, 1].$$

Equivalently, a geodesic metric space (X, d) is convex in the sense of Busemann provided that

$$d((1-t)x \oplus ty, (1-t)z \oplus tw) \le (1-t)d(x, z) + td(y, w)$$

for all $x, y, z, w \in X$ and $t \in [0, 1]$.

A reflexive and Busemann convex space is complete (see [13, Lemma 4.1]). We also mention that Busemann convex spaces are strictly convex with convex metric [15].

In this chapter, we present extensions of Theorems 9.2 and 9.4 by considering cyclic mappings in the setting of reflexive and Busemann convex spaces in order to study the existence of best proximity points. We also obtain a different version of Theorem 9.4 without the geometric property of a quasi-normal structure.

9.3 Best Proximity Points

9.3.1 Cyclic Relatively Nonexpansive Mappings

Let (X, d) be a metric space and A, B be two nonempty subsets of X. A mapping $T : A \cup B \to A \cup B$ is said to be a *cyclic mapping* provided that $T(A) \subseteq B$ and $T(B) \subseteq A$.

In [30], Kirk et al. established the following theorem which is an interesting extension of the Banach contraction principle.

Theorem 9.5 ([30, Theorem 1.1]) *Suppose that (A, B) is a nonempty and closed pair of subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ is a cyclic mapping for which there exists $k \in [0, 1)$ such that $d(Tx, Ty) \le kd(x, y)$*

for all $(x, y) \in A \times B$. *Then* $A \cap B$ *is nonempty and* T *has a unique fixed point in* $A \cap B$.

As a cyclic mapping does not have a fixed point necessarily, it is of considerable interest to find an element $x \in A \cup B$ that is as close to Tx as possible or equivalently, the error of $d(x, Tx)$ is minimum. Indeed, best proximity point theorems investigate the existence of such optimal approximate solutions, called best proximity points, of the fixed point equation $Tx = x$ when there is no exact solution for the cyclic mapping T.

Definition 9.6 Let A and B be nonempty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic mapping. A point $p \in A \cup B$ is said to be a *best proximity point* for the cyclic mapping T provided that

$$d(p, Tp) = \operatorname{dist}(A, B) := \inf\{d(x, y) \colon x \in A,\ \ y \in B\}.$$

In fact best proximity point theorems have been studied to find necessary conditions such that the *minimization problem*:

$$\min_{x \in A \cup B} d(x, Tx) \tag{9.4}$$

has at least one solution.

The first existence results of best proximity points for cyclic mappings was presented in [10] in Banach spaces and then in [17] in geodesic spaces. Before we state some of these results, we recall the following notions and notations.

We say that a pair (A, B) of subsets of a geodesic metric space (X, d) satisfies a property if both A and B satisfy that property. For example, (A, B) is convex if and only if both A and B are convex; $(A, B) \subseteq (C, D) \Leftrightarrow A \subseteq C,$ and $B \subseteq D$. We shall also adopt the following notations:

$$\delta_x(A) = \sup\{d(x, y) \colon y \in A\}, \quad \forall x \in X,$$

$$\delta(A, B) = \sup\{\delta_x(B) : x \in A\},$$

$$\operatorname{diam}(A) = \delta(A, A),$$

$$d^*(x, y) = d(x, y) - \operatorname{dist}(A, B), \quad \forall (x, y) \in A \times B.$$

The *closed and convex hull* of a set A will be denoted by $\overline{\operatorname{con}}(A)$ which is the smallest closed and convex subset of X containing the set A. Also, $\mathscr{B}(p, r)$ will denote the closed ball with center at $p \in X$ and radius $r > 0$.

The *metric projection operator* $\mathscr{P}_A : X \to 2^A$ is defined as

$$\mathscr{P}_A(x) := \{y \in A : d(x, y) = \operatorname{dist}(x, A)\},$$

where 2^A denotes the set of all subsets of A.

Given (A, B) a pair of nonempty subsets of X, then its proximal pair is the pair (A_0, B_0) given by

$$A_0 = \{x \in A \colon d(x, y') = \operatorname{dist}(A, B) \text{ for some } y' \in B\},$$

$$B_0 = \{y \in B \colon d(x', y) = \operatorname{dist}(A, B) \text{ for some } x' \in A\}.$$

Proximal pairs may be empty, but, in particular, if (A, B) is a nonempty, weakly compact and convex pair, then (A_0, B_0) is also nonempty weakly compact and convex.

In what follows, we provide the other sufficient conditions for non-emptiness of the pair (A_0, B_0) in Busemann convex spaces. To this end, we need the following lemma.

Lemma 9.1 *Let A be a nonempty, closed and convex subset of a reflexive and Busemann convex space X. Then the metric projection $\mathscr{P}_A : X \to 2^A$ is single-valued.*

Proof Let $x \in X$. For all $n \in \mathbb{N}$, define

$$A_n = \left\{a \in A : d(x, a) \leq \operatorname{dist}(x, A) + \frac{1}{n}\right\}.$$

Then A_n is closed for all $n \in \mathbb{N}$. Besides, if $a_1, a_2 \in A_n$ and $t \in (0, 1)$, then we have

$$d(x, ta_1 \oplus (1 - t)a_2) \leq td(x, a_1) + (1 - t)d(x, a_2) \leq \operatorname{dist}(x, A) + \frac{1}{n},$$

that is, A_n is convex. Therefore, $\{A_n\}$ is a decreasing sequence of nonempty, bounded, closed and convex subsets of X. Since X is reflexive, $\bigcap_{n\geq 1} A_n$ is nonempty. If $a^* \in A_n$ for all $n \in \mathbb{N}$ then $d(x, a^*) = \operatorname{dist}(x, A)$, that is, $a^* \in \mathscr{P}_A(x)$. Finally, from the strict convexity of X, we obtain $\mathscr{P}_A(x)$ is a singleton. This completes the proof. □

Proposition 9.1 *If (A, B) is a nonempty, closed and convex pair in a reflexive and Busemann convex space X such that B is bounded, then (A_0, B_0) is nonempty bounded closed and convex.*

Proof For all $n \in \mathbb{N}$, set

$$U_n := \left\{x \in A : \operatorname{dist}(\{x\}, B) \leq \operatorname{dist}(A, B) + \frac{1}{n}\right\}.$$

Clearly, U_n is nonempty and closed. Let $x_1, x_2 \in U_n$. Since X is a Busemann convex space, for all $y \in B$ and $t \in (0, 1)$, we have

$$d(tx_1 \oplus (1 - t)x_2, y) \leq td(x_1, y) + (1 - t)d(x_2, y),$$

which implies that

$$\operatorname{dist}(tx_1 \oplus (1-t)x_2, B) \leq \operatorname{dist}(A, B) + \frac{1}{n}$$

and so $tx_1 \oplus (1-t)x_2 \in U_n$. Thus U_n is convex for all $n \in \mathbb{N}$. Also, it is easy to see that the sequence $\{U_n\}$ is decreasing. In view of the fact that X is reflexive, $\bigcap_{n\geq 1} U_n$ is nonempty and, by Lemma 9.1, we obtain $A_0 = \bigcap_{n\geq 1} U_n$. Thus A_0 is nonempty closed and convex. Similarly, we can see that B_0 is also nonempty, closed and convex. The boundedness of B ensures that both A_0 and B_0 are bounded too. This completes the proof. □

Definition 9.7 A nonempty pair (A, B) in a metric space is said to be *proximinal* if

$$A = A_0, \quad B = B_0.$$

Here, we recall a geometric notion of proximal normal structure which was introduced in [10].

Definition 9.8 A convex pair (K_1, K_2) in a geodesic space X is said to have the *proximal normal structure* (PNS) if, for any bounded closed convex and proximinal pair $(H_1, H_2) \subseteq (K_1, K_2)$ for which $\operatorname{dist}(H_1, H_2) = \operatorname{dist}(K_1, K_2)$ and $\delta(H_1, H_2) > \operatorname{dist}(H_1, H_2)$, there exists $(x_1, x_2) \in H_1 \times H_2$ such that

$$\max\{\delta_{x_1}(H_2), \delta_{x_2}(H_1)\} < \delta(H_1, H_2).$$

Notice that the pair (K, K) has PNS if and only if K has the normal structure in the sense of Definition 9.1.

Let us illustrate the notion of PNS with the following examples.

Example 9.2 ([17, Proposition 3.5]) Every nonempty, closed and convex pair in a uniformly convex geodesic space X has the PNS.

Example 9.3 ([20, Theorem 3.5]) Every nonempty, compact and convex pair in a geodesic space X with convex metric has the PNS.

Definition 9.9 Let (A, B) be a nonempty pair of subsets of a metric space (X, d). A cyclic mapping $T : A \cup B \to A \cup B$ is said to be *relatively nonexpansive* if

$$d(Tx, Ty) \leq d(x, y), \quad \forall (x, y) \in A \times B. \tag{9.5}$$

Obviously, if, in above definition, $A = B$, then we get the class of nonexpansive self-mappings.

The next lemma has an important role in our coming discussions.

Lemma 9.2 *Let (X, d) be a reflexive and Busemann convex metric space and let (A, B) be a nonempty, closed and convex pair of subsets of X such that A is bounded. Assume that $T : A \cup B \to A \cup B$ is a cyclic mapping such that $d(Tx, Ty) = \operatorname{dist}(A, B)$ for all $(x, y) \in A \times B$ with $d(x, y) = \operatorname{dist}(A, B)$. Then* *there exists* $(K_1, K_2) \subseteq (A_0, B_0) \subseteq (A, B)$ *which is minimal with respect to being nonempty, closed, convex and* T*-invariant pair of subsets of* (A, B) *such that*

$$\mathrm{dist}(K_1, K_2) = \mathrm{dist}(A, B).$$

Moreover, the pair (K_1, K_2) *is proximinal and*

$$K_1 = \overline{\mathrm{con}}(T(K_2)), \quad K_2 = \overline{\mathrm{con}}(T(K_1)).$$

Proof By Proposition 9.1 the pair (A_0, B_0) is nonempty, bounded, closed and convex which is proximinal and $\mathrm{dist}(A, B) = \mathrm{dist}(A_0, B_0)$. Also if $x \in A_0$ is an arbitrary element, then there exists $y \in B_0$ such that $d(x, y) = \mathrm{dist}(A, B)$. It follows from the assumption on T that $d(Tx, Ty) = \mathrm{dist}(A, B)$ which implies that $Tx \in B_0$ and so $T(A_0) \subseteq B_0$. Similarly, $T(B_0) \subseteq A_0$, that is, T is cyclic on $A_0 \cup B_0$. Now let $\sum$ denote a set of all nonempty, bounded, closed, convex pair $(C, D) \subseteq (A, B)$ with $\mathrm{dist}(C, D) = \mathrm{dist}(A, B)$ which is T-invariant. Then $(A_0, B_0) \in \sum$ and so $\sum$ is nonempty. Assume that $\{(C_j, D_j)\}_j$ is a descending chain in $\sum$ and put

$$\mathscr{C} := \bigcap_j C_j, \quad \mathscr{D} := \bigcap_j D_j.$$

Since X is reflexive, the pair $(\mathscr{C}, \mathscr{D})$ is nonempty, closed and convex. Let $x \in \mathscr{C}$. Then $x \in C_j$ for all j. By the fact that any pair (C_j, D_j) is proximinal, and that X is strictly convex, there exists a unique $y \in D_j$ so that $d(x, y) = \mathrm{dist}(A, B)$ for all j. Hence, $y \in \mathscr{D}$ which ensures that

$$d(x, y) = \mathrm{dist}(A, B) = \mathrm{dist}(\mathscr{C}, \mathscr{D}).$$

that is, $(\mathscr{C}, \mathscr{D})$ is proximinal. Moreover,

$$T(\mathscr{C}) = T(\bigcap_j C_j) \subseteq \bigcap_j T(C_j) \subseteq \bigcap_j D_j = \mathscr{D}.$$

Similarly, $T(\mathscr{D}) \subseteq \mathscr{C}$ and so $(\mathscr{C}, \mathscr{D})$ is T-invariant. It now follows from Zorn's lemma that $\sum$ has a minimal element, namely (K_1, K_2). Since $((K_1)_0, , (K_2)_0) \subseteq (K_1, K_2)$ is nonempty, closed, convex and T-invariant, minimality of (K_1, K_2) implies that $(K_1)_0 = K_1$ and $(K_2)_0 = K_2$ which implies that (K_1, K_2) is proximinal. Furthermore, $T(K_1) \subseteq K_2$ and so, $\overline{\mathrm{con}}(T(K_1)) \subseteq K_2$ which deduces that

$$T(\overline{\mathrm{con}}(T(K_1))) \subseteq T(K_2) \subseteq \overline{\mathrm{con}}(T(K_2)).$$

Similarly, we have

$$T(\overline{\mathrm{con}}(T(K_2))) \subseteq \overline{\mathrm{con}}(T(K_1)),$$

that is, the pair $(\overline{\text{con}}(T(K_1)), \overline{\text{con}}(T(K_2))$ is T-invariant. Also, we have

$$\text{dist}((\overline{\text{con}}(T(K_1)), \overline{\text{con}}(T(K_2)) = \text{dist}(A, B).$$

Again, by the minimality of (K_1, K_2), we obtain $K_1 = \overline{\text{con}}(T(K_2))$ and $K_2 = \overline{\text{con}}(T(K_1))$. □

The following existence result of best proximity points for cyclic relatively nonexpansive mappings is a main result of [10] and [17].

Theorem 9.6 *Let (X, d) be a reflexive and Busemann convex space and let (A, B) be a nonempty, closed and convex pair of subsets of X such that A is bounded. Assume $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping. If (A, B) has the* PNS*, then T has a best proximity point.*

Proof From Lemma 9.2 there exists a pair $(K_1, K_2) \subseteq (A_0, B_0) \subseteq (A, B)$ which is minimal with respect to being nonempty bounded closed convex and T-invariant pair of subsets of (A, B) such that $\text{dist}(K_1, K_2) = \text{dist}(A, B)$. Also, the pair (K_1, K_2) is proximinal. Note that, if $\delta(K_1, K_2) = \text{dist}(K_1, K_2)$, then each point of K is a best proximity point of T and we are finished. So assume that $\delta(K_1, K_2) > \text{dist}(K_1, K_2)$. Since (A, B) has the PNS, there exists a point $(p, q) \in K_1 \times K_2$ and $\lambda \in (0, 1)$ for which

$$\max\{\delta_p(K_2), \delta_q(K_1)\} \leq \lambda\delta(K_1, K_2).$$

Let $(p', q') \in K_1 \times K_2$ be such that $d(p, q') = d(p', q) = \text{dist}(K_1, K_2)(= \text{dist}(A, B))$. Suppose that p_1 and q_1 are the midpoints of p, p' and q, q', respectively. Clearly, $d(p_1, q_1) = \text{dist}(A, B)$. By the fact that the metric d is convex, for all $y \in K_2$, we have

$$\begin{aligned} d(p_1, y) &\leq \frac{1}{2}(d(p', y) + d(p, y)) \\ &\leq \frac{1}{2}(\delta(K_1, K_2) + \lambda\delta(K_1, K_2)) \\ &= \frac{(1 + \lambda)}{2}\delta(K_1, K_2), \end{aligned}$$

which implies that $\delta_{p_1}(K_2) \leq \frac{(1+\lambda)}{2}\delta(K_1, K_2)$ (note that $\frac{(1+\lambda)}{2} < 1$). Similarly, we can see that $\delta_{q_1}(K_1) \leq \frac{(1+\lambda)}{2}\delta(K_1, K_2)$. Put

$$L_1 := \Big\{x \in K_1 : \delta_x(K_2) \leq \frac{(1 + \lambda)}{2}\delta(K_1, K_2) \text{ and, for its proximal point } y \in K_2,\ \delta_y(K_1) \leq \frac{(1 + \lambda)}{2}\delta(K_1, K_2)\Big\},$$

$$L_2 := \Big\{ y \in K_2 : \delta_y(K_1) \le \frac{(1+\lambda)}{2}\delta(K_1, K_2) \text{ and, for its proximal point } x \in K_1,\ \delta_x(K_2) \le \frac{(1+\lambda)}{2}\delta(K_1, K_2) \Big\}.$$

Notice that $(p_1, q_1) \in L_1 \times L_2$ and so, $\mathrm{dist}(L_1, L_2) = \mathrm{dist}(K_1, K_2)$. By an equivalent argument of Theorem 3.3 of [17], (L_1, L_2) is bounded, closed, convex and proximinal. We assert that T is cyclic on $L_1 \cup L_2$. To this end, assume that $x \in L_1$. Since (L_1, L_2) is proximinal, there exists $y \in L_2$ such that $d(x, y) = \mathrm{dist}(A, B)$. Let $v \in L_2$. Because of the fact that T is cyclic relatively nonexpansive,

$$d(Tx, Tv) \le d(x, v) \le \delta_x(K_2) \le \frac{(1+\lambda)}{2}\delta(K_1, K_2),$$

and so, $T(K_2) \subseteq \mathscr{B}(Tx; \frac{(1+\lambda)}{2}\delta(K_1, K_2)) \cap K_1$. Put

$$K_1' := \mathscr{B}\Big(Tx; \frac{(1+\lambda)}{2}\delta(K_1, K_2)\Big) \bigcap K_1,$$

and let K_2' is the set of all proximal points of K_1'. Then (K_1', K_2') is bounded, closed, convex, proximinal and T-invariant. Minimality of (K_1, K_2) ensures that $K_1' = K_1, K_2' = K_2$. Therefore, $\mathscr{B}\Big(Tx; \frac{(1+\lambda)}{2}\delta(K_1, K_2)\Big) \subseteq K_1$, that is, $\delta_{Tx}(K_1) \le \frac{(1+\lambda)}{2}\delta(K_1, K_2)$. Hence, $Tx \in L_2$ and so $T(L_1) \subseteq L_2$. Equivalently, we can see that $T(L_2) \subseteq L_1$ which implies that T is cyclic on $L_1 \cup L_2$. Again, by the minimality of (K_1, K_2) we conclude that $L_1 = K_1$ and $L_2 = K_2$. Thus $\delta_x(K_2) \le \frac{(1+\lambda)}{2}\delta(K_1, K_2)$ for any $x \in K_1$ which deduces that

$$\delta(K_1, K_2) = \sup_{x \in K_1} \delta_x(K_2) \le \frac{(1+\lambda)}{2}\delta(K_1, K_2),$$

which is a contradiction. This completes the proof. □

9.3.2 Cyclic Kannan Contractions

Recently, the class of Kannan contraction self-mappings was generalized as follows:

Definition 9.10 ([33]) Let (A, B) be a nonempty pair of subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is said to be the *cyclic Kannan contraction* if T is cyclic and

$$d(Tx, Ty) \le \alpha\{d(x, Tx) + d(y, Ty)\} + (1 - 2\alpha)\mathrm{dist}(A, B), \tag{9.6}$$

for some $\alpha \in [0, \frac{1}{2})$ and for all $(x, y) \in A \times B$.

The existence, uniqueness and convergence results of a best proximity point for the cyclic Kannan contraction mapping $T\colon A \cup B \to A \cup B$, where (A, B) is a nonempty closed and convex pair in a uniformly convex Banach space X, was proved in [33].

To extend this conclusion to geodesic metric spaces, we need the following concept.

Definition 9.11 ([38]) A nonempty pair (A, B) in a metric space (X, d) is said to satisfy the *property* UC if the following holds: If $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that

$$\lim_{n\to\infty} d(x_n, y_n) = \operatorname{dist}(A, B) = \lim_{n\to\infty} d(z_n, y_n),$$

then $\lim_{n\to\infty} d(x_n, z_n) = 0$.

It was announced in [11] that if (A, B) is a nonempty and closed pair of subsets of a uniformly convex Banach space X such that A is convex, then (A, B) has the property UC (see Lemma 3.8 of [11]). Other interesting examples can be found in [38].

Theorem 9.7 ([19, Theorem 3.1]) *Suppose that X is a reflexive and strictly convex geodesic metric space and suppose (A, B) is a nonempty pair of subsets of X such that A is closed and convex. Let $T : A \cup B \to A \cup B$ be a cyclic Kannan contraction. If the pair (A, B) has the property* UC*, then T has a unique best proximity point in A.*

Proof Let $r := \frac{\alpha}{1-\alpha}$. Then $r \in (0, 1)$. Now, for each $x \in A \cup B$ and $n \in \mathbb{N}$, we have

$$d^*(T^{2n-1}x, T^{2n}x) \le \alpha[d^*(T^{2n-2}x, T^{2n-1}x) + d^*(T^{2n-1}x, T^{2n}x)],$$

and so

$$\begin{aligned} d^*(T^{2n-1}x, T^{2n}x) &\le \frac{\alpha}{1-\alpha} d^*(T^{2n-2}x, T^{2n-1}x) \\ &= r d^*(T^{2n-2}x, T^{2n-1}x) \\ &\le r^{2n-1} d^*(x, Tx). \end{aligned}$$

Thus, for each $(x, y) \in A \times B$ and $n \in \mathbb{N}$, we conclude that

$$\begin{aligned} d^*(T^{2n}x, T^{2n}y) &\le \alpha[d^*(T^{2n-1}x, T^{2n}x) + d^*(T^{2n-1}y, T^{2n}y)] \\ &\le \alpha[r^{2n-1}d^*(x, Tx) + r^{2n-1}d^*(y, Ty)] \\ &= \alpha r^{2n-1}[d^*(x, Tx) + d^*(y, Ty)]. \end{aligned} \tag{9.7}$$

Suppose that $x \in A$ is an arbitrary but fixed element in A. Fix $l \in \mathbb{N}$ and let $m = l + k$ with $k \in \mathbb{N}$. It now follows from the relation (9.7) that

$$\begin{aligned} d^*(T^{2m}x, T^{2l+1}x) &= d^*(T^{2l}(T^{2k}x), T^{2l}(Tx)) \\ &\le \alpha r^{2l-1}[d^*(T^{2k}x, T^{2k+1}x) + d^*(Tx, T^2x)]. \end{aligned}$$

By Lemma 3 of [33], we have $d^*(T^{2n}x, T^{2n+1}x) \to 0$. So, we have

$$M(x) := \sup\{d^*(T^{2k}x, T^{2k+1}x) + d^*(Tx, T^2x) : k \in \mathbb{N}\},$$

exists. Therefore, we have $d^*(T^{2m}x, T^{2l+1}x) \le \alpha r^{2l-1}M(x)$. Now, for each $n \in \mathbb{N}$, there exists $l(n) \in \mathbb{N}$ such that $T^{2l}x \in \mathscr{B}(T^{2l(n)+1}x; \operatorname{dist}(A, B) + \frac{1}{n})$ for all $l \ge l(n)$. If we set $y_n := T^{2l(n)+1}x$, then $y_n \in B$ and $T^{2l}x \in \mathscr{B}(y_n; \operatorname{dist}(A, B) + \frac{1}{n})$ for all $l \ge l(n)$. Put $\mathscr{C}_n := \mathscr{B}(y_n; \operatorname{dist}(A, B) + \frac{1}{n})$ and set $\mathscr{D}_1 := A \cap \mathscr{C}_1$ and $\mathscr{D}_n := \mathscr{D}_{n-1} \cap \mathscr{C}_n$ for all $n \ge 2$. Hence, for each $n \in \mathbb{N}$, $\mathscr{D}_n$ is nonempty bounded and closed subset of X and by the fact that X is strictly convex metric space, every $\mathscr{D}_n$ is also convex. Moreover, since X is reflexive, we conclude that $\bigcap_{n\in\mathbb{N}} \mathscr{D}_n$ is nonempty. Assume that $p \in \bigcap_{n\in\mathbb{N}} \mathscr{D}_n$. Thus $d(y_n, p) \to \operatorname{dist}(A, B)$. Besides, we have

$$\begin{aligned} \lim_{n\to\infty} d(y_n, Ty_n) &= \lim_{n\to\infty} d(T^{2l(n)+1}x, T^{2l(n)+2}x) \\ &\le \limsup_{n\to\infty} d(T^{2n}x, T^{2n+1}x) = \operatorname{dist}(A, B), \end{aligned}$$

which implies that $d(y_n, Ty_n) \to \operatorname{dist}(A, B)$. Since (A, B) has the property UC, we must have $d(Ty_n, p) \to 0$ or $Ty_n \to p$. On the other hand, we have

$$\begin{aligned} d(p, Tp) &= \lim_{n\to\infty} d(Ty_n, Tp) \\ &\le \lim_{n\to\infty} \alpha[d(y_n, Ty_n) + d(p, Tp)] + (1 - 2\alpha)\operatorname{dist}(A, B) \\ &= \alpha d(p, Tp) + (1 - \alpha)\operatorname{dist}(A, B). \end{aligned}$$

Then $d(p, Tp) = \operatorname{dist}(A, B)$, that is, $p \in A$ is a best proximity point of the mapping T.

The uniqueness of the best proximity point for the mapping T in A can be obtained similarly from Theorem 5 of [33]. This completes the proof. □

9.3.3 *Cyclic Relatively Kannan Nonexpansive Mappings*

Here, we generalize the class of Kannan nonexpansive self-mappings as below.

Definition 9.12 Let (A, B) be a nonempty pair of subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is said to be the *cyclic relatively Kannan nonexpansive mapping* if T is cyclic and, for all $(x, y) \in A \times B$,

$$d(Tx, Ty) = \operatorname{dist}(A, B) \quad \text{if} \quad d(x, y) = \operatorname{dist}(A, B), \tag{9.8}$$

$$d(Tx, Ty) \leq \frac{1}{2}\{d(x, Tx) + d(y, Ty)\} \quad \text{if} \quad d(x, y) > \operatorname{dist}(A, B). \tag{9.9}$$

In special case, if $A = B$, then we get the class of Kannan nonexpansive self-mappings.

We also say that T is the *strongly cyclic relatively Kannan nonexpansive mapping* if T is cyclic which satisfies the condition (9.8) and

$$d(Tx, Ty) \leq \min\{d(x, Tx), d(y, Ty)\} \quad \text{if} \quad d(x, y) > \operatorname{dist}(A, B). \tag{9.10}$$

In this situation if $A = B$, then T is said to be the *strongly Kannan nonexpansive self-mapping*.

Obviously, the class of cyclic relatively Kannan nonexpansive mappings contains the class of strongly cyclic relatively Kannan nonexpansive mappings.

In order to study the existence of best proximity points for cyclic relatively Kannan nonexpansive mappings in geodesic spaces, we recall the following geometric concept which was introduced in [1].

Definition 9.13 A convex pair (K_1, K_2) in a geodesic space X is said to have the *proximal quasi-normal structure* (PQNS) if for any bounded closed and convex proximinal pair $(H_1, H_2) \subseteq (K_1, K_2)$ for which $\operatorname{dist}(H_1, H_2) = \operatorname{dist}(K_1, K_2)$ and $\delta(H_1, H_2) > \operatorname{dist}(H_1, H_2)$, there exists $(p_1, p_2) \in H_1 \times H_2$ such that

$$d(p_1, y) < \delta(H_1, H_2), \quad d(x, p_2) < \delta(H_1, H_2)$$

for all $(x, y) \in H_1 \times H_2$.

It is remarkable to note that, for a convex subset K of a geodesic space X, the pair (K, K) has the PQNS if and only if K has quasi-normal structure in the sense of Definition 9.2. Also, it is clear that

$$\text{PNS} \Longrightarrow \text{PQNS}.$$

To describe our main results of this section, we need the following important lemma.

Lemma 9.3 ([1, Lemma 3.7]) *Let (K_1, K_2) be a nonempty pair in a geodesic metric space (X, d). Then*

$$\delta(K_1, K_2) = \delta(\overline{\operatorname{con}}(K_1), \overline{\operatorname{con}}(K_2)).$$

Proof We have to prove that $\delta(\overline{\operatorname{con}}(K_1), \overline{\operatorname{con}}(K_2)) \leq \delta(K_1, K_2)$. Let $x \in K_2$. For all $y \in K_1$ we have $y \in \mathscr{B}(x; \delta_x(K_1))$. Then $K_1 \subseteq \bigcap_{x \in K_2} \mathscr{B}(x; \delta_x(K_1))$ and hence $\overline{\operatorname{con}}(K_1) \subseteq \bigcap_{x \in K_2} \mathscr{B}(x; \delta_x(K_1))$. Now if $z \in \overline{\operatorname{con}}(K_1)$, it is easy to see that $\overline{\operatorname{con}}(K_2) \subseteq \mathscr{B}(z; \delta(K_1, K_2))$. Thus we have

$$\overline{\mathrm{con}}(K_2) \subseteq \bigcap_{z \in \overline{\mathrm{con}}(K_1)} \mathscr{B}(z; \delta(K_1, K_2)),$$

and the result follows. This completes the proof. □

We now prove the following existence theorem.

Theorem 9.8 *Let (X, d) be a reflexive and Busemann convex space and let (A, B) be a nonempty closed and convex pair of subsets of X such that A is bounded. Assume $T : A \cup B \to A \cup B$ is a cyclic relatively Kannan nonexpansive mapping. If (A, B) has the* PQNS*, then T has a best proximity point.*

Proof By Lemma 9.2, there exists a pair $(K_1, K_2) \subseteq (A_0, B_0) \subseteq (A, B)$ which is minimal with respect to being nonempty, bounded, closed, convex and T-invariant pair of subsets of (A, B) such that $\mathrm{dist}(K_1, K_2) = \mathrm{dist}(A, B)$. Now, let r be a real positive number such that $r \geq \mathrm{dist}(A, B)$ and let $(p, q) \in K_1 \times K_2$ be such that

$$d(p, q) = \mathrm{dist}(A, B), \quad d(p, Tp) \leq r, \quad d(Tq, q) \leq r.$$

Define

$$K_1^r = \{x \in K_1 : d(x, Tx) \leq r\}, \quad K_2^r = \{x \in K_2 : d(Tx, x) \leq r\},$$

and set

$$C_1^r := \overline{\mathrm{con}}(T(K_1^r)), \quad C_2^r := \overline{\mathrm{con}}(T(K_2^r)).$$

Now, we claim that T is cyclic on $C_1^r \cup C_2^r$. First, we show that $C_1^r \subseteq K_2^r$. Let $x \in C_1^r$ be an arbitrary element. If $d(Tx, x) = \mathrm{dist}(A, B)$, then $x \in K_2^r$. So assume that $d(Tx, x) > \mathrm{dist}(A, B)$. Put $s := \sup\{d(Tw, Tx) : w \in K_1^r\}$. Then we have $T(K_1^r) \subseteq \mathscr{B}(Tx; s)$. This implies that

$$C_1^r = \overline{\mathrm{con}}(T(K_1^r)) \subseteq \mathscr{B}(Tx; s).$$

Since $x \in C_1^r$, we have $d(Tx, x) \leq s$. By the definition of s, for each $\varepsilon > 0$ there exists $w \in K_1^r$ such that $s - \varepsilon \leq d(Tw, Tx)$. Therefore, we have

$$\begin{aligned} d(Tx, x) - \varepsilon \leq s - \varepsilon &\leq d(Tw, Tx) \\ &\leq \frac{1}{2}[d(w, Tw) + d(Tx, x)] \\ &\leq \frac{1}{2}d(Tx, x) + \frac{1}{2}r. \end{aligned}$$

Thereby, $d(Tx, x) \leq r + 2\varepsilon$ which implies that $x \in K_2^r$. Thus $C_1^r \subseteq K_2^r$ and so

$$T(C_1^r) \subseteq T(K_2^r) \subseteq \overline{\mathrm{con}}(T(K_2^r)) = C_2^r.$$

By a similar manner, we can see that $T(C_2^r) \subseteq C_1^r$, that is, T is cyclic on $C_1^r \cup C_2^r$. We now prove that $\delta(C_1^r, C_2^r) \le r$. It follows from Lemma 9.3 that

$$\begin{aligned}\delta(C_1^r, C_2^r) &= \delta(\overline{\text{con}}(T(K_1^r), \overline{\text{con}}(T(K_2^r))) \\ &= \delta(T(K_1^r), T(K_2^r)) \\ &= \sup\{d(Tx, Ty) : x \in K_1^r, y \in K_2^r\} \\ &\le \sup\left\{\frac{1}{2}[d(x, Tx) + d(Ty, y)] : x \in K_1^r, y \in K_2^r\right\} \\ &\le r.\end{aligned}$$

Because of the fact that $p \in K_1^r, q \in K_2^r$ and $d(p, q) = \text{dist}(A, B)$, we obtain

$$\text{dist}(A, B) \le \text{dist}(C_2^r, C_1^r) \le d(Tq, Tp) = \text{dist}(A, B),$$

that is, $\text{dist}(C_2^r, C_1^r) = \text{dist}(A, B)$. Put

$$r_0 = \inf\{d(x, Tx) : x \in K_1 \cup K_2\}.$$

Then $r_0 \ge \text{dist}(A, B)$. Let $\{r_n\}$ be a nonnegative decreasing sequence such that $r_n \to r_0$. Thus $\{(C_1^{r_n}, C_2^{r_n})\}$ is a descending sequences of nonempty, bounded, closed and convex pair in X for which $(C_1^{r_n}, C_2^{r_n}) \subseteq (K_2, K_1)$ for all $n \in \mathbb{N}$. It follows from the reflexivity of the geodesic space X that

$$C_1^{r_0} = \bigcap_{n=1}^{\infty} C_1^{r_n} \ne \varnothing, \quad C_2^{r_0} = \bigcap_{n=1}^{\infty} C_2^{r_n} \ne \varnothing.$$

By a similar argument of Theorem 9.6, the pair $(C_2^{r_0}, C_1^{r_0})$ is T-invariant with $\text{dist}(C_2^{r_0}, C_1^{r_0}) = \text{dist}(A, B)$. Using the minimality of (K_1, K_2), we must have $C_2^{r_0} = K_1$ and $C_1^{r_0} = K_2$. Thus $d(x, Tx) \le r_0$ for all $x \in K_1 \cup K_2$. Now assume that $r_0 > \text{dist}(A, B)$. By the fact that (A, B) has PQNS, there exists $(p_1, q_1) \in K_1 \times K_2$ such that

$$d(p_1, y) < \delta(K_1, K_2) \le r_0, \quad d(x, q_1) < \delta(K_1, K_2) \le r_0,$$

for all $(x, y) \in K_1 \times K_2$. This ensures that

$$d(p_1, Tp_1) < \delta(K_1, K_2) \le r_0, \qquad d(Tq_1, q_1) < \delta(K_1, K_2) \le r_0,$$

which is impossible and so $r_0 = \text{dist}(A, B)$. In this case, we conclude that

$$d(x, Tx) = \text{dist}(A, B) = d(Ty, y), \quad \forall (x, y) \in K_1 \times K_2,$$

that is, every point of K_1 and K_2 is a best proximity point for the mapping T. This completes the proof. □

The following new fixed point result is a straightforward consequence of Theorem 9.8 which is an extension of Theorem 9.4 due to Wong.

Corollary 9.1 *Let A be a nonempty bounded closed and convex subset of a reflexive and Busemann convex space (X, d). Assume $T : A \to A$ is a Kannan nonexpansive mapping. If (A, B) has the quasi-normal structure, then T has a unique fixed point.*

The following example shows the usability of Theorem 9.8.

Example 9.4 Let $X = \mathbb{R}^2$ and d be the *river metric* on X defined with

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2|, & \text{if } x_1 = x_2, \\ |x_1 - x_2| + |y_1| + |y_2|, & \text{if } x_1 \neq x_2. \end{cases}$$

It is well known that $(\mathbb{R}^2, d)$ is a reflexive and Busemann convex space (see [14] for more information). Suppose $A = \{(0, x) : 0 \le x \le \frac{1}{2}\}$ and $B = \{(1, y) : 0 \le y \le 1\}$. Then (A, B) is a nonempty bounded closed and convex pair and it is easy to see that $\operatorname{dist}(A, B) = 1$. Moreover, we have

$$A_0 = A, \quad B_0 = \left\{(1, y) : 0 \le y \le \frac{1}{2}\right\}.$$

Now, define the cyclic mapping $T : A \cup B \to A \cup B$ with

$$T(0, x) = (1, x^2), \quad T(1, y) = (0, y),$$

where $(x, y) \in [0, \frac{1}{2}] \times [0, 1]$. Then, for all $\mathbf{x} = (0, x) \in A$ and $\mathbf{y} = (1, y) \in B$, we have

$$\begin{aligned} d(T\mathbf{x}, T\mathbf{y}) &= d((1, x^2), (0, y)) = 1 + x^2 + y \\ &\le \frac{1}{2}\{(1 + x + x^2) + (1 + 2y)\} \\ &= \frac{1}{2}\{d((0, x), (1, x^2)) + d((1, y), (0, y))\} \\ &= \frac{1}{2}\{d(\mathbf{x}, T\mathbf{x}) + d(\mathbf{y}, T\mathbf{y})\}, \end{aligned}$$

which concludes that T is a cyclic relatively Kannan nonexpansive mapping. Besides, since (A, B) is compact and convex pair, then (A, B) has the PNS and so has the PQNS. Therefore, all of the conditions of Theorem 9.8 hold and T has a best proximity point which is the point $((0, 0), (1, 0)) \in A \times B$.

9.4 Structure of Minimal Sets and Min-Max Property

Let (X, d) be a reflexive and Busemann convex space and let (A, B) be a nonempty closed and convex pair of subsets of X such that A is bounded. Let $T : A \cup B \to A \cup B$ be a cyclic mapping. We denote by Σ_T the set of all $(K_1, K_2) \subseteq (A_0, B_0) \subseteq (A, B)$ which is minimal with respect to being nonempty closed convex and T-invariant such that

$$\mathrm{dist}(K_1, K_2) = \mathrm{dist}(A, B).$$

We mention that from Lemma 9.2 if the cyclic mapping T satisfies the condition $d(Tx, Ty) = \mathrm{dist}(A, B)$ for any $(x, y) \in A \times B$ with $d(x, y) = \mathrm{dist}(A, B)$, then $\Sigma_T \neq \varnothing$.

The next geometric notion was introduced in [35].

Definition 9.14 Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \varnothing$. The pair (A, B) is said to have the P-*property* if

$$\begin{cases} d(x_1, y_1) = \mathrm{dist}(A, B) \\ d(x_2, y_2) = \mathrm{dist}(A, B) \end{cases} \implies d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

To present an example of the pairs having the P-property, we need the following concept.

Definition 9.15 For two geodesic segments $[x, y]$ and $[z, w]$ in a uniquely geodesic space (X, d), we say that $[x, y]$ is *parallel* to $[z, w]$ and we write $[x, y]||[z, w]$ provided that

$$d(x, z) = d(y, w) = d(m_1, m_2),$$

where $m_1 := \frac{1}{2}x \oplus \frac{1}{2}y$ and $m_2 := \frac{1}{2}z \oplus \frac{1}{2}w$.

The next interesting result holds in the setting of Busemann convex spaces.

Lemma 9.4 ([8]) *Let (X, d) be a Busemann convex space and $x, y, z, w \in X$ so that $[x, y]||[z, w]$. Then $[x, z]||[y, w]$.*

We now state the following conclusion related to the P-property.

Proposition 9.2 ([21], Lemma 4.3) *Let (A, B) be a nonempty closed and convex pair in a reflexive and Busemann convex space X so that A is bounded. Then (A, B) has the P-property.*

Proof From Lemma 9.1, (A_0, B_0) is nonempty bounded closed and convex. Let $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$ be such that $d(x_i, y_i) = \mathrm{dist}(A, B)$ for $i = 1, 2$. Set $m_1 := \frac{1}{2}x_1 \oplus \frac{1}{2}x_2$ and $m_2 := \frac{1}{2}y_1 \oplus \frac{1}{2}y_2$. We have

$$\mathrm{dist}(A, B) \leq d(m_1, m_2) \leq \frac{1}{2}[d(x_1, y_1) + d(x_2, y_2)] = \mathrm{dist}(A, B),$$

which implies that $[x_1, x_2]||[y_1, y_2]$. It now follows form Lemma 9.4 that $[x_1, y_1]||[x_2, y_2]$ and so

$$d(x_1, x_2) = d(y_1, y_2) = d(m_1', m_2'),$$

where $m_1' := \frac{1}{2}x_1 \oplus \frac{1}{2}y_1$ and $m_2' := \frac{1}{2}x_2 \oplus \frac{1}{2}y_2$ and the result follows. This completes the proof. □

Definition 9.16 Let (X, d) be a reflexive and Busemann convex space and let (A, B) be a nonempty, closed and convex pair of subsets of X such that A is bounded. Suppose $T : A \cup B \to A \cup B$ is a cyclic relatively Kannan nonexpansive mapping. The pair (A, B) has the *H-property* if, for any $(K_1, K_2) \in \Sigma_T$, we have

$$\max\{\mathrm{diam}(K_1), \mathrm{diam}(K_2)\} \leq \delta(K_1, K_2).$$

In what follows, we provide some sufficient conditions for the H-property. To this end, we need the following requirements.

Definition 9.17 ([12]) Let (A, B) be a nonempty pair of sets in a metric space (X, d). A point p in A (q in B) is said to be a *diametral point* with respect to B (w.r.t. A) if $\delta_p(B) = \delta(A, B)$ ($\delta_q(A) = \delta(A, B)$).

Lemma 9.5 *Let (X, d) be a reflexive and Busemann convex space and let (A, B) be a nonempty closed and convex pair of subsets of X such that A is bounded. Suppose $T : A \cup B \to A \cup B$ is a strongly cyclic relatively Kannan nonexpansive mapping. Let $(K_1, K_2) \in \Sigma_T$. Then each pair $(p, q) \in K_1 \times K_2$ with $d(p, q) = \mathrm{dist}(A, B)$ contains a diametral point (with respect to (K_1, K_2)).*

Proof Let $(p, q) \in K_1 \times K_2$ be such that $d(p, q) = \mathrm{dist}(A, B)$. Put $r_1 := \delta_p(K_2)$ and $r_2 := \delta_q(K_1)$. Suppose that (p, q) is a nondiametral pair. Then we have $r := \max\{r_1, r_2\} < \delta(K_1, K_2)$. Note that, from Lemma 9.2, the pair (K_1, K_2) is proximinal and

$$K_1 = \overline{\mathrm{con}}(T(K_2)), \quad K_2 = \overline{\mathrm{con}}(T(K_1)).$$

Let

$$\mathscr{C}_r(K_2) := K_1 \bigcap (\cap_{x \in K_2} \mathscr{B}(x; r)), \quad \mathscr{C}_r(K_1) := K_2 \bigcap (\cap_{x \in K_1} \mathscr{B}(x; r)).$$

Then $(p, q) \in C_r(K_2) \times C_r(K_1)$ and $(C_r(K_2), C_r(K_1)) \subseteq (K_1, K_2)$ is nonempty closed and convex. Besides, it is easy to verify that

$$(u, v) \in C_r(K_2) \times C_r(K_1) \Longleftrightarrow K_2 \subseteq \mathscr{B}(u; r), \ K_1 \subseteq \mathscr{B}(v; r).$$

Furthermore, we have

$$\operatorname{dist}(C_r(K_2), C_r(K_1)) \le d(p, q) = \operatorname{dist}(K_1, K_2) \le \operatorname{dist}(C_r(K_2), C_r(K_1)),$$

and this concludes that $\operatorname{dist}(C_r(K_2), C_r(K_1)) = \operatorname{dist}(K_1, K_2)(= \operatorname{dist}(A, B))$. We now assert that T is cyclic on $C_r(K_2) \cup C_r(K_1)$. Let $u \in C_r(K_2)$. Since T is a strongly cyclic relatively Kannan nonexpansive mapping, for all $v \in K_2$, we have

$$d(Tu, Tv) \le \max\Big\{\min\{d(u, Tu), d(v, Tv)\}, \operatorname{dist}(A, B)\Big\} \le r,$$

which ensures that $Tv \in \mathscr{B}(Tu; r)$. Hence $T(K_2) \subseteq \mathscr{B}(Tu; r)$ and so $K_1 = \overline{\text{con}}(T(K_2)) \subseteq \mathscr{B}(Tu; r)$, that is, $Tu \in C_r(K_1)$. Therefore, $T(C_r(K_2)) \subseteq C_r(K_1)$. Similarly, we have $T(C_r(K_1)) \subseteq C_r(K_2)$, which implies that T is cyclic on $C_r(K_2) \cup C_r(K_1)$. Again, by the minimality of (K_1, K_2), we obtain

$$C_r(K_2) = K_1, \qquad C_r(K_1) = K_2.$$

So, $K_1 \subseteq \bigcap_{v \in K_2} \mathscr{B}(v; r)$. Then, for each $u \in K_1$ we have $\delta_u(K_2) \le r$. Hence we have

$$\delta(K_1, K_2) = \sup_{u \in K_1} \delta_u(K_2) \le r,$$

which is impossible because of the fact that $r < \delta(K_1, K_2)$. This completes the proof. $\square$

Proposition 9.3 *Let (A, B) be a nonempty, bounded, closed and convex pair of subsets of a Busemann convex space X and $T : A \cup B \to A \cup B$ be a strongly cyclic relatively Kannan nonexpansive mapping. If X is uniformly convex, then (A, B) has the* H*-property.*

Proof Let $(K_1, K_2) \in \Sigma_T$ be such that $\operatorname{diam}(K_1) > \delta(K_1, K_2)$. Then there exist $x_1, x_2 \in K_1$ such that $d(x_1, x_2) \ge \frac{1}{2}\delta(K_1, K_2)$. Since the pair (K_1, K_2) is proximinal, we can find the elements $y_1, y_2 \in K_2$ so that $d(x_i, y_i) = \operatorname{dist}(A, B)(= \operatorname{dist}(K_1, K_2))$ for $i = 1, 2$. From Proposition 9.2, the pair (K_1, K_2) has the P-property which ensures that $d(x_1, x_2) = d(y_1, y_2)$ and so $d(y_1, y_2) \ge \frac{1}{2}\delta(K_1, K_2)$. Put

$$m_1 := \frac{1}{2}x_1 \oplus \frac{1}{2}y_1, \quad m_2 := \frac{1}{2}x_2 \oplus \frac{1}{2}y_2.$$

Then $(m_1, m_2) \in K_1 \times K_2$ and, by the fact that X is a Busemann convex space, we have

$$d(m_1, m_2) = d\left(\frac{1}{2}x_1 \oplus \frac{1}{2}y_1, \frac{1}{2}x_2 \oplus \frac{1}{2}y_2\right) \le \frac{1}{2}[d(x_1, y_1) + d(x_2, y_2)] = \operatorname{dist}(A, B).$$

Now, for all $y \in K_2$, we have

$$d(x_1, y) \leq \delta(K_1, K_2), \quad d(x_2, y) \leq \delta(K_1, K_2), \quad d(x_1, x_2) \geq \frac{1}{2}\delta(K_1, K_2).$$

Since X is uniformly convex, for $\varepsilon = \frac{1}{2}$ there exists $\eta \in (0, 1]$ for which

$$d(m_1, y) \leq (1 - \eta)\delta(K_1, K_2), \quad \forall y \in K_2,$$

and so $\delta_{m_1}(K_2) \leq (1 - \eta)\delta(K_1, K_2) < \delta(K_1, K_2)$. Equivalently, we can see that $\delta_{m_2}(K_1) < \delta(K_1, K_2)$. Hence, the proximal point $(m_1, m_2) \in K_1 \times K_2$ is a non-diametral pair which is a contradiction by Lemma 9.5. By a similar argument, if $\mathrm{diam}(K_2) > \delta(K_1, K_2)$, then we get a contradiction. □

Motivated by the results of a recent paper of the current author [22] which was discussed on the structure of minimal sets for cyclic relatively nonexpansive mappings, we introduce the following constant for cyclic relatively Kannan nonexpansive mappings.

Definition 9.18 Let (X, d) be a reflexive and Busemann convex space and let (A, B) be a nonempty disjoint closed and convex pair of subsets of X such that A is bounded. Let $T : A \cup B \to A \cup B$ be a cyclic relatively Kannan nonexpansive mapping and (A, B) has the H-property. We define

$$\nu_T := \inf\left\{\frac{\max\{\mathrm{diam}(K_1), \mathrm{diam}(K_2)\}}{\delta(K_1, K_2)} : (K_1, K_2) \in \Sigma_T\right\}.$$

It is clear that $\nu_T \in [0, 1]$.

Proposition 9.4 *Let (A, B) be a nonempty, disjoint, bounded, closed and convex pair of subsets of a Busemann convex space X and $T : A \cup B \to A \cup B$ be a strongly cyclic relatively Kannan nonexpansive mapping. If X is uniformly convex, then $\nu_T =$ 0.*

Proof From Proposition 9.3, (A, B) has the H-property. We consider two following cases:

Case 1. If $\nu_T = 1$, then, for any $(K_1, K_2) \in \Sigma_T$, we have

$$\max\{\mathrm{diam}(K_1), \mathrm{diam}(K_2)\} = \delta(K_1, K_2).$$

We may assume that $\mathrm{diam}(K_1) \leq \mathrm{diam}(K_2)$. Let $y_1, y_2 \in K_2$ be such that $d(y_1, y_2) \geq \frac{1}{2}\mathrm{diam}(K_2)$. It follows from the proximinality of (K_1, K_2) that there exist $x_1, x_2 \in K_1$ so that $d(x_i, y_i) = \mathrm{dist}(K_1, K_2)$ for $i = 1, 2$. Using Proposition 9.2, we observe that the pair (K_1, K_2) has the P-property and so $d(x_1, x_2) = d(y_1, y_2)$. Now, for any $x \in K_1$, we have

$$\begin{cases} d(x, y_1) \leq \delta(K_1, K_2), \\ d(x, y_2) \leq \delta(K_1, K_2), \\ d(y_1, y_2) \geq \frac{1}{2}\text{diam}(K_2) = \frac{1}{2}\delta(K_1, K_2). \end{cases}$$

Put $m_1 := \frac{1}{2}x_1 \oplus \frac{1}{2}y_1$ and $m_2 := \frac{1}{2}x_2 \oplus \frac{1}{2}y_2$. Then $(m_1, m_2) \in K_1 \times K_2$ and, by the fact that X is Busemann convex, $d(m_1, m_2) = \text{dist}(A, B)$. In view of the fact that X is uniformly convex, we obtain

$$\max\{\delta_{m_1}(K_2), \delta_{m_2}(K_1)\} < \delta(K_1, K_2),$$

which is impossible since from Lemma 9.5, the pair (m_1, m_2) contains a diametral point.

Case 2. Now, assume that $0 < \nu_T < 1$. Then for any $(K_1, K_2) \in \Sigma_T$ with $\delta(K_1, K_2) > \text{dist}(K_1, K_2)$, we have

$$\nu_T \delta(K_1, K_2) \leq \max\{\text{diam}(K_1), \text{diam}(K_2)\}.$$

By a similar manner of the Case 1, we can find a point $(m_1, m_2) \in K_1 \times K_2$ with $d(m_1, m_2) = \text{dist}(K_1, K_2)$ such that

$$\max\{\delta_{m_1}(K_2), \delta_{m_2}(K_1)\} < \delta(K_1, K_2)$$

and this is a contradiction by Lemma 9.5. Therefore, we must have $\nu_T = 0$. This completes the proof. □

We now ready to state the following best proximity point theorem for cyclic relatively Kannan nonexpansive mappings.

Theorem 9.9 *Let (X, d) be a reflexive and Busemann convex space and (A, B) be a nonempty disjoint closed and convex pair of subsets of X such that A is bounded. Suppose that $T : A \cup B \to A \cup B$ is a cyclic relatively Kannan nonexpansive mapping. If $\nu_T = 0$, then T has a best proximity point.*

Proof Let $\varepsilon > 0$ be given. Because of the fact that $\nu_T = 0$, there is an element $(K_1, K_2) \in \Sigma_T$ for which

$$\frac{\max\{\text{diam}(K_1), \text{diam}(K_2)\}}{\delta(K_1, K_2)} < \frac{\varepsilon}{\delta(A, B)}.$$

Thus we must have $\max\{\text{diam}(K_1), \text{diam}(K_2)\} < \varepsilon$ for all $\varepsilon > 0$ which implies that $\text{diam}(K_1) = \text{diam}(K_2) = 0$. Let $K_1 = \{p\}$. Since (K_1, K_2) is T-invariant and $\text{dist}(K_1, K_2) = \text{dist}(A, B)$, $K_2 = \{Tp\}$ and so $d(p, Tp) = \text{dist}(A, B)$, that is, $p \in K_1$ is a best proximity point for the mapping T. This completes the proof. □

By applying Proposition 9.4 and Theorem 9.9, the next result concludes, immediately.

Corollary 9.2 *Let (X, d) be a reflexive and Busemann convex space and (A, B) be a nonempty disjoint closed and convex pair of subsets of X such that A is bounded. Suppose that $T : A \cup B \to A \cup B$ is a strongly cyclic relatively Kannan nonexpansive mapping. If X is uniformly convex, then T has a best proximity point.*

Here, we present a notion of min-max property for cyclic mappings.

Definition 9.19 ([22]) Let (X, d) be a reflexive and Busemann convex space and let (A, B) be a nonempty disjoint closed and convex pair of subsets of X such that A is bounded. Assume that $T : A \cup B \to A \cup B$ is a cyclic mapping. We say that the pair (A, B) has the *min-max property* if, for any pair $(K_1, K_2) \in \Sigma_T$,

$$\delta(K_1, K_2) = \operatorname{dist}(K_1, K_2).$$

It is clear that, if in above definition the pair (A, B) has the min-max property, then the mapping T has a best proximity point. So, it is interesting to find some sufficient conditions to ensure that a consider pair having the min-max property.

Theorem 9.10 *Let (X, d) be a reflexive and Busemann convex space and (A, B) be a nonempty disjoint bounded closed and convex pair of subsets of X Suppose that $T : A \cup B \to A \cup B$ is a cyclic Kannan contraction mapping. Then (A, B) has the min-max property.*

Proof Let Γ denote the set of all nonempty closed convex and T-invariant pairs $(E, F) \subseteq (A, B)$. Then $(A, B) \in \Gamma$ and so $\Gamma \neq \varnothing$. Reflexivity of the geodesic space X implies that Γ has a minimal element. Assume that Γ_T denotes the set of all $(H_1, H_2) \subseteq (A, B)$ which is minimal with respect to being nonempty closed convex and T-invariant. We prove that $\Gamma_T = \Sigma_T$. It is sufficient to show that $\Sigma_T \supseteq \Gamma_T$. Let $(H_1, H_2) \in \Gamma_T$. We assert that

$$\delta(H_1, H_2) = \operatorname{dist}(A, B)$$

and this ensures that the pair (A, B) has the min-max property. As in the proof of Lemma 9.2, we have

$$\overline{\operatorname{con}}(T(H_2)) = H_1, \quad \overline{\operatorname{con}}(T(H_1)) = H_2.$$

Let $a \in H_1$ be an arbitrary element. Since T is a cyclic Kannan contraction, we have

$$\begin{aligned} d(Ta, Ty) &\leq \alpha\{d(a, Ta) + d(y, Ty)\} + (1 - 2\alpha)\operatorname{dist}(A, B) \\ &\leq 2\alpha\delta(H_1, H_2) + (1 - 2\alpha)\operatorname{dist}(A, B) \end{aligned}$$

for some $\alpha \in [0, \frac{1}{2})$ and for all $y \in H_2$ and so

$$Ty \in \mathscr{B}\Big(Ta; 2\alpha\delta(H_1, H_2) + (1 - 2\alpha)\operatorname{dist}(A, B)\Big)$$

for any $y \in H_2$. This implies that

$$H_1 = \overline{\text{con}}(T(H_2)) \subseteq \mathscr{B}\Big(Ta; 2\alpha\delta(H_1, H_2) + (1-2\alpha)\text{dist}(A, B)\Big).$$

Hence we have

$$\delta_{Ta}(H_1) \leq 2\alpha\delta(H_1, H_2) + (1-2\alpha)\text{dist}(A, B), \quad \forall a \in H_1,$$

and so

$$\begin{aligned}
\delta(H_1, H_2) &= \delta(H_1, \overline{\text{con}}(T(H_1))) = \delta(H_1, T(H_1)) \quad \text{(by Lemma 9.3)} \\
&= \sup_{a \in H_1} \delta_{Ta}(H_1) \\
&\leq 2\alpha\delta(H_1, H_2) + (1-2\alpha)\text{dist}(A, B).
\end{aligned}$$

Therefore, $\delta(H_1, H_2) = \text{dist}(A, B)$ and thus $\text{dist}(H_1, H_2) = \text{dist}(A, B)$, that is, $(H_1, H_2) \in \Sigma_T$ and the result follows. This completes the proof. □

The next corollary is a straightforward consequence of Theorem 9.10.

Corollary 9.3 *Every cyclic Kannan contraction mapping defined on a union of nonempty, disjoint, bounded, closed and convex subsets of a reflexive and Busemann convex space X has a best proximity point.*

Remark 9.1 By comparing Theorem 9.7 and Corollary 9.3 we find that the pair (A, B) in Corollary 9.3 does not have the condition of property UC but we need the boundedness condition of the pair (A, B), whereas in Theorem 9.7 we used the property UC in the process of the proof but without using the boundedness of neither A nor B.

Theorem 9.11 *Let (X, d) be a reflexive and Busemann convex space and (A, B) be a nonempty closed and convex pair of subsets of X such that A is bounded. Suppose that $T : A \cup B \to A \cup B$ is a strongly cyclic relatively Kannan nonexpansive mapping. If the pair (A, B) has the* PNS*, then (A, B) has the min-max property.*

Proof Let $(K_1, K_2) \in \Sigma_T$. If $\text{dist}(K_1, K_2) < \delta(K_1, K_2)$, then, by the fact that (A, B) has the PNS, there exists a point $(p, q) \in K_1 \times K_2$ such that $d(p, q) = \text{dist}(K_1, K_2)$ and

$$\max\{\delta_p(K_2), \delta_q(K_1)\} < \delta(K_1, K_2).$$

From Lemma 9.5, we see that the pair (p, q) contains a diametral point with respect to $\delta(K_1, K_2)$, which is a contradiction. So we must have $\delta(K_1, K_2) = \text{dist}(K_1, K_2)$ and the result follows. This completes the proof. □

9.5 On Dropping of PQNS for Cyclic Relatively Kannan Nonexpansive Mappings

In the current section of this chapter, we provide some best proximity point theorems for cyclic relatively nonexpansive mappings but without the essential condition of PQNS which was used in Theorem 9.8. We do that by considering some assumptions on the considered cyclic mapping.

Before, we state the main conclusions of this section, we recall the following equivalent of closed and convex hull of sets in geodesic spaces.

Lemma 9.6 ([6]) *Let A be a nonempty subset of a geodesic space X. Let $\mathscr{G}_1(A)$ denote the union of all geodesic segments with endpoints in A. Recursively, for each $n \geq 2$, put $\mathscr{G}_n(A) = \mathscr{G}_1(\mathscr{G}_{n-1}(A))$. Then we have*

$$\mathrm{con}(A) = \bigcup_{n=1}^{\infty} \mathscr{G}_n(A).$$

Remark 9.2 ([16]) It is worth noticing that in a Busemann convex space X the closure of $\mathrm{con}(A)$ is convex and so, coincides with $\overline{\mathrm{con}}(A)$.

Theorem 9.12 *Let (X, d) be a reflexive and Busemann convex space and (A, B) be a nonempty closed and convex pair of subsets of X such that A is bounded. Assume that $T : A \cup B \to A \cup B$ is a cyclic relatively Kannan nonexpansive mapping such that*

$$d(T^2x, Tx) < d(x, Tx), \quad \forall x \in A \cup B \quad \textit{with} \ \mathrm{dist}(A, B) < d(x, Tx). \tag{9.11}$$

Then T has a best proximity point.

Proof From Lemma 9.2 Σ_T is nonempty. Let $(K_1, K_2) \in \Sigma_T$. Consider an arbitrary element $x^* \in K_1$ and suppose $d(x^*, Tx^*) := r$. If $r = \mathrm{dist}(A, B)$, then x is a best proximity point for the mapping T and we are finished. So assume that $r > \mathrm{dist}(A, B)$. Define

$$H_1 := \{x \in K_1 : d(x, Tx) \leq r\}.$$

Then $x^* \in H_1$. Moreover, for $y^* := Tx^* \in K_2$ we have

$$d(y^*, Ty^*) = d(Tx^*, T^2x^*) \leq \frac{1}{2}\{d(x^*, Tx^*) + d(Tx^*, T^2x^*)\},$$

which deduces that $d(y^*, Ty^*) \leq d(x^*, Tx^*) \leq r$. Now define

$$H_2 := \{y \in K_2 : d(y, Ty) \leq r\}.$$

From the aforesaid discussion, $y^* \in H_2$ and thus $H_2 \neq \varnothing$. Let

$$G_1 := \overline{\text{con}}(T(H_2)), \quad G_2 := \overline{\text{con}}(T(H_1)).$$

We assert that $G_1 \subseteq H_1$. Suppose $p \in \overline{\mathscr{G}_1(T(H_2))}$ and $\varepsilon > 0$ is given. Then there exist $t_1 \in [0, 1]$ and $y_1, y_2 \in H_2$ for which

$$d(p, t_1 Ty_1 \oplus (1 - t_1)Ty_2) < \varepsilon.$$

This implies that

$$\begin{aligned} d(p, Tp) &\leq d(p, t_1 Ty_1 \oplus (1 - t_1)Ty_2) + d(t_1 Ty_1 \oplus (1 - t_1)Ty_2, Tp) \\ &< \varepsilon + t_1 d(Ty_1, Tp) + (1 - t_1)d(Ty_2, Tp) \\ &\leq \varepsilon + \frac{t_1}{2}\{d(y_1, Ty_1) + d(p, Tp)\} + \frac{(1 - t_1)}{2}\{d(y_2, Ty_2) + d(p, Tp)\} \\ &\leq \varepsilon + \frac{t_1}{2}r + \left(\frac{1 - t_1}{2}\right)r + \frac{1}{2}d(p, Tp). \end{aligned}$$

Therefore, $d(p, Tp) < 2\varepsilon + r$. Since $\varepsilon > 0$ is arbitrary chosen, we must have $d(p, Tp) \leq r$. Thus $p \in H_1$. Again by the fact that p is arbitrary chosen, we conclude that $\overline{\mathscr{G}_1(T(H_2))} \subseteq H_1$. Similarly, we can see that

$$\overline{\mathscr{G}_2(T(H_2))} = \overline{\mathscr{G}_1(\mathscr{G}_1(T(H_2)))} \subseteq H_1.$$

Continuing this process and by induction, we obtain $\overline{\mathscr{G}_n(T(H_2))} \subseteq H_1$ for all $n \in \mathbb{N}$. This implies that

$$\text{con}(T(H_2)) = \bigcup_{n=1}^{\infty} \mathscr{G}_n(T(H_2)) \subseteq \bigcup_{n=1}^{\infty} \overline{\mathscr{G}_n(T(H_2))} \subseteq H_1.$$

Hence, $G_1 = \overline{\text{con}}(T(H_2)) \subseteq H_1$. Equivalent argument implies that $G_2 = \overline{\text{con}}(T(H_1)) \subseteq H_2$. We now have

$$T(G_1) \subseteq T(H_1) \subseteq \overline{\text{con}}(T(H_1)) = G_2,$$

$$T(G_2) \subseteq T(H_2) \subseteq \overline{\text{con}}(T(H_2)) = G_1,$$

which ensures that T is cyclic on $G_1 \cup G_2$. It now follows from the minimality of (K_1, K_2) that $G_1 = K_1$ and $G_2 = K_2$. Therefore,

$$K_1 = G_1 \subseteq H_1 \subseteq K_1 \Rightarrow K_1 = H_1,$$

$$K_2 = G_2 \subseteq H_2 \subseteq K_2 \Rightarrow K_2 = H_2.$$

In view of the fact that $x^* \in K_1$ was chosen arbitrarily, we obtain $d(x, Tx) = r$ for any $x \in K_1$. Hence, for all $y \in K_2$, $Ty \in K_1$ and so

$$r = d(Ty, T^2y) \leq \frac{1}{2}\{d(y, Ty) + d(Ty, T^2y)\} \leq \frac{1}{2}\{r + r\} = r.$$

Thereby, $d(y, Ty) = r$ for any $y \in K_2$. On the other hand, from the condition (9.11), for any $u \in K_1 \cup K_2$ we have

$$r = d(Tu, T^2u) < d(u, Tu) = r,$$

which is impossible, This competes the proof. □

In the setting of normed linear spaces, we obtain the following result.

Corollary 9.4 ([18, Theorem 3.1]) *Let (A, B) be a nonempty weakly compact and convex pair of subsets of a normed linear space X. Suppose that $T : A \cup B \to A \cup B$ is a cyclic Kannan nonexpansive mapping such that*

$$\|T^2x - Tx\| < \|x - Tx\|, \quad \forall\, x \in A \cup B \;\; \textit{with} \;\; \mathrm{dist}(A, B) < \|x - Tx\|.$$

Then T has a best proximity point.

Remark 9.3 By comparing Theorems 9.12 and 9.8, we conclude that the considered pair (A, B) in Theorem 9.12 need not to have the geometric property of PQNS.

Remark 9.4 As we show in the following example, the reflexivity of the Busemann convex space X in Theorem 9.12 is sufficient but not a necessary condition. Now it is interesting to ask whether Theorem 9.12 satisfies whenever X is a nonreflexive Buseamann convex space.

Example 9.5 Consider the nonreflexive Banach space l_1 and $\{e_n\}$ be the canonical basis of l_1. Suppose that

$$A = \overline{\mathrm{con}}(\{e_{2n-1} + e_{2n} : n \in \mathbb{N}\}), \quad B = \overline{\mathrm{con}}(\{e_{2n} + e_{2n+1} : n \in \mathbb{N}\}).$$

Then (A, B) is a bounded closed convex and proximinal pair in l_1 with $\mathrm{dist}(A, B) = 2$ and $\delta(A, B) = 4$. Notice that (A, B) does not the PQNS. Indeed, for all $x \in A$, we have

$$x = \sum_{j=1}^{k} t_j (e_{2n_j - 1} + e_{2n_j}),$$

where $t_j \geq 0$ and $\sum_{j=1}^{k} t_j = 1$. Now, if we consider $y := \sum_{j=1}^{k} t_j (e_{2(n_k+j)} + e_{2(n_k+j)+1}) \in B$, then we have

$$\|x - y\| = 4 \sum_{j=1}^{k} t_j = 4 = \delta(A, B),$$

which concludes that (A, B) does not have the PQNS. Define the cyclic mapping $T : A \cup B \to A \cup B$ with

$$Tx = \begin{cases} e_2 + e_3, & \text{if } x \in A, \\ e_1 + e_2, & \text{if } x \in B. \end{cases}$$

Then

$$\|Tx - Ty\| = 2 \leq \frac{1}{2}\{\|x - Tx\| + \|y - Ty\|\}, \quad \forall (x, y) \in A \times B,$$

that is, T is a cyclic Kannan nonexpansive mapping. Besides,

$$\|T^2x - Tx\| = 2 < \|x - Tx\|, \quad \forall x \in A \cup B \text{ with } \|x - Tx\| > \operatorname{dist}(A, B).$$

Notice that T has best proximity points which are the points $p = e_1 + e_2 \in A$ and $q = e_2 + e_3 \in B$.

As a corollary of Theorem 9.12 we obtain the following fixed point result.

Corollary 9.5 *Let (X, d) be a reflexive and Busemann convex space and A be a nonempty bounded closed and convex subset of X. Assume that $T : A \to A$ is a Kannan nonexpansive mapping such that*

$$d(T^2x, Tx) < d(x, Tx), \quad \forall x \in A \text{ with } d(x, Tx) > 0.$$

Then T has a unique fixed point.

By using Theorem 9.12, we give the other sufficient conditions differ from the condition of PQNS appeared in Theorem 9.8 in order to study the existence of best proximity points for cyclic relatively Kannan nonexpansive mappings.

Theorem 9.13 *Let (X, d) be a reflexive and Busemann convex space and (A, B) be a nonempty closed and convex pair of subsets of X such that A is bounded. Assume that $T : A \cup B \to A \cup B$ is a cyclic relatively Kannan nonexpansive mapping such that for each nonempty closed and convex pair $(C_1, C_2) \subseteq (A, B)$ which is T-invariant and such that $\delta(C_1, C_2) > \operatorname{dist}(A, B)$, we have*

$$\inf\{d(x, Tx) : x \in C_1 \cup C_2\} < \delta(C_1, C_2). \tag{9.12}$$

Then T has a best proximity point.

Proof Let $(K_1, K_2) \in \Sigma_T$. Then, from Lemma 9.2, we have

$$\overline{\operatorname{con}}(T(K_2)) = K_1, \quad \overline{\operatorname{con}}(T(K_1)) = K_2.$$

Note that, if $\delta(K_1, K_2) = \operatorname{dist}(A, B)$, then each point of $K_1 \cup K_2$ is a best proximity point of T and we are finished.

Let us assume that $\delta(K_1, K_2) > \operatorname{dist}(A, B)$. Then there is a point $x^* \in K_1 \cup K_2$ for which $r := d(x^*, Tx^*) < \delta(K_1, K_2)$. By a similar argument of Theorem 9.12, we obtain

$$d(x, Tx) = r = d(y, Ty), \quad \forall (x, y) \in K_1 \times K_2.$$

Let (u, v) be arbitrary chosen in $K_1 \times K_2$. In view of the fact that T is a cyclic relatively Kannan nonexpansive mapping, we have

$$d(Tu, Tv) \leq \frac{1}{2}\{d(u, Tu) + d(v, Tv)\} = r.$$

This implies that

$$\delta(T(K_1), T(K_2)) = \sup_{(u,v)\in K_1\times K_2} d(Tu, Tv) \leq r.$$

It now follows from Lemma 9.3 that

$$\begin{aligned} r &< \delta(K_1, K_2) = \delta(\overline{\text{con}}(T(K_2)), \overline{\text{con}}(T(K_1))) \\ &= \delta(T(K_1), T(K_2)) \leq r, \end{aligned}$$

which is a contradiction. This completes the proof. □

If in the above theorem $A = B$, then we obtain the following existence and uniqueness fixed point result in reflexive and Busemann convex spaces.

Corollary 9.6 *Let (X, d) be a reflexive and Busemann convex space and A be a nonempty bounded closed and convex subset of X. Assume that $T : A \to A$ is a Kannan nonexpansive mapping such that for each closed and convex subset C of A which is T-invariant and* $\text{diam}(C) > 0$, *we have*

$$\inf\{d(x, Tx) : x \in C\} < \text{diam}(C).$$

Then T has a unique fixed point.

9.6 More on Minimal Invariant Pairs for Strongly Cyclic Relatively Kannan Nonexpansive Mappings

In the latest section of this chapter, we obtain more conclusions related to minimal invariant pairs for the class of strongly cyclic relatively Kannan nonexpansive mappings.

We begin with the following existence result of approximate best proximity point sequences for aforesaid mappings.

Lemma 9.7 *Let (X, d) be a reflexive and Busemann convex space and (A, B) be a nonempty closed and convex pair of subsets of X such that A is bounded. Let $T : A \cup B \to A \cup B$ be a strongly cyclic relatively Kannan nonexpansive mapping. Then there exists a sequence $\{y_n\}$ in B such that*

$$d(y_n, Ty_n) \to \text{dist}(A, B).$$

Proof Suppose that $(K_1, K_2) \in \Sigma_T$ and consider a fixed element $(x^*, y^*) \in K_1 \times K_2$ so that $d(x^*, y^*) = \text{dist}(A, B)(= \text{dist}(K_1, K_2))$. For any $\alpha \in [0, \frac{1}{2})$, define a mapping $T_\alpha : K_1 \cup K_2 \to K_1 \cup K_2$ by

$$T_\alpha(x) = \begin{cases} (1-2\alpha)y^* \oplus 2\alpha Tx, & \text{if } x \in K_1, \\ (1-2\alpha)x^* \oplus 2\alpha Tx, & \text{if } x \in K_2. \end{cases} \tag{9.13}$$

Since T is cyclic and (K_1, K_2) is a convex pair, T_α is also cyclic on $K_1 \cup K_2$. Besides, for any $(x, y) \in K_1 \times K_2$, we have

$$\begin{aligned} d(T_\alpha x, T_\alpha y) &= d((1-2\alpha)y^* \oplus 2\alpha Tx, (1-2\alpha)x^* \oplus 2\alpha Ty) \\ &\le 2\alpha d(Tx, Ty) + (1-2\alpha)d(x^*, y^*) \\ &\le 2\alpha \min\{d(x, Tx), d(y, Ty)\} + (1-2\alpha)\text{dist}(A, B) \\ &\le \alpha\{d(x, Tx) + d(y, Ty)\} + (1-2\alpha)\text{dist}(A, B), \end{aligned}$$

which ensures that T_α is a cyclic Kannan contraction for all $\alpha \in [0, \frac{1}{2})$ and so by Corollary 9.3 it has a best proximity point such as $y_\alpha \in K_2$, that is, $d(y_\alpha, T_\alpha(y_\alpha)) = \text{dist}(A, B)$. We now have

$$\begin{aligned} \text{dist}(A, B) &\le d(y_\alpha, T(y_\alpha)) \\ &\le d(y_\alpha, T_\alpha(y_\alpha)) + d(T_\alpha(y_\alpha), T(y_\alpha)) \\ &= \text{dist}(A, B) + d((1-2\alpha)x^* \oplus 2\alpha T(y_\alpha), T(y_\alpha)) \\ &\le \text{dist}(A, B) + (1-2\alpha)d(x^*, T(y_\alpha)) \\ &\le \text{dist}(A, B) + (1-2\alpha)\text{diam}(A). \end{aligned}$$

Letting $\alpha \to \frac{1}{2}^-$, since A is bounded, we obtain

$$d(y_\alpha, Ty_\alpha) \to \text{dist}(A, B),$$

and hence the lemma follows. This completes the proof. □

We recall that a subset A of a metric space (X, d) is said to be *boundedly compact* if every sequence in A has a convergent subsequence.

The next corollaries conclude, immediately.

Corollary 9.7 *Under the assumptions of Lemma 9.7, if moreover B is boundedly compact and $T|_B$ is continuous, then T has a best proximity point.*

Corollary 9.8 *Let (X, d) be a reflexive and Busemann convex space and let A be a nonempty bounded closed and convex subset of X. Let $T : A \to A$ be a strongly Kannan nonexpansive mapping, that is,*

$$d(Tx, Ty) \le \min\{d(x, Tx), d(y, Ty)\}, \quad \forall x, y \in A.$$

Then T has an approximate fixed point sequence, i.e., there exists a sequence $\{x_n\}$ in A such that $d(x_n, Tx_n) \to 0$.

To state the main result of this section, we introduce the following concept.

Definition 9.20 Let (A, B) be a nonempty pair in a metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic mapping. We say that the pair (A, B) is *T-proximal compact* if, for any approximate sequence $\{x_n\}$ in A, there exists a subsequence $\{x_{n_k}\}$ such that the sequence $\{(x_{n_k}, Tx_{n_k})\}$ is convergent in $A \times B$.

For example, if (A, B) is boundedly compact, then, for every cyclic mapping T defined on $A \cup B$, the pair (A, B) is T-proximal compact.

Theorem 9.14 (Compare with Theorem 4.5 of [19]) *Let (X, d) be a reflexive and Busemann convex space and (A, B) be a nonempty closed and convex pair of subsets of X such that A is bounded, (B, A) is T-proximal compact and satisfies the property* UC. *Let $T : A \cup B \to A \cup B$ be a strongly cyclic relatively Kannan nonexpansive mapping. Assume that $(K_1, K_2) \in \Sigma_T$ and $\{y_n\}$ is an approximate best proximity point sequence in the set K_2. Then, for any $(x^*, y^*) \in K_1 \times K_2$ with $d(x^*, y^*) =$* dist(A, B)*, we have*

$$\max\{\limsup_{n\to\infty} d(x^*, y_n), \limsup_{n\to\infty} d(Ty_n, y^*)\} = \delta(K_1, K_2).$$

Proof Suppose the contrary. Then there exist a point $(p, q) \in K_1 \times K_2$ with $d(p, q) = \text{dist}(A, B)$ and $r > 0$ with $r < \delta(K_1, K_2)$ such that

$$\max\{\limsup_{n\to\infty} d(p, y_n), \limsup_{n\to\infty} d(Ty_n, q)\} \le r.$$

Since T is a strongly cyclic relatively Kannan nonexpansive mapping,

$$d(Ty_n, T^2y_n) \le \min\{d(y_n, Ty_n), d(Ty_n, T^2y_n)\} \to \text{dist}(A, B).$$

In view of the fact that (B, A) satisfies the property UC, $\lim_{n\to\infty} d(y_n, T^2y_n) \to 0$. Put

$$\mathscr{L}_1 := \{x \in K_1 : \limsup_{n\to\infty} d(x, y_n) \le r\}, \quad \mathscr{L}_2 := \{y \in K_2 : \limsup_{n\to\infty} d(Ty_n, y) \le r\}.$$

Then $(p, q) \in \mathscr{L}_1 \times \mathscr{L}_2$ which ensures that $\text{dist}(\mathscr{L}_1, \mathscr{L}_2) = \text{dist}(A, B)$. Also, $(\mathscr{L}_1, \mathscr{L}_2)$ is closed. Moreover, if $u_1, u_2 \in \mathscr{L}_1$, then, by the fact that X is a Busemann convex space, for all $t \in [0, 1]$, we have

$$\limsup_{n\to\infty} d(tu_1 \oplus (1-t)u_2, y_n) \le \limsup_{n\to\infty} [td(u_1, y_n) + (1-t)d(u_2, y_n)] \le r,$$

which implies that $tu_1 \oplus (1-t)u_2 \in \mathscr{L}_1$ and so $\mathscr{L}_1$ is convex. Similarly, $\mathscr{L}_2$ is also convex. Now, assume that $x \in \mathscr{L}_1$. Then we have

$$\limsup_{n\to\infty} d(Ty_n, Tx) \le \limsup_{n\to\infty} \min\{d(y_n, Ty_n), d(x, Tx)\} = \mathrm{dist}(A, B) \le r,$$

which deduces that $Tx \in \mathscr{L}_2$. Thus $T(\mathscr{L}_1) \subseteq \mathscr{L}_2$. On the other hand, if $y \in \mathscr{L}_2$, then we have

$$\begin{aligned}
\limsup_{n\to\infty} d(Ty, y_n) &\le \limsup_{n\to\infty} [d(Ty, T^2y_n) + d(T^2y_n, y_n)\} \\
&= \limsup_{n\to\infty} d(Ty, T^2y_n) \\
&\le \limsup_{n\to\infty} \min\{d(y, Ty), d(Ty_n, T^2y_n)\} \\
&= \mathrm{dist}(A, B) \le r,
\end{aligned}$$

that is, $Ty \in \mathscr{L}_1$ and so $T(\mathscr{L}_2) \subseteq \mathscr{L}_1$ which ensures that T is cyclic on $\mathscr{L}_1 \cup \mathscr{L}_2$. Thereby, $(\mathscr{L}_1, \mathscr{L}_2)$ is a nonempty bounded closed convex and T-invariant pair with $\mathrm{dist}(\mathscr{L}_1, \mathscr{L}_2) = \mathrm{dist}(A, B)$. Minimality of (K_1, K_2) deduces that $K_1 = \mathscr{L}_1$ and $K_2 = \mathscr{L}_2$. Since (B, A) is T-proximal compact and the sequence $\{y_n\}$ is an approximate sequence in B, there exists a point $(z, w) \in K_1 \times K_2$ for which $y_{n_k} \to w$ and $Ty_{n_k} \to z$, where $\{y_{n_k}\}$ is a subsequence of the sequence $\{y_n\}$. In this case, $d(z, w) = \mathrm{dist}(A, B)$. Now, for all $(x, y) \in K_1 \times K_2$, we have

$$d(x, w) \le \limsup_{n\to\infty} d(x, y_n) \le r, \quad d(z, y) \le \limsup_{n\to\infty} d(Ty_n, y) \le r,$$

Therefore, we have $\delta_w(K_1) = \sup_{x\in K_1} d(x, w) \le r$ and $\delta_z(K_2) = \sup_{y\in K_2} d(z, y) \le r$ and thus

$$\max\{\delta_z(K_2), \delta_w(K_1)\} \le r < \delta(K_1, K_2),$$

which concludes that $(z, w) \in K_1 \times K_2$ does not contain a diametral point which is a contradiction with Lemma 9.5. This completes the proof. □

Theorem 9.15 *Under the conditions of Theorem 9.14 if, in addition, $y_n \to q \in K_2$, then T has a best proximity point.*

Proof Since (K_1, K_2) is proximinal, there exists a point $p \in K_1$ such that $d(p, q) = \mathrm{dist}(A, B)$. It now follows from Theorem 9.14 that

$$\max\{\limsup_{n\to\infty} d(p, y_n), \limsup_{n\to\infty} d(Ty_n, q)\} = \delta(K_1, K_2).$$

Moreover, we have

$$\limsup_{n\to\infty} d(Ty_n, q) \le \limsup_{n\to\infty} [d(Ty_n, y_n) + d(y_n, q)] = \mathrm{dist}(A, B).$$

Hence we have

$$
\begin{aligned}
\delta(K_1, K_2) &= \max\{\limsup_{n\to\infty} d(p, y_n), \limsup_{n\to\infty} d(Ty_n, q)\} \\
&\leq \max\{\limsup_{n\to\infty}[d(p, q) + d(q, y_n)], \limsup_{n\to\infty} d(Ty_n, q)\} \\
&= \operatorname{dist}(A, B)
\end{aligned}
$$

and the result follows. This completes the proof. □

The following result is the counterpart of Goebel–Karlovitz lemma [23] for strongly Kannan nonexpansive mappings.

Corollary 9.9 *Let (X, d) be a reflexive and Busemann convex space and A be a nonempty bounded closed and convex subset of X. Let $T : A \to A$ be a strongly Kannan nonexpansive mapping. Assume that K is a subset of A which is minimal with respect to being nonempty, closed, convex and T-invariant, and let $\{x_n\}$ be an approximate fixed point sequence in K. Then*

$$\lim_{n\to\infty} d(x^*, x_n) = \lim_{n\to\infty} d(x^*, Tx_n) = \operatorname{diam}(K), \quad \forall x^* \in K.$$

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Chapter 10
On Monotone Mappings in Modular Function Spaces

M. R. Alfuraidan, M. A. Khamsi, and W. M. Kozlowski

Abstract Because of its many diverse applications, fixed point theory has been a flourishing area of mathematical research for decades. Banach's formulation of the contraction mapping principle in the early twentieth century signaled the advent of an intense interest in the metric related aspects of the theory. The metric fixed point theory in modular function spaces is closely related to the metric theory, in that it provides modular equivalents of norm and metric concepts. Modular spaces are extensions of the classical Lebesgue and Orlicz spaces, and in many instances, conditions cast in this framework are more natural and more easily verified than their metric analogs. In this chapter, we study the existence and construction of fixed points for monotone nonexpansive mappings acting in modular functions spaces equipped with a partial order or a graph structure.

Keywords Di-graph · Fibonacci-Mann iteration · Fixed point · Krasnoselskii-Mann iteration · Modular function spaces · Modular uniform convexity · Monotone mappings

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10.1 Introduction

In this chapter, we study the existence and construction of fixed points for monotone nonexpansive mappings acting in modular functions spaces equipped with a partial order or a graph structure. Modular function spaces generalize many classes of function spaces including L^p, l^p, Orlicz, Musielak–Oricz, Orlicz–Lorentz, and several others. We refer the reader to the books by Kozlowski [22] and Musielak [28] and to the papers [20, 21] for the foundations of the theory of modular function spaces. The fixed point theory in such spaces, already initiated in [22], has been extensively investigated since the seminal 1990 paper by Khamsi, Kozlowski, and Reich [18]. The current status of the theory has been comprehensively treated in Kozlowski's two survey papers [23, 24] and in the 2015 book by Khamsi and Kozlowski [17].

The fixed point theory for contractive and nonexpansive mappings defined in Banach spaces has been extensively developed since the mid 1960s. The fixed point theory has been then extended to general metric spaces and independently to modular function spaces. We refer the interested reader to [17, Chap. 2] or to any standard textbook on metric fixed point theory, e.g., [11, 14].

In recent years, a new research stream has emerged. This new research is focused on dealing with fixed point theorems in metric spaces equipped with a partial order. Ran and Reurings [31] initiated this direction in relation to a class of matrix equations. The study of these matrix equations is motivated by applications including stochastic filtering, control theory, and dynamic programming, see the paper by El-Sayed and Ran [10]. Nieto and Rodriguez-Lopez [29] improved Ran and Reurings fixed point theorem and used similar arguments to find periodic solutions for a class of differential equations. In [12], Jachymski provided a more unifying approach to these extensions by equipping metric spaces with graphs rather than with partial orders. Khamsi and Khan in [13] used this approach to prove the convergence of the Krasnoselskii-Ishikawa iteration process to fixed points of a monotone nonexpansive mappings acting in L_1, i.e., mappings that are both monotone and nonexpansive on comparable (in the sense of partial order) elements. This direction has been further developed by Bachar and Khamsi [5] for considering common approximate fixed point theorems for monotone nonexpansive semigroups in Banach spaces. Dehaish and Khamsi proved in [7] analogues of Browder and Göhde fixed point theorems for monotone nonexpansive mappings acting in uniformly convex hyperbolic spaces and uniformly convex in every direction Banach spaces. The fixed point results of Ran and Reurings have been extended by Alfuraidan, Bachar, and Khamsi [2] to pointwise monotone contractions acting in modular function spaces. Dehaish and Khamsi in their 2016 paper [8] proved the existence of fixed points of monotone ρ-nonexpansive mappings in ρ-uniformly convex modular function spaces. The graph-focused research direction, initiated in [12], has been further developed by Alfuraidan and Khamsi [3], who proved a series of fixed point results for monotone G-nonexpansive mappings acting in a hyperbolic space with a graph. Also, Alfuraidan in [1] proved the existence of fixed points for G monotone pointwise contraction mappings in Banach spaces equipped with a graph. For more information on the results in the monotone fixed point theory, the reader is referred to a recent survey article by Bachar and Khamsi [6].

In this chapter, we demonstrate the existence of fixed points for monotone ρ-nonexpansive mappings acting in a convex and ρ-a.e. compact subset of a modular function space L_ρ equipped with a partial order. Our results and methods, inspired by [13], differ from the fixed point theorems proved in [8] because we do not assume uniform convexity of ρ. Also, we introduce for the first time methods of Γ_ρ nonexpansive mappings into the setting of modular function spaces, hence opening a new interesting research direction. It is important to keep in mind that the convergence results demonstrated in our paper define algorithms, which can be numerically implemented.

10.2 Preliminaries

For the basic definitions and properties of modular function spaces, we refer the readers to the books [17, 22].

Throughout this chapter, Δ stands for a nonempty set, Σ a nontrivial σ-algebra of subsets of Δ, $\mathscr{P}$ a δ-ring of subsets of Δ such that $P \cap S \in \mathscr{P}$ for any $P \in \mathscr{P}$ and $S \in \Sigma$. We will assume that there exists an increasing sequence $\{\Delta_n\} \subset \mathscr{P}$ such that $\Delta = \bigcup \Delta_n$. $\mathscr{M}_\infty$ will stand for the space of all extended measurable functions $f : \Delta \to [-\infty, \infty]$ for which there exists $\{g_n\} \subset \mathscr{E}$, with $|g_n| \leq |f|$ and $g_n(t) \to f(t)$, for all $t \in \Delta$, where $\mathscr{E}$ stands for the vector space of simple functions whose supports are in $\mathscr{P}$.

Definition 10.1 ([17, 22]) A convex and even function $\rho : \mathscr{M}_\infty \to [0, \infty]$ is called a *regular modular* if

(a) $\rho(f) = 0$ implies $f = 0\ \rho - a.e.$;

(b) $|f(t)| \leq |g(t)|$ for all $t \in \Delta$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathscr{M}_\infty$ (we will say that ρ is monotone);

(c) $|f_n(t)| \uparrow |f(t)|$ for all $t \in \Delta$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathscr{M}_\infty$ (ρ has the Fatou property).

Recall that a subset $A \in \Sigma$ is said to be *ρ-null* if $\rho(g\mathbf{1_A}) = \mathbf{0}$ for any $g \in \mathscr{E}$ and a property holds *ρ-almost everywhere* (shortly, ρ-*a.e.*) if the exceptional set is ρ-null. The notation $\mathbf{1_A}$ denotes the characteristic function of the set A. Consider the set

$$\mathscr{M} = \{f \in \mathscr{M}_\infty; |f(t)| < \infty\ \rho - a.e\}.$$

The *modular function space* L_ρ is defined as follows:

$$L_\rho = \{f \in \mathscr{M}; \rho(\lambda f) \to 0\ as\ \lambda \to 0\}.$$

In the following theorem, we recall some of the properties of modular spaces that will be used throughout this chapter:

Theorem 10.1 ([17, 22]) *Let ρ be a convex regular modular.*

(1) *If $\rho(\beta f_n) \to 0$ for some $\beta > 0$, then there exists a subsequence $\{f_{\psi(n)}\}$ such that $f_{\psi(n)} \to 0\ \rho - a.e.$*

(2) *If* $f_n \to f$ $\rho - a.e.$, *then* $\rho(g) \leq \liminf_{n\to\infty} \rho(g_n)$.

(3) *Recall that* ρ *satisfies the* Δ_2*-type condition if*

$$\omega(\alpha) = \sup \left\{ \frac{\rho(\alpha g)}{\rho(g)},\ 0 < \rho(g) < \infty \right\} < \infty$$

for any $\alpha \in [0, +\infty)$. *If* ρ *satisfies the* Δ_2*-type condition, then we have* $\rho(\alpha f_n) \to 0$ *if and only if* $\rho(\alpha f_n) \to 0$, *for any* $\alpha > 0$.

The following definition is needed since it connects the metric properties with its modular version:

Definition 10.2 ([17, 22]) Let ρ be a convex regular modular.
(1) $\{g_n\}$ is said to ρ-*converge* to g if $\lim_{n\to\infty} \rho(g_n - g) = 0$.
(2) A sequence $\{g_n\}$ is called a ρ-*Cauchy sequence* if $\lim_{n,m\to\infty} \rho(g_n - g_m) = 0$.
(3) A subset C of L_ρ is said to be ρ-*closed* if, for any sequence $\{g_n\}$ in C ρ-convergent to g implies that $g \in C$.
(4) A subset C of L_ρ is called ρ-*bounded* if its ρ-diameter $\sup\{\rho(g - h);\ g,\ h \in C\} < \infty$.

Note that despite the fact that ρ does not satisfy the triangle inequality in general, the ρ limit is unique and ρ-convergence may not imply ρ-Cauchy behavior. But it is interesting to know that ρ-balls $B_\rho(x, r) = \{y \in L_\rho;\ \rho(x - y) \leq r\}$ are ρ-closed and any ρ-Cauchy sequence in L_ρ is ρ-convergent, i.e., L_ρ is ρ-complete [17, 22].

Using Theorem 10.1, we get the following result:

Theorem 10.2 *Let* ρ *be a convex regular modular and* $\{g_n\} \subset L_\rho$ *be a sequence which* ρ*-converges to* g. *Then the following hold:*
(1) *If* $\{g_n\}$ *is monotone increasing, i.e.,* $g_n \leq g_{n+1}$ ρ*-a.e., for any* $n \geq 1$, *then* $g_n \leq g$ ρ*-a.e. for any* $n \geq 1$.
(2) *If* $\{g_n\}$ *is monotone decreasing, i.e.,* $g_{n+1} \leq g_n$ ρ*-a.e. for any* $n \geq 1$, *then* $g \leq g_n$ ρ*-a.e. for any* $n \geq 1$.

Next, we discuss a property called uniform convexity which plays an important part in metric fixed point theory.

Definition 10.3 ([17]) Let ρ be a convex regular modular.
(1) Let $r > 0$ and $\varepsilon > 0$. Define

$$\delta_\rho(r, \varepsilon) = \inf \left\{ 1 - \frac{1}{r}\, \rho\Big(\frac{f+g}{2}\Big) : (f, g) \in D(r, \varepsilon) \right\},$$

where

$$D(r, \varepsilon) = \{(f, g) \in L_\rho \times L_\rho : \rho(f) \leq r,\ \rho(g) \leq r,\ \rho(f - g) \geq \varepsilon r\}.$$

Then ρ is said to be *uniformly convex* (UC) if, for every $R > 0$ and $\varepsilon > 0$,

$$\delta_\rho(R, \varepsilon) > 0.$$

(2) ρ is said to be (UUC) if, for every $s \geq 0$ and $\varepsilon > 0$, there exists $\eta(s, \varepsilon) > 0$ such that

$$\delta_\rho(R, \varepsilon) > \eta(s, \varepsilon) > 0$$

for $R > s$.

(3) ρ is said to be *strictly convex* (SC) if, for any $g, h \in L_\rho$ with

$$\rho(g) = \rho(h), \quad \rho(\alpha\, g + (1-\alpha)h) = \alpha\, \rho(g) + (1-\alpha)\rho(h)$$

for some $\alpha \in (0, 1)$, we have $f = g$.

Note that the uniform convexity of ρ easily implies (SC).

Remark 10.1 It is known that, under suitable assumptions, the uniform convexity of the modular in Orlicz spaces is satisfied if the Orlicz function is uniformly convex [19, 33]. Examples of Orlicz functions that do not satisfy the Δ_2 condition and are uniformly convex are: $\varphi_1(t) = e^{|t|} - |t| - 1$ and $\varphi_2(t) = e^{t^2} - 1$ [26, 27].

Modular functions which are uniformly convex enjoy a property similar to reflexivity in Banach spaces.

Theorem 10.3 ([17, 19]) *Let ρ be a (UUC) convex regular modular. Then L_ρ has property (R), i.e., every nonincreasing sequence $\{C_n\}$ of nonempty, ρ-bounded, ρ-closed, convex subsets of L_ρ has nonempty intersection.*

Remark 10.2 Let ρ be a (UUC) convex regular modular and K be a ρ-bounded convex ρ-closed nonempty subset of L_ρ. Let $\{f_n\} \subset K$ be a monotone increasing sequence. Since order intervals in L_ρ are convex and ρ-closed, then the property (R) implies

$$\bigcap_{n \geq 1} \Big\{ f \in K : f_n \leq f \quad \rho - a.e. \Big\} \neq \varnothing.$$

In other words, there exists $f \in K$ such that $f_n \leq f$ ρ-a.e. for any $n \geq 1$. A similar conclusion holds for decreasing sequences.

The following lemma is useful throughout this chapter:

Lemma 10.1 ([16]) *Let ρ be a (UUC) convex regular modular. If there exists $R > 0$ and $\alpha \in (0, 1)$ with*

$$\limsup_{n\to\infty} \rho(f_n) \leq R, \quad \limsup_{n\to\infty} \rho(g_n) \leq R,$$

and

$$\lim_{n\to\infty} \rho(\alpha\, f_n + (1-\alpha)\, g_n) = R,$$

then $\lim\limits_{n\to\infty} \rho(f_n - g_n) \to 0$ holds.

The concept of ρ-type functions will prove to be an important tool dealing with the existence of fixed points.

Definition 10.4 Let ρ be a convex regular modular and C be a nonempty subset of L_ρ. A function $\tau : C \to [0, \infty]$ is called a *ρ-type* if there exists a sequence $\{g_m\}$ of elements of L_ρ such that

$$\tau(f) = \limsup_{m\to\infty} \rho(g_m - f)$$

for any $f \in C$. Let τ be a type. A sequence $\{f_n\}$ is called a *minimizing sequence* of τ in C if

$$\lim_{n\to\infty} \tau(f_n) = \inf\{\tau(f) : f \in C\}.$$

Now, we have the following amazing result about ρ-type functions in modular function spaces:

Lemma 10.2 ([16]) *Let ρ be a (UUC) convex regular modular. Let K be a ρ-bounded ρ-closed convex nonempty subset of L_ρ. Then any minimizing sequence of any ρ-type defined on K is ρ-convergent. Its limit is independent of the minimizing sequence.*

Before we finish this section, let us give the modular definitions of monotone Lipschitzian mappings. The definitions are straightforward generalizations of their norm and metric equivalents.

Definition 10.5 Let ρ be a convex regular modular. Let K be nonempty subset of L_ρ. A mapping $T : K \to K$ is said to be *monotone* if $T(f) \le T(g)$ ρ-a.e. whenever $f \le g$ ρ-a.e. for any $f, g \in K$. Moreover T is called:

(1) *monotone ρ-contraction* if T is monotone and there exists $K \in [0, 1)$ such that

$$\rho(T(g) - T(h)) \le K\ \rho(g - h)$$

for any g and h in K such that $g \le h$ ρ-a.e.

(2) *monotone ρ-nonexpansive* if T is monotone and

$$\rho(T(g) - T(h)) \le \rho(g - h)$$

for any g and h in K such that $g \le h$ ρ-a.e.

(3) *monotone asymptotically ρ-nonexpansive* if T is monotone and there exists $\{k_n\} \subset [1, +\infty)$ such that $\lim\limits_{n\to\infty} k_n = 1$ and

$$\rho(T^n(g) - T^n(h)) \le k_n\ \rho(g - h)$$

for any $g, h \in K$ such that $g \le h$ ρ-a.e. and $n \ge 1$.

(4) $f \in K$ is called a *fixed point* of T if $T(f) = f$.

10.3 Monotone Nonexpansive Mappings

Note that a monotone ρ-nonexpansive mapping does not have to be nonexpansive or even continuous. Hence, standard fixed point theorems as presented in [17] cannot be applied. Throughout this section, we drop ρ-a.e. whenever $f \leq g$ for any $f, g \in L_\rho$.

Definition 10.6 Let C be a nonempty convex subset of L_ρ and $T : C \to C$ be a monotone mapping. Fix $\lambda \in (0, 1)$ and $f_0 \in C$. The *Krasnoselskii-Mann iteration sequence* $\{f_n\}$ of elements $\{f_n\}$ in C is defined by

$$f_{n+1} = \lambda f_n + (1 - \lambda)T(f_n) \tag{10.1}$$

for each $n \in \mathbb{N}$.

Let us start with the following lemma which extends the L^1-result obtained in [13] to modular function spaces:

Lemma 10.3 *Let ρ be a convex regular modular, C be a nonempty convex subset of L_ρ and $T : C \to C$ be a monotone mapping. Fix $\lambda \in (0, 1)$ and $f_0 \in C$. Assume that the Krasnoselskii-Mann iteration sequence $\{f_n\}$ of elements $\{f_n\}$ is generated by (10.1) for any $f_0 \in C$. If $f_0 \leq T(f_0)$, then*

$$f_n \leq f_{n+1} \leq T(f_n) \leq T(f_{n+1}) \tag{10.2}$$

for each $n \in \mathbb{N}$.

Proof Let us note that, if $f \leq g$, then

$$f \leq \lambda f + (1 - \lambda)g \leq g. \tag{10.3}$$

Next, let us prove, by induction, that

$$f_n \leq T(f_n) \tag{10.4}$$

for each $n \in \mathbb{N}$. For $n = 1$, (10.4) follows from the assumption $f_0 \leq T(f_0)$. Assume now that $f_n \leq T(f_n)$. Observe that, using the inductive assumption, we get

$$f_n = \lambda f_n + (1 - \lambda) f_n \leq \lambda f_n + (1 - \lambda)T(f_n), \tag{10.5}$$

that is

$$f_n \leq f_{n+1} \leq T(f_n). \tag{10.6}$$

Since T is monotone, it follows that

$$T(f_n) \leq T(f_{n+1}), \tag{10.7}$$

which, combining with (10.6), gives us the required inequality $f_{n+1} \leq T(f_{n+1})$. This proves (10.4), which in turn allows us to conclude that

$$f_n = \lambda f_n + (1-\lambda) f_n \leq \lambda f_n + (1-\lambda) T(f_n) = f_{n+1}. \tag{10.8}$$

Combining (10.8) with (10.6) and (10.7), we get (10.2). This completes the proof.

Note that, if $T(f_0) \leq f_0$ holds, then we will have

$$T(f_{n+1}) \leq T(f_n) \leq f_{n+1} \leq f_n$$

for each $n \in \mathbb{N}$.

Theorem 10.4 *Let ρ be a convex regular modular. Let $C \subset E_\rho$ be nonempty, convex, ρ-bounded, compact with respect to the convergence ρ-a.e. Assume that $T : C \to C$ is a monotone ρ-nonexpansive mapping and also there exists $f_0 \in C$ such that f_0 and $T(f_0)$ are comparable. Let $\{f_n\}$ be the Krasnoselskii-Mann sequence defined by the formula (10.1) generated by $f_0 \in C$, T and $\lambda \in (0, 1)$. Then there exists f in C comparable to f_0 such that f is a fixed point of T. Moreover, $\|f_n - f\|_\rho \to 0$ and $f_n \to f$ ρ-a.e.*

Proof Without any loss of generality we can assume that $f_0 \leq T(f_0)$. From the ρ-a.e. compactness of C, it follows that there exists a subsequence $\{f_{n_k}\}$ and $f \in C$ such that $f_{n_k} \to f$ ρ-a.e.

Now, we claim that $f_n \to f$ ρ-a.e. Indeed, since $\{f_n\}$ is nondecreasing from Lemma 10.3, we get $f_{n_k} \leq f$ for any $n_k \geq 1$. This implies that $f_n \leq f$, for any $n \geq 1$. Let $g \in C$ be a ρ-a.e. limit of another subsequence of $\{f_n\}$. Then, for the same reason, we have $f_n \leq g$ for any $n \geq 1$. Using the properties of the partial order and the ρ-a.e. convergence, we obtain that $f \leq g$. Obviously, this implies that $f = g$. Therefore, $\{f_n\}$ has one ρ-a.e. cluster limit which implies $\{f_n\}$ ρ-a.e. converges to f. Moreover, we have $0 \leq f - f_n \leq f - f_0 \in E_\rho$ and then, by the Lebesgue dominated convergence theorem [22, Theorem 2.4.7], we have $\|f_n - f\|_\rho \to 0$, which implies that

$$\rho(\beta(f_n - f)) \to 0$$

for every $\beta > 0$. Since T is monotone ρ-nonexpansive and $f_n \leq f$, for any $n \geq 1$, we get

$$\rho(T(f_n) - T(f)) \leq \rho(f_n - f),$$

which implies $\rho(T(f_n) - T(f)) \to 0$. Since

$$f_{n+1} - f = \lambda(f_n - f) + (1-\lambda)(T(f_n) - f),$$

we get

$$\|T(f_n) - f\|_\rho = \frac{1}{1-\lambda}\|f_{n+1} - f - \lambda(f_n - f)\|_\rho$$
$$\leq \frac{1}{1-\lambda}(\|f_{n+1} - f\|_\rho + \lambda\|(f_n - f)\|_\rho)$$

for any $n \geq 1$ and hence $\|T(f_n) - f\|_\rho \to 0$. So, we have

$$\rho(T(f_n) - f) \to 0.$$

Thus, $\{T(f_n)\}$ ρ-converges to f and $T(f)$. Therefore, by the uniqueness of the ρ-limit, we have $T(f) = f$. This completes the proof.

As a consequence of our result we get the following corollary:

Corollary 10.1 *Let ρ be a convex regular modular. Let $C \subset E_\rho$ be nonempty, convex, ρ-bounded, compact with respect to the convergence ρ-a.e. Assume that $0 \in C$ and $T : C \to C$ is monotone ρ-nonexpansive mapping such that $0 \leq T(0)$ (resp., $T(0) \leq 0$). Let $\{f_n\}$ be the Krasnoselskii-Mann sequence defined by (10.1) with $f_0 = 0$. Then there exists $f \geq 0$ (resp., $f \leq 0$) such that f is a fixed point of T. Moreover, we have $\|f_n - f\|_\rho \to 0$ and $f_n \to f$ ρ-a.e.*

Next, we discuss the existence of fixed points of monotone asymptotically ρ-nonexpansive mappings.

First, recall that a map T is said to be *ρ-continuous* if $\{g_n\}$ ρ-converges to g implies $\{T(g_n)\}$ ρ-converges to $T(g)$. A similar result for asymptotically nonexpansive mappings in modular function spaces may be found in [16].

Theorem 10.5 *Let ρ be a (UUC) convex regular modular, K be a ρ-bounded ρ-closed convex nonempty subset of L_ρ and $T : K \to K$ be a ρ-continuous monotone asymptotically nonexpansive mapping. Assume there exists $f_0 \in K$ such that $f_0 \leq T(f_0)$ (resp., $T(f_0) \leq f_0$). Then T has a fixed point f such that $f_0 \leq f$ (resp., $f \leq f_0$).*

Proof Without loss of generality, assume $f_0 \leq T(f_0)$. Since T is monotone, the sequence $\{T^n(f_0)\}$ is monotone increasing. Remark 10.2 implies that $K_\infty = \{f \in K : f_n \leq f\}$ is not empty. Consider the ρ-type function $\varphi : K_\infty \to [0, +\infty)$ defined by

$$\varphi(h) = \limsup_{n\to\infty} \rho(T^n(f_0) - h)$$

for any $h \in K_\infty$. Let $\varphi_0 = \inf\{\varphi(h) : h \in K_\infty\}$ and $\{g_n\} \subset K_\infty$ be a minimizing sequence of φ. Lemma 10.2 implies that $\{g_n\}$ ρ-converges to $g \in K_\infty$.

Let us prove that g is a fixed point of T. First, notice that $\varphi(T^m(h)) \leq k_m\, \varphi(h)$ for any $h \in K_\infty$ and $m \geq 1$. In particular, we have $\varphi(T^m(g_n)) \leq k_m\, \varphi(g_n)$ for any $n, m \geq 1$. Clearly, the sequence $\{T^{n+p}(g_n)\}$ is a minimizing sequence in K_∞ for any $p \in \mathbb{N}$. Again, Lemma 10.2 forces $\{T^{n+p}(g_n)\}$ to ρ-converge to g for any $p \in$

$\mathbb{N}$. Since T is ρ-continuous and $\{T^n(g_n)\}$ is ρ-convergent to g, $\{T^{n+1}(g_n)\}$ is ρ-convergent to $T(g)$ and g. Since the ρ-limit of any ρ-convergent sequence is unique, we must have $T(g) = g$. Since $g \in K_\infty$, we have $f_0 \leq g$. This completes the proof.

Next, we discuss another iteration which will generate an approximate fixed point sequence of a monotone asymptotically ρ-nonexpansive mapping in modular function spaces.

Definition 10.7 ([32]) Let ρ be a convex regular modular, K be a convex nonempty subset of L_ρ and $T : K \to K$ be a mapping. Fix $f_0 \in K$ and $\alpha \in [0, 1]$. The *modified Mann iteration* is the sequence $\{f_n\}$ defined by

$$f_{n+1} = \alpha\, T^n(f_n) + (1-\alpha) f_n \tag{10.9}$$

for each $n \in \mathbb{N}$.

Now, we start by proving some Lemmas which will be helpful.

Lemma 10.4 *Let ρ be a convex regular modular, K be a convex nonempty subset of L_ρ and $T : K \to K$ be a mapping. Let $f_0 \in K$ be such that $f_0 \leq T(f_0)$ (resp., $T(f_0) \leq f_0$). Let $\{t_n\}$ be a sequence in $[0, 1]$ and consider the modified Mann iteration sequence $\{f_n\}$ generated by f_0 and $\alpha \in [0, 1]$. Let f be a fixed point of T such that $f_0 \leq f$ (resp., $f \leq f_0$). Then we have:*
(1) $f_0 \leq f_n \leq f$ (*resp.*, $f \leq f_n \leq f_0$).
(2) $T^n(f_0) \leq T^n(f_n) \leq f$ (*resp.*, $f \leq T^n(f_n) \leq T^n(f_0)$) *for each* $n \in \mathbb{N}$.

Proof Without loss of generality, assume $f_0 \leq T(f_0)$. Since T is monotone and f is a fixed point of T, we get (2) from (1).

Let us prove, by induction, (1). Indeed, we have $f_0 \leq T(f_0) \leq T(f) = f$ since T is monotone. Using the convexity of the order intervals, we conclude that $f_0 \leq f_1 \leq f$. Assume that $f_0 \leq f_n \leq f$. Again, using the monotonicity of T, we get

$$f_0 \leq T^n(f_0) \leq T^n(f_n) \leq T^n(f) = f,$$

which implies, by the convexity of the order intervals, that $f_0 \leq f_{n+1} \leq f$. By induction, we conclude that $f_0 \leq f_n \leq f$ for each $n \in \mathbb{N}$. This completes the proof.

Lemma 10.5 *Let ρ be a convex regular modular and K be a convex and ρ-bounded nonempty subset of L_ρ. Assume that the map $T : K \to K$ is monotone asymptotic nonexpansive with the associated constants $\{k_n\}$ satisfy the condition*

$$\sum_{n=1}^{\infty}(k_n - 1) < \infty.$$

Let $f_0 \in K$ be such that $f_0 \leq T(f_0)$ (resp., $T(f_0) \leq f_0$) and consider the modified Mann iteration sequence $\{f_n\}$ generated by f_0 and $\alpha \in (0, 1)$. Let f be a fixed point of T such that $f_0 \leq f$ (resp., $f \leq f_0$). Then $\lim_{n\to\infty} \rho(f_n - f)$ exists.

Proof Without loss of generality, assume that $f_0 \leq T(f_0)$. From the definition of $\{f_n\}$, we have

$$\begin{aligned}\rho(f_{n+1} - f) &\leq \alpha\ \rho(T^n(f_n) - f) + (1-\alpha)\ \rho(f_n - f)\\ &= \alpha\ \rho(T^n(f_n) - T^n(f)) + (1-\alpha)\ \rho(f_n - f)\end{aligned}$$

for any $n \geq 1$. Since T is monotone asymptotic nonexpansive, we get

$$\rho(f_{n+1} - f) \leq k_n\ \rho(f_n - f) = (k_n - 1)\ \rho(f_n - f) + \rho(f_n - f)$$

for any $n \geq 1$ and hence

$$\rho(f_{n+1} - f) - \rho(f_n - f) \leq (k_n - 1)\ \delta_\rho(K)$$

for any $n \in \mathbb{N}$, where $\delta_\rho(K) = \sup\{\rho(h - g) : h, g \in K\}$ is the ρ-diameter of K. Hence we have

$$\rho(f_{n+m} - f) - \rho(f_n - f) \leq \delta_\rho(K) \sum_{i=0}^{m-1} (k_{n+i} - 1)$$

for any $n, m \geq 1$. If we let $m \to \infty$, then we get

$$\limsup_{m\to\infty} \rho(f_m - f) \leq \rho(f_n - f) + \delta_\rho(K) \sum_{i=n}^{\infty} (k_i - 1)$$

for any $n \geq 1$. Next, if we let $n \to \infty$, then we get

$$\begin{aligned}\limsup_{m\to\infty} \rho(f_m - f) &\leq \liminf_{n\to\infty} \rho(f_n - f) + \delta_\rho(K) \liminf_{n\to\infty} \sum_{i=n}^{\infty} (k_i - 1)\\ &= \liminf_{n\to\infty} \rho(f_n - f).\end{aligned}$$

Therefore, we have

$$\limsup_{m\to\infty} \rho(f_m - f) = \liminf_{n\to\infty} \rho(f_n - f).$$

This completes the proof.

The next result shows that the sequence generated by the modified Mann iteration almost provides a fixed point. Similar results for such iteration in modular function spaces may be found in [9, 25].

Theorem 10.6 *Let ρ be a (UUC) convex regular modular, $K \subset L_\rho$ be a ρ-bounded ρ-closed convex nonempty subset and $T : K \to K$ be a ρ-continuous monotone asymptotically nonexpansive mapping with the associated constants $\{k_n\}$ satisfy the* *condition*

$$\sum_{n=1}^{\infty}(k_n - 1) < \infty.$$

Let $f_0 \in K$ *be such that* $f_0 \leq T(f_0)$ *(resp.* $T(f_0) \leq f_0$*) and let* $\alpha \in (0, 1)$. *Consider the modified Mann iteration sequence* $\{f_n\}$ *generated by* f_0 *and* α. *Then either* $\{f_n\}$ *is* ρ*-convergent or*

$$\lim_{n\to\infty} \rho(f_n - T^n(f_n)) = 0.$$

Proof Assume that $\{f_n\}$ is not ρ-convergent. Let us prove that

$$\lim_{n\to\infty} \rho(f_n - T^n(f_n)) = 0.$$

Without loss of generality, we assume $f_0 \leq T(f_0)$. Using Theorem 10.5, there exists a fixed point f of T such that $f_0 \leq f$. Using Lemma 10.5, we conclude that $\lim\limits_{n\to\infty} \rho(f_n - f)$ exists. Set $R = \lim\limits_{n\to\infty} \rho(f_n - f)$. Since $\{f_n\}$ is not ρ-convergent, we have $R > 0$ and

$$\begin{aligned}\limsup_{n\to\infty}\ \rho(T^n(f_n) - f) &= \limsup_{n\to\infty}\ \rho(T^n(f_n) - T^n(f))\\ &\leq \limsup_{n\to\infty} k_n\ \rho(f_n - f)\\ &= R.\end{aligned}$$

On the other hand, we have

$$\rho(f_{n+1} - f) \leq \alpha\ \rho(T^n(f_n) - f) + (1 - \alpha)\ \rho(f_n - f)$$

for any $n \geq 1$. Let $\mathscr{U}$ be a nontrivial ultrafilter over $\mathbb{N}$. Then we have

$$R = \lim_{\mathscr{U}} \rho(f_{n+1} - f) \leq \alpha\ \lim_{\mathscr{U}} \rho(T^n(f_n) - f) + (1 - \alpha)\ R.$$

Since $\alpha \neq 0$, we get $\lim\limits_{\mathscr{U}} \rho(T^n(f_n) - f) \geq R$. Hence, we have

$$R \leq \liminf_{n\to\infty} \rho(T^n(f_n) - f) \leq \lim_{\mathscr{U}} \rho(T^n(f_n) - f) \leq \limsup_{n\to\infty} \rho(T^n(f_n) - f) \leq R.$$

So $\lim\limits_{n\to\infty} \rho(T^n(f_n) - f) = R$. Using Lemma 10.1, we conclude that

$$\lim_{n\to\infty} \rho(f_n - T^n(f_n)) = 0,$$

which completes the proof.

Remark 10.3 In fact, the modified Mann sequence $\{f_n\}$ is an approximate fixed point sequence of T under suitable conditions. Indeed, assume ρ satisfies the Δ_2-type condition and T is uniformly ρ-Lipschitzian, i.e., there exists $\ell > 0$ such that

$$\rho(T^n(g) - T^n(h)) \leq \ell\ \rho(g - h)$$

for any $g, h \in K$ and $n \geq 1$. In this case, we have

$$\lim_{n\to\infty} \rho(f_n - T^m(f_n)) = 0$$

for any $m \geq 1$. Indeed, note that

$$\begin{aligned}\rho(f_n - T(f_n)) &\leq \omega(2)\ \rho\left(\frac{f_n - T(f_n)}{2}\right)\\ &\leq \omega(2)\ \rho(f_n - T^n(f_n)) + \omega(2)\ \rho(T^n(f_n) - T(f_n))\\ &\leq \omega(2)\ \rho(f_n - T^n(f_n)) + \omega(2)\ \ell\ \rho(T^{n-1}(f_n) - f_n)\end{aligned}$$

for any $n \geq 2$. From

$$\begin{aligned}\rho(T^{n-1}(f_n) - f_n) &\leq \omega(2)\ \rho\left(\frac{T^{n-1}(f_n) - f_n}{2}\right)\\ &\leq \omega(2)\rho(T^{n-1}(f_n) - T^{n-1}(f_{n-1})) + \omega(2)\rho(T^{n-1}(f_{n-1}) - f_n)\\ &\leq \omega(2)\ell\ \rho(f_n - f_{n-1}) + \omega(2)\rho(T^{n-1}(f_{n-1}) - f_n),\end{aligned}$$

$$\rho(f_n - f_{n-1}) = \alpha\rho(f_{n-1} - T^{n-1}(f_{n-1}))$$

and

$$\rho(T^{n-1}(f_{n-1}) - f_n) = (1-\alpha)\ \rho(f_{n-1} - T^{n-1}(f_{n-1})),$$

we get

$$\rho(T^{n-1}(f_n) - f_n) \leq \omega(2)(\ell + 1)\ \rho(f_{n-1} - T^{n-1}(f_{n-1})).$$

Hence, we have

$$\rho(f_n - T(f_n)) \leq \omega(2)\ \rho(f_n - T^n(f_n)) + \omega^2(2)\ (\ell + 1)^2\ \rho(f_{n-1} - T^{n-1}(f_{n-1}))$$

for any $n \geq 2$. Since $\lim_{n\to\infty} \rho(f_n - T^n(f_n)) = 0$, we conclude that

$$\lim_{n\to\infty} \rho(f_n - T(f_n)) = 0,$$

i.e., $\{f_n\}$ is an approximate fixed point sequence of T.

Finally, let us fix $m \geq 1$. Then we have

$$\begin{aligned}\rho(f_n - T^m(f_n)) &\leq \omega(m) \sum_{k=0}^{m-1} \rho(T^k(f_n) - T^{k+1}(f_n))\\ &\leq \omega(m) \sum_{k=0}^{m-1} \ell\, \rho(f_n - T(f_n)),\end{aligned}$$

which implies that $\rho(f_n - T^m(f_n)) \leq\ m\ \ell\ \omega(m)\ \rho(f_n - T(f_n))$ for any $m \geq 1$. Clearly, this implies

$$\lim_{n\to\infty} \rho(f_n - T^m(f_n)) = 0$$

for any $m \geq 1$.

When dealing with the modified Mann iteration sequence, it is unknown if the sequence is monotone like the sequence generated by Krasnoselskii-Mann iteration. For this reason, the authors in [4] introduced a new iteration which uses the Fibonacci sequence $\{\phi(n)\}$ defined by

$$\phi(0) = \phi(1) = 1, \quad \phi(n+1) = \phi(n) + \phi(n-1)$$

for any $n \geq 1$.

Definition 10.8 Let C be a nonempty convex subset of L_ρ and $T : C \to C$ be a monotone mapping. Fix $\lambda_n \in [0, 1]$ and $h_0 \in C$. The *Fibonacci-Mann iteration sequence* $\{h_n\}$ of elements in C is defined by

$$h_{n+1} = \alpha_n\, T^{\phi(n)}(h_n) + (1 - \alpha_n)\, h_n \tag{FMI}$$

for any $n \in \mathbb{N}$.

This new iteration scheme allowed the authors of [4] to prove the main results of Schu [32] for monotone asymptotically ρ-nonexpansive mappings defined in uniformly convex Banach spaces. A surprising fact since this class of mappings may fail to be continuous. Next, we discuss the behavior of the iteration (FMI) which will generate an approximate fixed point of monotone asymptotically ρ-nonexpansive mapping in modular function spaces.

The proof of the following lemma uses solely the partial order and is similar to the original proof done in [4] in the context of Banach spaces:

Lemma 10.6 ([4]) *Let ρ be convex regular modular, C be a convex nonempty subset of L_ρ and $T : C \to C$ be a monotone mapping. Let $h_0 \in C$ be such that $h_0 \leq T(h_0)$ (resp., $T(h_0) \leq h_0$). Let $\{\alpha_n\} \subset [0, 1]$ and consider the* (FMI) *sequence $\{h_n\}$ generated by h_0 and $\{\alpha_n\}$. Let $f \in C$ be a fixed point of T such that $h_0 \leq f$ (resp., $f \leq h_0$). Then we have the following:*

(1) $h_0 \leq h_n \leq h_{n+1} \leq T^{\phi(n)}(h_n) \leq f$ *(resp., $f \leq T^{\phi(n)}(h_n) \leq h_{n+1} \leq h_n \leq h_0$).*

(2) $h_0 \leq T^{\phi(n)}(h_0) \leq T^{\phi(n)}(h_n) \leq f$ *(resp., $f \leq T^{\phi(n)}(h_n) \leq T^{\phi(n)}(h_0) \leq h_0$) for any $n \in \mathbb{N}$.*

The next lemma is crucial throughout:

Lemma 10.7 *Let ρ be convex regular modular and C be a ρ-bounded and convex nonempty subset of L_ρ. Assume that $T : C \to C$ is monotone asymptotically ρ-nonexpansive mapping with the Lipschitz constants $\{k_n\}$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $h_0 \in C$ be such that $h_0 \leq T(h_0)$ (resp., $T(h_0) \leq h_0$). Let $\{\alpha_n\} \subset [0, 1]$ and consider the* (FMI) *sequence $\{h_n\}$ generated by h_0 and $\{\alpha_n\}$. Let $f \in C$ be a fixed point of T such that $h_0 \leq f$ (resp., $f \leq h_0$). Then $\lim_{n\to\infty} \rho(h_n - f)$ exists.*

Proof Without loss of generality, assume that $h_0 \leq T(h_0)$. Note that, since C is ρ-bounded, we must have $\limsup_{m\to\infty} \rho(h_m - f) \leq \delta_\rho(C) < +\infty$. From the definition of $\{h_n\}$, we have

$$\begin{aligned}
\rho(h_{n+1} - f) &\leq \alpha_n\, \rho(T^{\phi(n)}(h_n) - f) + (1 - \alpha_n)\, \rho(h_n - f)\\
&= \alpha_n\, \rho(T^{\phi(n)}(h_n) - T^{\phi(n)}(f)) + (1 - \alpha_n)\, \rho(h_n - f)\\
&\leq \alpha_n\, k_{\phi(n)}\, \rho(h_n - f) + (1 - \alpha_n)\, \rho(h_n - f)\\
&= \alpha_n\, (k_{\phi(n)} - 1)\, \rho(h_n - f) + \rho(h_n - f)\\
&\leq (k_{\phi(n)} - 1)\, \rho(h_n - f) + \rho(h_n - f)
\end{aligned}$$

for any $n \in \mathbb{N}$, where we used the fact that f is a fixed point of T, the definition of the Lipschitz constants $\{k_n\}$ and $\{\alpha_n\} \subset [0, 1]$. Hence we have

$$\rho(h_{n+1} - f) - \rho(h_n - f) \leq (k_{\phi(n)} - 1)\, \delta_\rho(C)$$

for any $n \in \mathbb{N}$, which implies

$$\rho(h_{n+m} - f) - \rho(h_n - f) \leq \delta_\rho(C) \sum_{i=0}^{m} (k_{\phi(n+i)} - 1)$$

for any $n, m \geq 1$. Let us rewrite this inequality as

$$\rho(h_{n+m} - f) \leq \rho(h_n - f) + \delta_\rho(C) \sum_{i=0}^{m} (k_{\phi(n+i)} - 1)$$

for any $n, m \geq 1$.

Next, we let $m \to \infty$ to get

$$\begin{aligned}
\limsup_{m\to\infty} \rho(h_m - f) &\leq \rho(h_n - f) + \delta_\rho(C) \sum_{i=n}^{\infty} (k_{\phi(i)} - 1)\\
&\leq \rho(h_n - f) + \delta_\rho(C) \sum_{i=n}^{\infty} (k_i - 1)
\end{aligned}$$

for any $n \geq 1$.

Finally, if we let $n \to \infty$, then we have

$$\begin{aligned}\limsup_{m\to\infty} \rho(h_m - f) &\leq \liminf_{n\to\infty} \rho(h_n - f) + \delta_\rho(C) \liminf_{n\to\infty} \sum_{i=n}^{\infty}(k_i - 1) \\ &= \liminf_{n\to\infty} \rho(h_n - f)\end{aligned}$$

since the series $\sum_{n=1}^{\infty}(k_n - 1)$ is convergent. Therefore, we have

$$\limsup_{m\to\infty} \rho(h_m - f) = \liminf_{n\to\infty} \rho(h_n - f).$$

This completes the proof.

The next result shows that the sequence generated by (FMI) has an approximate fixed point behavior which is crucial throughout

Proposition 10.1 *Let ρ be (UUC) convex regular modular and $C \subset L_\rho$ be a ρ-bounded and ρ-closed convex nonempty subset. Let $T : C \to C$ be a monotone asymptotically ρ-nonexpansive mapping with the associated constants $\{k_n\}$ satisfying the condition*

$$\sum_{n=1}^{\infty}(k_n - 1) < \infty.$$

Let $h_0 \in C$ be such that $h_0 \leq T(h_0)$ (resp., $T(h_0) \leq h_0$) and $f \in C$ be a fixed point of T such that $h_0 \leq f$ (resp., $f \leq h_0$). Let $\{\alpha_n\} \subset [a, b]$ with $0 < a \leq b < 1$ and consider the (FMI) *sequence $\{h_n\}$ generated by h_0 and $\{\alpha_n\}$. Then we have*

$$\lim_{n\to\infty} \rho(h_n - T^{\phi(n)}(h_n)) = 0.$$

Proof Without loss of generality, we assume $h_0 \leq T(h_0)$. From Lemma 10.4, we know that $h_n \leq h_{n+1} \leq f$. Using the properties of the modular ρ, we get

$$\rho(f - h_{n+1}) \leq \rho(f - h_n)$$

for any $n \in \mathbb{N}$, i.e., $\{\rho(f - h_n)\}$ is a decreasing sequence of positive numbers. Hence $R = \lim_{n\to\infty} \rho(h_n - f)$ exists. Assume that $R = 0$, i.e., $\{h_n\}$ ρ-converges to f. From Lemma 10.4, we get $h_n \leq T^{\phi(n)}(h_n) \leq f$, which implies

$$\rho(T^{\phi(n)}(h_n) - h_n) \leq \rho(f - h_n)$$

for any $n \in \mathbb{N}$. Hence we have

$$\lim_{n\to\infty} \rho(T^{\phi(n)}(h_n) - h_n) = 0.$$

Next, we assume $R > 0$. Then we have

$$\begin{aligned}\limsup_{n\to\infty} \rho(T^{\phi(n)}(h_n) - f) &= \limsup_{n\to\infty} \rho(T^{\phi(n)}(h_n) - T^{\phi(n)}(f))\\ &\leq \limsup_{n\to\infty} k_{\phi(n)}\, \rho(h_n - f)\\ &= R\end{aligned}$$

since $\lim\limits_{n\to\infty} k_n = 1$ and f is a fixed point of T.

On the other hand, we have

$$\rho(h_{n+1} - f) \leq \alpha_n\, \rho(T^{\phi(n)}(h_n) - f) + (1-\alpha_n)\, \rho(h_n - f)$$

for any $n \geq 1$. Let $\mathcal{U}$ be a nontrivial ultrafilter over $\mathbb{N}$. Then we have

$$R = \lim_{\mathcal{U}} \rho(h_{n+1} - f) \leq \alpha\, \lim_{\mathcal{U}} \rho(T^{\phi(n)}(h_n) - f) + (1-\alpha)\, R$$

with $\lim\limits_{\mathcal{U}} \alpha_n = \alpha \in [a, b]$. Since $\alpha \neq 0$, we get $\lim\limits_{\mathcal{U}} \rho(T^{\phi(n)}(h_n) - f) \geq R$. Since $\mathcal{U}$ was an arbitrary ultrafilter, we get

$$R \leq \liminf_{n\to\infty} \rho(T^{\phi(n)}(h_n) - f) \leq \limsup_{n\to\infty} \rho(T^{\phi(n)}(h_n) - f) \leq R.$$

So $\lim\limits_{n\to\infty} \rho(T^{\phi(n)}(h_n) - f) = R$. Since

$$\lim_{n\to\infty} \rho\Big(\alpha_n\, T^{\phi(n)}(h_n) + (1-\alpha_n)\, h_n - f\Big) = \lim_{n\to\infty} \rho(h_{n+1} - f) = R$$

and ρ is (UUC), by using Lemma 10.1, we conclude that

$$\lim_{n\to\infty} \rho(h_n - T^{\phi(n)}(h_n)) = 0,$$

which completes the proof.

Recall that the map $T : C \to C$ is said to be *ρ-compact* if $\{T(f_n)\}$ has a ρ-convergent subsequence for any sequence $\{f_n\}$ in C. The following result is the monotone version of Theorem 2.2 of [32].

Theorem 10.7 *Let ρ be (UUC) convex regular modular and $C \subset L_\rho$ be a ρ-bounded and ρ-closed convex nonempty subset of L_ρ. Let $T : C \to C$ be a monotone asymptotically ρ-nonexpansive mapping with the Lipschitz constants $\{k_n\}$. Assume that T^m is ρ-compact for some $m \geq 1$. Let $h_0 \in C$ be such that $h_0 \leq T(h_0)$ (resp., $T(h_0) \leq h_0$). Let $\{\alpha_n\} \subset [a, 1]$ with $0 < a \leq 1$ and consider the* (FMI) *sequence*

$\{h_n\}$ generated by h_0 and $\{\alpha_n\}$. Then $\{h_n\}$ ρ-converges to a fixed point f of T such that $h_0 \leq f$ (resp. $f \leq h_0$).

Proof Without loss of generality, we assume $h_0 \leq T(h_0)$. Since T is monotone, the sequence $\{T^n(h_0)\}$ is monotone increasing. Since T^m is ρ-compact, there exists a subsequence $\{T^{\varphi(n)}(h_0)\}$ which ρ-converges to $f \in C$.

Let us show that $\{T^n(h_0)\}$ ρ-converges to f and f is a fixed point of T. Using the properties of the ρ-a.e. partial order, we have $T^n(h_0) \leq f$ for any $n \in \mathbb{N}$. In particular, we have

$$T^{\varphi(n)}(h_0) \leq T^{\varphi(n)+1}(h_0) \leq f$$

for any $n \in \mathbb{N}$. Using the properties of the modular ρ, we get

$$\rho(f - T^{\varphi(n)+1}(h_0)) \leq \rho(f - T^{\varphi(n)}(h_0))$$

for any $n \in \mathbb{N}$. This implies that $\{T^{\varphi(n)+1}(h_0)\}$ ρ-converges to f. But we have

$$\rho(T(f) - T^{\varphi(n)+1}(h_0)) \leq k_1\ \rho(f - T^{\varphi(n)}(h_0))$$

for any $n \in \mathbb{N}$, which implies that $\{T^{\varphi(n)+1}(h_0)\}$ ρ-converges to $T(f)$ as well, which implies $T(f) = f$ from the uniqueness of the ρ-limit. It is clear from the properties of the modular ρ that $\{\rho(f - T^n(h_0))\}$ is a decreasing sequence of positive real numbers. Hence, we have

$$\lim_{n\to\infty} \rho(f - T^n(h_0)) = \lim_{n\to\infty} \rho(f - T^{\varphi(n)}(h_0)) = 0,$$

i.e., $\{T^n(h_0)\}$ ρ-converges to f.

Let us finish the proof of Theorem 10.7 by showing that $\{h_n\}$ ρ-converges to f. Since f is a fixed point of T which satisfies $h_0 \leq f$, Lemma 10.4 implies $T^{\phi(n)}(h_0) \leq T^{\phi(n)}(h_n) \leq f$, which implies

$$\rho(f - T^{\phi(n)}(h_n)) \leq \rho(f - T^{\phi(n)}(h_0))$$

for any $n \in \mathbb{N}$. Hence, $\{T^{\phi(n)}(h_n)\}$ ρ-converges to f. Since $\{h_n\}$ is monotone increasing and bounded above by f, we know that $\{\rho(f - h_n)\}$ is a decreasing sequence of positive real numbers. Hence, $\lim\limits_{n\to\infty} \rho(f - h_n) = R$ exists.

Let us prove that $R = 0$. Let $\mathcal{U}$ be a non-trivial ultrafilter over $\mathbb{N}$. Using the definition of $\{h_n\}$, we have

$$\rho(h_{n+1} - f) \leq \alpha_n\ \rho(T^{\phi(n)}(h_n) - f) + (1 - \alpha_n)\ \rho(h_n - f)$$

for any $n \in \mathbb{N}$. If we set $\lim\limits_{\mathcal{U}} \alpha_n = \alpha \in [a, 1]$, then we get

$$\lim_{\mathcal{U}} \rho(h_{n+1} - f) \leq \alpha\ \lim_{\mathcal{U}} \rho(T^{\phi(n)}(h_n) - f) + (1 - \alpha)\ \lim_{\mathcal{U}} \rho(h_n - f).$$

Since $\lim_{\mathcal{U}} \rho(h_{n+1} - f) = \lim_{\mathcal{U}} \rho(h_n - f) = R$ and $\lim_{\mathcal{U}} \rho(T^n(h_n) - f) = 0$, we get $R \leq (1 - \alpha)\, R$. Since $\alpha \neq 0$, we conclude that $R = 0$, i.e., $\{h_n\}$ ρ-converges to f. This completes the proof.

Before we investigate a weaker convergence of the (FMI) sequence, we will need the following result which may be seen as similar to the classical Opial condition [30]. First, we recall that a subset C of L_ρ is *ρ-a.e.-compact* if any sequence $\{f_n\}$ in C has a ρ-a.e.-convergent subsequence and its ρ-a.e.-limit is in C.

Proposition 10.2 *Let $C \subset L_\rho$ be a ρ-a.e.-compact and ρ-bounded convex nonempty subset of L_ρ and $\{f_n\}$ be a monotone increasing (resp., decreasing) bounded sequence in C. Set $C_\infty = \{h \in C : f_n \preceq h \ (resp.,\ h \preceq f_n) \text{ for any } n \in \mathbb{N}\}$. Consider the ρ-type function $\varphi : C_\infty \to [0, +\infty)$ defined by*

$$\varphi(h) = \lim_{n\to\infty} \rho(f_n - h).$$

Then $\{f_n\}$ is ρ-a.e. convergent to $f \in C_\infty$ and

$$\varphi(f) = \inf\{\varphi(h) : h \in C_\infty\}.$$

Moreover, if ρ is (UUC), then any minimizing sequence $\{h_n\}$ of φ in C_∞ ρ-converges to f. In particular, φ has a unique minimum point in C_∞.

Proof Without loss of generality, assume that $\{f_n\}$ is monotone increasing. Since C is ρ-a.e.-compact, there exists a subsequence $\{f_{\psi(n)}\}$ which is ρ-a.e. convergent to some $f \in C$. Using Theorem 10.2, we conclude that $f_n \leq f$ for any $n \in \mathbb{N}$. Hence, $f \in C_\infty$ which implies that C_∞ is nonempty. Let $h \in C_\infty$. Then the sequence $\{\rho(h - f_n)\}$ is a decreasing sequence of finite positive numbers since C is ρ-bounded. Hence, $\varphi(h) = \lim_{n\to\infty} \rho(f_n - h)$ exists. As we saw before, there exists a subsequence $\{f_{\psi(n)}\}$ of $\{f_n\}$ which ρ-a.e.-converges to $f \in C_\infty$.

Let us prove that $\{f_n\}$ ρ-a.e.-converges to f. Indeed, for any $n \geq \psi(0)$, there exists a unique $k_n \in \mathbb{N}$ such that $\psi(k_n) \leq n < \psi(k_n + 1)$. Clearly, we have $k_n \to \infty$ when $n \to \infty$. Moreover, we have $f_{\psi(k_n)} \leq f_n \leq f$ for any $n \in \mathbb{N}$. Since $\{f_{\psi(k_n)}\}$ ρ-a.e. converges to f, we conclude that $\{f_n\}$ also ρ-a.e. converges to f.

Next, let $h \in C_\infty$. Then we must have $f_n \leq f \leq h$, which implies

$$\rho(f - f_n) \leq \rho(h - f_n)$$

for any $n \in \mathbb{N}$. Hence $\varphi(f) \leq \varphi(h)$, i.e.,

$$\varphi(f) = \inf\{\varphi(h);\ h \in C_\infty\}.$$

The last part of Proposition 10.2 is a classical result which may be found in [16]. This completes the proof.

Now, we are ready to state a modular monotone version of Theorem 2.1 of [32].

Theorem 10.8 *Let ρ be (UUC) convex regular modular and $C \subset L_\rho$ be a ρ-a.e.-compact and ρ-bounded convex nonempty subset of L_ρ. Let $T : C \to C$ be a monotone asymptotically ρ-nonexpansive mapping with the Lipschitz constants $\{k_n\}$. Assume that $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$. Let $h_0 \in C$ be such that h_0 and $T(h_0)$ are ρ-a.e.-comparable. Let $\{\alpha_n\} \subset [a, b]$ with $0 < a \leq b < 1$ and consider the* (FMI) *sequence $\{h_n\}$ generated by h_0 and $\{\alpha_n\}$. Then $\{h_n\}$ is ρ-a.e.-convergent and its ρ-a.e.-limit is a fixed point of T ρ-a.e.-comparable to h_0.*

Proof Without loss of generality, assume that $h_0 \preceq T(h_0)$. In this case, we know that $\{T^n(h_0)\}$ is monotone increasing. Proposition 10.2 implies that $\{T^n(h_0)\}$ is ρ-a.e.-convergent to $f \in C_\infty$ with

$$C_\infty = \{h \in C : T^n(h_0) \leq h \ \text{ for any } n \in \mathbb{N}\}.$$

Since ρ is (UUC), f is the unique minimum point of the ρ-type $\varphi : C_\infty \to [0, +\infty)$ defined by

$$\varphi(h) = \lim_{n\to\infty} \rho(T^n(h_0) - h).$$

By the definition of $\{k_n\}$, we get

$$\varphi(f) \leq \varphi(T^m(f)) \leq k_m\, \varphi(f)$$

for any $m \geq 1$. Hence, $\{T^m(f)\}$ is a minimizing sequence of φ since $\lim\limits_{m\to\infty} k_m = 1$. Using Proposition 10.2, we conclude that $\{T^m(f)\}$ ρ-converges to f. Note that, since $T^n(h_0) \leq f$, we get $T^{n+1}(h_0) \leq T(f)$ for any $n \in \mathbb{N}$, which implies $f \leq T(f)$ for any $n \in \mathbb{N}$. Hence, $\{T^m(f)\}$ is monotone increasing and ρ-converges to f, which implies $T^m(f) \leq f$. Hence $T(f) = f$ holds, i.e., f is a fixed point of T. Using Lemma 10.4, we have

$$T^{\phi(n)}(h_0) \leq T^{\phi(n)}(h_n) \leq f$$

for any $n \in \mathbb{N}$, which implies that $\{T^{\phi(n)}(h_n)\}$ also ρ-a.e.-converges to f. Proposition 10.6 implies

$$\lim_{n\to\infty} \rho(h_n - T^{\phi(n)}(h_n)) = 0.$$

Using the properties of ρ-convergence and ρ-a.e.-convergence [17], there exists a sequence of increasing integers $\{j_n\}$ such that $\{h_{j_n} - T^{\phi(j_n)}(h_{j_n})\}$ ρ-a.e.-converges to 0. Therefore, we must have $\{h_{j_n}\}$ ρ-a.e.-converges to f. Since $\{h_n\}$ is monotone increasing and $h_n \leq f$, we conclude that $\{h_n\}$ ρ-a.e.-converges to f. This completes the proof of Theorem 10.8 by noting that f is a fixed point of T and $h_0 \leq f$.

10.4 Γ_ρ Nonexpansive Mappings

Now, we start this section with the graph theory terminology for the modular space mapping which will be studied throughout.

Let $C \subseteq L_\rho$. Let Γ be a digraph with the elements of C as its vertices and set of arcs $A(\Gamma)$ such that $(f, f) \in A(\Gamma)$ for any $f \in V(\Gamma)$. We also assume that Γ is simple, i.e., Γ has no multi-arcs. Therefore, we can detect Γ with the pair $(V(\Gamma), A(\Gamma))$. We use $\widetilde{\Gamma}$ to denote the graph attained from Γ by disregarding the direction of arcs.

Definition 10.9 A *Γ-interval* is any of the subsets $[s, \rightarrow) = \{f \in C : (s, f) \in A(\Gamma)\}$ and $(\leftarrow, t] = \{f \in C : (f, t) \in A(\Gamma)\}$ for any $s, t \in C$.

Definition 10.10 Let C be a nonempty subset of L_ρ. A mapping $T : C \to C$ is called:

(1) *Γ-monotone* if $(T(f), T(g)) \in A(\Gamma)$ whenever $(f, g) \in A(\Gamma)$ for any $f, g \in C$.

(2) *Γ_ρ-nonexpansive* if T is Γ-monotone and

$$\rho(T(f) - T(g)) \leq \rho(f - g)$$

whenever $(f, g) \in A(\Gamma)$ for any $f, g \in C$.

Definition 10.11 We say that the triple (C, ρ, Γ) has the *property* (Λ) if, for any sequence $\{f_n\}_{n\in\mathbb{N}}$ in C such that $(f_n, f_{n+1}) \in A(\Gamma)$ for any $n \geq 0$, a subsequence $\{f_{n_k}\}$ ρ-converges to f, then $(f_n, f) \in A(\Gamma)$ for all $n \geq 0$.

The following definition is introduced as an analog to the Lebesgue dominated convergence theorem:

Definition 10.12 We say that Γ satisfies the *Lebesgue dominated convergence property* if, for any $f_n,\ f \in E_\rho$ such that $(f_n, f_{n+1}) \in A(\Gamma)$, $(f_n, f) \in A(\Gamma)$ for all $n \geq 0$ and $f_n \to f$ ρ-a.e., then $\|f_n - f\|_\rho \to 0$.

Let $T : C \to C$ be a Γ_ρ-nonexpansive mapping. Throughout this section, we assume that Γ-intervals are convex and ρ-a.e. closed. Fix $\eta \in (0, 1)$. Let $f_0 \in C$ be such that $(f_0, T(f_0)) \in A(\Gamma)$. Define $f_1 = \eta f_0 + (1 - \eta)T(f_0)$. Since the set $[f_0, T(f_0)] = [f_0, \rightarrow) \bigcap (\leftarrow, T(f_0)]$ is convex, it follows that $f_1 \in [f_0, T(f_0)]$, i.e. (f_0, f_1) and $(f_1, T(f_0))$ are in $A(\Gamma)$. Since T is Γ_ρ-nonexpansive, we get $(T(f_0), T(f_1)) \in A(\Gamma)$ and

$$\rho(T(f_1) - T(f_0)) \leq \rho(f_1 - f_0).$$

By induction, we build a sequence $\{f_n\}$ in C such that the following hold for each $n \geq 0$:

(a) $f_{n+1} = \eta f_n + (1 - \eta)T(f_n)$;

(b) (f_n, f_{n+1}), $(f_n, T(f_n))$ and $(T(f_n), T(f_{n+1}))$ are in $A(\Gamma)$;

(c) $\rho(T(f_{n+1}) - T(f_n)) \leq \rho(f_{n+1} - f_n)$.

Such sequence $\{f_n\}$ is the Krasnoselskii-Mann iteration defined by 10.1.

Next, we present the graphical version of our results.

Theorem 10.9 *Let $C \subset E_\rho$ be nonempty, convex, ρ-bounded, compact with respect to the convergence ρ-a.e. Assume that (C, ρ, Γ) has the property (Λ) and Γ satisfies the Lebesgue dominated convergence property. Let $T : C \to C$ be a Γ_ρ-nonexpansive mapping. Assume there exists $f_0 \in C$ such that $(f_0, T(f_0)) \in E(\widetilde{\Gamma})$. Let $\{f_n\}$ be the Krasnoselskii-Mann sequence defined by (10.1) generated by f_0, T and $\eta \in (0, 1)$. Then there exists $f \in C$ with $(f, f_0) \in E(\widetilde{\Gamma})$ such that f is a fixed point of T. Moreover, $\|f_n - f\|_\rho \to 0$ and $f_n \to f$ ρ-a.e.*

Proof Without loss of any generality, we assume that $(f_0, T(f_0)) \in A(\Gamma)$. From the ρ-a.e. compactness of C it follows that there exists a subsequence $\{f_{n_k}\}$ and $f \in C$ such that $f_{n_k} \to f$ ρ-a.e.

Now, we claim that $f_n \to f$ ρ-a.e. By the properties (b) and (Λ), we get $(f_n, f) \in A(\Gamma)$, for any $n \geq 1$. Let $g \in C$ be a ρ-a.e. limit of another subsequence of $\{f_n\}$. Then, for the same reason, we have $(f_n, g) \in A(\Gamma)$ for any $n \geq 1$. Using the properties of the Γ-intervals, we obtain $(f, g) \in A(\Gamma)$. By similarity, we obtain $(g, f) \in A(\Gamma)$. Since Γ has no multi-arcs, then $f = g$. Therefore, $\{f_n\}$ has one ρ-a.e. cluster limit which implies $\{f_n\}$ ρ-a.e. converges to f. Since Γ satisfies the Lebesgue dominated convergence property, $\|f_n - f\|_\rho \to 0$, which implies that

$$\rho(\beta(f_n - f)) \to 0$$

for every $\beta > 0$. Since T is Γ_ρ-nonexpansive and $(f_n, f) \in A(\Gamma)$ for any $n \geq 0$, we get

$$\rho(T(f_n) - T(f)) \leq \rho(f_n - f),$$

which implies $\rho(T(f_n) - T(f)) \to 0$. Since

$$f_{n+1} - f = \eta(f_n - f) + (1 - \eta)(T(f_n) - f),$$

we get

$$\begin{aligned} \|T(f_n) - f\|_\rho &= \frac{1}{1-\eta}\|f_{n+1} - f - \eta(f_n - f)\|_\rho \\ &\leq \frac{1}{1-\eta}(\|f_{n+1} - f\|_\rho + \eta\|(f_n - f)\|_\rho) \end{aligned}$$

for any $n \geq 1$. Hence $\|T(f_n) - f\|_\rho \to 0$. So, we have

$$\rho(T(f_n) - f) \to 0.$$

Thus, $\{T(f_n)\}$ ρ-converges to f and $T(f)$. Therefore, by the uniqueness of the ρ-limit, we have $T(f) = f$. This completes the proof.

As a consequence of our result we obtain the following corollary:

Corollary 10.2 *Let $C \subset E_\rho$ be nonempty, convex, ρ-bounded, compact with respect to the convergence ρ-a.e. Assume that (C, ρ, Γ) has the property (Λ) and Γ satisfies the Lebesgue dominated convergence property. Let $T : C \to C$ be a Γ_ρ-nonexpansive mapping. Assume that $0 \in C$ such that $(0, T(0)) \in A(\Gamma)$. Let $\{f_n\}$ be the Krasnoselskii-Mann sequence defined by (10.1) generated by $f_0 = 0$, T and $\eta \in (0, 1)$. Then there exists $f \in C$ with $(0, f) \in A(\Gamma)$ such that f is a fixed point of T. Moreover, $\|f_n - f\|_\rho \to 0$ and $f_n \to f$ ρ-a.e.*

10.5 Synopsis

Before we close this chapter, we would like to invite the readers to join us in the journey taking all of us from a well-known base of classical fixed point theory in Banach and metric spaces to the world of the theory of fixed points of mappings defined in a class of spaces of measurable functions, i.e., modular function spaces. The results and methods of fixed point theory, applied to spaces of measurable functions, have been used extensively in the field of integral and differential equations. Indeed, since the 1930s, many prominent mathematicians like Orlicz and Birnbaum recognized that using the methods of L^p-norms alone created many complications and in some cases did not allow to solve some nonpower type integral equations. They considered spaces of functions with some growth properties different from the power type growth control provided by the L^p-norms. Using the apparatus of the modular function spaces, one can go much further: the operator itself is used for the construction of a function modular and hence of a space in which this operator has required properties. These techniques together with relevant modular function space fixed point theorems can be efficiently applied to solving many mathematical problems. As we said before, the aim of this chapter was to familiarize the readers with the main concepts and results of the fixed point theory for monotone Lipschitzian mappings defined within modular function spaces, as well as to encourage them to use these results in the course of their research activities.

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Chapter 11
Contributions to Fixed Point Theory of Fuzzy Contractive Mappings

Dhananjay Gopal

Abstract This chapter deals with a concise study of fixed point theorems concerning fuzzy contractive type mappings in fuzzy metric spaces. The results presented in detail were selected to illustrate the direction of research in the field from the past six decades up to most recent contribution in the subject.

Keywords Fuzzy metric space · Fuzzy contractive mappings · Fixed point · t-norms

11.1 Introduction

The concept of fuzzy set was initially investigated by Zadeh [52] as a new way to represent vagueness in everyday life. Subsequently it was developed extensively by many authors and used in almost all branches of science and engineering including mathematics. A fuzzy set on a set can be defined by assigning to each element of a set a value in [0, 1] representing its grade of membership in the fuzzy set. Mathematically, a fuzzy set A of X is a mapping $A : X \to [0, 1]$.

Several notions of fuzzy metric spaces have been introduced and discussed in different directions by various authors (see [34, 50]). In 1975, Kramosil and Michalek [26] gave a notion of fuzzy metric space which could be considered as a reformulation, in the fuzzy context, of the notion of probabilistic metric spaces due to Menger. However, in order to strengthen and to obtain a Housedroff topology (the so-called M-topology), George and Veeramani [14, 15] imposed some stronger conditions on fuzzy metric and modify the concept of fuzzy metric due to Kranmosil and Michalek.

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11.2 Fuzzy Metric Spaces

In order to define fuzzy metric spaces, we first define the following:

Definition 11.1 (Schweizer and Sklar [36]) A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a *continuous triangular norm* (*t-norm*) if the following conditions hold:

(a) $*$ is associative and commutative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0, 1]$;
(d) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0, 1]$.

Four basic examples of continuous t-norms are: $a *_1 b = \min\{a, b\}$, $a *_2 b = \frac{ab}{\max\{a,b,\lambda\}}$ for all $\lambda \in (0, 1)$, $a *_3 b = ab$, $a *_4 b = \max\{a + b - 1, 0\}$. (the Lukasiewicz t-norm, we will denote it by $*_L$). For all $a_1, a_2, \ldots, a_n \in [0, 1]$ and $n \in \mathbb{N}$, the product $a_1 * a_2 * \cdots * a_n$ will be denoted by $\prod_{i=1}^{n} a_i$.

A t-norm $*$ is said to be *positive* if $a * b > 0$ for all $a, b \in (0, 1]$.
A t-norm is said to be *nilpotent* if $a * b$ is continuous and, for each $a \in (0, 1)$, there exists $n \in \mathbb{N}$ such that $\prod_{i=1}^{n} a_i = 0$. For example, consider the Lukasiewicz t-norm for which we have $a * a * \cdots * a = 0$ for all $a \in (0, 1)$. For the details concerning t-norms we also refer [25].

Definition 11.2 (Kramosil and Michalek [26]) The triple $(X, M, *)$ is a fuzzy metric space if X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following axioms:

(KM1) $M(x, y, 0) = 0$;
(KM2) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$;
(KM3) $M(x, y, t) = M(y, x, t)$;
(KM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(KM5) The function $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous, for all $x, y, z \in X$ and $t, s > 0$.

In what follows, fuzzy metric spaces in the sense of Kramosil and Michalek [26] will be referred as KM-fuzzy metric space.

Example 11.1 ([46]) Let $X = \mathbb{R}$, the set of all real numbers. Define $a * b = ab$ for all $a, b \in [0, 1]$. For all all $x, y \in X$, $t \geq 0$, define

$$M(x, y, t) = \begin{cases} \frac{t}{t+d(x,y)}, & \text{if } x, y \in X, t > 0, \\ 0, & \text{if } x, y \in X, t = 0. \end{cases}$$

Then M is a KM-fuzzy metric on $\mathbb{R}$

Example 11.2 ([30]) Let X be a set with at least two elements. If we define the fuzzy set M by $M(x, x, t) = 1$ for all $x \in X, t > 0$, and

$$M(x, y, t) = \begin{cases} 0, & \text{if } t \leq 1, \\ 1, & \text{if } t > 1. \end{cases}$$

for all all $x \in X$, $x \neq y$, then $(X, M, *)$ is KM-fuzzy metric space under any continuous t-norm $*$.

In 1988, Grabiec [13] initiated the study of fixed point theory in fuzzy metric space and established fuzzy Banach contraction theorem and Fuzzy Edelstein contraction theorem. In order to obtain his theorems, he introduced the following notions:

Definition 11.3 ([13]) Let $(X, M, *)$ be a fuzzy metric space. Then

(i) a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be convergent to $x \in X$, if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.
(ii) a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be Cauchy (or G-Cauchy) if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for each $p \in \mathbb{N}$ and $t > 0$.

A fuzzy metric space in which every Cauchy (or G-Cauchy) sequence is convergent is called complete (or G-complete).

In [14, 46] it has been observed that the notion of G-completeness has disadvantage, it is a very strong notion of completeness, in fact, if d is the Euclidean metric in $\mathbb{R}$, then the induced fuzzy metric $(M_d, *)$ of Example 2.1 given in [46] is not G-complete. In order to strengthen and to obtain a Housedroff topology (the so-called M-topology), George and Veeramani [14, 15] imposed some stronger conditions on fuzzy metric due to Kranmosil and Michalek and gave the following concept of fuzzy metric.

Definition 11.4 (George and Veeramani [14]) The triple $(X, M, *)$ is called a *fuzzy metric space* if X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following axioms:

(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, y, t) = 1$ if and only if $x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(GV5) $M(x, y, .) : (0, \infty[\to [0, 1]$ is continuous for all $x, y, z \in X$ and $t, s > 0$.

The axiom (GV1) is justified by the authors because in the same way that a classical metric space does not take the value ∞ then M can not take the value 0. The axiom (GV2) is equivalent to the following:

$M(x, x, t) = 1$ for all $x \in X, t > 0$ and $M(x, y, t) < 1$ for all $x \neq y, t > 0$

The axiom (GV2) gives the idea that only when $x = y$ the degree of nearness of x and y is perfect, or simply 1, and then $M(x, x, t) = 1$ for each $x \in X$ and for each $t > 0$.

(we observe that the M in the Example 11.2 (above) does not satisfies axiom (GV2)). In this manner the value 0 and ∞ in the classical theory of metric space are identified with 1 and 0, respectively, in this fuzzy theory. Finally, in (GV5) the authors only assume that the variable t behave nicely, i.e., they assume that for fixed x and y, the function $t \to M(x, y, t)$ is continuous without any imposition for M as $t \to \infty$. In what follows, fuzzy metric spaces in the sense of (George and Veeramani [14]) will be referred to as GV-fuzzy metric space.

Example 11.3 Let $X = \mathbb{R}$. Define $a * b = ab$ for all $a, b \in [0, 1]$ and

$$M(x, y, t) = \left[exp\left(\frac{|x - y|}{t}\right)\right]^{-1}$$

for all $x, y \in X$ and $t \in (0, \infty)$. Then $(X, M, *)$ is a GV-fuzzy metric space.

The next example shows that every metric space induces a fuzzy metric space.

Example 11.4 Let (X, d) be a metric space. Define $a * b = ab$ for all $a, b \in [0, 1]$ and

$$M(x, y, t) = \frac{kt^n}{kt^n + md(x, y)}$$

for all $k, m, n \in N$. Then $(X, M, *)$ is a GV-fuzzy metric space. In particular, taking $k = m = n = 1$, we get

$$M(x, y, t) = \frac{t}{t + d(x, y)},$$

that is called a *standard fuzzy metric*.

George and Veeramani proved in [14, 15] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, r, t) : x \in X, 0 < r < 1, t > 0\}$, where $B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$ for all $x \in X$, $r \in (0, 1)$ and $t > 0$. If (X, d) is a metric space, then the topology generated by d coincides with the topology τ_{M_d} generated by the fuzzy metric M_d.

Definition 11.5 (George and Veeramani [14]) Let $(X, M, *)$ be fuzzy metric space. Then a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be a *Cauchy sequence* (or *M-Cauchy sequence*) if, for each $\varepsilon \in (0, 1)$ and $t > 0$, there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

A fuzzy metric space in which every Cauchy sequence (M-Cauchy sequence) is convergent is called *complete* (*M-complete*). It is called *compact* if every sequence contains a convergent subsequence.

Remark 11.1 (George and Veeramani [14]) The metric space (X, d) is complete if and only if the standard fuzzy metric space $(X, M_d, *)$ is complete.

Definition 11.6 ([22]) Let $(X, M, *)$ be a fuzzy metric space. The fuzzy metric $(M, *)$ (or the fuzzy metric space $(X, M, *)$) is said to be *non-Archimedean* or *strong* if it satisfies the following conditions:, for each $x, y, z \in X$ and $t > 0$

$$M(x, y, t) \geq M(x, z, t) * M(z, y, t).$$

11.3 Fuzzy Contractive Mappings

In order to obtain fuzzy version of classical Banach contraction theorem, Gregori and Sapena [19] introduced the following concepts:

Definition 11.7 Let $(X, M, *)$ be a fuzzy metric space. A mapping $f : X \to X$ is said to be *fuzzy contractive* if there exists $k \in (0, 1)$ such that

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k\left(\frac{1}{M(x, y, t)} - 1\right)$$

for each $x, y \in X$ and $t > 0$.

Definition 11.8 Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be *fuzzy contractive* if there exists $k \in (0, 1)$ such that

$$\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 \leq k\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right)$$

for all $t > 0, n \in \mathbb{N}$.

Recall that a sequence $\{x_n\}$ in a metric space (X, d) is said to be *contractive* if there exists $k \in (0, 1)$ such that $d(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.

Proposition 11.1 *Let $(X, M_d, *)$ be the standard fuzzy metric space induced by the metric d on X. The sequence $\{x_n\}$ in X is contractive in (X, d) iff $\{x_n\}$ is fuzzy contractive in $(X, M_d, *)$.*

Theorem 11.1 ([14]) *A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$.*

Theorem 11.2 (Fuzzy Banach contraction theorem) *Let $(X, M, *)$ be a complete fuzzy metric space (in the sense of George and Veeramani) in which fuzzy contractive sequences are Cauchy sequences. Let $f : X \to X$ be a contractive mapping being k the contractive constant. Then f has a unique fixed point.*

Proof Fix $x \in X$. Let $x_n = f^n(x)$ for each $n \in \mathbb{N}$. Then it follows that, for all $t > 0$,

$$\frac{1}{M(f(x), f^2(x), t)} - 1 \leq k\left(\frac{1}{M(x, x_1, t)} - 1\right)$$

and, by induction,

$$\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 \leq k \left(\frac{1}{M(x_n, x_{n+1}, t)} - 1 \right)$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ is a fuzzy contractive sequence. So it is a Cauchy sequence and hence x_n converges to y for some $y \in X$.

Now, we see that y is a fixed point for f. In fact, by Theorem 11.1, we have

$$\frac{1}{M(f(y), f(x_n), t)} - 1 \leq k \left(\frac{1}{M(y, x_n, t)} - 1 \right) \to 0$$

as $n \to \infty$. Then $\lim_{n\to\infty} M(f(y), f(x_n), t) = 1$ for each $t > 0$ and so $\lim_{n\to\infty} f(x_n) = f(y)$, i.e., $\lim_{n\to\infty} x_{n+1} = f(y)$ and then $f(y) = y$.

To show uniqueness, assume $f(z) = z$ for some $z \in X$. Then, for all $t > 0$, we have

$$\begin{aligned} \frac{1}{M(y, z, t)} - 1 &= \frac{1}{M(f(y), f(z), t)} - 1 \\ &\leq k \left(\frac{1}{M(y, z, t)} - 1 \right) \\ &\leq \ldots \leq k^n \left(\frac{1}{M(y, z, t)} - 1 \right) \to 0 \end{aligned}$$

as $n \to \infty$. Hence $M(y, z, t) = 1$ and then $y = z$. This completes the proof.

Now, suppose $(X, M_d, *)$ is a complete standard fuzzy metric space induced by the metric d on X. From Remark 11.1, (X, d) is complete and so, if $\{x_n\}$ is a fuzzy contractive sequence, by Proposition 11.1, it is contractive in (X, d) and hence convergent. So, from Theorem 11.2, we have the following corollary, which can be considered the fuzzy version of the classic Banach contraction theorem on complete metric space.

Corollary 11.1 *Let $(X, M_d, *)$ be a complete standard fuzzy metric space and $f : X \to X$ be a fuzzy contractive mapping. Then f has a unique fixed point.*

Theorem 11.3 (Fuzzy Banach contraction theorem) *Let $(X, M, *)$ be a G-complete fuzzy metric space (in the sense of Kramosil and Michalek) and $f : X \to X$ be a fuzzy contractive mapping. Then f has a unique fixed point.*

Proof Let $k \in (0, 1)$ and since f is fuzzy contractive, so f satisfies

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k \left(\frac{1}{M(x, y, t)} - 1 \right).$$

Fix $x \in X$. Let $x_n = f^n(x)$, $n \in \mathbb{N}$. We have seen in the proof of Theorem 11.2 that $\{x_n\}$ is a fuzzy contractive sequence satisfying

$$\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 \leq k \left(\frac{1}{M(x_n, x_{n+1}, t)} - 1 \right)$$

for each $n \in \mathbb{N}$. Thus we have

$$\begin{aligned} \frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 &\leq k^2 \left(\frac{1}{M(x_{n-1}, x_n, t)} - 1 \right) \\ &\leq \ldots \leq k^n \left(\frac{1}{M(x_1, x_2, t)} - 1 \right) \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

and so $\lim\limits_{n\to\infty} M\,(x_n, x_{n+1}, t) = 1$ for all $t > 0$. Then, for a fixed $p \in \mathbb{N}$, we have

$$\begin{aligned} M(x_n, x_{n+p}, t) \geq M\left(x_n, x_{n+1}, \frac{t}{p}\right) * \ldots * M\left(x_{n+p-1}, x_{n+p}, \frac{t}{p}\right) \\ \to \overbrace{1 * \ldots * 1}^{p} = 1 \end{aligned}$$

and so $\{x_n\}$ is a G-Cauchy sequence. Therefore, $\{x_n\}$ converges to y for some $y \in X$. Now, imitating the proof of Theorem 11.2, one can prove that y is the unique fixed point for f. This completes the proof.

Remark 11.2 In Theorem 11.3, it has been proved that each fuzzy contractive sequence is G-Cauchy sequence whereas, in Theorem 11.2, it was assumed that fuzzy contractive sequences are M-Cauchy sequence. This arises the following question:

Question (Gregori and Sapena [19]). Is a fuzzy contractive sequence a Cauchy sequence in George and Veeramani's sense?

The above problem generated much interest to fuzzy fixed point theorist to work on various aspects of fuzzy contractive mapping and associated fixed point. In this direction Tirado [43, 44] introduced the following:

Definition 11.9 We say that the mapping T is *Tirado's contraction* [43] (see also [30]) if the following condition is satisfied: there exists $k \in (0, 1)$ such that

$$1 - M(Tx, Ty, t) \leq k\,(1 - M(x, y, t))$$

for all $x, y \in X$ and $t > 0$. The constant k is called the *contractive constant* of T.

Tirado [43] proved the following theorem as a consequence of his study.

Theorem 11.4 *Let $(X, M, *_L)$ be a complete fuzzy metric space. If T is a Tirado's contraction on X, then T has a unique fixed point.*

On the other hand, Mihet [29] introduced the concept of point convergent and improve the result of Gregori and Sapena [19].

Definition 11.10 Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be *pointwise convergent* to $x \in X$ (we write $x_n \to_p x$) if there exists $t > 0$ such that

$$\lim_{n\to\infty} M(x_n, x, t) = 1.$$

It is easy to see that, endowed with the point convergence, a GV-fuzzy metric space $(X, M, *)$ is a space with the convergence in the sense of Fréchet, that is, one of the following holds:

(a) Every sequence in X has at most one limit point.
(b) Every constant sequence, $x_n = x$, $\forall n \in \mathbb{N}$, is convergent and $\lim_{n\to\infty} x_n = x$.
(c) Every subsequence of a convergent sequence is also convergent and has the same limit as the whole sequence.

Remark 11.3 It is worth noting that if the point convergence in a fuzzy metric space $(X, M, *)$ is Fréchet, then (GV2) holds (so the uniqueness of the limit in the point convergence characterizes, in a sense, a fuzzy metric space in the sense of George and Veeramani). Indeed, let $x, y \in X$ with $x \neq y$. If $M(x, y, t) = 1$ for some $t > 0$, then the sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ defined as $x, y, x, y, \cdots$ has two distinct limits, for the equality $M(x, x, t) = M(y, x, t) = 1$ implies $x_n \to_p x$, while $M(x, y, t) = M(y, y, t) = 1$ implies $x_n \to_p y$.

In the next example, we will see that there exist p-convergent but not convergent sequences.

Example 11.5 Let $\{x_n\}_{n\in\mathbb{N}} \subset (0, \infty)$ with $x_n \to 1$ and $X = \{x_n\} \cup \{1\}$. Define $M(x_n, x_n, t) = 1$ for all $n \in \mathbb{N}$ and $t > 0$, $M(1, 1, t) = 1$ for all $t > 0$, $M(x_n, x_m, t) = \min\{x_n, x_m\}$ for all $n, m \in \mathbb{N}$ and $t > 0$ and

$$M(x_n, 1, t) = \begin{cases} \min\{x_n, t\}, & \text{if } 0 < t < 1, \\ x_n, & \text{if } t > 1, \end{cases}$$

for all $n \in \mathbb{N}$. Then (X, M, T_M), where $T_M(a, b) = \min\{a, b\}$, is a fuzzy metric space (see [20], Example 2]). Since $\lim_{n\to\infty} M(x_n, 1, \frac{1}{2}) = \frac{1}{2}$, $\{x_n\}$ is not convergent. Nevertheless, it is p-convergent to 1 for $\lim_{n\to\infty} M(x_n, 1, 2) = 1$.

Theorem 11.5 *Let $(X, M, *)$ be a GV-fuzzy metric space and $f : X \to X$ be a fuzzy contractive mapping. Suppose that, for some $x \in X$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by $x_n = f^n(x)$ of its iterates has a p-convergent subsequence. Then f has a unique fixed point.*

It should be noted that a similar theorem does not hold in KM-fuzzy metric spaces. This is illustrated in the following:

Example 11.6 Let X be the set $\mathbb{N} = \{1, 2, \ldots\}$. We define (for $p \neq q$) the fuzzy mapping M by

$$M(p,q,t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 - 2^{-\min\{p,q\}}, & \text{if } 0 < t \leq 1, \\ 1, & \text{if } t > 1. \end{cases}$$

As $1 - 1/2^{-\min(p,r)} \geq \min\{1 - 1/2^{-\min(r,q)}$ and $1 - 1/2^{-\min(p,q)}\}$ for all $p, q, r \in \mathbb{N}$, (X, M, T_M) is a KM-fuzzy metric space satisfying $M(x, y, t) \neq 0$ for all $x, y \in X$ and $t > 0$. The mapping $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x + 1$ is fuzzy contractive. Indeed, if $t > 1$, then we have

$$\frac{1}{M(f(p), f(q), t)} - 1 = 0 \leq \frac{1}{2}\left(\frac{1}{M(p,q,t)} - 1\right)$$

for all $p, q \in \mathbb{N}$, while, if $0 < t \leq 1$ and $p < q$, then we have

$$\begin{aligned} \frac{1}{M(f(p), f(q), t)} - 1 &= \frac{1}{2^{p+1} - 1} \\ &\leq \frac{1}{2^{p+1} - 2} = \frac{1}{2}\left(\frac{1}{M(p,q,t)} - 1\right). \end{aligned}$$

As $\lim_{n\to\infty} M(f^n(x), 1, s) = 1$ for all $x \in X$ and $s > 1$, it follows that $x_n \to_p 1$. Nevertheless, 1 is not a fixed point of f.

Remark 11.4 (1) We note that in Example 11.5, as well as in Example 11.4, there are essentially no nonconstant convergent sequences.

(2) It will be natural to continue the study of these convergence spaces, by finding some more examples and introducing a similar concept for Cauchy sequence, p-completeness, etc. Also, it would be interesting to compare different types of contraction maps in fuzzy metric spaces.

On the other hand, Yun et al. [51] introduced the notion of minimal slop of a map between fuzzy metric spaces and studied various properties of fuzzy contractive mapping which complement the above question proposed by Gregori and Sapena [19].

11.4 Fuzzy Ψ-Contractive Mappings

In 2008, Mihet [30] provided a partial answer to the above question proposed by Gregori and Sapena in affirmative by introducing the notion of fuzzy Ψ-contractive mapping as follows:

Definition 11.11 ([30]) Let Ψ be the class of all mapping $\psi : [0, 1] \to [0, 1]$ such that ψ is continuous, nondecreasing and $\psi(t) > t$ for all $t \in (0, 1)$. Let $(X, M, *)$ be a fuzzy metric space and $\psi \in \Psi$.

(1) A mapping $f : X \to X$ is called a *fuzzy ψ-contractive mapping* if the following implication takes place:

$$M(x, y, t) > 0 \Longrightarrow M(f(x), f(y), t) \geq \psi(M(x, y, t)).$$

(2) A *fuzzy ψ-contractive sequence* in a fuzzy metric space $(X, M, *)$ is any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that

$$M(x_{n+2}, x_{n+1}, t) \geq \psi(M(x_{n+1}, x_n, t))$$

for all $n \in \mathbb{N}$ and $t > 0$.

Example 11.7 Let $X = [0, \infty)$, $a * b = \min\{a, b\}$ $\forall a, b \in [0, 1]$ and

$$M(x, y, t) = \begin{cases} 0, & \text{if } t \leq |x - y|, \\ 1, & \text{if } t > |x - y|. \end{cases}$$

It is well known that $(X, M, *)$ is KM-fuzzy metric space. Let ψ be a mapping in Ψ. Since $\psi(1) = 1$ and

$$\begin{aligned} M(x, y, t) > 0 &\Longrightarrow M(x, y, t) = 1 \\ &\Longrightarrow \psi(M(x, y, t)) = 1. \end{aligned}$$

It follows that any fuzzy contractive mapping on $(X, M, *)$ satisfying

$$|x - y| < t \Longrightarrow |f(x) - f(y)| < t,$$

that is,

$$|f(x) - f(y)| \leq |x - y|, \quad \forall x, y \in X.$$

Conversely, if $f : X \to X$ is such that $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in X$, then f is a fuzzy ψ contractive mapping for all $\psi \in \Psi$ such that $\psi(0) = 0$. Thus the mapping $f : X \to X$, $f(x) = x + 1$, $g(x) = x$ are fuzzy ψ_k-contractive on $(X, M, *)$.

Remark 11.5 (Mihet [28], Example 3.4) The sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by $x_n = n + 1$ in the fuzzy metric space considered in the above Example 11.7, although fuzzy ψ_k-contractive, is not an M-Cauchy sequence.

We note that, for every $k \in (0, 1)$, the mapping $\psi_k : [0, 1] \to [0, 1]$ defined by $\psi_k(t) = \dfrac{t}{t + k(1 - t)}$ is in Ψ and a ψ_k-fuzzy contractive mapping is a fuzzy contractive mapping in the sense of Geogori and Sepena [19].

Theorem 11.6 *Let $(X, M, *)$ be an M-complete non-Archimedean fuzzy metric space and $f : X \to X$ be a a fuzzy ψ-contractive mapping. If there exists $x \in X$ such that $M(x, f(x), t) > 0$ for all $t > 0$, then f has a fixed point.*

Proof Let $x \in X$ be such that $M(x, f(x), t) > 0$ for all $t > 0$ and $x_n = f^n(x)$ for each $n \in \mathbb{N}$. Then we have

$$\begin{aligned} M(x_1, x_2, t) &\geq \psi(M(x_0, x_1, t)) \\ &\geq M(x_0, x_1, t) > 0, \quad \forall t > 0. \end{aligned}$$

Hence we have

$$\begin{aligned} M(x_2, x_3, t) &\geq \psi(M(x_1, x_2, t)) \\ &\geq M(x_1, x_2, t) > 0, \quad \forall t > 0. \end{aligned}$$

By induction, $M(x_{n+1}, x_{n+2}, t) \geq M(x_n, x_{n+1}, t) > 0$ for all $t > 0$. Therefore, for every $t > 0$, $M(x_n, x_{n+1}, t)_{n\in\mathbb{N}}$ is a nondecreasing sequence of numbers in $(0, 1]$. Fix a $t > 0$ and denote $\lim_{n\to\infty} M(x_n, x_{n+1}, t)$ by l. We have $l \in (0, 1]$ (for $M(x_0, x_1, t) > 0$) and since $M(x_n, x_{n+1}, t) \geq \psi(M(x_{n-1}, x_n, t))$ and ψ is continuous, $l \geq \psi(l)$. This implies $l = 1$ and so

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1, \ \forall t > 0.$$

If $\{x_n\}$ is not an M-Cauchy sequence, then there are $\varepsilon \in (0, 1)$ and $t > 0$ such that, for each $k \in \mathbb{N}$, there exist $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and

$$M(x_{m(k)}, x_{n(k)}, t) \leq 1 - \varepsilon.$$

Let, for each $k \geq 1$, $m(k)$ be the least integer exceeding $n(k)$ satisfying the above property, that is, $M(x_{m(k)-1}, x_{n(k)-1}, t) > 1 - \varepsilon$ and $M(x_{m(k)}, x_{n(k)}, t) \leq 1 - \varepsilon$. Then, for each positive integer $k \geq 1$,

$$\begin{aligned} 1 - \varepsilon &\geq M(x_{m(k)}, x_{n(k)}, t) \\ &\geq *(M(x_{m(k)-1}, x_{n(k)}, t), M(x_{m(k)-1}, x_{m(k)}, t)) \\ &\geq *(1 - \varepsilon, M(x_{m(k)-1}, x_{m(k)}, t)). \end{aligned}$$

Since $\lim_{k\to\infty} *(1 - \varepsilon, M(x_{m(k)-1}, x_{m(k)}, t)) = *(1 - \varepsilon, 1) = 1 - \varepsilon$, it follows that

$$\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}, t) = 1 - \varepsilon.$$

Let us denote $M(x_{n(k)}, x_{n(k)+1}, t)$ by z_n. Then we have

$$
\begin{aligned}
M(x_{m(k)}, x_{n(k)}, t) &\geq *^2(z_n, M(x_{m(k)+1}, x_{n(k)+1}, t), z_m) \\
&\geq *^2(z_n, M(x_{m(k)}, x_{n(k)}, t), z_m).
\end{aligned}
$$

Letting $k \to \infty$, we obtain

$$
\begin{aligned}
1 - \varepsilon &\geq *^2(1, \psi(1 - \varepsilon), 1) \\
&= \psi(1 - \varepsilon) > 1 - \varepsilon,
\end{aligned}
$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence.

If $\lim\limits_{n\to\infty} x_n = y$, then, from $M(f(y), f(x_n), t) \geq \psi(M(y, x_n, t))$, it follows that $x_{n+1} \to f(y)$. From here, we deduce that

$$
M(y, f(y), t) \geq *^2(M(y, x_n, t), M(x_n, x_{n+1}, t), M(x_{n+1}, f(y), t)) \xrightarrow[n\to\infty]{} 1
$$

for all $t > 0$ and hence $f(y) = y$. This completes the proof.

Theorem 11.7 *Let $(X, M, *)$ be an M-complete non-Archimedean fuzzy metric space satisfying the condition $M(x, y, t) > 0$ for all $t > 0$ and $f : X \to X$ be a fuzzy ψ-contractive mapping. Then f has a unique fixed point.*

Example 11.8 Let $X = (0, \infty)$, $a * b = ab$ for all $a, b \in [0, 1]$ and

$$
M(x, y, t) = \frac{\min(x, y)}{\max(x, y)}
$$

for all $t \in (0, \infty)$ and $x, y > 0$. Then $(X, M, *)$ is an M-complete non-Archimedean fuzzy metric space. Since $\sqrt{t} > t$ for all $t \in (0, 1)$, the mapping $f : X \to X$ defined by $f(x) = \sqrt{x}$ is a fuzzy ψ-contractive mapping with $\psi(t) = \sqrt{t}$. Thus all the conditions of Theorem 11.7 are satisfied and so the fixed point of f is $x = 1$.

Some other generalizations of results of Geogori and Sepena [19] and Mihet [30] can be found in [1, 16, 43, 47, 48].

11.5 α-ϕ-Fuzzy Contractive Mappings

We start this section by introducing the notions of α-ϕ-fuzzy contractive and α-admissible mappings in fuzzy metric spaces.

Denote by Φ the family of all right continuous functions $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(r) < r$ for all $r > 0$.

Remark 11.6 Note that, for every function $\phi \in \Phi$, $\lim_{n\to+\infty} \phi^n(r) = 0$ for each $r > 0$, where $\phi^n(r)$ denotes the nth iterate of ϕ.

Definition 11.12 ([17]) Let $(X, M, *)$ be a fuzzy metric space in the sense of George and Veeramani. We say that $f : X \to X$ is an *α-ϕ-fuzzy contractive mapping* if there exist two functions $\alpha : X \times X \times (0, +\infty) \to [0, +\infty)$ and $\phi \in \Phi$ such that

$$\alpha(x, y, t)\left(\frac{1}{M(fx, fy, t)} - 1\right) \leq \phi\left(\frac{1}{M(x, y, t)} - 1\right) \tag{11.1}$$

for all $x, y \in X$ and $t > 0$.

Remark 11.7 If $\alpha(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ and $\phi(r) = kr$ for all $r > 0$ and for some $k \in (0, 1)$, then Definition 11.12 reduces to the definition of the fuzzy contractive mapping given by Gregori and Sapena [19]. It follows that a fuzzy contractive mapping is an α-ϕ-fuzzy contractive mapping, but the converse is not necessarily true (see Example 11.9 given below).

Definition 11.13 Let $(X, M, *)$ be a fuzzy metric space in the sense of George and Veeramani. We say that $f : X \to X$ is *α-admissible* if there exists a function $\alpha : X \times X \times (0, +\infty) \to [0, +\infty)$ such that, for all $t > 0$,

$$x, y \in X, \quad \alpha(x, y, t) \geq 1 \Longrightarrow \alpha(fx, fy, t) \geq 1.$$

Definition 11.14 (Di Bari and Vetro [9]) Let $(X.M, *)$ be a fuzzy metric space in the sense of George and Veeramani. The fuzzy metric M is said to be *triangular* if the following condition holds:

$$\left(\frac{1}{M(x, y, t)} - 1\right) \leq \left(\frac{1}{M(x, z, t)} - 1\right) + \left(\frac{1}{M(y, z, t)} - 1\right) \tag{11.2}$$

for all $x, y, z \in X$ and $t > 0$.

Now, we are ready to state and prove our first result of this section.

Theorem 11.8 ([17]) *Let $(X, M, *)$ be a G-complete fuzzy metric space in the sense of George and Veeramani. Let $f : X \to X$ be an α-ϕ-fuzzy contractive mapping satisfying the following conditions:*

(a) *f is α-admissible;*
(b) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1$ for all $t > 0$;*
(c) *f is continuous.*

Then f has a fixed point, that is, there exists $x^* \in X$ such that $fx^* = x^*$.

Proof Let $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1$ for all $t > 0$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = fx_n$, for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point of f. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since f is α-admissible, we have

$$\alpha(x_0, x_1, t) = \alpha(x_0, fx_0, t) \geq 1 \Longrightarrow \alpha(fx_0, fx_1, t) = \alpha(x_1, x_2, t) \geq 1.$$

By induction, we get

$$\alpha(x_n, x_{n+1}, t) \geq 1 \tag{11.3}$$

for all $n \in \mathbb{N}$ and $t > 0$. By (11.3), we have

$$\begin{aligned}\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right) &= \left(\frac{1}{M(fx_{n-1}, fx_n, t)} - 1\right)\\ &\leq \alpha(x_{n-1}, x_n, t)\left(\frac{1}{M(fx_{n-1}, fx_n, t)} - 1\right).\end{aligned}$$

Using (11.1) with $x = x_{n-1}$ and $y = x_n$ from the above inequality, by the property of ϕ ($\phi(r) < r$ for all $r > 0$), we obtain

$$\begin{aligned}\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right) &\leq \phi\left(\frac{1}{M(x_{n-1}, x_n, t)} - 1\right)\\ &< \left(\frac{1}{M(x_{n-1}, x_n, t)} - 1\right).\end{aligned}$$

Consequently, $M(x_n, x_{n+1}, t) > M(x_{n-1}, x_n, t)$ for all $n \in \mathbb{N}$ and thus $M(x_{n-1}, x_n, t)$ is an increasing sequence of positive real numbers in [0,1].

Let $S(t) = \lim_{n\to+\infty} M(x_{n-1}, x_n, t)$. Now, we show that $S(t) = 1$ for all $t > 0$. We suppose that there is $t_0 > 0$ such that $S(t_0) < 1$. Then, from

$$\left(\frac{1}{M(x_n, x_{n+1}, t_0)} - 1\right) \leq \phi\left(\frac{1}{M(x_{n-1}, x_n, t_0)} - 1\right)$$

as $n \to +\infty$, using the right continuity of the function ϕ, we deduce that

$$\frac{1}{S(t_0)} - 1 \leq \phi\left(\frac{1}{S(t_0)} - 1\right) < \frac{1}{S(t_0)} - 1,$$

which is a contradiction and so we get $\lim_{n\to+\infty} M(x_{n-1}, x_n, t) = 1$ for all $t > 0$. Then, for a fixed $p \in \mathbb{N}$, we have

$$\begin{aligned}M(x_n, x_{n+p}, t) \geq\ & M\left(x_n, x_{n+1}, \frac{t}{p}\right) * M\left(x_{n+1}, x_{n+2}, \frac{t}{p}\right)\\ & * \cdots\cdots * M\left(x_{n+p-1}, x_{n+p}, \frac{t}{p}\right) \to \overbrace{1 * \cdots\cdots * 1}^{p} = 1\end{aligned}$$

as $n \to +\infty$ and thus $\{x_n\}$ is a G-Cauchy sequence. Therefore, $\{x_n\}$ converges to x^* for some $x^* \in X$. Now, the continuity of f implies that $fx_n \to fx^*$ and so $\lim_{n\to+\infty} M(fx_n, fx^*, t) = 1$ for all $t > 0$. It follows that

$$\lim_{n\to+\infty} M(x_{n+1}, fx^*, t) = \lim_{n\to+\infty} M(fx_n, fx^*, t) = 1$$

for all $t > 0$, that is, $x_n \to fx^*$. By the uniqueness of the limit, we get $x^* = fx^*$, that is, x^* is a fixed point of f. This completes the proof.

In the next theorem, we omit the continuity hypothesis of f:

Theorem 11.9 ([17]) *Let $(X, M, *)$ be a G-complete fuzzy metric space in the sense of George and Veeramani. Let M be triangular and $f : X \to X$ be an α-ϕ-fuzzy contractive mapping satisfying the following conditions:*

(a) *f is α-admissible;*
(b) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1$ for all $t > 0$;*
(c) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x, t) \geq 1$ for all $n \in \mathbb{N}$.*

Then f has a fixed point.

Proof Following the proof of Theorem 11.8, we get that $\{x_n\}$ is a G-Cauchy sequence in the G-complete fuzzy metric space $(X, M, *)$. Then, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to +\infty$. On the other hand, from (11.3) and the hypothesis (c), we have

$$\alpha(x_n, x^*, t) \geq 1 \tag{11.4}$$

for all $n \in \mathbb{N}$ and $t > 0$. Now, using, successively, (11.2), (11.4) and (11.1), also in view of (GV-3), we obtain

$$\begin{aligned}\left(\frac{1}{M(fx^*, x^*, t)} - 1\right) &\leq \left(\frac{1}{M(fx^*, fx_n, t)} - 1\right) + \left(\frac{1}{M(x_{n+1}, x^*, t)} - 1\right)\\ &\leq \alpha(x_n, x^*, t)\left(\frac{1}{M(fx_n, fx^*, t)} - 1\right) + \left(\frac{1}{M(x_{n+1}, x^*, t)} - 1\right)\\ &\leq \phi\left(\frac{1}{M(x_n, x^*, t)} - 1\right) + \left(\frac{1}{M(x_{n+1}, x^*, t)} - 1\right).\end{aligned}$$

Letting $n \to +\infty$, since ϕ is continuous at $r = 0$, we obtain

$$\left(\frac{1}{M(fx^*, x^*, t)} - 1\right) = 0,$$

that is, $fx^* = x^*$. This completes the proof.

The following example shows that the generalization given by Definition 11.12 offers many possibilities to study the existence of a fixed point for a mapping:

Example 11.9 Let $X = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0, 2\}$, $a * b = ab$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and $t > 0$. Clearly, $(X, M, *)$ is a G-complete fuzzy metric space. Define the mapping $f : X \to X$ by

$$fx = \begin{cases} \dfrac{x^2}{4}, & \text{if } x \in X \setminus \{2\}, \\ \\ 2, & \text{if } x = 2, \end{cases}$$

and the function $\alpha : X \times X \times (0, +\infty) \to [0, +\infty)$ by

$$\alpha(x, y, t) = \begin{cases} 1. & \text{if } x, y \in X \setminus \{2\}, \\ \\ 0, & \text{otherwise,} \end{cases}$$

for all $t > 0$. Clearly, f is an α-ϕ-contractive mapping with $\phi(r) = r/2$ for all $r \geq 0$. In fact, if at least one between x and y is equal to 2, then $\alpha(x, y, t) = 0$ and so (11.1) holds trivially. Otherwise, if both x and y are in $X \setminus \{2\}$, then $\alpha(x, y, t) = 1$ and so (11.1) becomes

$$\left(\frac{1}{M(fx, fy, t)} - 1\right) \leq \frac{1}{2}\left(\frac{1}{M(x, y, t)} - 1\right),$$

which is always true since $x + y \leq 2$.

Now, let $x, y \in X$ such that $\alpha(x, y, t) \geq 1$ for all $t > 0$, this implies that $x, y \in X \setminus \{2\}$ and, by the definitions of f and α, we have

$$fx = \frac{x^2}{4} \in X \setminus \{2\}, \quad fy = \frac{y^2}{4} \in X \setminus \{2\}, \quad \alpha(fx, fy, t) = 1, \quad \forall t > 0,$$

that is, f is α-admissible. Further, there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1$ for all $t > 0$. Indeed, for $x_0 = 1$, we have $\alpha(1, f(1), t) = 1$.

Finally, let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to +\infty$. By the definition of the function α, it follows that $x_n \in X \setminus \{2\}$ for all $n \in \mathbb{N}$ and hence $x \in X \setminus \{2\}$. Therefore $\alpha(x_n, x, t) = 1$ for all $n \in \mathbb{N}$. Thus all the hypotheses of Theorem 11.8 are satisfied. Here 0 and 2 are two fixed points of f. However, f is not a fuzzy contractive mapping [19]. To see this consider $x = 2$ and $y = 1$, then we have

$$\left(\frac{1}{M(fx, fy, t)} - 1\right) = \frac{7}{4t} \not\leq \frac{k}{t} = k\left(\frac{1}{M(x, y, t)} - 1\right)$$

since $k \in (0, 1)$.

Remark 11.8 Let $(X, M, *)$ be a fuzzy metric space in the sense of George and Veeramani. A sequence $\{x_n\}_{n \in \mathbb{N}}$ is said to be fuzzy contractive if there exists $k \in (0, 1)$ such that

$$\left(\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1\right) \leq k\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right)$$

for all $n \in \mathbb{N}$ and for all $t > 0$. In the conclusions of their paper, Yun et al. [51] observed that every fuzzy contractive sequence is Cauchy in both George and Veeramani sense and Grabiec sense. Here, in proving Theorems 11.8 and 11.9, we used the G-completeness of the fuzzy metric space $(X, M, *)$. Thus it will be interesting to see whether these results will remain true in a M-complete fuzzy metric space.

Now, we give a sufficient condition to obtain the uniqueness of the fixed point in the previous theorems. Precisely, we consider the following hypothesis:

(H) for all $x, y \in X$ and $t > 0$, there exists $z \in X$ such that $\alpha(x, z, t) \geq 1$ and $\alpha(y, z, t) \geq 1$.

Theorem 11.10 ([17]) *Adding the condition (H) to the hypotheses of Theorem 11.8 (resp. Theorem 11.9), we obtain the uniqueness of the fixed point of f.*

Proof Suppose that x^* and y^* are two fixed points of f. If $\alpha(x^*, y^*, t) \geq 1$, then, by (11.1), we conclude easily that $x^* = y^*$. Assume that $\alpha(x^*, y^*, t) < 1$, it follows from (H) that there exists $z \in X$ such that

$$\alpha(x^*, z, t) \geq 1 \quad \text{and} \quad \alpha(y^*, z, t) \geq 1. \tag{11.5}$$

Since f is α-admissible, from (11.5), we get

$$\alpha(x^*, f^n z, t) \geq 1 \quad \text{and} \quad \alpha(y^*, f^n z, t) \geq 1 \tag{11.6}$$

for all $n \in \mathbb{N}$ and $t > 0$. Using (11.1) and (11.6), we have

$$\begin{aligned}
\left(\frac{1}{M(x^*, f^n z, t)} - 1\right) &= \left(\frac{1}{M(fx^*, f(f^{n-1}z), t)} - 1\right) \\
&\leq \alpha(x^*, f^{n-1}z, t)\left(\frac{1}{M(fx^*, f(f^{n-1}z), t)} - 1\right) \\
&\leq \phi\left(\frac{1}{M(x^*, f^{n-1}z, t)} - 1\right).
\end{aligned}$$

This implies that

$$\left(\frac{1}{M(x^*, f^n z, t)} - 1\right) \leq \phi^n\left(\frac{1}{M(x^*, z, t)} - 1\right), \quad \forall n \in \mathbb{N}.$$

Then, letting $n \to +\infty$, we have

$$f^n z \to x^*. \tag{11.7}$$

Similarly, for $n \to +\infty$, we get also

$$f^n z \to y^*. \tag{11.8}$$

Using (11.7) and (11.8), the uniqueness of the limit gives us $x^* = y^*$. This completes the proof.

In view of Remark 11.7 and to show the usefulness of our theorems, we prove the following classical theorem of Gregori and Sapena [19]:

Theorem 11.11 *Let $(X, M, *)$ be a G-complete fuzzy metric space in the sense of George and Veeramani. Let $f : X \to X$ be a fuzzy contractive mapping. Then f has a unique fixed point.*

Proof Let $\alpha : X \times X \times (0, +\infty) \to [0, +\infty)$ be the function defined by $\alpha(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$. Define also $\phi : [0, +\infty) \to [0, +\infty)$ by $\phi(r) = kr$ for all $r > 0$. Then f is an α-ϕ-contractive mapping. It is easy to show that all the hypotheses of Theorems 11.8 and 11.10 are satisfied. Consequently, f has a unique fixed point. This completes the proof.

Following [6, 8, 35], we show that the obtained theorems are also useful to deduce easily some fixed point results in ordered fuzzy metric spaces. We begin by giving the following two definitions:

Definition 11.15 Let $\preceq$ be an order relation on X. We say that $f : X \to X$ is a *nondecreasing mapping* with respect to $\preceq$ if $x \preceq y$ implies $fx \preceq fy$.

Definition 11.16 Let $(X, \preceq)$ be a partially ordered set and $(X, M, *)$ be a fuzzy metric space in the sense of George and Veeramani. We say that $f : X \to X$ is a *fuzzy order ϕ-contractive mapping* if there exists $\phi \in \Phi$ such that the following implication holds:

$$x, y \in X, \quad x \preceq y \Longrightarrow \left(\frac{1}{M(fx, fy, t)} - 1\right) \leq \phi\left(\frac{1}{M(x, y, t)} - 1\right), \quad \forall t > 0.$$

Theorem 11.12 *Let $(X, \preceq)$ be a partially ordered set and $(X, M, *)$ be a G-complete fuzzy metric space in the sense of George and Veeramani. Let $\phi \in \Phi$ be such that $f : X \to X$ is a fuzzy order ϕ-contractive mapping and suppose that the following conditions hold:*

(a) *f is a nondecreasing mapping with respect to $\preceq$;*
(b) *there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, $M(x_0, fx_0, t) > 0$ for all $t > 0$;*
(c) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x \in X$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then f has a fixed point.

Proof Define the function $\alpha : X \times X \times (0, +\infty) \to [0, +\infty)$ by

$$\alpha(x, y, t) = \begin{cases} 1, & \text{if } x \preceq y, \\ 0, & \text{otherwise,} \end{cases}$$

for all $t > 0$. The reader can show easily that f is α-ϕ-contractive and α-admissible. Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to +\infty$. By the definition of α, we have $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. From (c), this implies that $x_n \preceq x$ for all $n \in \mathbb{N}$, which gives us that $\alpha(x_n, x, t) = 1$ for all $n \in \mathbb{N}$ and $t > 0$. Thus all the hypotheses of Theorem 11.9 are satisfied and f has a fixed point. This completes the proof.

11.6 β-ψ-Fuzzy Contractive Mappings

In this section, we present the notions of β-ψ-fuzzy contractive and β-admissible mappings in fuzzy metric spaces due to Gopal et al. [17].

Let Ψ be the class of all functions $\psi : [0, 1] \to [0, 1]$ such that

(a) ψ is non-decreasing and left continuous;
(b) $\psi(r) > r$ for all $r \in (0, 1)$.

It can easily be shown (see, e.g., [47]) that, if $\psi \in \Psi$, then $\psi(1) = 1$ and $\lim_{n\to+\infty} \psi^n(r) = 1$ for all $r \in (0, 1)$.

Definition 11.17 Let $(X, M, *)$ be a fuzzy metric space. We say that $f : X \to X$ is a *β-ψ-fuzzy contractive mapping* if there exist two functions $\beta : X \times X \times (0, +\infty) \to (0, +\infty)$ and $\psi \in \Psi$ such that

$$M(x, y, t) > 0 \implies \beta(x, y, t)M(fx, fy, t) \geq \psi(M(x, y, t)) \tag{11.9}$$

for all $t > 0$ and $x, y \in X$ with $x \neq y$.

Remark 11.9 If $\beta(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$, then Definition 11.17 reduces to the definition of the fuzzy ψ-contractive mapping given by Mihet [30]. It follows that a fuzzy ψ-contractive mapping is a β-ψ-fuzzy contractive mapping; but the converse is not true always (see Example 11.10 given below).

Definition 11.18 Let $(X, M, *)$ be a fuzzy metric space. We say that $f : X \to X$ is *β-admissible* if there exists a function $\beta : X \times X \times (0, +\infty) \to (0, +\infty)$ such that, for all $t > 0$,

$$x, y \in X, \quad \beta(x, y, t) \leq 1 \implies \beta(fx, fy, t) \leq 1.$$

Theorem 11.13 *Let $(X, M, *)$ be a M-complete non-Archimedean fuzzy metric space and $f : X \to X$ be a β-ψ-fuzzy contractive mapping satisfying the following conditions:*

(a) *f is β-admissible;*
(b) *there exists $x_0 \in X$ such that $\beta(x_0, fx_0, t) \leq 1$ for all $t > 0$;*

(c) *for each sequence* $\{x_n\}$ *in* X *such that* $\beta(x_n, x_{n+1}, t) \leq 1$ *for all* $n \in \mathbb{N}$ *and* $t > 0$*, there exists* $k_0 \in \mathbb{N}$ *such that* $\beta(x_{m+1}, x_{n+1}, t) \leq 1$ *for all* $m, n \in \mathbb{N}$ *with* $m > n \geq k_0$ *and* $t > 0$*;*
(d) *if* $\{x_n\}$ *is a sequence in* X *such that* $\beta(x_n, x_{n+1}, t) \leq 1$ *for all* $n \in \mathbb{N}$ *and* $t > 0$ *and* $x_n \to x$ *as* $n \to +\infty$*, then* $\beta(x_n, x, t) \leq 1$ *for all* $n \in \mathbb{N}$ *and* $t > 0$.

Then f *has a fixed point.*

Proof Let $x_0 \in X$ such that $\beta(x_0, fx_0, t) \leq 1$ for all $t > 0$. Define the sequence $\{x_n\}$ in X by $x_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point of f. Assume $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since f is β-admissible, we have

$$\beta(x_0, fx_0, t) = \beta(x_0, x_1, t) \leq 1 \implies \beta(fx_0, fx_1, t) = \beta(x_1, x_2, t) \leq 1.$$

By induction, we get

$$\beta(x_n, x_{n+1}, t) \leq 1 \tag{11.10}$$

for all $n \in \mathbb{N}$ and $t > 0$. Now, applying (11.9) with $x = x_{n-1}$ and $y = x_n$ and using (11.10), we obtain

$$\begin{aligned} M(x_n, x_{n+1}, t) &= M(fx_{n-1}, fx_n, t) \\ &\geq \beta(x_{n-1}, x_n, t)M(fx_{n-1}, fx_n, t) \\ &\geq \psi(M(x_{n-1}, x_n, t)). \end{aligned}$$

By induction, we get

$$M(x_n, x_{n+1}, t) \geq \psi^n(M(x_0, x_1, t)), \quad \forall\, n \in \mathbb{N}.$$

Since $\lim_{n\to+\infty} \psi^n(r) = 1$ for all $r \in (0, 1)$, we deduce that

$$\lim_{n\to+\infty} M(x_n, x_{n+1}, t) = 1, \quad \forall t > 0.$$

Now, if the sequence $\{x_n\}$ is not an M-Cauchy sequence, then there are $\varepsilon \in (0, 1)$, $t > 0$ and $k_0 \in \mathbb{N}$ (by (c)) such that, for each $k \in \mathbb{N}$ with $k \geq k_0$, there exist $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and

$$M(x_{m(k)}, x_{n(k)}, t) \leq 1 - \varepsilon \text{ and } \beta(x_{m(k)}, x_{n(k)}, t) \leq 1.$$

Let, for each $k \geq 1$, $m(k)$ be the least positive integer exceeding $n(k)$ satisfying the above property, that is,

$$M(x_{m(k)-1}, x_{n(k)}, t) > 1 - \varepsilon \text{ and } M(x_{m(k)}, x_{n(k)}, t) \leq 1 - \varepsilon.$$

Then, for each positive integer $k \geq k_0$, we have

$$
\begin{aligned}
1-\varepsilon &\geq M(x_{m(k)}, x_{n(k)}, t) \\
&\geq M(x_{m(k)-1}, x_{n(k)}, t) * M(x_{m(k)-1}, x_{m(k)}, t) \quad \text{(by (NA))} \\
&\geq (1-\varepsilon) * M(x_{m(k)-1}, x_{m(k)}, t).
\end{aligned}
$$

Since $\lim_{n\to+\infty}(1-\varepsilon) * M(x_{m(k)-1}, x_{m(k)}, t) = (1-\varepsilon) * 1 = 1-\varepsilon$, it follows that

$$
\lim_{n\to+\infty} M(x_{m(k)}, x_{n(k)}, t) = 1-\varepsilon.
$$

Now, by (NA) and the condition (c), we get

$$
\begin{aligned}
M(x_{m(k)}, x_{n(k)}, t) &\geq M(x_{m(k)}, x_{m(k)+1}, t) * M(x_{m(k)+1}, x_{n(k)}, t) \\
&\geq M(x_{m(k)}, x_{m(k)+1}, t) * M(x_{m(k)+1}, x_{n(k)+1}, t) * M(x_{n(k)+1}, x_{n(k)}, t) \\
&= M(x_{m(k)}, x_{m(k)+1}, t) * M(fx_{m(k)}, fx_{n(k)}, t) * M(x_{n(k)+1}, x_{n(k)}, t) \\
&\geq M(x_{m(k)}, x_{m(k)+1}, t) * \beta(fx_{m(k)}, fx_{n(k)}, t) * M(fx_{m(k)}, fx_{n(k)}, t) \\
&\quad * M(x_{n(k)+1}, x_{n(k)}, t) \\
&\geq M(x_{m(k)}, x_{m(k)+1}, t) * \psi(M(x_{m(k)}, x_{n(k)}, t)) * M(x_{n(k)}, x_{n(k)+1}, t).
\end{aligned}
$$

Letting $k \to +\infty$, we obtain

$$
1-\varepsilon \geq 1 * \psi(1-\varepsilon) * 1 = \psi(1-\varepsilon) > 1-\varepsilon,
$$

which is a contradiction and so $\{x_n\}$ is a Cauchy sequence. Since X is M-complete, there exists $x^* \in X$ such that $\lim_{n\to+\infty} x_n = x^*$.

On the other hand, from (11.10) and the hypothesis (d), we have

$$
\beta(x_n, x^*, t) \leq 1, \quad \forall t > 0.
$$

Now, by (NA) and (11.9), we get

$$
\begin{aligned}
M(fx^*, x^*, t) &\geq M(fx^*, fx_n, t) * M(x_{n+1}, x^*, t) \\
&\geq \beta(x_n, x^*, t) M(fx_n, fx^*, t) * M(x_{n+1}, x^*, t) \\
&\geq \psi(M(x_n, x^*, t)) * M(x_{n+1}, x^*, t).
\end{aligned}
$$

Letting $n \to +\infty$ and since $\psi(1) = 1$, we conclude that $fx^* = x^*$. This completes the proof.

The following example shows the usefulness of Definition 11.17:

Example 11.10 Let $X = (0, +\infty)$, $a * b = ab$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{\min\{x,y\}}{\max\{x,y\}}$ for all $x, y \in X$ and $t > 0$. Clearly, $(X, M, *)$ is a M-complete non-Archimedean fuzzy metric space. Define the mapping $f : X \to X$ by

$$fx = \begin{cases} \sqrt{x}, & \text{if } x \in (0, 1], \\ 2, & \text{otherwise,} \end{cases}$$

and the function $\beta : X \times X \times (0, +\infty) \to (0, +\infty)$ by

$$\beta(x, y, t) = \begin{cases} 1, & \text{if } x, y \in (0, 1], \\ 2, & \text{otherwise,} \end{cases}$$

for all $t > 0$. It is easy to show that f is a β-ψ-contractive mapping with $\psi(r) = \sqrt{r}$ for all $r \in [0, 1]$. Clearly, f is β-admissible. Further, there exists $x_0 \in X$ such that $\beta(x_0, fx_0, t) \leq 1$ for all $t > 0$. Indeed, for $x_0 = 1$, we have $\beta(1, f(1), t) = 1$.

Finally, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}$, $x_n \to x \in X$ as $n \to +\infty$ and let $k_0 = 1$ be such that, for all $m, n \in \mathbb{N}, m > n \geq k_0$. By the definition of the function β, it follows that $x_n \in (0, 1]$ for all $n \in \mathbb{N}$. Now, if $x > 1$, we get

$$M(x_n, x, t) = \frac{\min\{x_n, x\}}{\max\{x_n, x\}} = \frac{x_n}{x} \leq \frac{1}{x} < 1,$$

which contradicts (1) of Definition 11.3 since $\lim_{n\to+\infty} M(x_n, x, t) = 1$ for all $t > 0$. Consequently, we obtain that $x \in (0, 1]$. Therefore, $\beta(x_n, x, t) = 1$ and $\beta(x_{m+1}, x_{n+1}, t) = 1$ for all $m, n \in \mathbb{N}$. Thus all the hypotheses of Theorem 11.13 are satisfied. Here 1 and 2 are two fixed points of f. However, f is not a fuzzy ψ-contractive mapping [30]. To see this, consider $x = \frac{1}{2}$ and $y = 3$. Then we have

$$M(fx, fy, t) = \frac{\sqrt{1/2}}{2} \ngeq \sqrt{\frac{1/2}{3}} = \sqrt{M(x, y, t)} = \psi(M(x, y, t)).$$

To ensure the uniqueness of the fixed point, we will consider the following hypothesis:

(J) For all $x, y \in X$ and $t > 0$, there exists $z \in X$ such that

$$\beta(x, z, t) \leq 1 \text{ and } \beta(y, z, t) \leq 1.$$

Theorem 11.14 *Adding the condition (J) to the hypotheses of Theorem 11.13, we obtain the uniqueness of the fixed point of f.*

Proof The proof can be completed using a similar technique as given in the proof of Theorem 11.10. Therefore, to avoid repetitions, we omit the details.

Remark 11.10 Motivated by Samet et al. [35], we proposed the concept of α-ϕ-fuzzy contractive mapping, which is weaker than the corresponding concept of fuzzy contractive mapping [19] and the concept of β-ψ-fuzzy contractive mapping, which is weaker than the corresponding concept of fuzzy-ψ-contractive mapping [30]. Moreover, we proved two theorems which ensure the existence and uniqueness of fixed points of these new types of contractive mappings. The new concepts lead to further investigations and applications. For example, using the recent ideas in the literature [12], it is possible to extend our results to the case of coupled fixed points in fuzzy metric spaces.

11.7 Fuzzy $\mathscr{H}$-Contractive Mappings and α Type Fuzzy $\mathscr{H}$-Contractive Mappings

Recently, Wardowski [49] introduced the concept of fuzzy $\mathscr{H}$-contractive mappings, as a generalization of that of fuzzy contractive mappings, and established the conditions guaranteeing the existence and uniqueness of fixed point for this type of contractions in M-complete fuzzy metric spaces in the sense of George and Veeramani.

Definition 11.19 Let $\mathscr{H}$ be the family of the mappings $\eta\colon (0, 1] \to [0, \infty)$ satisfying the following conditions:

(H1) η transforms $(0, 1]$ onto $[0, \infty)$;
(H2) η is strictly decreasing.

Then the mapping $f : X \to X$ is called a *fuzzy $\mathscr{H}$-contractive mapping* (see Wardowski [49]) with respect to $\eta \in \mathscr{H}$ if there exists $k \in (0, 1)$ satisfying the following condition:

$$\eta(M(fx, fy, t)) \leq k\eta(M(x, y, t))$$

for all $x, y \in X$ and $t > 0$.

Proposition 11.2 *Let $(X, M, *)$ be a fuzzy metric space and let $\eta \in \mathscr{H}$. A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is an M-Cauchy sequence if and only if, for every $\varepsilon > 0$ and $t > 0$, there exist $n_0 \in \mathbb{N}$ such that*

$$\eta\,(M(x_m, x_n, t)) < \varepsilon, \quad \forall m, n \geq n_0.$$

Proposition 11.3 *Let $(X, M, *)$ be a fuzzy metric space and let $\eta \in \mathscr{H}$. A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is convergent to $x \in X$ if and only if $\lim_{n\to\infty} \eta(M(x_n, x, t)) = 0, \ \forall t > 0$.*

Theorem 11.15 ([49]) *Let $(X, M, *)$ be an M-complete fuzzy metric space and $f\colon X \to X$ be a fuzzy $\mathscr{H}$-contractive mapping with respect to $\eta \in \mathscr{H}$ such that*

(a) $\prod_{i=1}^{k} M(x, fx, t_i) \neq 0$ *for all* $x \in X$ *and* $k \in \mathbb{N}$ *and a sequence* $(t_i)_{i\in\mathbb{N}} \subset (0, \infty)$ *with* $t_i \to 0$;
(b) *if* $r * s > 0$, *then* $\eta(r * s) \leq \eta(r) + \eta(s)$ *for all* $r, s \in \{M(x, fx, t) : x \in X, t > 0\}$;
(c) $\{\eta(M(x, fx, t_i))\colon i \in \mathbb{N}\}$ *is bounded for all* $x \in X$ *and any sequence* $(t_i)_{i\in\mathbb{N}} \subset (0.\infty)$ *with* $t_i \to 0$.

Then f has a unique fixed point $x^* \in X$ and, for each $x_0 \in X$, the sequence $(f^n x_0)_{n\in\mathbb{N}}$ converges to x^*.

In a recent note, Gregori and Minana [24] observed that the main idea of Wardowski [49] is correct and different from the known concepts in the literature but they also showed that the class of fuzzy $\mathcal{H}$-contractive mappings are included in the class of fuzzy Ψ-contractive mappings, as well as they point out some drawbacks of the conditions used in the above Theorem 11.15.

Remark 11.11 (See Gregori and Miñana [24]) If $\eta \in \mathcal{H}$, then the mappings $\eta \cdot k\colon (0,1] \to [0,\infty)$ and $\eta^{-1}\colon [0,\infty) \to (0,1]$, defined in its obvious sense, are two bijective continuous mappings which are strictly decreasing.

In view of the above remark, we observe that every fuzzy $\mathcal{H}$-contractive mapping is a fuzzy ψ-contraction with $\psi(t) = \eta^{-1}(k\eta(t))$ for all $t \in (0,1]$ (see [24]).

In this direction of research work, a recent paper of Mihet [32] provides a larger perspective and further scope to study fixed points of fuzzy $\mathcal{H}$-contractive mappings.

Most recently, Beg et al. [10] introduced a new concept of α-fuzzy $\mathcal{H}$-contractive mapping which is essentially weaker than the class of fuzzy contractive mapping and stronger than the concept of α-ϕ-fuzzy contractive mapping. For this type of contractions, the existence and uniqueness of fixed point in fuzzy M-complete metric spaces have been established.

Definition 11.20 Let $(X, M, *)$ be a fuzzy metric space. We say that $f : X \to X$ is an *α-fuzzy-$\mathcal{H}$-contractive mapping* with respect to $\eta \in \mathcal{H}$ if there exists a function $\alpha : X \times X \times (0,\infty) \to [0,\infty)$ such that

$$\alpha(x, y, t)\eta\left(M(fx, fy, t)\right) \leq k\eta\left(M(x, y, t)\right) \tag{11.11}$$

for all $x, y \in X$ and $t > 0$.

Remark 11.12 If $\alpha(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$, then Definition 11.20 reduces to the Definition 18 but converse is not necessarily true (see Example 11.11 given bellow).

Definition 11.21 Let $(X, M, *)$ be a fuzzy metric space. We say that $f : X \to X$ is *α-admissible* if there exists a function $\alpha : X \times X \times (0,+\infty) \to [0,+\infty)$ such that

$$\alpha(x, y, t) \geq 1 \Longrightarrow \alpha(fx, fy, t) \geq 1$$

for all $x, y \in X$ and $t > 0$.

Now, we are ready to state and prove the following:

Theorem 11.16 *Let $(X, M, *)$ be a M-complete fuzzy metric space, where $*$ is positive. Let $f : X \to X$be an α-fuzzy-$\mathcal{H}$-contractive mapping with respect to $\eta \in \mathcal{H}$ satisfying the following conditions:*

(a) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0, t) \geq 1,\ t > 0$*;*
(b) f *is* α*-admissible;*
(c) $\eta(r * s) \leq \eta(r) + \eta(s),\ r, s \in (0, 1]$*;*
(d) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}, t) \geq 1$ *for each* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} x_n = x$, *then* $\alpha(x_n, x, t) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $t > 0$.

Then f *has a fixed point* $x^* \in X$. *Moreover, the sequence* $\{f^n x_0\}_{n\in N}$ *converges to* x^*.

Proof Let $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1,\ t > 0$. Define the sequence $\{x_n\}_{n\in\mathbb{N}}$ in X by $x_{n+1} = fx_n,\ n \in \mathbb{N} \cup \{0\}$. If $x_{n+1} = x_n$, for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point of f. So, assume that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N}$. Since f is α-admissible, we have

$$\alpha(x_0, x_1, t) = \alpha(x_0, fx_0, t) \geq 1 \Longrightarrow \alpha(fx_0, fx_1, t) = \alpha(x_1, x_2, t) \geq 1, \quad \forall t > 0.$$

By induction, we get

$$\alpha(x_n, x_{n+1}, t) \geq 1 \tag{11.12}$$

for all $n \in \mathbb{N}$ and $t > 0$. Now, applying (11.11) and using (11.12), we obtain the following:

$$\begin{aligned}
\eta\left(M(x_{n+1}, x_{n+2}, t)\right) &= \eta\left(M(fx_n, f_{n+1}, t)\right) \\
&\leq \alpha(x_n, x_{n+1}, t)\eta\left(M(fx_n, f_{n+1}, t)\right) \\
&\leq k\eta\left(M(x_n, x_{n+1}, t)\right) \\
&\leq k\alpha(x_{n-1}, x_n, t)\eta\left(M(fx_{n-1}, fx_n, t)\right) \\
&\leq kk\eta\left(M(x_{n-2}, x_{n-1}, t)\right) \\
&\leq \cdots \\
&\leq k^{n+1}\eta\left(M(x_0, x_1, t)\right), \quad \forall t > 0.
\end{aligned}$$

Since $k \in (0, 1)$ and η is strictly decreasing, we have

$$\eta\left(M(x_{n+1}, x_{n+2}, t)\right) < \eta\left(M(x_0, x_1, t)\right), \quad \forall t > 0,$$

and

$$M(x_{n+1}, x_{n+2}, t) \geq M(x_0, x_1, t) > 0, \quad \forall n \in N,\ t > 0. \tag{11.13}$$

Now, let us consider any $m, n \in N$ with $m < n$ and let $\{a_i\}_{i\in N}$ be a strictly decreasing sequence of positive numbers such that $\sum_{i=1}^{\infty} a_i = 1$. From (GV-4), (GV-2) and the positivity of $*$, we get

$$
\begin{aligned}
M(x_m, x_n, t) &\geq M\left(x_m, x_m, t - \sum_{i=m}^{n-1} a_i t\right) * M\left(x_m, x_n, \sum_{i=m}^{n-1} a_i t\right) \\
&= M\left(x_m, x_n, \sum_{i=m}^{n-1} a_i t\right) \\
&\geq M\left(x_m, x_{m+1}, a_m t\right) * M\left(x_{m+1}, x_{n+2}, a_{m+1} t\right) * \cdots * M\left(x_{n-1}, x_n, a_{n-1} t\right)
\end{aligned}
$$

for all $t > 0$. By the condition (c) and (11.13), we get

$$
\begin{aligned}
\eta\left(M(x_m, x_n, t)\right) &\leq \eta\left(\prod_{i=m}^{n-1} M(x_i, x_{i+1}, a_i t)\right) \\
&\leq \sum_{i=m}^{n-1} \eta\left(M(x_i, x_{i+1}, a_i t)\right) \\
&\leq \sum_{i=m}^{n-1} k^i \eta\left(M(x_0, x_1, t)\right)
\end{aligned}
$$

for all $m, n \in N$ with $m < n$ and $t > 0$. The above sum is finite, i.e., for any $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that

$$
\eta\left(M(x_m, x_n, t)\right) \leq \sum_{i=m}^{n-1} k^i \eta\left(M(x_0, x_1, t)\right) < \varepsilon
$$

for all $m, n \in N$ with $m < n$ and $t > 0$. Thus, by Proposition 11.2, it follows that $\{x_n\}_{n\in\mathbb{N}}$ is an M-Cauchy sequence in X. By the completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Due to Proposition 11.3, we have

$$
\lim_{n\to\infty} \eta(M(x_n, x^*, t)) = 0, \quad \forall t > 0.
$$

Now, applying the condition (d) and (11.11), we obtain

$$
\begin{aligned}
\eta\left(M(x_{n+1}, fx^*, t)\right) &= \eta\left(M(fx_n, fx^*, t)\right) \\
&\leq \alpha(x_n, x^*, t)\eta\left(M(fx_n, fx^*, t)\right) \\
&\leq k\eta\left(M(x_n, x^*, t)\right), \quad \forall t > 0,
\end{aligned}
$$

which implies that

$$
\lim_{n\to\infty} \eta\left(M(x_{n+1}, fx^*, t)\right) = 0, \quad \forall t > 0,
$$

i.e.,

$$
fx^* = \lim_{n\to\infty} x_{n+1} = x^*.
$$

So, x^* is a fixed point of f. This completes the proof.

The following examples shows the usefulness of the above theorem:

Example 11.11 Let $X = \mathbb{R}$, $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and $t > 0$. Clearly, $(X, M, *)$ is an M-complete fuzzy metric space. Define the mapping $f : X \to X$ by

$$f(x) = \begin{cases} \frac{x^2}{4}, & \text{if } x \in [0, 1], \\ 2, & \text{otherwise.} \end{cases}$$

Also, define $\eta(s) = \frac{1}{s} - 1$, $s \in (0, 1]$ and $\alpha : X \times X \times (0, \infty) \to [0, \infty)$ by

$$\alpha(x, y, t) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, f is an α-fuzzy-$\mathscr{H}$-contractive mapping with $k = \frac{1}{2}$.

Now, let $x, y \in X$ such that $\alpha(x, y, t) \geq 1$ for all $t > 0$. This implies that $x, y \in [0, 1]$ and, by the definitions of f and α, we have

$$f(x) = \frac{x^2}{4} \in [0, 1], \quad f(y) = \frac{y^2}{4} \in [0, 1], \quad \alpha(fx, fy, t) = 1, \quad \forall t > 0,$$

i.e., f is α-admissible. Further, there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1$ for all $t > 0$. Indeed, for any $x_0 \in [0, 1]$, we have $\alpha(x_0, fx_0, t) = 1$ for all $t > 0$.

Finally, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$. By the definition of the function α, it follows that $x_n \in [0, 1]$ for each $n \in \mathbb{N}$ and hence $x \in [0, 1]$. Therefore, $\alpha(x_n, x, t) = 1$ for each $n \in \mathbb{N}$. So, all the hypothesis of Theorem 11.16 are satisfied. Here, 0 and 2 are two fixed point of f. However, f is not a fuzzy $\mathscr{H}$-contractive mapping [49]. To see this, consider $x = 2$ and $y = 1$. Then, since $k \in (0, 1)$, we have

$$\eta(M(fx, fy, t)) = \frac{7}{4t} > \frac{k}{t} = k\eta(M(x, y, t)), \quad \forall t > 0.$$

Now, we give a sufficient condition to obtain the uniqueness of the fixed point in the previous theorem. Precisely, we consider the following hypothesis:

(U) For all $x, y \in X$ and $t > 0$, there exists $z \in X$ such that

$$\alpha(x, z, t) \geq 1 \text{ and } \alpha(y, z, t) \geq 1.$$

Theorem 11.17 *Adding the condition* (U) to the hypothesis of Theorem *11.16, we obtain the uniqueness of the fixed point of f.*

Proof Suppose that x^* and y^* are two fixed points of f. If $\alpha(x^*, y^*, t) \geq 1$ for some $t > 0$, then by (11.11), we conclude easily that $x^* = y^*$.

Assume $\alpha(x^*, y^*, t) < 1$ for all $t > 0$. Then, by (U), there exists $z \in X$ such that

$$\alpha(x^*, z, t) \geq 1 \text{ and } \alpha(y^*, z, t) \geq 1, \quad \forall t > 0. \tag{11.14}$$

Since f is α-admissible, and by (11.14), we get

$$\alpha(x^*, f^n z, t) \geq 1 \text{ and } \alpha(y^*, f^n z, t) \geq 1 \tag{11.15}$$

for all $n \in \mathbb{N}$ and $t > 0$. Now, applying (11.11) and (11.15), we have

$$M(x^*, f^n z, t) = M(fx^*, f(f^{n-1}z), t)$$

and

$$\begin{aligned}\eta\left(M(x^*, f^n z, t)\right) &= \eta\left(M(fx^*, f(f^{n-1}z), t)\right)\\ &\leq \alpha(x^*, f^{n-1}z, t)\eta\left(M(fx^*, f(f^{n-1}z), t)\right)\\ &\leq k\eta\left(M(x^*, f^{n-1}z, t)\right)\\ &\leq \cdots \leq k^n\eta\left(M(x^*, z, t)\right)\end{aligned}$$

for all $n \in \mathbb{N}$ and $t > 0$. By letting $n \to \infty$ in the last relation, we get

$$\lim_{n\to\infty} \eta\left(M(x^*, f^n z, t)\right) = 0, \quad \forall t > 0,$$

and

$$\lim_{n\to\infty} f^n z = x^*.$$

Similarly, we have

$$\lim_{n\to\infty} f^n z = y^*.$$

Finally, the uniqueness of the above limits gives us $x^* = y^*$. This completes the proof.

The assumption that $*$ is positive can be further relaxed in Theorem 11.16. In fact, we can prove the following:

Theorem 11.18 *Let $(X, M, *)$ be a M-complete strong fuzzy metric space for a nilpotent t-norm $*_L$, and let $f : X \to X$ be an α-fuzzy-$\mathscr{H}$-contractive mapping with respect to $\eta \in H$ satisfying the following conditions:*

(a) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, t) \geq 1$ for all $t > 0$;*
(b) *f is α-admissible;*
(c) *$\eta(r * s) \leq \eta(r) + \eta(s)$ for all $r, s \in \{M(x, fx, t) : x \in X, t > 0\}$;*
(d) *each subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of a sequence $\{x_n\}_{n\in\mathbb{N}} = \{f^n x_0\}_{n\in\mathbb{N}}$ has a following property:*

$$\alpha(x_{n_k}, x_{n_l}, t) \geq 1$$

for all $k, l \in \mathbb{N}$ *with* $k > l$ *and* $t > 0$*;*

(e) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}, t) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $t > 0$ *and* $\lim_{n\to\infty} x_n = x$, *then* $\alpha(x_n, x, t) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $t > 0$.

Then f *has a fixed point* $x^* \in X$. *Moreover, the sequence* $\{f^n x_0\}_{n\in\mathbb{N}}$ *converges to* x^*.

Proof Let $x_0 \in X$ and $\alpha(x_0, fx_0, t) \geq 1$ for all $t > 0$. Define a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n = fx_{n-1} = f^n x_0$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point of f. So, assume $x_n \neq x_{n-1}$ for each $n \in \mathbb{N}$. Since f is α-admissible, we have

$$\alpha(x_0, fx_0, t) = \alpha(x_0, x_1, t) \geq 1 \Longrightarrow \alpha(fx_0, fx_1, t) = \alpha(x_1, x_2, t) \geq 1$$

for all $t > 0$. By induction, we get

$$\alpha(x_n, x_{n+1}, t) \geq 1, \;\; n \in \mathbb{N}, \quad \forall t > 0.$$

By (11.11), we have

$$\begin{aligned} \eta\left(M(x_1, x_2, t)\right) &= \eta\left(M(fx_0, fx_1, t)\right) \\ &\leq \alpha(x_0, x_1, t)\eta\left(M(fx_0, fx_1, t)\right) \\ &\leq k\eta\left(M(x_0, x_1, t)\right), \quad \forall t > 0. \end{aligned}$$

Inductively, we have

$$\eta\left(M(x_n, x_{n+1}, t)\right) \leq k\eta\left(M(x_{n-1}, x_n, t)\right) \leq \cdots \leq k^n\eta\left(M(x_0, x_1, t)\right) \tag{11.16}$$

for all $n \in \mathbb{N}$ and $t > 0$. Since η is strictly decreasing and $k \in (0, 1)$, we have

$$M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, t)$$

for all $n \in \mathbb{N}$ and $t > 0$. So, for every $t > 0$, the sequence $\{M(x_n, x_{n+1}, t)\}_{n\in\mathbb{N}}$ is nondecreasing and bounded, it is convergent, i.e.,

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = p, \quad \forall t > 0.$$

Let us prove, by the contradiction, that $p = 1$. Suppose that $p < 1$. Letting $n \to \infty$ in (11.16), since η is continuous, we have

$$\lim_{n\to\infty} \eta\left(M(x_n, x_{n+1}, t)\right) \leq k \lim_{n\to\infty} \eta\left(M(x_{n-1}, x_n, t)\right), \quad \forall t > 0.$$

So, we obtain a contradiction $\eta(p) \leq k\eta(p)$ and conclude that $p = 1$, i.e.,

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1, \quad \forall t > 0. \tag{11.17}$$

Let us prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Suppose the contrary. Then there exist $\varepsilon > 0$, $t_0 > 0$ and $s_0 \in \mathbb{N}$ such that, for each $s \in \mathbb{N}$ and $s \geq s_0$, there exist $m(s), n(s) \in \mathbb{N}$, $m(s) > n(s) \geq s$ such that

$$\eta\left(M(x_{m(s)}, x_{n(s)}, t_0)\right) \geq \varepsilon$$

and, by the condition (d),

$$\alpha(x_{m(s)-1}, x_{n(s)-1}, t_0) \geq 1.$$

Let, for each $s \geq 1$, $m(s)$ be the least positive integer exceeding $n(s)$ satisfying the above property, i.e., $\eta\left(M(x_{m(s)-1}, x_{n(s)}, t_0)\right) < \varepsilon$ and $\eta\left(M(x_{m(s)}, x_{n(s)}, t_0)\right) \geq \varepsilon$ for each $s \in \mathbb{N}$. Since η is continuous, there exists $0 < \varepsilon_1 < 1$ such that $\eta(\varepsilon_1) = \varepsilon$, i.e.,

$$M(x_{m(s)-1}, x_{n(s)}, t_0) > \varepsilon_1, \quad \forall s \in \mathbb{N}. \tag{11.18}$$

Then we have

$$\begin{aligned}\varepsilon &\leq \eta\left(M(x_{m(s)}, x_{n(s)}, t_0)\right)\\ &\leq \alpha(x_{m(s)-1}, x_{n(s)-1}, t_0)\, \eta\left(M(x_{m(s)}, x_{n(s)}, t_0)\right)\\ &\leq k\,\eta\left(M(x_{m(s)-1}, x_{n(s)-1}, t_0)\right), \quad \forall s \in \mathbb{N}.\end{aligned} \tag{11.19}$$

Since fuzzy metric is strong, we obtain

$$\begin{aligned}M(x_{m(s)-1}, x_{n(s)-1}, t_0) &\geq *_L \left\{M(x_{m(s)-1}, x_{n(s)}, t_0), M(x_{n(s)}, x_{n(s)-1}, t_0)\right\}\\ &= \max\left\{M(x_{m(s)-1}, x_{n(s)}, t_0) + M(x_{n(s)}, x_{n(s)-1}, t_0) - 1, 0\right\}\end{aligned} \tag{11.20}$$

for each $s \in \mathbb{N}$. Take ε_1 defined in (11.18). Then, by (11.17), there exist $s_0 \in \mathbb{N}$ such that

$$M(x_{n(s)}, x_{n(s)-1}, t_0) > 1 - \varepsilon_1, \quad \forall s > s_0. \tag{11.21}$$

Now, by (11.18) and (11.21), we get

$$M(x_{m(s)-1}, x_{n(s)}, t_0) + M(x_{n(s)}, x_{n(s)-1}, t_0) > 1, \quad \forall s > s_0. \tag{11.22}$$

So, applying (11.19), (11.20), (11.22) and the condition (c), we get

$$\begin{aligned}\varepsilon &\leq \eta(M(x_{m(s)}, x_{n(s)}, t_0))\\ &\leq k\eta\left(M(x_{m(s)-1}, x_{n(s)-1}, t_0)\right)\\ &\leq k\left[\eta\left(M(x_{m(s)-1}, x_{n(s)}, t)\right) + \eta\left(M(x_{n(s)}, x_{n(s)-1}, t)\right)\right]\end{aligned}$$

for each $s > s_0$. Letting $s \to \infty$ in the above expression, we get

$$\varepsilon \leq k\,\varepsilon < \varepsilon.$$

So, we get a contradiction. Hence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X.

The rest of the proof follows similar lines to Theorem 11.16. This completes the proof.

Remark 11.13 In the paper of Wardowski ([49]) one could find the following open question:

"Can the condition (a) in Theorem 15 (i.e., Theorem 3.2 in [49]) be omitted for nilpotent t-norms?"

If $\alpha(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ in Theorem 18, then a partial answer to this question is obtained. Namely, in narrowed space (strong fuzzy metric space), we could expand the class of the t-norms, i.e., in that case Theorem 18 holds for the nilpotent t-norm $* = *_L$.

Open Problem. Can the assumption of strong fuzzy metric in Theorem 18 be omitted/further relaxed?

11.8 Fuzzy $\mathscr{Z}$-Contractive Mappings

Most recently Shukla et al. [40] unified different classes of fuzzy contractive mappings by introducing a new class of fuzzy contractive mappings called as Fuzzy $\mathscr{Z}$-contractive mappings.

First, we define the $\mathscr{Z}$-contraction in GV-fuzzy metric spaces. Denote by $\mathscr{Z}$ the family of all functions $\zeta\colon (0, 1] \times (0, 1] \to \mathbb{R}$ satisfying the following condition:

$$\zeta(t, s) > s$$

for all $t, s \in (0, 1)$.

Example 11.12 Consider the following functions ζ from $(0, 1] \times (0, 1]$ into $\mathbb{R}$ defined by

(1) $\zeta(t, s) = \psi(s)$, where $\psi\colon (0, 1] \to (0, 1]$ is a function such that $s < \psi(s)$ for all $s \in (0, 1)$;

(2) $\zeta(t, s) = \dfrac{1}{s + t} + t$;

(3) $\zeta(t, s) = \dfrac{s}{t}$.

Then, in all the cases, $\zeta \in \mathscr{Z}$.

Remark 11.14 By the above definition, it is obvious that $\zeta(t, t) > t$ for all $0 < t < 1$.

Definition 11.22 Let $(X, M, *)$ be a fuzzy metric space and $f\colon X \to X$ be a mapping. Suppose that there exists $\zeta \in \mathscr{Z}$ such that

$$M(fx, fy, t) \geq \zeta(M(fx, fy, t), M(x, y, t)) \tag{11.23}$$

for all $x, y \in X$ with $fx \neq fy$ and $t > 0$. Then f is called a *fuzzy $\mathscr{Z}$-contractive mapping* with respect to the function $\zeta \in \mathscr{Z}$.

Example 11.13 Every Tirado's contraction with contractive constant k is a fuzzy $\mathscr{Z}$-contraction with respect to the function $\zeta_f \in \mathscr{Z}$ defined by $\zeta_f(t, s) = 1 + k(s - 1)$ for all $s, t \in (0, 1]$.

Example 11.14 Every fuzzy contractive mapping with contractive constant k is a fuzzy $\mathscr{Z}$-contraction with respect to the function $\zeta_{GS} \in \mathscr{Z}$ defined by $\zeta_{GS}(t, s) = \frac{s}{k+(1-k)s}$ for all $s, t \in (0, 1]$.

Example 11.15 In view of Remark 11.11, every $\mathscr{H}$-contractive mapping with respect to $\eta \in \mathscr{H}$ is a fuzzy $\mathscr{Z}$-contraction with respect to the function $\zeta_W \in \mathscr{Z}$ defined by $\zeta_W(t, s) = \eta^{-1}(k\eta(s))$ for all $s, t \in (0, 1]$.

Example 11.16 Every ψ-contractive mapping is a fuzzy $\mathscr{Z}$-contraction with respect to the function ζ_M defined by $\zeta_M(t, s) = \psi(s)$ for all $s, t \in (0, 1]$.

Example 11.17 Let $X = \mathbb{R}$ and d be the usual metric on X. Then $(X, M_d, *_m)$ is a complete fuzzy metric space, where $M_d = \dfrac{t}{t + d(x, y)}$ for all $x, y \in X, t > 0$, is the standard fuzzy metric induced by d (see [14]). Let $f : X \to X$ be Edelstein's mapping (contractive mapping) on metric space (X, d), i.e., $d(fx, fy) < d(x, y)$ for all $x, y \in X$, then f is a fuzzy $\mathscr{Z}$-contractive mapping with respect to the function $\zeta_m \in \mathscr{Z}$ defined by

$$\zeta_m(t, s) = \begin{cases} \frac{s+t}{2}, & \text{if } t > s; \\ 1, & \text{otherwise.} \end{cases}$$

Indeed, the above fact remains true, if instead $\frac{s+t}{2}$ (i.e., the arithmetic mean of s and t) for $t > s$, one take geometric or harmonic mean of s and t.

Remark 11.15 If $(X, M, *)$ is an arbitrary fuzzy metric space and $f : X \to X$ be a Edelstein's mapping on $(X, M, *)$, i.e., $M(fx, fy, t) > M(x, y, t)$ for all $x, y \in X$ and $t > 0$. Then f is a fuzzy $\mathscr{Z}$-contractive mapping with respect to the function $\zeta_m \in \mathscr{Z}$ defined in the previous example. Therefore, we conclude that for any given fuzzy Edelstein's mapping we always have $\zeta(= \zeta_m) \in \mathscr{Z}$ such that the fuzzy Edelstein mapping is a fuzzy $\mathscr{Z}$-contractive mapping and so the contractive mappings considered by Tirado [43], Gregori and Sapena [19], Wardowski [49] and Miheţ [30] are included in this new class. Although there are fuzzy $\mathscr{Z}$-contractive mapping which do not belong to any of these considered classes (see, e.g., Example 11.18, Example 11.20 and Example 11.22).

The following example shows that a fuzzy $\mathscr{Z}$-contractive mapping may not have a fixed point even in an M-complete fuzzy metric space:

Example 11.18 Let $X = \mathbb{N}$ and define the fuzzy set M on $X \times X \times (0, \infty)$ by $M(n, m, t) = \min\left\{\frac{n}{m}, \frac{m}{n}\right\}$ for all $n, m \in X$ and $t > 0$. Then $(X, M, *_p)$ is an M-complete fuzzy metric space. Define a mapping $f : X \to X$ by $fn = n + 1$ for all $n \in X$. Then f is a fuzzy $\mathscr{Z}$-contractive mapping with respect to the function $\zeta_m \in \mathscr{Z}$ defined in Example 11.17. Notice that f is a fixed point free mapping on X.

The above example motivates us for the consideration of a space having some additional property so that the existence of fixed point of fuzzy $\mathscr{Z}$-contractive mapping can be ensured.

Definition 11.23 Let $(X, M, *)$ be a fuzzy metric space, $f : X \to X$ a mapping and $\zeta \in \mathscr{Z}$. Then we say that the quadruple (X, M, f, ζ) has the *property* (S) if, for any Picard sequence $\{x_n\}$ with initial value $x \in X$, i.e., $x_n = f^n x$ for all $n \in \mathbb{N}$ such that $\inf_{m>n} M(x_n, x_m, t) \leq \inf_{m>n} M(x_{n+1}, x_{m+1}, t)$ for all $n \in \mathbb{N}$ and $t > 0$ implies that

$$\lim_{n\to\infty} \inf_{m>n} \zeta(M(x_{n+1}, x_{m+1}, t), M(x_n, x_m, t)) = 1$$

for all $t > 0$.

The following example shows that there exists a function ζ such that the mappings introduced by Tirado [43] forms a quadruple (X, M, f, ζ), which satisfies the property (S), where $(X, M, *)$ is an arbitrary fuzzy metric space:

Example 11.19 Let $(X, M, *)$ be an arbitrary fuzzy metric space and $f : X \to X$ be a fuzzy Tirado-contraction. Then the quadruple (X, M, f, ζ) has the property (S) with $\zeta(t, s) = 1 + k(s - 1)$ for all $t, s \in (0, 1]$. Indeed, if $x \in X$ and $\{x_n\}$ be a Picard sequence with initial value x such that

$$\inf_{m>n} M(x_n, x_m, t) \leq \inf_{m>n} M(x_{n+1}, x_{m+1}, t)$$

for all $n \in \mathbb{N}$ and $t > 0$, then $\lim_{n\to\infty} \inf_{m>n} M(x_n, x_m, t)$ must exists for all $t > 0$. Suppose that $\lim_{n\to\infty} \inf_{m>n} M(x_n, x_m, t) = a(t)$ for all $t > 0$, then $a(t) \leq 1$. By the definition of ζ, for every $t > 0$, we have

$$\lim_{n\to\infty} \inf_{m>n} \zeta(M(x_{n+1}, x_{m+1}, t), M(x_n, x_m, t)) = 1 + k(a(t) - 1).$$

Also, by the contractivity condition, we obtain $1 + ka(t) \leq k + a(t)$ and so $1 \leq a(t)$. It shows that $a(t) = 1$ for all $t > 0$, i.e.,

$$\lim_{n\to\infty} \inf_{m>n} \zeta(M(x_{n+1}, x_{m+1}, t), M(x_n, x_m, t)) = 1.$$

The following theorem generalizes Theorem 11.4 (see also Corollary 3.9 in [30]) for arbitrary t-norms:

Theorem 11.19 *Let $(X, M, *)$ be an M-complete fuzzy metric space and $f : X \to X$ be a fuzzy $\mathscr{Z}$-contraction. If the quadruple (X, M, f, ζ) has the property (S), then f has a unique fixed point $u \in X$.*

Proof First, we show that if the fixed point of f exists, then it is unique. Suppose that u, v are two distinct fixed point of f, i.e., $fu = u$ and $fv = v$ and there exists $s > 0$ such that $M(u, v, s) < 1$. Then, by the condition (11.23) and the definition of ζ, we have

$$M(u, v, s) = M(fu, fv, s) \geq \zeta(M(fu, fv, s), M(u, v, s)) > M(u, v, s).$$

This contradiction shows that $M(u, v, t) = 1$ for all $t > 0$ and so $u = v$. It proves the uniqueness.

Now, we show the existence of fixed point of f. Let $x_0 \in X$ and define the Picard sequence $\{x_n\}$ by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$.

If $x_n = x_{n-1}$ for any $n \in \mathbb{N}$, then $fx_{n-1} = x_n = x_{n-1}$ is a fixed point of f. Therefore, we assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$, i.e., no consecutive terms of the sequence $\{x_n\}$ are equal.

Further, if $x_n = x_m$ for some $n < m$, then, as no consecutive terms of the sequence $\{x_n\}$ are equal from (11.23), we have

$$M(x_{n+1}, x_{n+2}, t) \geq \zeta(M(x_{n+1}, x_{n+2}, t), M(x_n, x_{n+1}, t)) > M(x_n, x_{n+1}, t),$$

i.e., $M(x_n, x_{n+1}, t) < M(x_{n+1}, x_{n+2}, t)$.

Similarly, one can prove that

$$M(x_n, x_{n+1}, t) < M(x_{n+1}, x_{n+2}, t) < \cdots < M(x_m, x_{m+1}, t).$$

Since $x_n = x_m$, we have $x_{n+1} = fx_n = fx_m = x_{m+1}$ and so the above inequality yields a contradiction. Thus we can assume that $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$.

Now, for $t > 0$, let

$$a_n(t) = \inf_{m>n} M(x_n, x_m, t).$$

Then it follows from (11.23) and the definition of ζ that

$$\begin{aligned} M(x_{n+1}, x_{m+1}, t) &= M(fx_n, fx_m, t) \\ &\geq \zeta(M(fx_n, fx_m, t), M(x_n, x_m, t)) \\ &> M(x_n, x_m, t) \end{aligned} \tag{11.24}$$

for each $t > 0$. Therefore, for all $n < m$, we have

$$M(x_n, x_m, t) < M(x_{n+1}, x_{m+1}, t) \text{ for all } n < m.$$

Taking infimum over $m(> n)$ in the above inequality, we obtain

$$\inf_{m>n} M(x_n, x_m, t) \leq \inf_{m>n} M(x_{n+1}, x_{m+1}, t),$$

i.e., $a_n(t) \leq a_{n+1}(t)$ for all $n \in \mathbb{N}$. Thus $\{a_n(t)\}$ is bounded and monotonic for all $t > 0$.

Suppose that $\lim\limits_{n\to\infty} a_n(t) = a(t)$ for all $t > 0$. We claim that $a(t) = 1$ for all $t > 0$. If $s > 0$ and $a(s) < 1$, then, using the fact that the quadruple (X, M, f, ζ) having the property (S), we obtain

$$\lim_{n\to\infty} \inf_{m>n} \zeta(M(x_n, x_m, s), M(x_{n+1}, x_{m+1}, s)) = 1. \tag{11.25}$$

From the inequality (11.24), we have

$$\inf_{m>n} M(x_{n+1}, x_{m+1}, s) \geq \inf_{m>n} \zeta(M(fx_n, fx_m, s), M(x_n, x_m, s)) \geq \inf_{m>n} M(x_n, x_m, s),$$

i.e.,

$$a_{n+1}(s) \geq \inf_{m>n} \zeta(M(fx_n, fx_m, s), M(x_n, x_m, s)) \geq a_n(s).$$

Letting $n \to \infty$ and using (11.25) in the above inequality, we obtain

$$\lim_{n\to\infty} \inf_{m>n} M(x_n, x_m, s) = a(s) = 1.$$

This contradiction verifies our claim. By the definition of a_n, we have $\lim\limits_{n,m\to\infty} M(x_n, x_m, t) = 1$ for all $t > 0$. Hence $\{x_n\}$ is an M-Cauchy sequence and, by M-completeness of X, there exists $u \in X$ such that

$$\lim_{n\to\infty} M(x_n, u, t) = 1, \quad \forall t > 0. \tag{11.26}$$

Now, we show that u is a fixed point of f. Suppose that $fu \neq u$. Without loss of generality, we can assume that $x_n \neq u$ and $x_n \neq Tu$ for all $n \in \mathbb{N}$, and so, there exists $s > 0$ such that $M(u, fu, s) < 1$, $M(x_n, u, s) < 1$ and $M(x_{n+1}, fu, s) = M(fx_n, fu, s) < 1$ for all $n \in \mathbb{N}$. Then we have

$$M(x_n, u, s) < \zeta(M(fx_n, fu, s), M(x_n, u, s)) \leq M(fx_n, fu, s) = M(x_{n+1}, fu, s).$$

Letting $n \to \infty$ and using (11.26), we obtain $1 \leq M(u, fu, s)$. This contradiction shows that $M(u, fu, t) = 1$ for all $t > 0$ and so $fu = u$. Thus the existence of fixed point follows. This completes the proof.

Remark 11.16 Example 11.13 and Example 11.19 shows that the above theorem generalizes Theorem 11.4 for arbitrary t-norms.

The following example shows that this generalization is proper:

Example 11.20 Let $\{x_n\}$ be a strictly increasing sequence of real numbers such that $0 < x_n \leq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = 1$. Let $X = \{x_n : n \in \mathbb{N}\} \cup \{1\}$ and define a fuzzy set M on $X \times X \times (0, \infty)$ by:

$$M(x, y, t) = \begin{cases} 1, & \text{if } x = y; \\ \min\{x, y\}, & \text{otherwise,} \end{cases}$$

for all $x, y \in X$ and $t \in (0, \infty)$. Then $(X, M, *_m)$ is an M-complete fuzzy metric space. Define a function $\zeta\colon (0, 1] \times (0, 1] \to \mathbb{R}$ by

$$\zeta(t, s) = \begin{cases} t, & \text{if } t > s, \\ \sqrt{s}, & \text{if } t \leq s, \end{cases}$$

for all $s, t \in (0, 1]$ and a mapping $f\colon X \to X$ by

$$f x_n = x_{n+1} \text{ and } f1 = 1$$

for all $n \in \mathbb{N}$. Then $\zeta \in \mathscr{Z}$ and the quadruple (X, M, f, ζ) has the property (S). Furthermore, the mapping f is a fuzzy $\mathscr{Z}$-contractive mapping with respect to the function ζ. Thus all the conditions of Theorem 11.19 are satisfied and we can conclude the existence of fixed point of f. Indeed, $x = 1$ is the unique fixed point of f.

Remark 11.17 In view of the above example, we can conclude that the mapping f is not Tirado's contraction. For instance, take the sequence $\{x_n\}$ defined by $x_n = 1 - \dfrac{1}{2n^2}$ for all $n \in \mathbb{N}$ in the above example. Then we have

$$1 - M(f x_n, f x_{n+1}, t) = 1 - M(x_{n+1}, x_{n+2}, t) = 1 - x_{n+1},$$

$$1 - M(x_n, x_{n+1}, t) = 1 - x_n$$

for all $t > 0$. Therefore, for sufficient large n, there exists no k such that $k \in [0, 1)$ and

$$1 - M(f x_n, f x_{n+1}, t) \leq k[1 - M(x_n, x_{n+1}, t)], \quad \forall t > 0.$$

Thus Theorem 11.19 is an actual generalization of the fixed point result of Tirado [43], i.e., Theorem 11.4.

Next, we introduce another condition (S') which is weaker than the condition (S).

Definition 11.24 Let $(X, M, *)$ be a fuzzy metric space, $f\colon X \to X$ be a mapping and $\zeta \in \mathscr{Z}$. Then we say that the quadruple (X, M, f, ζ) has the *property* (S') if, for any Picard sequence $\{x_n\}$ with initial value $x \in X$, i.e., $x_n = f^n x$ for all $n \in \mathbb{N}$ such that $\inf_{m>n} M(x_n, x_m, t) \leq \inf_{m>n} M(x_{n+1}, x_{m+1}, t)$ for all $n \in \mathbb{N}$ and $t > 0$ and $0 < \lim_{n\to\infty} \inf_{m>n} M(x_n, x_m, t) < 1$ for all $t > 0$, we have

$$\lim_{n\to\infty}\inf_{m>n}\zeta(M(x_{n+1},x_{m+1},t),M(x_n,x_m,t))=1,\quad \forall t>0.$$

The following example verifies the fact that condition (S') is weaker than condition (S):

Example 11.21 Let $\varepsilon>0$ be fixed and $X=[\varepsilon,\infty)$. Define a fuzzy set M on $X\times X\times(0,\infty)$ by

$$M(x,y,t)=\begin{cases}1, & \text{if } x=y;\\ \dfrac{1}{1+\max\{x,y\}}, & \text{otherwise,}\end{cases}$$

for all $x,y\in X$ and $t\in(0,\infty)$. Then $(X,M,*_m)$ is a fuzzy metric space. Define a mapping $f\colon X\to X$ by $fx=2x$ for all $x\in X$. Suppose that $\zeta\colon(0,1]\times(0,1]\to\mathbb{R}$ is defined by $\zeta(t,s)=\psi(s)$ for all $t,s\in(0,1]$, where $\psi\in\Psi$ is such that $\psi(0)=0$. Then it is easy to see that the quadruple (X,M,f,ζ) satisfies the condition (S') trivially.

On the other hand, the quadruple (X,M,f,ζ) does not satisfy the condition (S). Indeed, for any $x\in X$ and $t>0$, we have

$$\inf_{m>n}M(f^nx,f^mx,t)=\inf_{m>n}M(2^nx,2^mx,t)=0<1.$$

Therefore, $\inf_{m>n}M(f^nx,f^mx,t)\le\inf_{m>n}M(f^{n+1}x,f^{m+1}x,t)$ for all $n\in\mathbb{N}$ and $t>0$, but we have

$$\lim_{n\to\infty}\inf_{m>n}\zeta(M(x_{n+1},x_{m+1},t),M(x_n,x_m,t))=\lim_{n\to\infty}\inf_{m>n}\psi(M(x_n,x_m,t))=0\neq 1.$$

In the next theorem, we see that the condition (S') enables us to extend the result of Mihet [30] for fuzzy $\mathscr{Z}$-contraction, but with an additional assumption to Theorem 11.19:

Theorem 11.20 *Let $(X,M,*)$ be an M-complete fuzzy metric space, $f\colon X\to X$ be a fuzzy $\mathscr{Z}$-contraction and the quadruple (X,M,f,ζ) has the property (S'). In addition, suppose that $\lim_{n\to\infty}\inf_{m>n}M(f^nx,f^mx,t)>0$ for all $x\in X$ and $t>0$. Then f has a unique fixed point $u\in X$.*

Proof Because of $\lim_{n\to\infty}\inf_{m>n}M(f^nx,f^mx,t)>0$ for all $x\in X$ and $t>0$, following the lines of the proof of Theorem 11.19 and using the property (S'), we obtain the required result. This completes the proof.

In the next example, we show that the class of fuzzy $\mathscr{Z}$-contractions is wider than that of fuzzy ψ-contractions and verify the merit of fuzzy $\mathscr{Z}$-contractive mappings over fuzzy ψ-contractive mappings. For this, we use the idea of Example 11.20.

Example 11.22 Let $X = \{x_n : n \in \mathbb{N}\} \cup \{1\}$, where $\{x_n\}$ is an arbitrary sequence such that $x_n \in (0, 1)$, $x_n < x_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = 1$. Define a fuzzy set M on $X \times X \times (0, \infty)$ by

$$M(x, y, t) = \begin{cases} 1, & \text{if } x = y, \\ \min\{x, y\}, & \text{otherwise,} \end{cases}$$

for all $x, y \in X$ and $t \in (0, \infty)$. Then $(X, M, *_m)$ is an M-complete fuzzy metric space. Define a mapping $f : X \to X$ by $fx_n = x_{n+1}$ for all $n \in \mathbb{N}$ and $f1 = 1$. Then we claim that T is not a fuzzy ψ-contraction. On the contrary, suppose that T is a fuzzy ψ-contraction. Therefore, there exists $\psi \in \Psi$ such that $\psi(M(x_n, x_m, t)) \leq M(fx_n, fx_m, t)$ for all $n, m \in \mathbb{N}$ with $n < m$, i.e.,

$$x_n < \psi(x_n) \leq x_{n+1}. \tag{11.27}$$

Since $\psi \in \Psi$, we can choose the sequence $\{x_n\}$ such that, for any $x_1 \in (0, 1)$, $x_{n+1} = \dfrac{x_n + \psi(x_n)}{2}$ for all $n \in \mathbb{N}$. Then, by (11.27), we obtain

$$x_n < \psi(x_n) \leq \frac{x_n + \psi(x_n)}{2}.$$

The above inequalities contradict the definition of ψ. Therefore, f is not a fuzzy ψ-contraction. On the other hand, we have shown in Example 11.20 that the mapping f is a fuzzy $\mathscr{Z}$-contractive mapping as well as, the condition (S') is satisfied. Now the existence and uniqueness of fixed point of f is assured by Theorem 11.20. Indeed, 1 is the unique fixed point of f.

Corollary 11.2 *Let $(X, M, *)$ be an M-complete fuzzy metric space, $f : X \to X$ be a fuzzy ψ-contractive mapping and $\lim_{n\to\infty} \inf_{m>n} M(f^n x, f^m x, t) > 0$ for all $x \in X$, $t > 0$. Then f has a unique fixed point $u \in X$.*

Proof In view of Example 11.16 we need only to show that the quadruple (X, M, f, ζ) have the property (S'), where $\zeta(t, s) = \psi(s)$. Suppose that $x \in X$ and $\{x_n\}$ is a Picard sequence with the initial value x such that $\inf_{m>n} M(x_n, x_m, t) \leq \inf_{m>n} M(x_{n+1}, x_{m+1}, t)$ and, for all $t > 0$, $0 < \lim_{n\to\infty} \inf_{m>n} M(x_n, x_m, t) = a(t) < 1$. Then, by the definition of ψ, it follows that, for all $t > 0$,

$$\lim_{n\to\infty} \inf_{m>n} \zeta(M(x_{n+1}, x_{m+1}, t), M(x_n, x_m, t)) = \psi(a(t)).$$

Also, by the ψ-contractivity, we obtain $\psi(a(t)) \leq a(t)$ and so $a(t) = 1$, i.e.,

$$\lim_{n\to\infty} \inf_{m>n} \zeta(M(x_{n+1}, x_{m+1}, t), M(x_n, x_m, t)) = 1, \quad \forall t > 0.$$

Therefore, the quadruple (X, M, f, ζ) has the property (S'). This completes the proof.

Remark 11.18 Since the class of fuzzy ψ-contractions consists of the class of fuzzy contractive mappings [19], Tirado's contraction [43] and Wardowski's contraction [49], therefore, fixed point results for these contractions can be obtained by the above corollary.

Remark 11.19 It is clear from the definition that every fuzzy $\mathscr{Z}$-contractive mapping is a fuzzy Edelstein's mapping (contractive mapping). Also, Remark 11.15 shows that, for every fuzzy Edelstein's mapping f, there exists a function $\zeta_{\text{mean}} \in \mathscr{Z}$ such that f is a fuzzy $\mathscr{Z}$-contractive mapping with $\zeta_{\text{mean}} \in \mathscr{Z}$. In view of existence of fixed point of mapping f, notice that, for a fuzzy Edelstein's mapping, the quadruple $(X, M, f, \zeta_{\text{mean}})$ need not have the property (S), e.g., in Example 11.18, f is a fuzzy Edelstein's mapping but the quadruple (X, M, f, ζ) does not possess the property (S). Indeed, in this example, for any Picard sequence $\{x_n\}$ with initial value $x \in X$, we have

$$\inf_{m>n} M(x_n, x_m, t) \leq \inf_{m>n} M(x_{n+1}, x_{m+1}, t)$$

for all $n \in \mathbb{N}$ and $t > 0$, but

$$\lim_{n\to\infty} \inf_{m>n} \zeta_{\text{mean}}(M(x_{n+1}, x_{m+1}, t), M(x_n, x_m, t)) = 0 \neq 1, \quad \forall t > 0.$$

Therefore, the condition (S) of Theorem 11.19 is not satisfied. Also, one can see that the condition: $\lim\limits_{n\to\infty} \inf\limits_{m>n} M(f^n x, f^m x, t) > 0$ for all $x \in X$ and $t > 0$ of Theorem 11.20 is not satisfied, while the condition (S') is satisfied.

Remark 11.20 Motivated by the results of Tirado [43] and Miheţ [30], we introduced the class of fuzzy $\mathscr{Z}$-contractive mappings and showed that the mappings of this new class have a unique fixed point on an arbitrary M-complete fuzzy metric space having the properties (S) and (S'). With suitable examples, we showed that the class of fuzzy $\mathscr{Z}$-contractive mappings is weaker than the existing ones in the literature. Further, it will be interesting to apply this new approach in general settings, e.g., in fuzzy metric-like setting (see [38, 39]) as well as it will be interesting to generalize the class of fuzzy $\mathscr{Z}$-contractive mappings for weaker contractive conditions, e.g., (ε, δ)-type contractive conditions (see [31]).

11.9 Conclusions

The notion of fuzzy metric spaces are introduced for the first time by I. Kramosil and J. Michalek in 1975, thus releasing axioms to the fuzzy metric spaces requires a function of the distance has supremum 1, in the relation to the axiomatic of probability of metric spaces. The modified definition of the fuzzy metric spaces introduces A.

George and P. Veeramani In 1994, which relieves axiomatic of the fuzzy metric spaces and it is desired that the infimum of the function of the distance is 0, in relation to the probability approximation space. Today, they are studying the fuzzy metric spaces in terms of both definitions.

Recently, many authors (see, for example, [2, 4, 5, 7, 11, 18, 21, 33, 42] and referenced mentioned their in) observed that the various contraction mappings in metric spaces may be exactly translated into probabilistic or fuzzy metric spaces endowed with special $t-$norms, such as minimum $t-$norm.

Starting with famous Banach contraction principle, a huge number of mathematicians started to formulate better contractive conditions for which fixed point exists. In this chapter, we have identified some of the first but no less important contraction conditions that have been formulated by well-known mathematicians, Grabiec, Gregori-Sapena, Tirado, Mihet, Wardowski, and made them in the framework of the fuzzy metric space.

It is our hope that the material presented in this chapter will be enough to stimulate scientists and students to investigate further this challenging field.

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Chapter 12
Common Fixed Point Theorems for Four Maps

Muhammad Nazam, Choonkil Park, and Muhammad Arshad

Abstract In this chapter, we manifest some coincidence and common fixed point theorems for four maps satisfying Círíc type and Hardy–Rogers type (α, F)-contractions on α-complete metric spaces. We apply these results to infer several new and old corresponding results in ordered metric spaces and graphic metric spaces. These results also generalize some results obtained previously. We present an example and an application to support our results.

Keywords α-Complete metric space · (α, F)-Contraction · Common fixed point · Four maps · Coincidence point

12.1 Introduction and Preliminaries

After the famous Banach's Contraction Principle, a large number of researchers revealed many fruitful generalizations of Banach's fixed point theorem. One of these generalizations is known as F-contraction presented by Wardowski [21]. Wardowski [21] evinced that every F-contraction defined on complete metric space has a unique fixed point. The concept of F-contraction proved another milestone in fixed point theory and numerous research papers on F-contraction have been published (see for instant [1, 2, 5, 12–14, 16, 18, 22]).

In 2012, Samet et al. [20] investigated the idea of (α, ψ)-contractive and α-admissible mappings and evinced some significant fixed point results for such kind of mappings defined on complete metric spaces. Subsequently, Salimi et al. [19] and

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Hussain et al. [10, 11] improved the concept of α-admissible mapping and proved some important (common) fixed point theorems.

Recently, Cosentino et al. [4] established a fixed point result for Hardy–Rogers type F-contraction and Minak et al. [17] presented a fixed point result for Círíc type generalized F-contraction. We bring into use the idea of Círíc type and Hardy–Rogers type (α, F)-contractions comprising four self-mappings defined on metric space. We present some fixed point results for four maps satisfying such kind of contractions on α-complete metric space. We apply our results to infer several new and old results. We present ordered metric and graphic metric versions of these theorems as consequences. We apply our result to show the existence of common solution of the system of Volterra type integral equations.

We denote $(0, \infty)$ by $\mathbf{R}^+$, $[0, \infty)$ by $\mathbf{R}_0^+$, $(-\infty, +\infty)$ by $\mathbf{R}$ and the set of natural numbers by $\mathbf{N}$. Wardowski [21] investigated a nonlinear function $F : \mathbf{R}^+ \to \mathbf{R}$ complying with the following axioms:

(F_1) F is strictly increasing;

(F_2) For each sequence $\{r_n\}$ of positive numbers $\lim_{n\to\infty} r_n = 0$ if and only if $\lim_{n\to\infty} F(r_n) = -\infty$;

(F_3) There exists $\theta \in (0, 1)$ such that $\lim_{\alpha\to 0^+} (\alpha)^\theta F(\alpha) = 0$.

We denote by Δ_F the set of all functions satisfying the conditions (F_1)–(F_3).

Example 12.1 ([21]) Let $F : \mathbf{R}^+ \to \mathbf{R}$ be the functions defined by
(1) $F(r) = \ln(r)$;
(2) $F(r) = r + \ln(r)$;
(3) $F(r) = \ln(r^2 + r)$;
(4) $F(r) = -\frac{1}{\sqrt{r}}$.

It is easy to check that the functions (1)–(4) (d) are members of Δ_F.

In [21], Wardowski utilized function F in an excellent manner and gave the following remarkable result:

Theorem 12.1 ([21]) *Let (M, d) be a complete metric space and $T : M \to M$ be a mapping satisfying*

$$(d(T(r_1), T(r_2)) > 0 \Longrightarrow \tau + F(d(T(r_1), T(r_2)) \leq F(d(r_1, r_2))) \tag{12.1}$$

for all $r_1, r_2 \in M$ and some $\tau > 0$. Then T has a unique fixed point $\upsilon \in M$ and, for every $r_0 \in M$, the sequence $\{T^n(r_0)\}$ for all $n \in \mathbf{N}$ is convergent to υ.

Remark 12.1 ([21, Remark 2.1]) In metric spaces, a mapping giving fulfillment to F-contraction is always a Banach contraction and hence a continuous mapping.

Definition 12.1 ([20]) Let $S : M \to M$ be a mapping and $\alpha : M \times M \to \mathbf{R}_0^+$ be a function. S is said to be an *α-admissible mapping* if

$$\alpha(r_1, r_2) \geq 1 \text{ implies } \alpha(S(r_1), S(r_2)) \geq 1$$

for all $r_1, r_2 \in M$.

Definition 12.2 ([20]) Let $S : M \to M$ be a mapping and $\alpha : M \times M \to \mathbf{R}_0^+$ be a function. The mapping S is said to be a *triangular α-admissible mapping* if the following conditions hold:

(a) $\alpha(r_1, r_2) \geq 1$ implies $\alpha(S(r_1), S(r_2)) \geq 1$;

(b) $\alpha(r_1, r_3) \geq 1$ and $\alpha(r_3, r_2) \geq 1$ imply $\alpha(r_1, r_2) \geq 1$ for all $r_1, r_2, r_3 \in M$.

Definition 12.3 ([3]) Let $f, g : M \to M$ be mappings and $\alpha : M \times M \to \mathbf{R}_0^+$ be a function. The pair (f, g) is said to be:

(1) a *weakly α-admissible pair* of mappings if

$$\alpha(f(r), gf(r)) \geq 1, \quad \alpha(g(r), fg(r)) \geq 1$$

for all $r \in M$;

(2) a *partially weakly α-admissible pair* of mappings if $\alpha(f(r), gf(r)) \geq 1$ for all $r \in M$.

Let $f^{-1}(r) = \{m \in M : \ f(m) = r\}$.

Definition 12.4 ([3]) Let $f, g, h : M \to M$ be three mappings such that $f(M) \cup g(M) \subseteq h(M)$ and $\alpha : M \times M \to \mathbf{R}_0^+$ be a function. The pair (f, g) is said to be:

(1) a *weakly α-admissible pair* of mappings with respect to h if $\alpha(f(r_1), g(r_2)) \geq 1$ for all $r_1 \in M$ and $r_2 \in h^{-1}f(r_1)$ and $\alpha(g(r_1), f(r_2)) \geq 1$ for all $r_2 \in h^{-1}g(r_1)$;

(2) a *partially weakly α-admissible pair* of mappings with respect to h if $\alpha(f(r_1), g(r_2)) \geq 1$ for all $r_1 \in M$ and $r_2 \in h^{-1}f(r_1)$.

Remark 12.2 Note that

(1) if $g = f$ in Definition 12.4, then f is weakly α-admissible (partially weakly α-admissible) with respect to h;

(2) if $H = I_M$ (the identity mapping on M), then Definition 12.4 reduces to Definition 12.3.

Definition 12.5 Let $f, g, h : M \to M$ be three mappings such that $f(M) \cup g(M) \subseteq h(M)$ and $\alpha : M \times M \to \mathbf{R}_0^+$ be a function. The pair (f, g) is said to be *triangular weakly α-admissible pair* of mappings with respect to h if the following conditions hold:

(a) $\alpha(f(r_1), g(r_2)) \geq 1$ for all $r_1 \in M$ and $r_2 \in h^{-1}f(r_1)$ and $\alpha(g(r_1), f(r_2)) \geq 1$ for all $r_2 \in h^{-1}g(r_1)$;

(b) $\alpha(r_1, r_3) \geq 1$ and $\alpha(r_3, r_2) \geq 1$ imply $\alpha(r_1, r_2) \geq 1$ for all $r_1, r_2, r_3 \in M$.

Definition 12.6 Let $f, g, h : M \to M$ be three mappings such that $f(M) \cup g(M) \subseteq h(M)$ and $\alpha : M \times M \to \mathbf{R}_0^+$ be a function. The pair (f, g) is said to be *triangular partially weakly α-admissible pair* of mappings with respect to h if the following conditions hold:

(a) $\alpha(f(r_1), g(r_2)) \geq 1$ for all $r_1 \in M$ and $r_2 \in h^{-1}f(r_1)$;

(b) $\alpha(r_1, r_3) \geq 1, \alpha(r_3, r_2) \geq 1$ imply $\alpha(r_1, r_2) \geq 1$ for all $r_1, r_2, r_3 \in M$.

Example 12.2 Let $M = [0, \infty)$ and define the functions by

$$f(r) = \begin{cases} r, & \text{if } r \in [0,1], \\ c, & \text{if } r \in (1,\infty), \end{cases} \quad g(r) = \begin{cases} r^{\frac{1}{3}}, & \text{if } r \in [0,1], \\ c, & \text{if } r \in (1,\infty), \end{cases}$$

$$S(r) = \begin{cases} r^3, & \text{if } r \in [0,1], \\ c, & \text{if } r \in (1,\infty), \end{cases} \quad T(r) = \begin{cases} r^5, & \text{if } r \in [0,1], \\ c, & \text{if } r \in (1,\infty), \end{cases}$$

where c is a constant. Define a mapping $\alpha : M \times M \to \mathbf{R}_0^+$ by $\alpha(r_1, r_2) = \pi^{r_2 - r_1}$ for all $r_1, r_2 \in M$. Then the pair (f, g) is a triangular weakly α-admissible pair of mappings with respect to T and (g, f) is a triangular weakly α-admissible pair of mappings with respect to S. Indeed, if $\begin{cases} \alpha(r_1, r_2) \geq 1, \\ \alpha(r_2, r_3) \geq 1, \end{cases}$ then $\begin{cases} r_1 - r_2 \leq 0, \\ r_2 - r_3 \leq 0, \end{cases}$ which implies that $r_1 - r_3 \leq 0$. Hence, $\alpha(r_1, r_3) = \pi^{r_3 - r_1} \geq 1$.

To prove that (f, g) is a partially weakly α-admissible pair of mappings with respect to T, let $r_1, r_2 \in M$ be such that $r_2 \in T^{-1} f(r_1)$, that is, $T(r_2) = f(r_1)$ and thus we have $r_2^5 = r_1$ or $r_2 = r_1^{\frac{1}{5}}$. Since $g(r_2) = r_1^{\frac{1}{15}} \geq r_1 = f(r_1)$ for all $r_1 \in [0, 1]$, $\alpha(fr_1, gr_2) = \pi^{gr_2 - fr_1} \geq 1$. Hence, (f, g) is a partially weakly α-admissible pair of mappings with respect to T. Similarly, it can be proved that (g, f) is a partially weakly α-admissible pair of mappings with respect to S.

Recently, Hussain et al. [11] introduced the concept of α-completeness for a metric space, which is weaker than the concept of completeness.

Definition 12.7 ([11]) Let (M, d) be a metric space and $\alpha : M \times M \to \mathbf{R}_0^+$ be a function. The metric space M is said to be α-*complete* if every Cauchy sequence $\{r_n\}$ in M such that $\alpha(r_n, r_{n+1}) \geq 1$ for all $n \in \mathbf{N}$ converges in M.

Remark 12.3 If M is a complete metric space, then M is also an α-complete metric space. But the converse is not true (see [15, Example 1.17]).

Definition 12.8 Let (M, d) be a metric space and $\alpha : M \times M \to \mathbf{R}_0^+$, $T : M \to M$ be two mappings. We say that T is an α-continuous mapping on (M, d) if, for any $r \in M$ and a sequence $\{r_n\}$,

$$\lim_{n\to\infty} d(r_n, r) = 0 \ \text{ and } \ \alpha(r_n, r_{n+1}) \geq 1 \ \text{ imply } \ \lim_{n\to\infty} d(T(r_n), T(r)) = 0.$$

Example 12.3 Let $M = [0, \infty)$ and $d : M \times M \to [0, \infty)$ be defined by $d(r_1, r_2) = |r_1 - r_2|$ for all $r_1, r_2 \in M$. Define the functions by

$$T(r) = \begin{cases} \sin(\pi r), & \text{if } r \in [0,1], \\ \cos(\pi r) + 2, & \text{if } r \in (1,\infty), \end{cases} \quad \alpha(r_1, r_2) = \begin{cases} r_1^3 + r_2^3 + 1, & \text{if } r_1, r_2 \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Then T is not continuous on M, however, T is α-continuous.

Definition 12.9 ([3]) Let (M, d) be a metric space. The pair $\{f, g\}$ is said to be *α-compatible* if $\lim_{n\to\infty} d(fg(r_n), gf(r_n)) = 0$, whenever $\{r_n\}$ is a sequence in M such that $\alpha(r_n, r_{n+1}) \geq 1$ and

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} g(r_n) = t$$

for some $t \in M$.

Remark 12.4 If (f, g) is a compatible pair, then (f, g) is also an α-compatible pair. But the converse is not true.

Definition 12.10 ([8]) Let f and T be self-mappings defined on a nonempty set M. If $f(r) = T(r)$ for some $r \in M$, then r is called a *coincidence point* of f and T. Two self-mappings f and T defined on M are said to be *weakly compatible* if they commute at their coincidence points, That is, if $f(r) = T(r)$ for some $r \in M$, then $fT(r) = Tf(r)$.

Example 12.4 Let $M = \mathbf{R}$ and $T, f : M \to M$ be the mappings given by

$$T(r) = 6r - 5, \qquad f(r) = 5r - 4$$

for all $r \in M$. Then f, T are weakly compatible mappings for coincidence point $r = 1$.

Definition 12.11 Let (M, d) be a metric space and $\alpha : M \times M \to \mathbf{R}_0^+$ be a function. The space (M, d) is said to be *α-regular* if there exists a sequence $\{r_n\}$ in M such that, if $r_n \to r$ and $\alpha(r_n, r_{n+1}) \geq 1$, then $\alpha(r_n, r) \geq 1$ for all $n \in \mathbf{N}$.

Lemma 12.1 *Let (M, d) be a metric space. Assume that there exist two sequences $\{r_n\}, \{s_n\}$ such that*

$$\lim_{n\to\infty} d(r_n, s_n) = 0, \quad \lim_{n\to\infty} r_n = t$$

for some $t \in M$. Then $\lim_{n\to\infty} s_n = t$.

Proof Due to the triangular inequality, we have

$$d(s_n, t) \leq d(s_n, r_n) + d(r_n, t)$$

and so the result follows after applying limit as $n \to \infty$.

12.2 Main Results

Let (M, d) be metric space, $f, g, S, T : M \to M$ be mappings and $\alpha : M \times M \to [0, \infty)$ be a function. We define the set $\gamma_{f,g,\alpha}$ by

$$\gamma_{f,g,\alpha} = \{(r_1, r_2) \in M \times M : \alpha(Sr_1, Tr_2) \geq 1,\ d(f(r_1), g(r_2)) > 0\}.$$

Let

$$\mathscr{M}_1(r_1, r_2) = \max \Big\{ d(S(r_1), T(r_2)), d(f(r_1), S(r_1)), d(g(r_2), T(r_2)), \frac{d(S(r_1), g(r_2)) + d(f(r_1), T(r_2))}{2} \Big\}.$$

The following theorem is one of our main results:

Theorem 12.2 *Let M be a nonempty set and $\alpha : M \times M \to [0, \infty)$ be a function. Let (M, d) be an α-complete metric space and f, g, S, T be α-continuous self-mappings on (M, d) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. Suppose that, for all $(r_1, r_2) \in \gamma_{f,g,\alpha}$ and some $F \in \Delta_F$ and $\tau > 0$, the inequality*

$$\tau + F(d(f(r_1), g(r_2))) \leq F(\mathscr{M}_1(r_1, r_2)) \tag{12.2}$$

holds. Assume that the pairs (f, S), (g, T) are α-compatible and the pairs (f, g) and (g, f) are triangular partially weakly α-admissible pairs of mappings with respect to T and S, respectively. Then the pairs (f, S) and (g, T) have a coincidence point υ in M. Moreover, if $\alpha(S\upsilon, T\upsilon) \geq 1$, then υ is a common point of the mappings f, g, S, T.

Proof Let $r_0 \in M$ be an arbitrary point. Since $f(M) \subseteq T(M)$, there exists $r_1 \in M$ such that $f(r_0) = T(r_1)$. Since $g(r_1) \in S(M)$, we can choose $r_2 \in M$ such that $g(r_1) = S(r_2)$. In general, r_{2n+1} and r_{2n+2} are chosen in M such that $f(r_{2n}) = T(r_{2n+1})$ and $g(r_{2n+1}) = S(r_{2n+2})$. Define a sequence $\{j_n\}$ in M such that

$$j_{2n+1} = f(r_{2n}) = T(r_{2n+1})$$

and

$$j_{2n+2} = g(r_{2n+1}) = S(r_{2n+2})$$

for all $n \geq 0$. Since $r_1 \in T^{-1}(fr_0), r_2 \in S^{-1}(gr_1)$ and (f, g) and (g, f) are triangular partially weakly α-admissible pairs of mappings with respect to T and S, respectively, we have

$$\alpha(Tr_1 = fr_0, gr_1 = Sr_2) \geq 1$$

and

$$\alpha(gr_1 = Sr_2, fr_2 = Tr_3) \geq 1.$$

Continuing this way, we obtain $\alpha(Tr_{2n+1}, Sr_{2n+2}) = \alpha(j_{2n+1}, j_{2n+2}) \geq 1$ for all $n \geq 0$.

Now, we prove that $\lim_{l\to\infty} d(j_l, j_{l+1}) = 0$. Define $d_l = d(j_l, j_{l+1})$. Suppose that $d_{l_0} = 0$ for some l_0. Then $j_{l_0} = j_{l_0+1}$. If $l_0 = 2n$, then $j_{2n} = j_{2n+1}$ gives $j_{2n+1} = j_{2n+2}$. Indeed, from the contractive condition (12.2), we get

$$F\left(d(j_{2n+1}, j_{2n+2})\right) = F\left(d(f(r_{2n}), g(r_{2n+1}))\right) \le F\left(\mathscr{M}_1(r_{2n}, r_{2n+1})\right) - \tau$$

for all $n \in \mathbf{N} \cup \{0\}$, where

$$\begin{aligned}
&\mathscr{M}_1(r_{2n}, r_{2n+1}) \\
&= \max\left\{ \begin{array}{l} d(S(r_{2n}), T(r_{2n+1})), d(f(r_{2n}), S(r_{2n})), d(g(r_{2n+1}), T(r_{2n+1})), \\ \frac{d(S(r_{2n}), g(r_{2n+1})) + d(f(r_{2n}), T(r_{2n+1}))}{2} \end{array} \right\} \\
&= \max\left\{ \begin{array}{l} d(j_{2n}, j_{2n+1}), d(j_{2n+1}, j_{2n}), d(j_{2n+2}, j_{2n+1}), \\ \frac{d(j_{2n}, j_{2n+2}) + d(j_{2n+1}, j_{2n+1})}{2} \end{array} \right\} \\
&= \max\left\{ d(j_{2n}, j_{2n+1}), d(j_{2n+1}, j_{2n+2}) \right\}.
\end{aligned}$$

Since $d(j_{2n}, j_{2n+1}) = 0$, $\mathscr{M}(r_{2n}, r_{2n+1}) = d(j_{2n+1}, j_{2n+2})$ and so

$$F\left(d(j_{2n+1}, j_{2n+2})\right) \le F\left(d(j_{2n+1}, j_{2n+2})\right) - \tau,$$

which is a contradiction due to F_1. Thus, $j_{2n+1} = j_{2n+2}$. Similarly, if $l_0 = 2n + 1$, then $j_{2n+1} = j_{2n+2}$ gives $j_{2n+2} = j_{2n+3}$. Continuing this process, we find that j_l is a constant sequence for $l \ge l_0$. Hence, $\lim_{l\to\infty} d(j_l, j_{l+1}) = 0$ holds true.

Suppose that $d_l = d(j_l, j_{l+1}) > 0$ for each l. We claim that

$$\lim_{l\to\infty} F\left(d(j_l, j_{l+1})\right) = -\infty.$$

Let $l = 2n$. Since $\alpha(Sr_{2n}, Tr_{2n+1}) \ge 1$ and $d(f(r_{2n}), g(r_{2n-1})) > 0$, $(r_{2n}, r_{2n-1}) \in \gamma_{f,g,\alpha}$ Using (12.2), we obtain

$$F\left(d(j_{2n}, j_{2n+1})\right) \le F\left(d(j_{2n-1}, j_{2n})\right) - \tau \tag{12.3}$$

for all $n \in \mathbf{N}$. Similarly, for $l = 2n - 1$

$$F\left(d(j_{2n-1}, j_{2n})\right) \le F\left(d(j_{2n-2}, j_{2n-1})\right) - \tau \tag{12.4}$$

for all $n \in \mathbf{N}$. Hence, from (12.3) and (12.4), we have

$$F\left(d(j_n, j_{n+1})\right) \le F\left(d(j_{n-1}, j_n)\right) - \tau \tag{12.5}$$

for all $n \in \mathbf{N}$. By (12.5), we obtain

$$F\left(d(j_n, j_{n+1})\right) \le F\left(d(j_{n-2}, j_{n-1})\right) - 2\tau.$$

Repeating these steps, we get

$$F\left(d(j_n, j_{n+1})\right) \leq F\left(d(j_0, j_1)\right) - n\tau. \tag{12.6}$$

From (12.6), we obtain $\lim_{n\to\infty} F\left(d(j_n, j_{n+1})\right) = -\infty$. Since $F \in \Delta_F$,

$$\lim_{n\to\infty} d(j_n, j_{n+1}) = 0. \tag{12.7}$$

From the property (F_3) of F-contraction, there exists $\kappa \in (0, 1)$ such that

$$\lim_{n\to\infty} \left((d(j_n, j_{n+1}))^{\kappa} F\left(d(j_n, j_{n+1})\right)\right) = 0. \tag{12.8}$$

By (12.6), for all $n \in \mathbf{N}$, we obtain

$$(d(j_n, j_{n+1}))^{\kappa} \left(F\left(d(j_n, j_{n+1})\right) - F\left(d(j_0, j_1)\right)\right) \leq -\left(d(j_n, j_{n+1})\right)^{\kappa} n\tau \leq 0. \tag{12.9}$$

Considering (12.7), (12.8) and letting $n \to \infty$ in (12.9), we have

$$\lim_{n\to\infty} \left(n\left(d(j_n, j_{n+1})\right)^{\kappa}\right) = 0. \tag{12.10}$$

Since (12.10) holds, there exists $n_1 \in \mathbf{N}$ such that $n\left(d(j_n, j_{n+1})\right)^{\kappa} \leq 1$ for all $n \geq n_1$ or

$$d(j_n, j_{n+1}) \leq \frac{1}{n^{\frac{1}{\kappa}}} \tag{12.11}$$

for all $n \geq n_1$. Using (12.11), it follows that, for $m > n \geq n_1$

$$\begin{aligned} d(j_n, j_m) &\leq d(j_n, j_{n+1}) + d(j_{n+1}, j_{n+2}) + d(j_{n+2}, j_{n+3}) + \cdots + d(j_{m-1}, j_m) \\ &\leq d(j_n, j_{n+1}) + d(j_{n+1}, j_{n+2}) + d(j_{n+2}, j_{n+3}) + \cdots + d(j_{m-1}, j_m) \\ &= \sum_{i=n}^{m-1} d(j_i, j_{i+1}) \leq \sum_{i=n}^{\infty} d(j_i, j_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\kappa}}}$ entails $\lim_{n,m\to\infty} d(j_n, j_m) = 0$. Hence, $\{j_n\}$ is a Cauchy sequence in (M, d). Since $\{j_n\}$ is a Cauchy sequence in the α-complete metric space M and $\alpha(j_n, j_{n+1}) \geq 1$, there exists $\upsilon \in M$ such that

$$\lim_{n\to\infty} d(j_{2n+1}, \upsilon) = \lim_{n\to\infty} d(Tr_{2n+1}, \upsilon) = \lim_{n\to\infty} d(fr_{2n}, \upsilon) = 0$$

and

$$\lim_{n\to\infty} d(j_{2n}, \upsilon) = \lim_{n\to\infty} d(Sr_{2n}, \upsilon) = \lim_{n\to\infty} d(gr_{2n-1}, \upsilon) = 0.$$

Hence,

$$Sr_{2n} \to \upsilon, \quad fr_{2n} \to \upsilon$$

as $n \to \infty$. Since (f, S) is an α-compatible pair and $\alpha(j_{2n}, j_{2n+1}) \geq 1$, we have

$$\lim_{n\to\infty} d(fSr_{2n}, Sfr_{2n}) = 0.$$

Moreover, from $\lim_{n\to\infty} d(fr_{2n}, \upsilon) = 0$, $\lim_{n\to\infty} d(Sr_{2n}, \upsilon) = 0$ and α-continuity of mappings f and S, we obtain

$$\lim_{n\to\infty} d(fSr_{2n}, f\upsilon) = 0 = \lim_{n\to\infty} d(Sfr_{2n}, S\upsilon).$$

By the triangular inequality, we have

$$\begin{aligned} d(f\upsilon, S\upsilon) &\leq d(f\upsilon, Sfr_{2n}) + d(Sfr_{2n}, S\upsilon) \\ &\leq d(f\upsilon, fSr_{2n}) + d(fSr_{2n}, Sfr_{2n}) + d(Sfr_{2n}, S\upsilon). \end{aligned} \tag{12.12}$$

Applying the limit as $n \to \infty$ in (12.12), we obtain $d(f\upsilon, S\upsilon) \leq 0$, which yields that $f\upsilon = S\upsilon$. Thus, υ is a coincidence point of f and S. Arguing in a similar manner, we can prove that $g\upsilon = T\upsilon$. Let $\alpha(T\upsilon, S\upsilon) \geq 1$ and assume that $d(f\upsilon, g\upsilon) > 0$. Since $\upsilon \in \gamma_{f,g,\alpha}$, using the contractive condition (12.2), we have

$$F(d(f(\upsilon), g(\upsilon)) \leq F(\mathscr{M}_1(\upsilon, \upsilon) - \tau, \tag{12.13}$$

where

$$\begin{aligned} \mathscr{M}_1(\upsilon, \upsilon) &= \max\left\{ \begin{array}{l} d(S(\upsilon), T(\upsilon)), d(f(\upsilon), S(\upsilon)), d(\upsilon), T(\upsilon)), \\ \frac{d(S(\upsilon), g(\upsilon)) + d(f(\upsilon), T(\upsilon))}{2} \end{array} \right\} \\ &= \max\left\{ \begin{array}{l} d(f(\upsilon), g(\upsilon)), d(f(\upsilon), S(\upsilon)), d(g(\upsilon), T(\upsilon)), \\ \frac{d(f(\upsilon), g(\upsilon)) + d(f(\upsilon), g(\upsilon))}{2} \end{array} \right\} \\ &= d(f(\upsilon), g(\upsilon)). \end{aligned}$$

Using (12.13), we deduce that $f\upsilon = g\upsilon$. Hence, $f\upsilon = g\upsilon = T\upsilon = S\upsilon$, that is, υ is a coincidence point of f, g, S, T.

We show that υ is a common fixed point of f, g, S and T. Since S is α-continuous,

$$\lim_{n\to\infty} Sf(r_{2n}) = S(\upsilon) = \lim_{n\to\infty} S^2(r_{2n+2}).$$

Since the pair (f, S) is α-compatible,

$$\lim_{n\to\infty} d(fS(r_{2n}), Sf(r_{2n}) = 0$$

and by Lemma 12.1

$$\lim_{n\to\infty} fS(r_{2n}) = S(\upsilon).$$

Now, put $r_1 = S(r_{2n})$ and $r_2 = r_{2n+1}$ in (12.2) and suppose on contrary that $d(S(\upsilon), \upsilon) > 0$. Then we obtain

$$F(d(fS(r_{2n}), g(r_{2n+1})) \leq F(\mathscr{M}_1(S(r_{2n}), r_{2n+1})) - \tau, \tag{12.14}$$

where

$$\mathscr{M}_1(S(r_{2n}), r_{2n+1}) = \max\left\{ \begin{array}{l} d(S^2(r_{2n}), T(r_{2n+1})), d(fS(r_{2n}), S^2(r_{2n})), d(g(r_{2n+1}), T(r_{2n+1})), \\ \frac{d(S^2(r_{2n}), g(r_{2n+1})) + d(fS(r_{2n}), T(r_{2n+1}))}{2} \end{array} \right\}.$$

Applying the limit as $n \to \infty$ in (12.14) and using the continuity of F, we have

$$F(d(S(\upsilon), \upsilon) \leq F(d(S(\upsilon), \upsilon) - \tau < F(d(S(\upsilon), \upsilon),$$

which is a contradiction. Hence, $d(S(\upsilon), \upsilon) = 0$ implies $S(\upsilon) = \upsilon$. Thus, $f\upsilon = g\upsilon = T\upsilon = S\upsilon = \upsilon$, that is, υ is a common fixed point of the mappings f, g, S, T. This completes the proof.

Remark 12.5 If we suppose that $\alpha(\upsilon, \omega) \geq 1$ for each common fixed point of the mappings f, g, S, T, then υ is unique. Indeed, if ω is another fixed point of f, g, S, T and assume on contrary that $d(f\upsilon, g\omega) > 0$. Then, from (12.2), we have

$$F(d(\upsilon, \omega)) = F(d(S(\upsilon), T(\omega))) \leq F\left(\mathscr{M}_1(\upsilon, \omega)\right) - \tau, \tag{12.15}$$

where

$$\mathscr{M}_1(\upsilon, \omega) = \max\left\{ \begin{array}{l} d(S(\upsilon), T(\omega)), d(f(\upsilon), S(\upsilon)), d(g(\omega), T(\omega)), \\ \frac{d(S(\upsilon), g(\omega)) + d(f(\upsilon), T(\omega))}{2} \end{array} \right\}.$$

Thus, from (12.15), we have

$$F(d(\upsilon, \omega)) < F\left(d(\upsilon, \omega)\right),$$

which is a contradiction. Hence, $\upsilon = \omega$ and υ is a unique common fixed point of four mappings f, g, S, T.

The following example elucidates Theorem 12.2:

Example 12.5 Let $M = [0, \infty)$ and define $d : M \times M \to \mathbf{R}_0^+$ by $d\,(r_1, r_2) = |r_1 - r_2|$. Define $\alpha : M \times M \to [0, \infty)$ by $\alpha(r_1, r_2) = e^{r_1 - r_2}$ for all $r_1, r_2 \in M$ with $r_1 \geq r_2$. Then (M, d) is an α-complete metric space. Define the mappings $f, g, S, T : M \to M$ by

$$f(r) = \ln\left(1 + \frac{r}{6}\right), \quad g(r) = \ln\left(1 + \frac{r}{7}\right),$$
$$S(r) = e^{7r} - 1, \qquad T(r) = e^{6r} - 1$$

for all $r \in M$. Clearly, f, g, S, T are α-continuous self mappings complying with $f(M) = T(M) = g(M) = S(M)$. We note that the pair (f, S) is α-compatible. Indeed, let $\{r_n\}$ be a sequence in M satisfying $\alpha(r_n, r_{n+1}) \geq 1$ and

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} S(r_n) = t$$

for some $t \in M$. Then we have

$$\lim_{n\to\infty} |f(r_n) - t| = \lim_{n\to\infty} |S(r_n) - t| = 0,$$

equivalently,

$$\lim_{n\to\infty} \left|\ln\left(1 + \frac{r_n}{6}\right) - t\right| = \lim_{n\to\infty} |e^{7r_n} - 1 - t| = 0,$$

which implies

$$\lim_{n\to\infty} |r_n - (6e^t - 6)| = \lim_{n\to\infty} \left|r_n - \frac{\ln(t+1)}{7}\right| = 0.$$

The uniqueness of the limit gives that $6e^t - 6 = \frac{\ln(t+1)}{7}$ and thus $t = 0$ is only possible solution. Due to α-continuity of f, S, we have

$$\begin{aligned}\lim_{n\to\infty} d(fS(r_n), Sf(r_n)) &= \lim_{n\to\infty} |fS(r_n) - Sf(r_n)| \\ &= |f(t) - S(t)| = |0 - 0| = 0\end{aligned}$$

for $t = 0 \in M$. Similarly, the pair (g, T) is α-compatible. To prove that (f, g) is a partially weakly α-admissible pair of mappings with respect to T, let $r_1, r_2 \in M$ be such that $r_2 \in T^{-1}(f(r_1))$, that is, $T(r_2) = f(r_1)$ and thus we have $e^{6r_2} - 1 = \ln\left(1 + \frac{r_1}{6}\right)$ or $r_2 = \frac{\ln\left(1+\ln\left(1+\frac{r_1}{6}\right)\right)}{6}$. Since

$$f(r_1) = \ln\left(1 + \frac{r_1}{6}\right) \geq \ln\left(1 + \frac{\ln\left(1 + \ln\left(1 + \frac{r_1}{6}\right)\right)}{42}\right) = \ln\left(1 + \frac{r_2}{7}\right) = g(r_2),$$

$\alpha(fr_1, gr_2) = e^{fr_1 - gr_2} \geq 1$. Hence, (f, g) is a partially weakly α-admissible pair of mappings with respect to T. To prove that (g, f) is a partially weakly α-admissible pair of mappings with respect to S, let $r_1, r_2 \in M$ be such that $r_2 \in S^{-1}(g(r_1))$, that is, $S(r_2) = g(r_1)$ and thus we have $e^{7r_2} - 1 = \ln\left(1 + \frac{r_1}{7}\right)$ or $r_2 = \frac{\ln\left(1+\ln\left(1+\frac{r_1}{7}\right)\right)}{7}$. Since

$$g(r_1) = \ln\left(1 + \frac{r_1}{7}\right) \geq \ln\left(1 + \frac{\ln\left(1 + \ln\left(1 + \frac{r_1}{7}\right)\right)}{42}\right) = \ln\left(1 + \frac{r_2}{6}\right) = f(r_2),$$

$\alpha(gr_1, fr_2) = e^{gr_1 - fr_2} \geq 1$. Hence, (g, f) is a partially weakly α-admissible pair of mappings with respect to S.

Now, for each $r_1, r_2 \in M$, consider

$$\begin{aligned} d(f(r_1), g(r_2)) &= |f(r_1) - g(r_2)| = \left|\ln\left(1 + \frac{r}{6}\right) - \ln\left(1 + \frac{r}{7}\right)\right| \\ &\leq \left(\frac{r}{6} - \frac{r}{7}\right) = \left(\frac{1}{42}\right)|7r - 6r| \leq \left(\frac{1}{42}\right)|e^{7r} - e^{6r}| \\ &= \left(\frac{1}{42}\right) d(T(r_1), S(r_2)) \leq \left(\frac{1}{42}\right) \mathscr{M}_1(r_1, r_2). \end{aligned}$$

The above inequality can be written as

$$\ln(42) + \ln\left(d(f(r_1), g(r_2))\right) \leq \ln\left(\mathscr{M}_1(r_1, r_2)\right).$$

Define the function $F : \mathbf{R}^+ \to \mathbf{R}$ by $F(r) = \ln(r)$, for all $r \in \mathbf{R}^+ > 0$. Hence, for all $r_1, r_2 \in M$ such that $d(f(r_1), g(r_2)) > 0$, $\tau = \ln(42)$, we obtain

$$\tau + F\left(d(f(r_1), g(r_2))\right) \leq F\left(\mathscr{M}(r_1, r_2)\right).$$

Thus, the contractive condition (12.2) is satisfied for all $r_1, r_2 \in M$. Hence, all the hypotheses of Theorem 12.2 are satisfied. So the mappings f, g, S, T have a unique common fixed point $r = 0$.

The following corollary is a generalization of [9, Theorem 3.1]:

Corollary 12.1 *Let M be a nonempty set and $\alpha : M \times M \to [0, \infty)$ be a function. Let (M, d) be an α-complete metric space and f, g, S, T be α-continuous self-mappings on (M, d) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. Suppose that, for all $(r_1, r_2) \in \gamma_{f,g,\alpha}$, the inequality*

$$d(f(r_1), g(r_2)) \leq \mathscr{M}_1(r_1, r_2) \tag{12.16}$$

holds. Assume that the pairs (f, S), (g, T) are α-compatible and the pairs (f, g) and (g, f) are triangular partially weakly α-admissible pairs of mappings with respect to T and S, respectively. Then the pairs (f, S), (g, T) have a coincidence point υ_1 in M. Moreover, if $\alpha(S\upsilon_1, T\upsilon_1) \geq 1$, then υ_1 is a common point of the mappings f, g, S, T.

Proof For all $(r_1, r_2) \in \gamma_{f,g,\alpha}$, we have

$$d(f(r_1), g(r_2)) \leq \mathscr{M}_1(r_1, r_2).$$

It follows that

$$\tau + \ln(d(f(r_1), g(r_2))) \leq \ln\left(\mathscr{M}_1(r_1, r_2)\right),$$

where $\tau = \ln\left(\frac{1}{k}\right) > 0$. Then the contraction condition (12.16) reduces to (12.2) with $F(r) = \ln(r)$ and application of Theorem 12.2 ensures the existence of fixed point. This completes the proof.

In the following theorem, we omit the assumption of α-continuity of f, g, T, S and replace the α-compatibility of the pairs (f, S) and (g, T) by weak compatibility of the pairs.

Theorem 12.3 *Let (M, d) be an α-regular and α-complete metric space and f, g, S, T be self-mappings on (M, d) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$, and $T(M)$ and $S(M)$ are closed subsets of M. Suppose that, for all $(r_1, r_2) \in \gamma_{f,g,\alpha}$ and some $F \in \Delta_F$ and $\tau > 0$ the inequality*

$$\tau + F(d(f(r_1), g(r_2))) \leq F\left(\mathscr{M}_1(r_1, r_2)\right) \tag{12.17}$$

holds. Assume that the pairs (f, S), (g, T) are weakly compatible and the pairs (f, g) and (g, f) are triangular partially weakly α-admissible pairs of mappings with respect to T and S, respectively. Then the pairs (f, S), (g, T) have a coincidence point υ in M. Moreover, if $\alpha(S\upsilon, T\upsilon) \geq 1$, then υ is a coincidence point of f, g, S, T.

Proof In the proof of Theorem 12.2, we know that there exists $\upsilon \in M$ such that

$$\lim_{l\to\infty} d(j_l, \upsilon) = 0.$$

Since $T(M)$ is a closed subset of M and $\{j_{2n+1}\} \subseteq T(M)$, $\upsilon \in T(M)$. Thus, there exists $\omega_1 \in M$ such that $\upsilon = T(\omega_1)$ and

$$\lim_{n\to\infty} d(j_{2n+1}, T(\omega_1)) = \lim_{n\to\infty} d(Tr_{2n+1}, T(\omega_1)) = 0.$$

Similarly, there exists $\omega_2 \in M$ such that $\upsilon = T(\omega_1) = S(\omega_2)$ and

$$\lim_{n\to\infty} d(j_{2n}, S(\omega_2)) = \lim_{n\to\infty} d(Sr_{2n}, S(\omega_2)) = 0.$$

Now, since $\lim_{n\to\infty} d(Tr_{2n+1}, S(\omega_2)) = 0$, the α-regularity of M implies that $\alpha(Tr_{2n+1}, S(\omega_2)) \geq 1$ and, from the contractive condition (12.17), we have

$$F\left(d(f(\omega_2), g(r_{2n+1}))\right) \leq F\left(\mathscr{M}_1(\omega_2, r_{2n+1})\right) - \tau \tag{12.18}$$

for all $n \in \mathbf{N} \cup \{0\}$, where

$$
\begin{aligned}
&\mathscr{M}_1(\omega_2, r_{2n+1})\\
&= \max\left\{\begin{array}{l} d(S(\omega_2), T(r_{2n+1})), d(f(\omega_2), S(\omega_2)), d(g(r_{2n+1}), T(r_{2n+1})), \\ \frac{d(S(\omega_2), g(r_{2n+1})) + d(f(\omega_2), T(r_{2n+1}))}{2} \end{array}\right\}\\
&= \max\left\{\begin{array}{l} d(\upsilon, j_{2n+1}), d(f(\omega_2), \upsilon), d(j_{2n+2}, j_{2n+1}), \\ \frac{d(\upsilon, j_{2n+2}) + d(f(\omega_2), j_{2n+1})}{2} \end{array}\right\}.
\end{aligned}
$$

When $n \to \infty$ in (12.18), we obtain $f(\omega_2) = \upsilon = S(\omega_2)$. Weakly compatibility of f and S gives $f(\upsilon) = fS(\omega_2) = Sf(\omega_2) = S(\upsilon)$, which shows that υ is a coincidence point of f and S. Similarly, it can be shown that υ is a coincidence point of the pair (g, T).

The rest of the proof follows from similar arguments as in the proof of Theorem 12.2. This completes the proof.

If we set $S = T$ in Theorem 12.2, then we obtain the following result:

Corollary 12.2 *Let M be a nonempty set and $\alpha : M \times M \to [0, \infty)$ be a function. Let (M, d) be an α-complete metric space and f, g, T be self-mappings on (M, d) such that $f(M) \cup g(M) \subseteq T(M)$ and $T(M)$ is α-continuous. Suppose that, for all $r_1, r_2 \in M$ with $\alpha(Tr_1, Tr_2) \geq 1$, $d(f(r_1), g(r_2)) > 0$ and for some $F \in \Delta_F$ and $\tau > 0$, the inequality*

$$\tau + F(d(f(r_1), g(r_2))) \leq F\left(\mathscr{M}_1(r_1, r_2)\right)$$

holds, where

$$\mathscr{M}_1(r_1, r_2) = \max\Big\{d(T(r_1), T(r_2)), d(f(r_1), T(r_1)), d(g(r_2), T(r_2)), \frac{d(T(r_1), g(r_2)) + d(f(r_1), T(r_2))}{2}\Big\}.$$

Assume that either the pair (f, T) is α-compatible and f is α-continuous or (g, T) is α-compatible and g is α-continuous. Then the pairs (f, T) and (g, T) have a coincidence point υ in M provided the pair (f, g) is a triangular weakly α-admissible pair of mappings with respect to T. Moreover, if $\alpha(T\upsilon, T\upsilon) \geq 1$, then υ is a common point of the mappings f, g, T.

If we set $S = T$ and $f = g$ in Theorem 12.2, then we obtain the following result:

Corollary 12.3 *Let M be a nonempty set and $\alpha : M \times M \to [0, \infty)$ be a function. Let (M, d) be an α-complete metric space and f, T be α-continuous self-mappings on (M, d) such that $f(M) \subseteq T(M)$. Suppose that, for all $r_1, r_2 \in M$ with $\alpha(Tr_1, Tr_2) \geq 1$, $d(f(r_1), f(r_2)) > 0$ and for some $F \in \Delta_F$ and $\tau > 0$, the inequality*

$$\tau + F(d(f(r_1), f(r_2))) \leq F\left(\mathscr{M}_1(r_1, r_2)\right)$$

holds, where

$$\mathscr{M}_1(r_1, r_2) = \max \Big\{ d(T(r_1), T(r_2)), d(f(r_1), T(r_1)), d(f(r_2), T(r_2)), \frac{d(T(r_1), f(r_2)) + d(f(r_1), T(r_2))}{2} \Big\}.$$

Assume that the pair (f, T) is α-compatible. Then the mappings f, T have a coincidence point in M provided that f is a triangular weakly α-admissible mapping with respect to T. Moreover, if $\alpha(T\upsilon, T\upsilon) \geq 1$, then f, T have a common point υ.

Corollary 12.4 *Let (M, d) be an α-regular and α-complete metric space and f, g, T be self-mappings on (M, d) such that $f(M) \subseteq T(M)$, $g(M) \subseteq T(M)$ and $T(M)$ is a closed subset of M. Suppose that, for all $r_1, r_2 \in M$ with $\alpha(Tr_1, Tr_2) \geq 1$, $d(f(r_1), g(r_2)) > 0$ and for some $F \in \Delta_F$ and $\tau > 0$, the inequality*

$$\tau + F(d(f(r_1), g(r_2))) \leq F\,(\mathscr{M}_1(r_1, r_2))$$

holds, where

$$\mathscr{M}_1(r_1, r_2) = \max \Big\{ d(T(r_1), T(r_2)), d(f(r_1), T(r_1)), d(g(r_2), T(r_2)), \frac{d(T(r_1), g(r_2)) + d(f(r_1), T(r_2))}{2} \Big\}.$$

Assume that the pairs (f, T), (g, T) are weakly compatible and the pair (f, g) is a triangular weakly α-admissible pair of mapping with respect to T. Then the pairs (f, T), (g, T) have a coincidence point υ in M. Moreover, if $\alpha(T\upsilon, T\upsilon) \geq 1$, then υ is a coincidence point of the mappings f, g, T.

Corollary 12.5 *Let (M, d) be an α-regular and α-complete metric space and f, T be self-mappings on (M, d) such that $f(M) \subseteq T(M)$ and $T(M)$ is closed subset of M. Suppose that, for all $r_1, r_2 \in M$ with $\alpha(Tr_1, Tr_2) \geq 1$, $d(f(r_1), f(r_2)) > 0$ and for some $F \in \Delta_F$ and $\tau > 0$, the inequality*

$$\tau + F(d(f(r_1), f(r_2))) \leq F\,(\mathscr{M}_1(r_1, r_2))$$

holds, where

$$\mathscr{M}_1(r_1, r_2) = \max \Big\{ d(T(r_1), T(r_2)), d(f(r_1), T(r_1)), d(f(r_2), T(r_2)), \frac{d(T(r_1), f(r_2)) + d(f(r_1), T(r_2))}{2} \Big\}.$$

Assume that the pair (f, T) is weakly compatible and f is a triangular weakly α-admissible mapping with respect to T. Then the pair (f, T) has a coincidence point υ in M.

If we set $S = T = I_M$ (: the identity mapping) in Theorems 12.2 and 12.3, then we obtain the following result:

Corollary 12.6 *Let* (M, d) *be an* α*-complete metric space and* f, g *be self-mappings on* (M, d)*. Suppose that, for all* $r_1, r_2 \in M$ *with* $\alpha(r_1, r_2) \geq 1$, $d(f(r_1), f(r_2)) > 0$ *and for some* $F \in \Delta_F$ *and* $\tau > 0$*, the inequality*

$$\tau + F(d(f(r_1), f(r_2))) \leq F(\mathscr{M}_1(r_1, r_2))$$

holds, where

$$\mathscr{M}_1(r_1, r_2) = \max\Big\{d(r_1), r_2), d(f(r_1), r_1), d(g(r_2), r_2), \frac{d(r_1, g(r_2)) + d(f(r_1), r_2)}{2}\Big\}.$$

Assume that the pair (f, g) *is a triangular weakly* α*-admissible pair of mappings. Then* f, g *have a common fixed point* υ *in* M *provided either* f *or* g *is* α*-continuous or* M *is* α*-regular.*

The following theorem shows that the arguments given in the proof of Theorem 12.2 hold equally if we replace $\mathscr{M}_1(r_1, r_2)$ with $\mathscr{M}_i(r_1, r_2)$ $(i = 2, 3, 4, 5, 6)$:

Theorem 12.4 *Let* M *be a nonempty set and* $\alpha : M \times M \to [0, \infty)$ *be a function. Let* (M, d) *be an* α*-complete metric space and* f, g, S, T *be* α*-continuous self-mappings on* (M, d) *such that* $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$*. Suppose that, for all* $(r_1, r_2) \in \gamma_{f,g,\alpha}$ *and for some* $F \in \Delta_F$ *and* $\tau > 0$*, the inequality*

$$\tau + F(d(f(r_1), g(r_2))) \leq F(\mathscr{M}_i(r_1, r_2)) \tag{12.19}$$

holds for each $i = 2, 3, 4, 5, 6$*, where*

$$\begin{aligned}
\mathscr{M}_2(r_1, r_2) &= a_1 d(S(r_1), T(r_2)) + a_2 d(f(r_1), S(r_1)) + a_3 d(g(r_2), T(r_2)) \\
&\quad + a_4[d(S(r_1), g(r_2)) + d(f(r_1), T(r_2))] \\
&\quad \text{with } a_i \geq 0 (i = 1, 2, 3, 4) \text{ such that } a_1 + a_2 + a_3 + 2a_4 < 1, \\
\mathscr{M}_3(r_1, r_2) &= a_1 d(S(r_1), T(r_2)) + a_2 d(f(r_1), S(r_1)) + a_3 d(g(r_2), T(r_2)) \\
&\quad \text{with } a_1 + a_2 + a_3 < 1, \\
\mathscr{M}_4(r_1, r_2) &= k \max\{d(f(r_1), S(r_1)), d(g(r_2), T(r_2))\} \text{ with } k \in [0, 1), \\
\mathscr{M}_5(r_1, r_2) &= a_1(r_1, r_2) d(S(r_1), T(r_2)) + a_2(r_1, r_2) d(f(r_1), S(r_1)) \\
&\quad + a_3(r_1, r_2) d(g(r_2), T(r_2)) \\
&\quad + a_4(r_1, r_2)[d(S(r_1), g(r_2)) + d(f(r_1), T(r_2))] \\
&\quad \text{where } a_i(r_1, r_2)(i = 1, 2, 3, 4) \text{ are nonnegative functions such that} \\
&\quad \sup_{r_1, r_2 \in M} \{a_1(r_1, r_2) + a_2(r_1, r_2) + a_3(r_1, r_2) + 2a_4(r_1, r_2)\} = \mu < 1,
\end{aligned}$$

$$\begin{aligned}\mathscr{M}_6(r_1, r_2) = a_1 d(S(r_1), T(r_2)) &+ \frac{a_2 + a_3}{2}[d(f(r_1), S(r_1)) + d(g(r_2), T(r_2))] \\ &+ \frac{a_4 + a_5}{2}[d(S(r_1), g(r_2)) + d(f(r_1), T(r_2))] \\ &\text{with } a_1 + a_2 + a_3 + a_4 + a_5 < 1.\end{aligned}$$

Assume that the pairs (f, S), (g, T) are α-compatible and the pairs (f, g) and (g, f) are triangular partially weakly α-admissible pairs of mappings with respect to T and S, respectively. Then the pairs (f, S), (g, T) have a coincidence point υ in M. Moreover, if $\alpha(S\upsilon, T\upsilon) \geq 1$, then υ is a common point of the mappings f, g, S, T.

Proof In the beginning part of the proof of Theorem 12.2, for all $(r_1, r_2) \in \gamma_{f,g,\alpha}$, for some $F \in \Delta_F$ and $\tau > 0$, from the contractive condition (12.19), we get

$$F\,(d(j_{2n}, j_{2n+1})) = F\,(d(f(r_{2n}), g(r_{2n+1}))) \leq F\,(\mathscr{M}_2(r_{2n}, r_{2n+1})) - \tau \tag{12.20}$$

for all $n \in \mathbf{N} \cup \{0\}$, where

$$\begin{aligned}&\mathscr{M}_2(r_{2n}, r_{2n+1}) \\ &= a_1 d(S(r_{2n}), T(r_{2n+1})) + a_2 d(f(r_{2n}), S(r_{2n})) + a_3 d(g(r_{2n+1}), T(r_{2n+1})) \\ &\quad + a_4[d(S(r_{2n}), g(r_{2n+1})) + d(f(r_{2n}), T(r_{2n+1}))] \\ &= a_1 d(j_{2n-1}, j_{2n}) + a_2 d(j_{2n}, j_{2n-1}) + a_3 d(j_{2n+1}, j_{2n}) \\ &\quad + a_4[d(j_{2n-1}, j_{2n+1}) + d(j_{2n}, j_{2n})] \\ &= (a_1 + a_2 + a_4) d(j_{2n-1}, j_{2n}) + (a_3 + a_4) d(j_{2n}, j_{2n+1}).\end{aligned}$$

Now, by (12.20) we have

$$\begin{aligned}&F\,(d(j_{2n}, j_{2n+1})) \\ &\leq F\,((a_1 + a_2 + a_4) d(j_{2n-1}, j_{2n}) + (a_3 + a_4) d(j_{2n}, j_{2n+1})) - \tau.\end{aligned} \tag{12.21}$$

Since F is strictly increasing, (12.22) implies

$$d(j_{2n}, j_{2n+1}) \leq (a_1 + a_2 + a_4) d(j_{2n-1}, j_{2n}) + (a_3 + a_4) d(j_{2n}, j_{2n+1}),$$

$$(1 - a_3 - a_4) d(j_{2n}, j_{2n+1}) \leq (a_1 + a_2 + a_4) d(j_{2n-1}, j_{2n}),$$

$$d(j_{2n}, j_{2n+1}) \leq \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} d(j_{2n-1}, j_{2n}).$$

Since $a_1 + a_2 + a_3 + 2a_4 < 1$, we have

$$d(j_{2n}, j_{2n+1}) \leq \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} d(j_{2n-1}, j_{2n}) < d(j_{2n-1}, j_{2n}).$$

Thus, from (12.22), we obtain

$$F\left(d(j_{2n}, j_{2n+1})\right) \leq F\left(d(j_{2n-1}, j_{2n})\right) - \tau \tag{12.22}$$

for all $n \in \mathbf{N}$. Similarly, we have

$$F\left(d(j_{2n-1}, j_{2n})\right) \leq F\left(d(j_{2n-2}, j_{2n-1})\right) - \tau \tag{12.23}$$

for all $n \in \mathbf{N}$. Hence, from (12.22) and (12.23), we have

$$F\left(d(j_n, j_{n+1})\right) \leq F\left(d(j_{n-1}, j_n)\right) - \tau. \tag{12.24}$$

The inequality (12.24) leads us to remark that $\{j_n\}$ is a Cauchy sequence and the remaining part of the proof follows from the finishing part of the proof of Theorem 12.2.

The case $\mathscr{M}_3(r_1, r_2)$: In the beginning part of the proof of Theorem 12.2, for all $(r_1, r_2) \in \gamma_{f,g,\alpha}$ and for some $F \in \Delta_F$ and $\tau > 0$, from the contractive condition (12.19), we get

$$F\left(d(j_{2n}, j_{2n+1})\right) = F\left(d(f(r_{2n}), g(r_{2n+1}))\right) \leq F\left(\mathscr{M}_3(r_{2n}, r_{2n+1})\right) - \tau \tag{12.25}$$

for all $n \in \mathbf{N} \cup \{0\}$, where

$$\begin{aligned}
&\mathscr{M}_3(r_{2n}, r_{2n+1}) \\
&= a_1 d(S(r_{2n}), T(r_{2n+1})) + a_2 d(f(r_{2n}), S(r_{2n})) + a_3 d(g(r_{2n+1}), T(r_{2n+1})) \\
&= a_1 d(j_{2n-1}, j_{2n}) + a_2 d(j_{2n}, j_{2n-1}) + a_3 d(j_{2n+1}, j_{2n}) \\
&= (a_1 + a_2) d(j_{2n-1}, j_{2n}) + a_3 d(j_{2n}, j_{2n+1}).
\end{aligned}$$

Now, from (12.25), we have

$$F\left(d(j_{2n}, j_{2n+1})\right) \leq F\left((a_1 + a_2) d(j_{2n-1}, j_{2n}) + a_3 d(j_{2n}, j_{2n+1})\right) - \tau. \tag{12.26}$$

Since F is strictly increasing, (12.26) implies

$$d(j_{2n}, j_{2n+1}) \leq (a_1 + a_2) d(j_{2n-1}, j_{2n}) + a_3 d(j_{2n}, j_{2n+1}),$$

$$(1 - a_3) d(j_{2n}, j_{2n+1}) \leq (a_1 + a_2) d(j_{2n-1}, j_{2n}),$$

$$d(j_{2n}, j_{2n+1}) \leq \frac{a_1 + a_2}{1 - a_3} d(j_{2n-1}, j_{2n}).$$

Since $a_1 + a_2 + a_3 < 1$,

$$d(j_{2n}, j_{2n+1}) \leq \frac{a_1 + a_2}{1 - a_3} d(j_{2n-1}, j_{2n}) < d(j_{2n-1}, j_{2n}).$$

Thus, from (12.26), we obtain

$$F\left(d(j_{2n}, j_{2n+1})\right) \leq F\left(d(j_{2n-1}, j_{2n})\right) - \tau \tag{12.27}$$

for all $n \in \mathbf{N}$. Similarly, we have

$$F\left(d(j_{2n-1}, j_{2n})\right) \leq F\left(d(j_{2n-2}, j_{2n-1})\right) - \tau \tag{12.28}$$

for all $n \in \mathbf{N}$. Hence, from (12.27) and (12.28), we have

$$F\left(d(j_n, j_{n+1})\right) \leq F\left(d(j_{n-1}, j_n)\right) - \tau. \tag{12.29}$$

The inequality (12.29) leads us to note that $\{j_n\}$ is a Cauchy sequence and the remaining part of the proof follows from the finishing part of the proof of Theorem 12.2.

The case $\mathscr{M}_4(r_1, r_2)$: In the beginning part of the proof of Theorem 12.2, for all $(r_1, r_2) \in \gamma_{f,g,\alpha}$ and for some $F \in \Delta_F$ and $\tau > 0$, from the contractive condition (12.19), we get

$$F\left(d(j_{2n}, j_{2n+1})\right) = F\left(d(f(r_{2n}), g(r_{2n+1}))\right) \leq F\left(\mathscr{M}_4(r_{2n}, r_{2n+1})\right) - \tau$$

for all $n \in \mathbf{N} \cup \{0\}$, where

$$\begin{aligned}\mathscr{M}_4(r_{2n}, r_{2n+1}) &= k \max\left\{d(f(r_{2n}), S(r_{2n})), d(g(r_{2n+1}), T(r_{2n+1}))\right\} \\ &= k \max\left\{d(j_{2n}, j_{2n-1}), d(j_{2n+1}, j_{2n})\right\}.\end{aligned}$$

The remaining part of the proof follows from the proof of Theorem 12.2. Similar arguments hold for $\mathscr{M}_5(r_1, r_2)$ and $\mathscr{M}_6(r_1, r_2)$. This completes the proof.

12.3 Results in Ordered Metric Spaces

In this section, we present some common fixed point theorems on metric spaces endowed with an arbitrary binary relation, especially, a partial order relation which can be regarded as consequences of the results presented in the previous section.
Let (M, d) be a metric space and let $\prec$ be a binary relation over M.

Definition 12.12 ([3]) Let f and g be two self-mappings on M and $\prec$ be a binary relation over M. A pair (f, g) is said to be:

(1) *weakly $\prec$-increasing* if $f(r) \prec gf(r)$ and $g(r) \prec fg(r)$ for all $r \in M$;

(2) *partially weakly $\prec$-increasing* if $f(r) \prec gf(r)$ for all $r \in M$.

Definition 12.13 Let $f, g, h : M \to M$ be three mappings such that $f(M) \cup g(M) \subseteq h(M)$. The pair (f, g) is said to be a *transitive weakly $\prec$-increasing pair* of mappings with respect to h if the following conditions hold:

(a) $f(r_1) \prec g(r_2)$ for all $r_1 \in M$ and $r_2 \in h^{-1}f(r_1)$ and $g(r_1) \prec f(r_2))$ for all $r_2 \in h^{-1}g(r_1)$;

(b) $r_1 \prec r_3, r_3 \prec r_2$ imply $r_1 \prec r_2$ for all $r_1, r_2, r_3 \in M$.

Definition 12.14 Let $f, g, h : M \to M$ be three mappings such that $f(M) \cup g(M) \subseteq h(M)$. The pair (f, g) is said to be a *transitive partially weakly $\prec$-increasing pair* of mappings with respect to h if the following conditions hold:

(a) $f(r_1) \prec g(r_2)$ for all $r_1 \in M$ and $r_2 \in h^{-1}f(r_1)$;

(b) $r_1 \prec r_3, r_3 \prec r_2$ imply $r_1 \prec r_2$ for all $r_1, r_2, r_3 \in M$.

Let $\prec$ be a binary relation over M and let

$$\alpha(r_1, r_2) = \begin{cases} 1, & \text{if } r_1 \prec r_2, \\ 0, & \text{otherwise.} \end{cases}$$

By this assumption, we see that the above definitions are special cases from the definitions of weak α-admissibility and partially weak α-admissibility.

Definition 12.15 ([15]) Let (M, d) be a metric space. It is said to be *$\prec$-complete* if every Cauchy sequence $\{r_n\}$ in M such that $r_n \prec r_{n+1}$ converges in M.

Definition 12.16 ([15]) Let (M, d) be a metric space and $T : M \to M$ be a mapping. We say that T is an *$\prec$-continuous mapping* on (M, d) if, for any $r \in M$ and a sequence $\{r_n\}$,

$$\lim_{n\to\infty} d(r_n, r) = 0, \quad r_n \prec r_{n+1}, \quad \forall n \in \mathbf{N}, \quad \text{imply} \quad \lim_{n\to\infty} d(T(r_n), T(r)) = 0.$$

Definition 12.17 ([3]) Let (M, d) be a metric space. The pair (f, g) is said to be an *$\prec$-compatible* if $\lim_{n\to\infty} d(fg(r_n), gf(r_n)) = 0$, whenever $\{r_n\}$ is a sequence in M such that $r_n \prec r_{n+1}$ and

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} g(r_n) = t$$

for some $t \in M$.

Definition 12.18 The metric space (M, d) is said to be *$\prec$-regular* if there exists a sequence $\{r_n\}$ in M such that

$$r_n \to r, \quad r_n \prec r_{n+1}, \quad \forall n \in \mathbf{N}, \quad \text{imply} \quad r_n \prec r$$

for all $n \in \mathbf{N}$.

Now, we are able to remodel Theorems 12.2 and 12.3 in the framework of ordered metric spaces.

Theorem 12.5 *Let (M, d) be an $\prec$-complete metric space and f, g, S, T be $\prec$-continuous self-mappings on (M, d) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. Suppose that, for all $r_1, r_2 \in M$ with $S(r_1) \prec T(r_2)$, $d(f(r_1), g(r_2)) > 0$ and for some $F \in \Delta_F$ and $\tau > 0$, the inequality*

$$\tau + F(d(f(r_1), g(r_2))) \leq F(\mathscr{M}_1(r_1, r_2))$$

holds. Assume that the pairs (f, S), (g, T) are $\prec$-compatible and the pairs (f, g) and (g, f) are transitive partially weakly $\prec$-increasing pairs of mappings with respect to T and S, respectively. Then the pairs (f, S), (g, T) have a coincidence point υ in M. Moreover, if $S\upsilon \prec T\upsilon$, then υ is a common point of the mappings f, g, S, T.

Proof Define

$$\alpha(r_1, r_2) = \begin{cases} 1, & \text{if } r_1 \prec r_2, \\ 0, & \text{otherwise} \end{cases}$$

and the proof follows from the proof of Theorem 12.2.

Theorem 12.6 *Let (M, d) be an $\prec$-regular and $\prec$-complete metric space. Let f, g, S, T be $\prec$-continuous self-mappings on (M, d) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$ and $T(M)$ and $S(M)$ are closed subsets of M. Suppose that, for all $r_1, r_2 \in M$ with $S(r_1) \prec T(r_2)$, $d(f(r_1), g(r_2)) > 0$ and for some $F \in \Delta_F$ and $\tau > 0$, the inequality*

$$\tau + F(d(f(r_1), g(r_2))) \leq F(\mathscr{M}_1(r_1, r_2))$$

holds. Assume that the pairs (f, S), (g, T) are weakly compatible and the pairs (f, g) and (g, f) are transitive partially weakly $\prec$-increasing pairs of mappings with respect to T and S, respectively. Then the pairs (f, S), (g, T) have a coincidence point υ in M. Moreover, if $S\upsilon \prec T\upsilon$, then υ is a coincidence point of the mappings f, g, S, T.

Proof Define

$$\alpha(r_1, r_2) = \begin{cases} 1, & \text{if } r_1 \prec r_2; \\ 0, & \text{otherwise} \end{cases}$$

and the proof follows from the proofs of Theorems 12.2 and 12.3.

12.4 Results in Metric Spaces Endowed with a Graph

Consistent with Jachymski [6], let (Md) be a metric space and Δ denotes the diagonal of the Cartesian product $M \times M$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with M and the set $E(G)$ of its edges contains all loops. We assume that G has no parallel edges and so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [7]) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N \in \mathbf{N}$) is a sequence $\{x_i\}_{i=1}^{N}$ of $N+1$ vertices such that $x_0 = x$ and $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, 3, \ldots, N$.

Recently, some results have appeared in the setting of metric spaces which are endowed with a graph. The first result in this direction was given by Jachymski [6].

Definition 12.19 ([3]) Let f and g be two self-mappings on a graphic metric space (M, d). A pair (f, g) is said to be:

(1) *weakly G-increasing* if $(f(r), gf(r)) \in E(G)$ and $(g(r), fg(r)) \in E(G)$ for all $r \in M$;

(2) *partially weakly G-increasing* if $(f(r), gf(r)) \in E(G)$ for all $r \in M$.

Definition 12.20 Let $f, g, h : M \to M$ be three mappings such that $f(M) \cup g(M) \subseteq h(M)$. The pair (f, g) is said to be a *transitive weakly G-increasing pair* of mappings with respect to h if the following conditions hold:

(a) $(f(r_1), g(r_2)) \in E(G)$ for all $r_1 \in M$ and $r_2 \in h^{-1}f(r_1)$ and $(g(r_1), f(r_2)) \in E(G)$ for all $r_2 \in h^{-1}g(r_1)$;

(b) $(r_1, r_3) \in E(G)$ and $(r_3, r_2) \in E(G)$ imply $(r_1, r_2) \in E(G)$ for all $r_1, r_2, r_3 \in M$.

Definition 12.21 Let $f, g, h : M \to M$ be three mappings such that $f(M) \cup g(M) \subseteq h(M)$. The pair (f, g) is said to be a *transitive partially weakly G-increasing pair* of mappings with respect to h if the following conditions hold:

(a) $(f(r_1), g(r_2)) \in E(G)$ for all $r_1 \in M$ and $r_2 \in h^{-1}f(r_1)$;

(b) $(r_1, r_3) \in E(G)$ and $(r_3, r_2) \in E(G)$ imply $(r_1, r_2) \in E(G)$ for all $r_1, r_2, r_3 \in M$.

Let (M, d) be a graphic metric space and let

$$\alpha(r_1, r_2) = \begin{cases} 1, & \text{if } (r_1, r_2) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

By this assumption, we see that the above definitions are special cases of the definitions of weak α-admissibility and partially weak α-admissibility.

Definition 12.22 ([15]) Let (M, d) be a graphic metric space. It is said to be *G-complete* if and only if every Cauchy sequence $\{r_n\}$ in M such that $(r_n, r_{n+1}) \in E(G)$ converges in M.

Definition 12.23 ([15]) Let (M, d) be a graphic metric space and $T : M \to M$ be a mapping. We say that T is a *G-continuous mapping* on (M, d) if, for any $r \in M$ and a sequence $\{r_n\}$,

$$\lim_{n\to\infty} d(r_n, r) = 0, \quad (r_n, r_{n+1}) \in E(G), \ \forall n \in \mathbf{N}, \quad \text{imply} \quad \lim_{n\to\infty} d(T(r_n), T(r)) = 0.$$

Definition 12.24 ([3]) Let (M, d) be a graphic metric space. The pair (f, g) is said to be *G-compatible* if $\lim_{n\to\infty} d(fg(r_n), gf(r_n)) = 0$, whenever $\{r_n\}$ is a sequence in M such that $(r_n, r_{n+1}) \in E(G)$ and

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} g(r_n) = t$$

for some $t \in M$.

Definition 12.25 The graphic metric space (M, d) is said to be G-regular if for any sequence $\{r_n\}$ in M, the following condition holds:

$$\text{if} \ \ r_n \to r \ \ \text{and} \ \ (r_n, r_{n+1}) \in E(G), \ \ \forall n \in \mathbf{N}, \ \ \text{then} \ \ (r_n, r) \in E(G), \ \ \forall n \in \mathbf{N}.$$

Now, we are able to remodel Theorems 12.2 and 12.3 in the framework of graphic metric spaces.

Theorem 12.7 *Let (M, d) be a G-complete graphic metric space and f, g, S, T be G-continuous self-mappings on (M, d) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$. Suppose that, for all $r_1, r_2 \in M$ with $(S(r_1), T(r_2)) \in E(G)$, $d(f(r_1), g(r_2)) > 0$ and for some $F \in \Delta_F$ and $\tau > 0$, the inequality*

$$\tau + F(d(f(r_1), g(r_2))) \le F\left(\mathscr{M}_1(r_1, r_2)\right)$$

holds. Assume that the pairs (f, S), (g, T) are G-compatible and the pairs (f, g) and (g, f) are transitive partially weakly G-increasing pairs of mappings with respect to T and S, respectively. Then the pairs (f, S), (g, T) have a coincidence point υ in M. Moreover, if $(S\upsilon, T\upsilon) \in E(G)$, then υ is a common point of the mappings f, g, S, T.

Proof Define

$$\alpha(r_1, r_2) = \begin{cases} 1, & \text{if } (r_1, r_2) \in E(G), \\ 0, & \text{otherwise} \end{cases}$$

and the proof follows from the proof of Theorem 12.2.

Theorem 12.8 *Let (M, d) be a G-regular and G-complete graphic metric space. Let f, g, S, T be G-continuous self-mappings on (M, d) such that $f(M) \subseteq T(M)$, $g(M) \subseteq S(M)$ and $T(M)$ and $S(M)$ are closed subsets of M. Suppose that, for all*

$r_1, r_2 \in M$ with $(S(r_1), T(r_2)) \in E(G)$, $d(f(r_1), g(r_2)) > 0$ and for some $F \in \Delta_F$ and $\tau > 0$, the inequality

$$\tau + F(d(f(r_1), g(r_2))) \leq F(\mathscr{M}_1(r_1, r_2))$$

holds. Assume that the pairs (f, S), (g, T) are weakly compatible and the pairs (f, g) and (g, f) are transitive partially weakly G-increasing pairs of mappings with respect to T and S respectively. Then the pairs (f, S), (g, T) have a coincidence point υ in M. Moreover, if $(S\upsilon, T\upsilon) \in E(G)$, then υ is a coincidence point of the mappings f, g, S, T.

Proof Define

$$\alpha(r_1, r_2) = \begin{cases} 1, & \text{if } (r_1, r_2) \in E(G); \\ 0, & \text{otherwise} \end{cases}$$

and the proof follows from the proofs of Theorems 12.2 and 12.3.

Corollaries 12.2, 12.3, 12.4, 12.5 and 12.6 given above hold equally good in ordered metric spaces and graphic metric spaces.

12.5 Application

Let $M = C([a, b], \mathbf{R})$ be the space of all continuous real valued functions defined on $[a, b]$. Let the function $d : M \times M \to [0, \infty)$ be defined by

$$d(u, v) = \sup_{t \in [a,b]} |u(t) - v(t)| \tag{12.30}$$

for all $u, v \in C([a, b], \mathbf{R})$ and define $\alpha : M \times M \to [0, \infty)$ by

$$\alpha(u(t), v(t)) = \begin{cases} 1, & \text{if } t \in [a, b]; \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, (M, d) is an α-complete metric space.

Now, we apply Theorem 12.2 to show the existence of common solution of the system of Volterra type integral equations given by

$$u(t) = p(t) + \int_a^t K(t, r, S(u(t)))dr, \tag{12.31}$$

$$w(t) = p(t) + \int_a^t J(t, r, T(v(t)))dr \tag{12.32}$$

for all $t \in [a, b]$ and $a > 0$, where $p : M \to \mathbf{R}$ is a continuous function and $K, J : [a, b] \times [a, b] \times M \to \mathbf{R}$ are lower semi continuous operators. Now we prove the following theorem to ensure the existence of solution of system of the integral equations (12.31) and (12.32).

Theorem 12.9 *Let $M = C([a, b], \mathbf{R})$ and define the mappings $f, g : M \to M$ by*

$$fu(t) = p(t) + \int_a^t K(t, r, S(u(t)))dr,$$

$$gu(t) = p(t) + \int_a^t J(t, r, T(v(t)))dr$$

for all $t \in [a, b]$ and $a > 0$, where $p : M \to \mathbf{R}$ is a continuous function and $K, J : [a, b] \times [a, b] \times M \to \mathbf{R}$ are lower semi continuous operators. Assume the following conditions are satisfied:

(H1) *there exists a continuous function $H : M \to [0, \infty)$ such that*

$$|K(t, r, S) - J(t, r, T)| \leq H(r)|S(u(t)) - T(v(t))|$$

for each $t, r \in [a, b]$ and $S, T \in M$;

(H2) *there exists $\tau > 0$ and for each $r \in M$, we have*

$$\int_a^t H(r)dr \leq e^{-\tau}$$

for all $t \in [a, b]$;

(H3) *there exists a sequence $\{r_n\}$ in M such that* $\lim_{n\to\infty} d(fS(r_n), Sf(r_n)) = 0$ *and* $\lim_{n\to\infty} d(gT(r_n), Tg(r_n)) = 0$, *whenever $\alpha(r_n, r_{n+1}) \geq 1$ and*

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} S(r_n) = t, \quad \lim_{n\to\infty} g(r_n) = \lim_{n\to\infty} T(r_n) = t$$

for some $t \in M$;

(H4) $\alpha(f(r_1), g(r_2)) \geq 1$ *for all $r_1 \in M$ and $r_2 \in h^{-1} f(r_1)$;*

(H5) $\alpha(r_1, r_3) \geq 1$ *and* $\alpha(r_3, r_2) \geq 1$ *imply* $\alpha(r_1, r_2) \geq 1$ *for all $r_1, r_2, r_3 \in M$.*

Then the system of integral equations given in (12.31) and (12.32) has a solution.

Proof By the assumptions (H1) and (H2), we have

$$\begin{aligned}
d(fu(t), gv(t)) &= \sup_{t\in[a,b]} |fu(t) - gv(t))| \\
&= \sup_{t\in[a,b]} \int_a^t |K(t, r, S(u(t)) - J(t, r, T(v(t))))|dr \\
&\leq \sup_{t\in[a,b]} \int_a^t H(r)|S(u(t)) - T(v(t))|dr \\
&\leq \sup_{t\in[a,b]} |S(u(t)) - T(v(t))| \int_a^t H(r)dr \\
&= d(S(u(t)), T(v(t))) \int_a^t H(r)dr \\
&\leq d(S(u(t)), T(v(t)))e^{-\tau} \leq \mathscr{M}_1(u(t), v(t))e^{-\tau}.
\end{aligned}$$

Consequently, we have

$$d(fu(t), gv(t)) \leq e^{-\tau}\mathscr{M}_1(u(t), v(t)),$$

which implies

$$\tau + \ln(d(fu(t), gv(t))) \leq \ln(\mathscr{M}_1(u(t), v(t))).$$

Taking $F(r) = \ln(r)$, we can show that all the hypotheses of Theorem 12.2 are satisfied. Hence, the system of integral equations given in (12.31) and (12.32) has a unique common solution. This completes the proof.

12.6 Conclusion

We have seen that the concepts of α-complete metric space, α-continuity of a mapping, and α-compatibility of a pair of mappings are weaker than the concepts of complete metric space, continuity of a mapping, and compatibility of a pair of mappings, respectively. Therefore, Theorems 12.2 and 12.3 and the corresponding corollaries enrich the fixed point theory on F-contraction under weaker conditions.

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Chapter 13
Measure of Noncompactness in Banach Algebra and Its Application on Integral Equations of Two Variables

Anupam Das and Bipan Hazarika

Abstract The aim of this chapter is to introduce a class of measure of noncompactness satisfying certain conditions. We apply it to establish a few theorems on existence of solution integral equations of two variables in Banach algebra. Further, we explain the results with the help of examples.

Keywords Measure of noncompactness · Fixed point theorem · Functional Integral Equations · Banach Algebra.

Mathematics Subject Classification 45G05 · 26A33 · 74H20

13.1 Introduction

The measure of noncompactness plays a very significant role in fixed point theory. The measure of noncompactness was first introduced by Kuratowski [22]. There are different types of measure of noncopactness in metric and topological spaces. We refer to the reader [8] for details on measure of noncompactness. On the other hand, the measure of noncompactness has applications in different types of integral equations and differential equations (see [2–7, 16–19, 21, 24–28]).

Assuming that measure of noncompactness used in the study satisfies certain condition, the existence of solution of the integral equations in two variables has been proved. The results that are going to be proved in this chapter are generalization of the results of the other papers and monographs [11, 13, 14].

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The main idea of this investigation depends on the indication of a class of measure of noncompactness in Banach algebras satisfying certain condition called condition(m). We discussed the measure of noncompactness satisfying condition (m) in the Banach algebras $C(I \times I)$ and $BC(\mathbb{R}_+ \times \mathbb{R}_+)$.

Suppose that E is a real Banach space with the norm $\| \, . \, \|$. Let $B(y, d)$ be a closed ball in E centered at y and with radius d. If X is a nonempty subset of E then by $\overline{X}$ and Conv$X$ we denote the closure and convex closure of X. Moreover, let $\mathscr{M}_E$ denote the family of all nonempty and bounded subsets of E and $\mathscr{N}_E$ its subfamily consisting of all relatively compact sets. We denote by $\mathbb{R}$ the set of real numbers, $\mathbb{R}_+ = [0, \infty)$ and $I = [0, 1]$.

13.2 Measure of Noncompactness

Measure of noncompactness is an important tool in Banach spaces. It can be used in fixed point theory, differential equations, integral equations, integro-differential, functional equations, etc.

13.2.1 Preliminaries

Let M and S be subsets of a metric space (X, d) and $\varepsilon > 0$. Then, the set S is called *ε-net* of M if, for any $x \in M$, there exists $s \in S$, such that $d(x, s) < \varepsilon$. If S is finite, then the ε-net S of M is called *finite ε-net*. The set M is said to be *totally bounded* if it has a finite ε-net for every $\varepsilon > 0$. A subset M of a metric space X is said to be *compact* if every sequence (x_n) in M has a convergent subsequence and the limit of that subsequence is in M. The set M is called *relatively compact* if the closure $\overline{M}$ of M is a compact set. If a set M is relatively compact, then M is totally bounded. If the metric space (X, d) is complete, then the set M is relatively compact if and only if it is totally bounded.

If $x \in X$ and $r > 0$, then the *open ball* with center at x and radius r is denoted by $B(x, r)$, where $B(x, r) = \{y \in X : d(x, y) < r\}$. If X is a normed space, then we denote by B_X the *closed unit ball* in X and by S_X the unit sphere in X.

Let $\mathscr{M}_X$ or, simply, $\mathscr{M}$ be the family of all nonempty and bounded subsets of a metric space (X, d) and let $\mathscr{M}_X^c$ or simply $\mathscr{M}^c$ be the subfamily of $\mathscr{M}_X$ consisting of all closed sets. Further, let $\mathscr{N}_X$ or simply $\mathscr{N}$ be the family of all nonempty and relatively compact subsets of (X, d). Let $d_H : \mathscr{M} \times \mathscr{M} \to \mathbb{R}$ be the function defined by

$$d_H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\},$$

where $A, B \in \mathscr{M}_X$. The function d_H is called the *Hausdorff distance* and $d_H(A, B)$ is the *Hausdorff distance* of two sets A, B.

Let X and Y be infinite-dimensional complex Banach spaces and denote the set of bounded linear operators from X into Y by $B(X, Y)$. We put $B(X) = B(X, X)$. For T in $B(X, Y)$, $N(T)$ and $R(T)$ denote the null space and the range space of T, respectively. A linear operator L from X to Y is called *compact* (or *completely continuous*) if $D(L) = X$ for the domain of L and, for every sequence $(x_n) \in X$ such that $\| x_n \| \leq C$, the sequence $(L(x_n))$ has a subsequence which converges in Y. A compact operator is bounded. An operator L in $B(X, Y)$ is *of finite rank* if $\dim R(L) < \infty$. An operator of finite rank is clearly compact. Let $F(X, Y)$, $C(X, Y)$ denote the set of all finite rank and compact operators from X to Y, respectively. Set

$$F(X) = F(X, X), \quad C(X) = C(X, X).$$

If E is a subset of X, then the intersection of all convex sets that contain F is called *convex cover* or *convex hull* of F denoted by $co(E)$.

Let Q be a nonempty and bounded subset of a normed space X. Then, the convex closure of Q denoted by $Co(Q)$ is the smallest convex and closed subset of X that contains Q. Note that $Co(Q) = \overline{co}(Q)$.

13.2.2 Kuratowski Measure of Noncompactness

Definition 13.1 ([8]) Let (X, d) be a metric space and Q a bounded subset of X. Then, the *Kuratowski measure of noncompactness* (α-measure or set measure of noncompactness) of Q, denoted by $\alpha(Q)$, is the infimum of the set of all numbers $\varepsilon > 0$ such that Q can be covered by a finite number of sets with diameters $\varepsilon > 0$, that is,

$$\alpha(Q) = \inf \left\{ \varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} S_i,\, S_i \subset X, \operatorname{diam}(S_i) < \varepsilon \ (i = 1, 2, ..., n), n \in \mathbb{N} \right\}.$$

The function α is called *Kuratowski's measure of noncompactness*, which was introduced by Kuratowski [22]. Clearly, we have

$$\alpha(Q) \leq \operatorname{diam}(Q) \text{ for each bounded subset } Q \text{ of } X.$$

Lemma 13.1 ([8]) *Let Q, Q_1 and Q_2 be bounded subsets of a complete metric space (X, d). Then,*
(1) $\alpha(Q) = 0$ *if and only if* $\overline{Q}$ *is compact (regularity).*
(2) $\alpha(Q) = \alpha(\overline{Q})$ *(invariance under passage to the closure).*
(3) $Q_1 \subset Q_2$ *implies* $\alpha(Q_1) \leq \alpha(Q_2)$ *(monotonicity).*
(4) $\alpha(Q_1 \cup Q_2) = \max\{\alpha(Q_1), \alpha(Q_2)\}$ *(maximum property).*
(5) $\alpha(Q_1 \cap Q_2) \leq \min\{\alpha(Q_1), \alpha(Q_2)\}$.

Lemma 13.2 ([8]) *Let Q, Q_1 and Q_2 be bounded subsets of a normed space X. Then,*
(1) $\alpha(Q_1 + Q_2) \leq \alpha(Q_1) + \alpha(Q_2)$.
(2) $\alpha(Q + x) = \alpha(Q)$ *for each* $x \in X$.
(3) $\alpha(\lambda Q) = |\lambda|\, \alpha(Q)$ *for each* $\lambda \in \mathbb{F}$, *where* $\mathbb{F}$ *is the field of scalars.*
(4) $\alpha(Q) = \alpha(Co(Q))$.

We recall the following definition of a measure of noncompactness given in [10].

Definition 13.2 A function $\mu : \mathscr{M}_E \to [0, \infty)$ is called a *measure of noncompactness* in E if it satisfies the following conditions:
(a) for all $X \in \mathscr{M}_E$, we have $\mu(X) = 0$ implies that X is precompact;
(b) the family $\ker \mu = \{X \in \mathscr{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathscr{N}_E$;
(c) $X \subseteq Y \implies \mu(X) \leq \mu(Y)$;
(d) $\mu\left(\overline{X}\right) = \mu(X)$;
(e) $\mu(\mathrm{Conv}\, X) = \mu(X)$;
(f) $\mu(\lambda X + (1-\lambda) Y) \leq \lambda \mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$;
(g) if $X_n \in \mathscr{M}_E$, $X_n = \overline{X}_n$, $X_{n+1} \subset X_n$ for each $n = 1, 2, 3, \ldots$ and $\lim\limits_{n\to\infty} \mu(X_n) = 0$, then $\bigcap\limits_{n=1}^{\infty} X_n \neq \varnothing$.

The family $\ker \mu$ is said to be the *kernel of measure* μ. Observe that the intersection set X_∞ from (g) is a member of the family $\ker \mu$. In fact, since $\mu(X_\infty) \leq \mu(X_n)$ for any $n = 1, 1, 3, \ldots$, we infer that $\mu(X_\infty) = 0$. This gives $X_\infty \in \ker \mu$.

Definition 13.3 A measure μ is said to be *sublinear* if it satisfies the following conditions:
(a) $\mu(\lambda X) = |\lambda|\, \mu(X)$ for all $\lambda \in \mathbb{R}$;
(b) $\mu(X + Y) \leq \mu(Y) + \mu(Y)$.

A sublinear measure of noncompactness μ satisfying the condition:

$$\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$$

and such that $\ker \mu = \mathscr{N}_E$ is said to be *regular*.

13.2.3 Hausdorff Measure of Noncompactness

Definition 13.4 ([9]) Let (X, d) be a metric space, Q be a bounded subset of X and $B(x, r) = \{y \in X : d(x, y) < r\}$. Then, the *Hausdorff measure of noncompactness* $\chi(Q)$ of Q is defined by

$$\chi(Q) := \inf\left\{\varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i), x_i \in X, r_i < \varepsilon \quad (i = 1, 2, \ldots, n), n \in \mathbb{N}\right\}.$$

The definition of the Hausdorff measure of noncompactness of the set Q is not supposed that centers of the balls that cover Q belong to Q. Hence, it can equivalently be stated as follows:

$$\chi(Q) = \inf \{\varepsilon > 0 : Q \text{ has a finite } \varepsilon - \text{net in } X\}.$$

Then, the following results were obtained in [8, 9].

Lemma 13.3 ([9]) *Let Q, Q_1 and Q_2 be bounded subsets of the complete metric space (X, d). Then,*
(1) $\chi(Q) = 0$ *if and only if $\overline{Q}$ is compact.*
(2) $\chi(Q) = \chi(\overline{Q})$.
(3) $Q_1 \subset Q_2$ *implies* $\chi(Q_1) \leq \chi(Q_2)$.
(4) $\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\}$.
(5) $\chi(Q_1 \cap Q_2) \leq \min\{\chi(Q_1), \chi(Q_2)\}$.

Now, we point out the well-known result of Goldenštein et al. [20].

Let X be a Banach space with a Schauder basis $\{e_1, e_2, ...\}$. Then, each element $x \in X$ has a unique representation $x = \sum\limits_{i=1}^{\infty} \phi_i(x)e_i$, where the functions ϕ_i are the basis functionals. Let $P_n : X \to X$ be the projector onto the linear span of $\{e_1, e_2, ..., e_n\}$, that is, $P_n(x) = \sum\limits_{i=1}^{n} \phi_i(x)e_i$.

Theorem 13.1 ([9]) *Let X be a BK-space with Schauder basis (b_n), $Q \in \mathscr{M}_X$, $P_n : X \to X$ $(n \in \mathbb{N})$ be the projector onto the linear span of $\{e_1, e_2, ..., e_n\}$ and $\mathscr{I}$ be the identity operator on X. Then,*

$$\frac{1}{a} \limsup_{n\to\infty} \left(\sup_{x\in Q} \| (\mathscr{I} - P_n)(x) \| \right) \leq \chi(Q) \leq \limsup_{n\to\infty} \left(\sup_{x\in Q} \| (\mathscr{I} - P_n)(x) \| \right),$$

where $a = \limsup\limits_{n\to\infty} \| \mathscr{I} - P_n \|$.

We say that a norm $\| . \|$ on a sequence space is *monotone* if $x, \overline{x} \in X$ with $|x_k| \leq |\overline{x}_k|$ for all k implies $\| x \| \leq \| \overline{x} \|$.

Theorem 13.2 ([8]) *Let X be a BK-space with AK and monotone norm, $Q \in \mathscr{M}_X$ and $P_n : X \to X$ $(n \in \mathbb{N})$ be the operator (projection) defined by $P_n(x_1, x_2, ...) = x^{[n]} = (x_1, x_2, ..., x_n, 0, 0, ...)$ for all $x = (x_1, x_2, ...) \in X$. Then,*

$$\chi(Q) = \lim_{n\to\infty} \left(\sup_{x\in Q} \| (\mathscr{I} - P_n)(x) \| \right).$$

Now, let us assume that Ω is a nonempty subset of a Banach space E and $F : \Omega \to E$ is a continuous operator which transforms bounded subsets of Ω onto bounded ones. Suppose that μ is a measure of noncompactness given in E.

Definition 13.5 ([22]) We say that T satisfies the *Darbo condition* with a constant k with respect to a measure of noncompactness μ provided $\mu(TX) \leq k\mu(X)$ for each $X \in \mathscr{M}_E$ such that $X \subset \Omega$. If $k < 1$, then T is called a *contraction* with respect to μ.

We assume that the space E has the structure of Banach algebra. For given subsets X, Y of a Banach algebra E, let us denote

$$XY = \{xy : x \in X, y \in Y\}.$$

The measure of noncompactness μ defined on a Banach algebra E is said to be satisfy the *condition* (m) if, for arbitrary sets $X, Y \in \mathscr{M}_E$, the following condition is satisfied:

$$\mu(XY) \leq \| X \| \mu(Y) + \| Y \| \mu(X).$$

We recall following important theorems:

Theorem 13.3 (Shauder [1]) *Let D be a nonempty, closed, and convex subset of a Banach space E. Then every compact, continuous map $T : D \to D$ has at least one fixed point.*

Theorem 13.4 (Darbo [15]) *Let D be a nonempty, bounded, closed, and convex subset of a Banach space E. Let $T : D \to D$ be a continuous mapping. Assume that there is a constant $k \in [0, 1)$ such that*

$$\mu(TM) \leq k\mu(M), \quad M \subseteq D.$$

Then, T has a fixed point.

Theorem 13.5 ([12]) *Assume that Ω is nonempty, bounded, closed, and convex subset of the Banach algebra E, and operators P and T transform continuously the set Ω into E in such way that $P(\Omega)$ and $T(\Omega)$ are bounded. Moreover, we assume that the operator $S = P.T$ transforms Ω into itself. If the operators P and T satisfy on the set Ω the Darbo condition with respect to the measure of noncompactness μ with the constants k_1 and k_2, respectively, then the operator S satisfies on Ω the Darbo condition with the constant $\| P(\Omega) \| k_2 + \| T(\Omega) \| k_1$. Particularly, if $\| P(\Omega) \| k_2 + \| T(\Omega) \| k_1 < 1$, then S is a contraction with respect to the measure of noncompactness μ and has at least one fixed point in the set Ω.*

This condition (m) was used in the paper [12] for measures of noncompactness defined on the Banach algebra $C(I)$. Particularly, the Hausdorff measure of noncompactness χ [12] satisfies condition (m).

The space $C(I \times I)$ represents the Banach space of real functions defined and continuous on $I \times I$ with the norm

$$\| x \| = \sup\{|x(t, s)| : t, s \in I\},$$

where $x \in C(I \times I)$. With respect to the usual product of functions, this space has the structure of Banach algebra.

For arbitrary fixed $\varepsilon > 0$, set $X \in \mathscr{M}_{C(I\times I)}$ and $x \in X$, we denote by $\omega(x, \varepsilon)$ the *modulus continuity* of x, i.e.,

$$\omega(x, \varepsilon) = \sup\{|x(t, s) - x(u, v)| : t, s, u, v \in I, |t - u| \le \varepsilon, |s - v| \le \varepsilon\}.$$

Also, let

$$\omega(X, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in X\}$$

and

$$\omega_0(X) = \lim_{\varepsilon\to 0} \omega(X, \varepsilon).$$

It can be shown that $\omega_0(X)$ is a measure of noncompactness in $C(I \times I)$. Also, $\omega_0(X)$ satisfies condition (m).

Now, we introduce another measure of noncompactness in the Banach algebra $C(I \times I)$ which satisfies the condition (m) on $\mathscr{M}_{C(I\times I)}$. Let a set $X \in C(I \times I)$ and $x \in X$. Also, consider the following quantity:

$$d(x) = \sup\{|x(t, s) - x(u, v)| - [x(t, s) - x(u, v)] : t, s, u, v \in I, u \le t, v \le s\}.$$

Further, let

$$d(X) = \sup\{d(x) : x \in X\}.$$

Finally, we denote

$$\mu_d(X) = \omega_0(X) + d(X). \tag{13.1}$$

It can be shown that μ_d is a measure of noncompactness on the space $C(I \times I)$.

Theorem 13.6 *The measure of noncompactness μ_d satisfies condition(m) on the subfamily of $\mathscr{M}_{C(I\times I)}$ consisting of sets of function being nonnegative on $I \times I$.*

Proof Let X, Y be any arbitrary sets in $\mathscr{M}_{C(I\times I)}$ such that the functions belonging to X, Y are nonnegative on $I \times I$. Further, let $x \in X,\ y \in Y$ be arbitrary fixed and $t, s, \bar{t}, \bar{s} \in I$ with $\bar{t} \le t,\ \bar{s} \le s$. Then, we have

$$\begin{aligned}
&\left|x(t, s)y(t, s) - x(\bar{t}, \bar{s})y(\bar{t}, \bar{s})\right| - \left[x(t, s)y(t, s) - x(\bar{t}, \bar{s})y(\bar{t}, \bar{s})\right]\\
&\le \left|x(t, s)y(t, s) - x(t, s)y(\bar{t}, \bar{s})\right| + \left|x(t, s)y(\bar{t}, \bar{s}) - x(\bar{t}, \bar{s})y(\bar{t}, \bar{s})\right|\\
&\quad - \left\{\left[x(t, s)y(t, s) - x(t, s)y(\bar{t}, \bar{s})\right] + \left[x(t, s)y(\bar{t}, \bar{s}) - x(\bar{t}, \bar{s})y(\bar{t}, \bar{s})\right]\right\}\\
&= |x(t, s)| \left|y(t, s) - y(\bar{t}, \bar{s})\right| + \left|y(\bar{t}, \bar{s})\right| \left|x(t, s) - x(\bar{t}, \bar{s})\right|\\
&\quad - x(t, s)\left[y(t, s) - y(\bar{t}, \bar{s})\right] - y(\bar{t}, \bar{s})\left[x(t, s) - x(\bar{t}, \bar{s})\right]\\
&= |x(t, s)| \left\{\left|y(t, s) - y(\bar{t}, \bar{s})\right| - \left[y(t, s) - y(\bar{t}, \bar{s})\right]\right\}\\
&\quad + \left|y(\bar{t}, \bar{s})\right| \left\{\left|x(t, s) - x(\bar{t}, \bar{s})\right| - \left[x(t, s) - x(\bar{t}, \bar{s})\right]\right\}\\
&\le \| x \| \, d(y) + \| y \| \, d(x).
\end{aligned}$$

This gives

$$d(XY) \leq \| X \| d(Y) + \| Y \| d(X).$$

Since $\omega_0(X)$ satisfy the condition (m), we get

$$\mu_d(XY) \leq \| X \| \mu_d(Y) + \| Y \| \mu_d(X),$$

i.e., μ_d satisfy the condition (m). This completes the proof.

Now, consider the Banach space $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ which consists of the set of real continuous and bounded functions on $\mathbb{R}_+ \times \mathbb{R}_+$ with respect to the norm:

$$\| x \| = \sup \{|x(t, s)| : t, s \geq 0\}, \; x(t, s) \in BC(\mathbb{R}_+ \times \mathbb{R}_+).$$

Let X be a fixed nonempty and bounded subset of the space $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ and τ be a fixed positive number. For $x \in X$ and $\varepsilon > 0$, denote by $\omega^T(x, \varepsilon)$ the modulus of the continuity function x on the interval $[0, \tau]$, i.e.,

$$\omega^\tau(x, \varepsilon) = \sup \{|x(t, s) - x(u, v)| : t, s, u, v \in [0, \tau], |t - u| \leq \varepsilon, |s - v| \leq \varepsilon\}.$$

Further, we define

$$\omega^\tau(X, \varepsilon) = \sup \{\omega^\tau(x, \varepsilon) : x \in X\}.$$

$$\omega_0^\tau(X) = \lim_{\varepsilon \to 0} \omega^\tau(X, \varepsilon)$$

and

$$\omega_0^\infty(X) = \lim_{\tau \to \infty} \omega_0^\tau(X).$$

Also, let

$$a(X) = \lim_{\tau \to \infty} \sup_{x \in X} \{\sup \{|x(t, s)| : t, s \geq \tau\}\}.$$

We denote

$$\mu_a(X) = \omega_0^\infty(X) + a(X). \tag{13.2}$$

It can be shown that μ_a is a measure of noncompactness of μ_a. Let us mention that kernel of the measure μ_a consists of all sets $X \in \mathscr{M}_{BC(\mathbb{R}_+ \times \mathbb{R}_+)}$ such that for any $\varepsilon > 0$ there exists $\tau > 0$ such that $|x(t, s)| \leq \varepsilon$ for all $x \in X$ and $t, s \geq \tau$.

Theorem 13.7 *The measure of noncompactness μ_a satisfies the condition (m).*

Proof This theorem can be proved in the same way as Theorem 13.6. This completes the proof.

13.3 Existence of Solution of a Functional Integral Equation with Two Variables in $C(I \times I)$

Consider the following integral equation:

$$x(t,s) = f(t,s,x(t,s))\left(p(t,s) + \int_0^t \int_0^s G(t,s,v,w,x(v,w))dvdw\right), \quad (13.3)$$

where $t, s, v, w \in I = [0, 1]$. The Eq. (13.3) can be written in the following form:

$$x(t,s) = (Fx)(t,s)(Vx)(t,s), \quad (13.4)$$

where

$$(Fx)(t,s) = f(t,s,x(t,s))$$

and

$$(Vx)(t,s) = p(t,s) + \int_0^t \int_0^s G(t,s,v,w,x(v,w))dvdw,$$

where $t, s, v, w \in I$.

Consider the following assumptions:

(a) $p \in C(I \times I)$ and p is nonnegative and nondecreasing function on $I \times I$;

(b) The function $f : I \times I \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(I \times I \times \mathbb{R}_+) \subseteq \mathbb{R}_+$. Moreover, the function $f(t,s,x)$ is nondecreasing with respect to $t, s \in I$ for any fixed $x \in \mathbb{R}_+$ and the function $f(t,s,x)$ is nondecreasing on $\mathbb{R}_+$ for any fixed $t, s \in I$;

(c) There exists $0 < K < 1$ such that

$$|f(t,s,x) - f(t,s,y)| \leq K\,|x - y|\,,$$

for all $t, s \in I$ and $x, y \in [-r, r]$;

(d) The function $G : I \times I \times I \times I \times \mathbb{R} \to \mathbb{R}$ is continuous such that $G : I \times I \times I \times I \times \mathbb{R}_+ \to \mathbb{R}_+$ and $G(t,s,v,w,x)$ is nondecreasing with respect to each variable t, s, v, w and x separately;

(e) There exists a continuous and nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $|G(t,s,v,w,x)| \leq \phi(|x|)$ for all $t, s, v, w \in I$ and $x \in \mathbb{R}$;

(f) There exists a positive solution r_0 of the inequality

$$\left(rK + \overline{F}\right)(\parallel p \parallel + \phi(r)) \leq r,$$

where $\overline{F} = \max\{|f(t,s,0)| : t, s \in I\}$. Moreover, the number r_0 such that

$$K\,(\parallel p \parallel + \phi(r_0)) < 1.$$

Theorem 13.8 *Under the hypothesis (i)–(vi), Eq. (13.3) has at least one solution in* $C(I \times I)$.

Proof By the assumption (b), we observe that the operator F transforms the Banach space $C(I \times I)$ into itself and is continuous. Again, by the assumptions (a) and (d), we observe that the operator V transforms the Banach space $C(I \times I)$ into itself and is continuous.

On the other hand, for fixed $x \in C(I \times I)$ and $t, s \in I$, we get

$$|(Fx)(t,s)| \leq |f(t,s,x(t,s)) - f(t,s,0)| + |f(t,s,0)| \leq K \parallel x \parallel + \overline{F}. \quad (13.5)$$

Moreover, we obtain

$$\begin{aligned} |(Vx)(t,s)| &\leq |p(t,s)| + \int_0^t \int_0^s |G(t,s,v,w,x(v,w))| \, dvdw \\ &\leq \parallel p \parallel + \int_0^t \int_0^s \phi(\parallel x \parallel) dvdw, \end{aligned}$$

i.e.,

$$|(Vx)(t,s)| \leq \parallel p \parallel + \phi(\parallel x \parallel). \quad (13.6)$$

It can be seen that, using (13.5), (13.6) and the assumption (f), there exists a positive number r_0 such that operator $W = F.V$ maps the ball $B_{r_0} = \{x : \parallel x \parallel \leq r_0\}$ into itself. On the other hand, we observe that, from (13.5), (13.6), the following inequalities are satisfied:

$$\parallel FB_{r_0} \parallel \leq r_0 K + \overline{F} \quad (13.7)$$

and

$$\parallel VB_{r_0} \parallel \leq \parallel p \parallel + \phi(r_0). \quad (13.8)$$

Further, let the set Q consisting of all nonnegative functions $x \in B_{r_0}$. Then, by the assumptions, we infer that the operator W maps Q into itself. Moreover, form (13.7) and (13.8), we get

$$\parallel FQ \parallel \leq r_0 K + \overline{F}$$

and

$$\parallel VQ \parallel \leq \parallel p \parallel + \phi(r_0).$$

Since the operator F is continuous on Q by the assumptions (b) and (c) and the operator V is also continuous on Q by the assumptions (a), (d), and (e).

Now, fix a nonempty subset X of the subset Q, choose a number $\varepsilon > 0$ and take t_1, s_1, t_2, s_2 such that $|t_2 - t_1| \leq \varepsilon, \ |s_2 - s_1| \leq \varepsilon$. Then, we have

$$\begin{aligned}|(Fx)(t_2, s_2) - (Fx)(t_1, s_1)| &\leq |f(t_2, s_2, x(t_2, s_2)) - f(t_1, s_1, x(t_1, s_1))| \\ &\leq |f(t_2, s_2, x(t_2, s_2)) - f(t_2, s_2, x(t_1, s_1))| \\ &\quad + |f(t_2, s_2, x(t_1, s_1)) - f(t_1, s_1, x(t_1, s_1))| \\ &\leq K\,|x(t_2, s_2)) - x(t_1, s_1))| + \omega_{r_0}(f, \varepsilon) \\ &\leq K\omega(x, \varepsilon) + \omega_{r_0}(f, \varepsilon),\end{aligned}$$

where

$$\begin{aligned}\omega_{r_0}(f, \varepsilon) = \sup\{&|f(t_2, s_2, x) - f(t_1, s_1, x)| : t_1, s_1, t_2, s_2 \in I,\ |t_2 - t_1| \leq \varepsilon, \\ &|s_2 - s_1| \leq \varepsilon,\ x \in [-r_0, r_0]\}.\end{aligned}$$

Hence, we have

$$\omega(Fx, \varepsilon) \leq K\omega(x, \varepsilon) + \omega_{r_0}(f, \varepsilon)$$

and, consequently,

$$\omega_0(FX) \leq K\omega_0(X). \tag{13.9}$$

Again, we have

$$\begin{aligned}&|(Vx)(t_2, s_2) - (Vx)(t_1, s_1)| \\ &\leq |p(t_2, s_2) - p(t_1, s_1)| \\ &\quad + \Big| \int_0^{t_2} \int_0^{s_2} G(t_2, s_2, v, w, x(v, w))dvdw - \int_0^{t_1} \int_0^{s_1} G(t_1, s_1, v, w, x(v, w))dvdw \Big| \\ &\leq \omega(p, \varepsilon) + \Big| \int_0^{t_2} \int_0^{s_2} G(t_2, s_2, v, w, x(v, w))dvdw \\ &\quad - \int_0^{t_1} \int_0^{s_1} G(t_1, s_1, v, w, x(v, w))dvdw \Big|.\end{aligned}$$

This gives

$$\begin{aligned}\omega(VX, \varepsilon) \leq \omega(p, \varepsilon) + \Big| &\int_0^{t_2} \int_0^{s_2} G(t_2, s_2, v, w, x(v, w))dvdw \\ &- \int_0^{t_1} \int_0^{s_1} G(t_1, s_1, v, w, x(v, w))dvdw \Big|\end{aligned}$$

and, consequently,

$$\omega_0(VX) = 0. \tag{13.10}$$

Taking an arbitrary function $x \in X$ and $t_1 \leq t_2,\ s_1 \leq s_2$, we get

$$
\begin{aligned}
&|(Vx)(t_2, s_2) - (Vx)(t_1, s_1)| - [(Vx)(t_2, s_2) - (Vx)(t_1, s_1)] \\
&\leq |p(t_2, s_2) - p(t_1, s_1)| - [p(t_2, s_2) - p(t_1, s_1)] \\
&\quad + \left| \int_0^{t_2} \int_0^{s_2} G(t_2, s_2, v, w, x(v, w)) dv dw - \int_0^{t_1} \int_0^{s_1} G(t_1, s_1, v, w, x(v, w)) dv dw \right| \\
&\quad - \left[\int_0^{t_2} \int_0^{s_2} G(t_2, s_2, v, w, x(v, w)) dv dw - \int_0^{t_1} \int_0^{s_1} G(t_1, s_1, v, w, x(v, w)) dv dw \right].
\end{aligned}
$$

Therefore, we have

$$d(Vx) = 0$$

and hence

$$d(VX) = 0. \tag{13.11}$$

Similarly, we can show that

$$d(FX) \leq Kd(X). \tag{13.12}$$

From (13.9), (13.10), (13.11), (13.12) and the definition of the measure of noncompactness μ_d, we get

$$\mu_d(FX) \leq K\mu_d(X)$$

and

$$\mu_d(VX) = 0.$$

In view of Theorem 13.5 that the operator W is a contraction with respect to μ_d on the set Q. Thus, W has a fixed point x in Q. Thus, the integral Eq. (13.4) has a solution in $C(I \times I)$. This completes the proof.

Example 13.1 Consider the following system of integral equations:

$$x(t, s) = \left[\frac{ts}{t^2 s^2 + 15} + \frac{x(t, s)}{2} \right] \left[t^2 s^2 e^{-2ts} + \int_0^t \int_0^s \{vwts + x(v, w)\} \, dv dw \right], \tag{13.13}$$

where $t, s \in I = [0, 1]$. It can be seen that this equation is a particular case of the Eq. (13.3), where

$$f(t, s, x) = \frac{ts}{t^2 s^2 + 15} + \frac{x}{2},$$

$$p(t, s) = t^2 s^2 e^{-2ts}$$

and

$$G(t, s, v, w, x) = tsvwx.$$

It can be easily seen that the Eq. (13.13) satisfies the assumption of Theorem 13.8 with

$$K = \frac{1}{2},\ f(t, s, 0) = \frac{ts}{t^2 s^2 + 15},\ \overline{F} = \frac{1}{15}$$

and

$$\phi(x) = x.$$

Accordingly, we have $\| p \| = \frac{1}{2e}$. Thus, the inequality (f), of the assumptions, has the form

$$\left(\frac{r}{2} + \frac{1}{15}\right)\left(\frac{1}{2e} + r\right) \leq r,$$

i.e.,

$$(0.5r + 0.067)\,(0.183 + r) \leq r.$$

We check that $r = \frac{1}{2}$ satisfies the above inequality, i.e., $r_0 = \frac{1}{2}$. Also, we have

$$K\,(\| p \| + \phi(r_0)) = \frac{1}{2}\left(\frac{1}{2e} + \frac{1}{2}\right) < 1.$$

Therefore, the Eq. (13.13) has a solution belonging to $B_{\frac{1}{2}}$ and hence in $C(I \times I)$.

13.4 Existence of Solution of a Functional Integral Equation with Two Variables in $BC(\mathbb{R}_+ \times \mathbb{R}_+)$

Consider the following integral equation:

$$x(t, s) = (Vx)(t, s)(Ux)(t, s), \tag{13.14}$$

where $t, s \in \mathbb{R}_+$ and the operators V and U are defined on $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ in the following way:

$$(Vx)(t, s) = p_1(t, s) + f_1(t, s, x(t, s)) \int_0^t \int_0^s h_1(t, s, v, w, x(v, w))dvdw$$

and

$$(Ux)(t, s) = p_2(t, s) + f_2(t, s, x(t, s)) \int_0^t \int_0^s h_2(t, s, v, w, x(v, w))dvdw.$$

Consider the following assumptions:

(a) $p_i \in BC(\mathbb{R}_+ \times \mathbb{R}_+)$ and $p_i(t, s) \to 0$ as $t, s \to \infty (i = 1, 2)$;

(b) The function $f_i : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and $f_i(t, s, 0) \to 0$ as $t, s \to \infty$ for $i = 1, 2$;

(c) There exists constants $K_i > 0$ such that

$$|f_i(t,s,x) - f_i(t,s,y)| \leq K_i\, |x-y|$$

for all $t, s \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$ $(i = 1, 2)$;

(d) The function $h_1 : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a continuous nondecreasing function $G_1 : \mathbb{R}_+ \to \mathbb{R}_+$ and a continuous function $g_1 : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|h_1(t,s,v,w,x)| \leq g_1(t,s,v,w)G_1(|x|)$$

for all $t, s, v, w \in \mathbb{R}_+$ and $x \in \mathbb{R}$;

(e) The function $h_2 : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a continuous nondecreasing function $G_2 : \mathbb{R}_+ \to \mathbb{R}_+$ and a continuous function $g_2 : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|h_2(t,s,v,w,x)| \leq g_2(t,s,v,w)G_2(|x|)$$

for all $t, s, v, w \in \mathbb{R}_+$ and $x \in \mathbb{R}$;

(f) The function

$$(t,s) \to \int_0^t \int_0^s g_1(t,s,v,w)\,dvdw$$

is bounded on $\mathbb{R}_+ \times \mathbb{R}_+$;

(g) The function

$$(t,s) \to \int_0^t \int_0^s g_2(t,s,v,w)\,dvdw$$

is bounded on $\mathbb{R}_+ \times \mathbb{R}_+$;

(h) Also,

$$\overline{F}_i = \sup\{|f_i(t,s,0)| : t, s \in \mathbb{R}_+\}$$

and

$$\overline{G}_i = \sup\left\{\int_0^t \int_0^s g_i(t,s,v,w)\,dvdw : t, s \in \mathbb{R}_+\right\}$$

for $i = 1, 2$;

(i) There exists a positive solution r_0 of the inequality

$$\left[p + K\overline{G}_1 r G_1(r) + \overline{FG}_1 G_1(r)\right]\left[p + K\overline{G}_2 r G_2(r) + \overline{FG}_2 G_2(r)\right] \leq r$$

such that

$$pK\left[\overline{G}_1 G_1(r_0) + \overline{G}_2 G_2(r_0)\right] + 2K\overline{F}\overline{G}_1 G_1(r_0)\overline{G}_2 G_2(r_0) \\ + 2K^2 r_0 \overline{G}_1 G_1(r_0)\overline{G}_2 G_2(r_0) < 1,$$

where

$$p = \max\{\| p_1 \|, \| p_1 \|\}, \overline{F} = \max\left\{\overline{F}_1, \overline{F}_2\right\} \text{ and } K = \max\{K_1, K_2\}.$$

Theorem 13.9 *Under the assumptions* (a)-(i)*, the Eq. (13.14) has at least one solution* $x(t,s)$ *in the Banach algebra* $BC(\mathbb{R}_+ \times \mathbb{R}_+)$.

Proof Suppose x is a fixed function from $BC(\mathbb{R}_+ \times \mathbb{R}_+)$. From the assumption (i)–(iv), it is obvious that Vx is a continuous function on $\mathbb{R}_+ \times \mathbb{R}_+$. For arbitrary fixed $t, s \in \mathbb{R}_+$, we get

$$\begin{aligned}
&|(Vx)(t,s)| \\
&\leq |p_1(t,s)| + |f_1(t,s,x(t,s))| \int_0^t \int_0^s |h_1(t,s,v,w,x(v,w))|\, dvdw \\
&\leq |p_1(t,s)| + [|f_1(t,s,x(t,s)) - f_1(t,s,0)| + |f_1(t,s,0)|] \\
&\quad \times \int_0^t \int_0^s g_1(t,s,v,w) G_1(|x(v,w)|) dvdw \\
&\leq |p_1(t,s)| + [K_1|x(t,s)| + |f_1(t,s,0)|] G_1(\| x \|) \int_0^t \int_0^s g_1(t,s,v,w) dvdw.
\end{aligned}$$

Hence, we have

$$|(Vx)(t,s)| \leq \| p_1 \| + K_1 \overline{G}_1 \| x \| G_1(\| x \|) + \overline{F}_1 \overline{G}_1 G_1(\| x \|).$$

Thus, the function Vx is bounded on $\mathbb{R}_+ \times \mathbb{R}_+$. Therefore, it can be concluded that V transforms the Banach algebra $BC(\mathbb{R}_+ \times \mathbb{R}_+)$ to itself. Moreover, we have

$$\| Vx \| \leq p + K\overline{G}_1 \| x \| G_1(\| x \|) + \overline{F}\overline{G}_1 G_1(\| x \|). \tag{13.15}$$

Similarly, it can be shown that $Ux \in BC(\mathbb{R}_+ \times \mathbb{R}_+)$ and

$$\| Ux \| \leq p + K\overline{G}_2 \| x \| G_2(\| x \|) + \overline{F}\overline{G}_2 G_2(\| x \|). \tag{13.16}$$

By linking the estimates (13.15), (13.16) and the assumption (a), it can be seen that there exists a number $r_0 > 0$ such that the operator W transforms the ball B_{r_0} into itself, where W is defined by

$$(Wx)(t,s) = (Vx)(t,s)(Ux)(t,s)$$

for all $x \in BC(\mathbb{R}_+ \times \mathbb{R}_+)$ and $t, s \in \mathbb{R}_+$. Moreover, r_0 satisfies the second inequality of the assumption (a). From the above statement and the estimates (13.15), (13.16), we get

$$\| VB_{r_0} \| \leq p + K\overline{G}_1 r_0 G_1(r_0) + \overline{FG}_1 G_1(r_0)$$

and

$$\| UB_{r_0} \| \leq p + K\overline{G}_2 r_0 G_2(r_0) + \overline{FG}_2 G_2(r_0).$$

Consider a fixed nonempty subset X of B_{r_0}. Now, choose arbitrary numbers $T > 0$ and $\varepsilon > 0$. Then, for all $x \in X$ and $t, s, \bar{t}, \bar{s} \in [0, T]$ with $\left|t - \bar{t}\right| \leq \varepsilon, \ |s - \bar{s}| \leq \varepsilon$, we get

$$\begin{aligned}
&\left|(Vx)(t,s) - (Vx)(\bar{t},\bar{s})\right| \\
&\leq \left|p_1(t,s) - p_1(\bar{t},\bar{s})\right| \\
&\quad + \left|f_1(t,s,x(t,s)) - f_1(\bar{t},\bar{s},x(\bar{t},\bar{s}))\right| \int_0^t \int_0^s |h_1(t,s,v,w,x(v,w))|\, dvdw \\
&\quad + |f_1(\bar{t},\bar{s},x(\bar{t},\bar{s}))| \Big| \int_0^t \int_0^s h_1(t,s,v,w,x(v,w))dvdw \\
&\quad - \int_0^{\bar{t}} \int_0^{\bar{s}} h_1(\bar{t},\bar{s},v,w,x(v,w))dvdw \Big| \\
&\leq \left|p_1(t,s) - p_1(\bar{t},\bar{s})\right| \\
&\quad + \left[\left|f_1(t,s,x(t,s)) - f_1(t,s,x(\bar{t},\bar{s}))\right| + \left|f_1(t,s,x(\bar{t},\bar{s})) - f_1(\bar{t},\bar{s},x(\bar{t},\bar{s}))\right|\right] \\
&\quad \times \int_0^t \int_0^s g_1(t,s,v,w) G_1(|x(v,w)|) dvdw \\
&\quad + \left[\left|f_1(\bar{t},\bar{s},x(\bar{t},\bar{s})) - f_1(\bar{t},\bar{s},0)\right| + \left|f_1(\bar{t},\bar{s},0)\right|\right] \\
&\quad \times \left| \int_0^t \int_0^s h_1(t,s,v,w,x(v,w))dvdw - \int_0^{\bar{t}} \int_0^{\bar{s}} h_1(\bar{t},\bar{s},v,w,x(v,w))dvdw \right| \\
&\leq \omega^T(p_1,\varepsilon) + \left[K_1 \left|x(\bar{t},\bar{s}) - x(t,s)\right| + \omega^T_{r_0}(f_1,\varepsilon)\right] G_1(r_0) \int_0^T \int_0^T g_1(t,s,v,w)dvdw \\
&\quad + \left[K_1 \left|x(\bar{t},\bar{s})\right| + \overline{F}_1\right] \Big| \int_0^t \int_0^s h_1(t,s,v,w,x(v,w))dvdw \\
&\quad - \int_0^{\bar{t}} \int_0^{\bar{s}} h_1(\bar{t},\bar{s},v,w,x(v,w))dvdw \Big|
\end{aligned}$$

$$\begin{aligned}
&\leq \omega^{\tau}(p_1,\varepsilon) + \left[K_1 \omega^{\tau}(x,\varepsilon) + \omega^{\tau}_{r_0}(f_1,\varepsilon)\right] G_1(r_0) \overline{G}_1 \\
&\quad + \left(K_1 r_0 + \overline{F}_1\right) \Big| \int_0^t \int_0^s h_1(t,s,v,w,x(v,w))dvdw \\
&\quad - \int_0^{\bar{t}} \int_0^{\bar{s}} h_1(\bar{t},\bar{s},v,w,x(v,w))dvdw \Big|,
\end{aligned}$$

where

$$\omega_d^{\tau}(f_1,\varepsilon)=\sup\left\{\begin{array}{c}\left|f_1\left(t,s,y\right)-f_1\left(\bar{t},\bar{s},y\right)\right|:t,\bar{t},s,\bar{s}\in[0,\tau],\\ \left|t-\bar{t}\right|\leq\varepsilon,|s-\bar{s}|\leq\varepsilon,\,y\in[-d,d]\end{array}\right\}.$$

It is obvious that $\omega_{r_0}^{\tau}(f_1,\varepsilon)\to 0$ as $\varepsilon\to 0$ because of the uniform continuity of f_1 on $[0,\tau]\times[0,\tau]\times[-r_0,r_0]$. Similarly, it can be seen that

$$\left|\int_0^t\int_0^s h_1(t,s,v,w,x(v,w))dvdw-\int_0^{\bar{t}}\int_0^{\bar{s}} h_1(\bar{t},\bar{s},v,w,x(v,w))dvdw\right|\to 0$$

as $\varepsilon\to 0$. Thus, we have

$$\omega_0^{\infty}(VX)\leq K\overline{G}_1G_1(r_0)\omega_0^{\infty}(X)$$

and

$$\omega_0^{\infty}(UX)\leq K\overline{G}_2G_2(r_0)\omega_0^{\infty}(X).$$

In view of the assumptions (a), (b), we get

$$a(VX)\leq K\overline{G}_1G_1(r_0)a(X)$$

and

$$a(UX)\leq K\overline{G}_2G_2(r_0)a(X).$$

Therefore, we have

$$\mu_a(VX)\leq K\overline{G}_1G_1(r_0)\mu_a(X)$$

and

$$\mu_a(UX)\leq K\overline{G}_2G_2(r_0)\mu_a(X).$$

By Theorem 13.5, it can be seen that $W=VU$ is a contraction operator with respect to measure of noncompactness μ_a with the constant L given by

$$\begin{aligned}L&=pK\left[\overline{G}_1G_1(r_0)+\overline{G}_2G_2(r_0)\right]+2K\overline{F}\overline{G}_1G_1(r_0)\overline{G}_2G_2(r_0)\\&\quad+2K^2r_0\overline{G}_1G_1(r_0)\overline{G}_2G_2(r_0)\\&<1.\end{aligned}$$

Further, consider the sequence of sets $\left(B_{r_0}^n\right)$, where $B_{r_0}^1=\text{Conv }W(B_{r_0})$, $B_{r_0}^2=\text{Conv }W(B_{r_0}^1)$ and so on. Observe that all sets of this sequence are nonempty bounded closed and convex. Moreover, $B_{r_0}^{n+1}\subset B_{r_0}^n\subset B_{r_0}$ for each $n=1,2,\ldots$. Thus, we have

$$\mu_a\left(B_{r_0}^n\right)<L^n\mu_a\left(B_{r_0}\right).$$

This gives $\lim_{n\to\infty} \mu_a\left(B^n_{r_0}\right) = 0$. So the set $Y = \bigcap_{n=1}^{\infty} B^n_{r_0}$ nonempty bounded closed and convex and $Y \in \ker \mu_a$. The operator W maps set Y into itself.

Now, to show that W is continuous on Y. Fix $\varepsilon > 0$ and take $x, y \in X$ such that $\| x - y \| \le \varepsilon$. Since $Y \in \ker \mu_a$ therefore we can find a number $\tau > 0$ such that, for each $z \in Y$ and $t, s \ge \tau$, we have that $|z(t, s)| \le \varepsilon$. Since $W : Y \to Y$, we have that $Wx, Wy \in Y$. Thus, for all $t, s \ge \tau$, we get

$$|(Wx)(t, s) - (Wy)(t, s)| \le |(Wx)(t, s)| + |(Wy)(t, s)| \le 2\varepsilon.$$

On the other hand, take an arbitrary $t \in [0, \tau]$. Now, we have

$$\begin{aligned}
&|(Wx)(t, s) - (Wy)(t, s)| \\
&\le |(Ux)(t, s)|\,|(Vx)(t, s) - (Vy)(t, s)| + |(Vy)(t, s)|\,|(Ux)(t, s) - (Uy)(t, s)| \\
&\le \| UB_{r_0} \|\,|(Vx)(t, s) - (Vy)(t, s)| + \| VB_{r_0} \|\,|(Ux)(t, s) - (Uy)(t, s)|\,.
\end{aligned}$$

Further, we get

$$\begin{aligned}
&|(Vx)(t, s) - (Vy)(t, s)| \\
&\le |f_1(t, s, x(t, s)) - f_1(t, s, y(t, s))| \int_0^t \int_0^s |h_1(t, s, v, w, x(v, w))|\, dvdw \\
&\quad + |f_1(t, s, y(t, s))| \int_0^t \int_0^s |h_1(t, s, v, w, x(v, w)) - h_1(t, s, v, w, y(v, w))|\, dvdw \\
&\le K_1 |x(t, s) - y(t, s)| \int_0^t \int_0^s g_1(t, s, v, w) G_1(|x|) dvdw \\
&\quad + [K_1 |y(t, s)| + |f_1(t, s, 0)|] \int_0^t \int_0^s \overline{\omega}^T_{r_0}(h_1, \varepsilon) dvdw \\
&\le K\varepsilon \overline{G}_1 G_1(r_0) + \left(Kr_0 + \overline{F}\right) \tau^2 \overline{\omega}^{\tau}_{r_0}(h_1, \varepsilon),
\end{aligned}$$

where

$$\overline{\omega}^{\tau}_d(h_1, \varepsilon) = \sup \left\{ \begin{array}{c} |h_1(t, s, v, w, x) - h_1(t, s, v, w, y)| : t, s, v, w \in [0, \tau], \\ |x - y| \le \varepsilon, x, y \in [-d, d] \end{array} \right\}.$$

It is obvious that $\overline{\omega}^{\tau}_d(h_1, \varepsilon) \to 0$ as $\varepsilon \to 0$. Similarly, it can be shown that

$$|(Ux)(t, s) - (Uy)(t, s)| \le K\varepsilon \overline{G}_2 G_2(r_0) + \left(Kr_0 + \overline{F}\right) T^2 \overline{\omega}^{\tau}_{r_0}(h_2, \varepsilon),$$

where

$$\overline{\omega}^{\tau}_d(h_2, \varepsilon) = \sup \left\{ \begin{array}{c} |h_2(t, s, v, w, x) - h_2(t, s, v, w, y)| : t, s, v, w \in [0, \tau], \\ |x - y| \le \varepsilon, x, y \in [-d, d] \end{array} \right\}.$$

As $\varepsilon \to 0$ gives $\overline{\omega}_d^T(h_2, \varepsilon) \to 0$. Therefore, W is continuous on the set Y. Using Schauder's fixed point theorem, it can be concluded that $W : Y \to Y$ has at least one fixed point x in the set $Y \subset BC(\mathbb{R}_+ \times \mathbb{R}_+)$. This completes the proof.

Example 13.2 Consider the following system of integral equations:

$$x(t,s) = (Vx)(t,s)\,(Ux)(t,s), \tag{13.17}$$

where

$$(Vx)(t,s) = \frac{ts}{t^2s^2+4} + \left[x(t,s) + e^{-ts}\right] \int_0^t \int_0^s \frac{vw\sqrt{|x(v,w)|}}{\left(v^2+1\right)\left(w^2+1\right)(t+1)(s+1)} dvdw$$

and

$$(Ux)(t,s) = tse^{-2ts} + \frac{1}{\sqrt{2\pi}} \arctan\left(ts + x(t,s)\right) \int_0^t \int_0^s e^{-v(t+1)-w(s+1)} x^2(v,w)dvdw,$$

where $t, s \in \mathbb{R}_+$. It can be seen that this equation is a particular case of the Eq. (13.14), where

$$p_1(t,s) = \frac{ts}{t^2s^2+4}, \quad p_2(t,s) = tse^{-2ts},$$

$$f_1(t,s,x(t,s)) = x(t,s) + e^{-ts},$$

$$f_2(t,s,x(t,s)) = \frac{1}{\sqrt{2\pi}} \arctan\left(ts + x(t,s)\right),$$

$$h_1(t,s,v,w,x(v,w)) = \frac{vw\sqrt{|x(v,w)|}}{\left(v^2+1\right)\left(w^2+1\right)(t+1)(s+1)}$$

and

$$h_2(t,s,v,w,x(v,w)) = e^{-v(t+1)-w(s+1)} x^2(v,w).$$

It is obvious that $p_1(t,s) \to 0$ as $t, s \to \infty$ and $\| p_1 \| = \frac{1}{4}$. Similarly, it can be shown that $p_2(t,s) \to 0$ as $t, s \to \infty$ and $\| p_2 \| = \frac{1}{2e}$. Again, f_1, f_2 are continuous functions with

$$f_1(t,s,0) = e^{-ts}, \quad f_2(t,s,0) = \frac{1}{\sqrt{2\pi}} \arctan(ts).$$

As $t, s \to \infty$, it gives

$$f_1(t,s,0) \to 0, \quad f_2(t,s,0) \to 0.$$

Moreover, the functions $f_1(t, s, x)$ and $f_2(t, s, x)$ satisfy the assumption (c) with the constants $K_1 = 1$ and $K_2 = \frac{1}{\sqrt{2\pi}}$. Hence

$$\overline{F}_1 = 1, \quad \overline{F}_2 = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Therefore,

$$p = \frac{1}{4}, \quad K = 1, \quad \overline{F} = 1.$$

Again,

$$g_1(t, s, v, w) = \frac{vw}{\left(v^2+1\right)\left(w^2+1\right)(t+1)(s+1)},$$

$$g_2(t, s, v, w) = e^{-v(t+1)-w(s+1)}, \quad G_1(x) = \sqrt{x}, \quad G_2(x) = x^2.$$

It can be observed that h_1, h_2, g_1, and g_2 are continuous. Also, G_1 and G_2 are continuous nondecreasing functions. Again,

$$\int_0^t \int_0^s g_1(t, s, v, w) = \frac{1}{4}\frac{\log\left(t^2+1\right)\log\left(s^2+1\right)}{(t+1)(s+1)}$$

and

$$\int_0^t \int_0^s g_2(t, s, v, w) = \frac{\left(1-e^{-t-t^2}\right)\left(1-e^{-s-s^2}\right)}{(t+1)(s+1)}.$$

Thus $\overline{G}_1 = \frac{1}{4}$ and $\overline{G}_2 = 1$. The inequality (a), of the assumptions, has the form:

$$\left(\frac{1}{4} + \frac{1}{4}r\sqrt{r} + \frac{1}{4}\sqrt{r}\right)\left(\frac{1}{4} + r^3 + r^2\right) \leq r.$$

It is easy to observe that $r = \frac{1}{4}$ (i.e., $r_0 = \frac{1}{4}$) is a solution of the above inequality, also satisfying the second inequality of assumption (i).

Finally, it can be concluded that all the assumptions (a)–(i) of Theorem 13.9 are satisfied and so the integral Eq. (13.17) has a solution $x(t, s)$ belonging to the ball $B_{\frac{1}{4}} \subset BC(\mathbb{R}_+ \times \mathbb{R}_+)$.

13.5 Conclusion

In our present investigation, we have established the existence of the solution of a functional integral equation of two variables, which is of the form of the product of two operators in the Banach algebra $C([0, 1] \times [0, 1])$ and $BC(\mathbb{R}_+ \times \mathbb{R}_+)$. Also we have illustrated our results with the help of an example. Moreover, due to our existence theorem for Eqs. (13.4) and (13.14) of two variables, we therefore conclude that our existence result is more general than the one obtained earlier by Banaś and Olszowy [12]. Also, one can apply these results for fractional differential equations and fractional integral equations for single and more than one variable.

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Chapter 14
Generalization of Darbo-Type Fixed Point Theorem and Applications to Integral Equations

Hemant Kumar Nashine, Rabha W. Ibrahim, Reza Arab, and M. Rabbani

Abstract We propose a new notation of μ-set contractive mappings for two classes of functions involving a measure of noncompactness in Banach space and Darbo-type fixed point and n-tupled fixed point results. These results include and extend the results of Falset and Latrach [Falset, J. G., Latrach, K.: On Darbo–Sadovskii's fixed point theorems type for abstract measures of (weak) noncompactness, Bull. Belg. Math. Soc. Simon Stevin 22 (2015), 797–812.] The results are also correlated with the classical generalized Banach fixed point theorems. Finally, we apply these results to two different Volterra integral equations in Banach algebras with an illustration.

Keywords Fixed point · Set contractive map · Measure of noncompactness · Darbo theorem · Volterra integral equation

14.1 Introduction and Preliminaries

To understand the work in the underlying area, we start out by listing some notations and preliminaries that we shall need to express our results.

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Throughout the paper,

$\mathbb{R}$ = the set of real numbers,
$\mathbb{N}$ = the set of natural numbers,
$\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

Let $(E, \|.\|)$ be a real Banach space with zero element θ. Let $\mathscr{B}(x, r)$ denote the closed ball centered at x with radius r. The symbol $\mathscr{B}_r$ stands for the ball $\mathscr{B}(\theta, r)$. For X, a nonempty subset of E, we denote by $\overline{X}$ and $ConvX$ the closure and the convex closure of X, respectively. Moreover, let us denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

We use the following definition of the measure of noncompactness (MNC) given in [9].

Definition 14.1 A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}^+$ is said to be the *measure of noncompactness* (MNC) in E if it satisfies the following conditions:

(1^0) The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{N}_E$;
(2^0) (Monotonicity) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$;
(3^0) (Invariance under closure) $\mu(\overline{X}) = \mu(X)$;
(4^0) (Invariance under passage to the convex hull) $\mu(ConvX) = \mu(X)$;
(5^0) (Convexity) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$;
(6^0) (Cantor's generalized intersection property) If (X_n) is a decreasing sequence of nonempty, closed sets in $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, \ldots)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty and compact.

The family $ker\mu$ defined in axiom (1^0) is called the *kernel* of the MNC μ.

One of the properties of the MNC is $X_\infty \in ker\mu$. Indeed, from the inequality $\mu(X_\infty) \leq \mu(X_n)$ for $n = 1, 2, 3, \ldots$, we infer that $\mu(X_\infty) = 0$.

The Kuratowski MNC is the map $\alpha : \mathfrak{M}_E \to \mathbb{R}^+$ with

$$\alpha(\mathscr{Q}) = \inf \left\{ \epsilon > 0 : \mathscr{Q} \subset \bigcup_{k=1}^{n} S_k, S_k \subset E, diam(S_k) < \epsilon \ (k \in \mathbb{N}) \right\}. \tag{14.1}$$

In 1955, Darbo [11] used the notation of Kuratowski measure of noncompactness α to prove the fixed point theorem and generalized topological Schauder fixed point theorem [9] and classical Banach fixed point theorem [8].

Theorem 14.1 ([9]) *Let X be a closed, convex subset of a Banach space E. Then every compact, continuous map $T : \mathscr{X} \to X$ has at least one fixed point.*

Theorem 14.2 ([11]) *Let X be a nonempty, bounded, closed and convex subset of a Banach space E, and μ be the Kuratowski MNC on E. Let $T : \Omega \to \Omega$ be a continuous and μ-set contraction operator, that is, there exists a constant $k \in [0, 1)$ with*

$$\mu(TM) \leq k\mu(M)$$

for any nonempty subset M of X. Then T has a fixed point.

Following this result, various authors proved several Darbo-type fixed point and coupled theorems by using different types of control functions. Here, we mention the paper discussed in [2–7, 11, 13, 15, 22, 23, 23, 24, 37]. In this work, we establish some new results of Darbo's integral type which generalizes and includes work mentioned in [2–4, 11, 13] as well. We apply these results to get solutions of two different types of Volterra integral equations in Banach algebras followed by an illustration.

14.2 Generalized Darbo-Type Fixed Point Theorems

We start the section with the following notation:

Definition 14.2 ([26]) Let $\varDelta_F$ be a family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ such that

($\varDelta_1$) F is continuous and strictly increasing;

($\varDelta_2$) for each sequence $\{t_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} t_n = 0$ if and only if

$$\lim_{n\to\infty} F(t_n) = -\infty.$$

$\varDelta_{G,\beta}$ denotes the set of pairs (G, β), where $G : \mathbb{R}^+ \to \mathbb{R}$ and $\beta : [0, \infty) \to [0, 1)$ such that

($\varDelta_3$) for each sequence $\{t_n\} \subseteq \mathbb{R}^+$, $\limsup_{n\to\infty} G(t_n) \geq 0$ if and only if

$$\limsup_{n\to\infty} t_n \geq 1;$$

($\varDelta_4$) for each sequence $\{t_n\} \subseteq [0, \infty)$, $\limsup_{n\to\infty} \beta(t_n) = 1$ implies

$$\lim_{n\to\infty} t_n = 0;$$

($\varDelta_5$) for each sequence $\{t_n\} \subseteq \mathbb{R}^+$, $\sum_{n=1}^{\infty} G(\beta(t_n)) = -\infty$.

Set $\mathbf{I} = \{f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+;\ f$ is a Lebesgue integrable mapping which is summable and nonnegative and satisfies $\int_0^\epsilon f(t)dt > 0$, for each $\epsilon > 0\}$.

Our first main result is as follows:

Theorem 14.3 *Let $\varOmega$ be a nonempty, bounded, closed and convex subset of a Banach space E, and $T : \varOmega \to \varOmega$ be a continuous operator. If there exist $F \in \varDelta_F$, $(G, \beta) \in \varDelta_{G,\beta}$ and a continuous and strictly increasing mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\mu(TM) > 0 \Longrightarrow F\Big(\int_0^{\mu(TM)+\varphi(\mu(TM))} f(s)ds\Big) \leq F\Big(\int_0^{\mu(M)+\varphi(\mu(M))} f(s)ds\Big) + G\Big(\beta\Big(\int_0^{\mu(M)+\varphi(\mu(M))} f(s)ds\Big)\Big) \tag{14.2}$$

for all $M \subseteq \Omega$, where μ is an arbitrary MNC and $f \in \boldsymbol{I}$, then T has at least one fixed point in Ω.

Proof Starting with the assumption $\Omega_0 = \Omega$, we define a sequence $\{\Omega_n\}$ such that $\Omega_{n+1} = Conv(T\Omega_n)$, for $n \in \mathbb{N}^*$. If $\mu(\Omega_{n_0}) + \varphi(\mu(\Omega_{n_0})) = 0$, that is, $\mu(\Omega_{n_0})$=0 for some natural number $n_0 \in \mathbb{N}$, then Ω_{n_0} is compact. Thus, we conclude the result from Theorem 14.1, and hence we assume that $\mu(\Omega_n) + \varphi(\mu(\Omega_n)) > 0$, for all $n \in \mathbb{N}^*$. From (14.2) and (4^0) of Definition 14.1, we have

$$\begin{aligned}
&F\Big(\int_0^{\mu(\Omega_{n+1})+\varphi(\mu(\Omega_{n+1}))} f(s)ds\Big)\\
&= F\Big(\int_0^{\mu(Conv(T\Omega_n))+\varphi(\mu(Conv(T\Omega_n)))} f(s)ds\Big)\\
&= F\Big(\int_0^{\mu(T\Omega_n)+\varphi(\mu(T\Omega_n))} f(s)ds\Big)\\
&\leq F\Big(\int_0^{\mu(\Omega_n)+\varphi(\mu(\Omega_n))} f(s)ds\Big) + G\Big(\beta\Big(\int_0^{\mu(\Omega_n)+\varphi(\mu(\Omega_n))} f(s)ds\Big)\Big)\\
&\leq F\Big(\int_0^{\mu(\Omega_{n-1})+\varphi(\mu(\Omega_{n-1}))} f(s)ds\Big) + G\Big(\beta\Big(\int_0^{\mu(\Omega_n)+\varphi(\mu(\Omega_n))} f(s)ds\Big)\Big)\\
&\quad + G\Big(\beta\Big(\int_0^{\mu(\Omega_{n-1})+\varphi(\mu(\Omega_{n-1}))} f(s)ds\Big)\Big)\\
&\leq \cdots\\
&\leq F\Big(\int_0^{\mu(\Omega_0)+\varphi(\mu(\Omega_0))} f(s)ds\Big) + \sum_{i=0}^{n} G(\beta\Big(\int_0^{\mu(\Omega_i)+\varphi(\mu(\Omega_i))} f(s)ds\Big)\Big),
\end{aligned}$$

that is,

$$\begin{aligned}
&F\Big(\int_0^{\mu(\Omega_{n+1})+\varphi(\mu(\Omega_{n+1}))} f(s)ds\Big)\\
&\leq F\Big(\int_0^{\mu(\Omega_0)+\varphi(\mu(\Omega_0))} f(s)ds\Big) + \sum_{i=0}^{n} G\Big(\beta\Big(\int_0^{\mu(\Omega_i)+\varphi(\mu(\Omega_i))} f(s)ds\Big)\Big) \qquad (14.3)
\end{aligned}$$

for all $n \in \mathbb{N}$. From the properties of $(G, \beta) \in \Delta_{G,\beta}$, $F(\int_0^{\mu(\Omega_{n+1})+\varphi(\mu(\Omega_{n+1}))} f(s)ds) \to -\infty$ as $n \to \infty$ and, by (Δ_2), we have

$$\lim_{n\to\infty} \int_0^{\mu(\Omega_{n+1})+\varphi(\mu(\Omega_{n+1}))} f(s)ds = 0,$$

and hence

$$\lim_{n\to\infty} \mu(\Omega_{n+1}) + \varphi(\mu(\Omega_{n+1})) = 0.$$

Therefore, we have

$$\lim_{n\to\infty} \mu(\Omega_n) = 0.$$

Now, from (6^0) of Definition 14.1, we have $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is a nonempty, closed, convex set and $X_\infty \subseteq X_n$ for all $n \in \mathbb{N}$. Also, $T(X_\infty) \subset X_\infty$ and $X_\infty \in ker\mu$. Therefore, by Theorem 14.1, T has a fixed point u in the set X_∞ and hence $u \in X$. This completes the proof.

Theorem 14.4 *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and $T : \Omega \to \Omega$ be a continuous operator. If there exist $F \in \Delta_F$, $(G, \beta) \in \Delta_{G,\beta}$ and a continuous and strictly increasing mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\begin{aligned}\mu(TM) > 0 \implies \quad & F(\mu(TM) + \varphi(\mu(TM))) \\ & \leq F(\mu(M) + \varphi(\mu(M))) + G(\beta(\mu(M) + \varphi(\mu(M))))\end{aligned}$$

for all $M \subseteq \Omega$, where μ is an arbitrary MNC, then T has at least one fixed point in Ω.

Corollary 14.1 *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and $T : \Omega \to \Omega$ be a continuous operator. If there exist $\tau > 0$, $F \in \Delta_F$ and a continuous and a strictly increasing mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\begin{aligned}\mu(TM) > 0 \implies \quad & \tau + F\Big(\int_0^{\mu(TM)+\varphi(\mu(TM))} f(s)ds\Big) \\ & \leq F\Big(\int_0^{\mu(M)+\varphi(\mu(M))} f(s)ds\Big)\end{aligned} \tag{14.4}$$

for all $M \subseteq \Omega$, where μ is an arbitrary MNC and $f \in \boldsymbol{I}$, then T has at least one fixed point in X.

Proof If we consider $G(t) = \ln t$ for all $t > 0$, $\beta(t) = \lambda \in (0, 1)$ and $\tau = -\ln\lambda > 0$ in (14.2) of Theorems 14.3, we have (14.4) and the result follows from Theorem 14.3.

If we consider $F(t) = \ln t$ and $\tau = \ln(\frac{1}{\lambda})$ for all $\lambda \in (0, 1)$ in (14.4) of Corollary 14.1, then we obtain the following result.

Corollary 14.2 *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and $T : \Omega \to \Omega$ be a continuous operator. If there exists a continuous and strictly increasing mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\mu(TM) > 0 \implies \int_0^{\mu(TM)+\varphi(\mu(TM))} f(s)ds \leq \lambda\Big[\int_0^{\mu(M)+\varphi(\mu(M))} f(s)ds\Big] \tag{14.5}$$

for all $M \subseteq \Omega$, where μ is an arbitrary MNC and $f \in \boldsymbol{I}$, then T has at least one fixed point in X.

Remark 14.1 Put $f(t) = 1$ and $\varphi(t) = t$ for all $t \in [0, +\infty)$ in Corollary 14.2. Then we have

$$\begin{aligned}\mu(TM) &= \frac{1}{2}[\mu(TM) + \varphi(\mu(TM))] = \frac{1}{2}\int_0^{\mu(TM)+\varphi(\mu(TM))} f(s)ds \\ &\leq \frac{\lambda}{2}\int_0^{\mu(M)+\varphi(\mu(M))} f(s)ds \\ &= \frac{\lambda}{2}[\mu(M) + \varphi(\mu(M))] \\ &= \lambda\,\mu(M)\end{aligned}$$

and so we get Darbo's fixed point theorem.

Proposition 14.1 *Let X be a nonempty, bounded, closed and convex subset of a Banach space E and $T : X \to X$ be a continuous operator such that*

$$\begin{aligned}&\mu(TM) > 0 \\ &\Longrightarrow F\Big(\int_0^{diam(TM)+\varphi(diam(TM))} f(s)ds\Big) \qquad (14.6)\\ &\leq F\Big(\int_0^{diam(M)+\varphi(diam(M))} f(s)ds\Big) + G\Big(\beta\Big(\int_0^{diam(M)+\varphi(diam(M))} f(s)ds\Big)\Big)\end{aligned}$$

for all $M \subseteq X$ and $f \in \boldsymbol{I}$, $F \in \Delta_F$ and $(G, \beta) \in \Delta_{G,\beta}$ and a continuous mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. Then T has a unique fixed point in X.

Proof Following Theorem 14.3 and Proposition 3.2 [13], T has a fixed point in X.

To prove the uniqueness, we suppose that there exist two distinct fixed points $\zeta, \xi \in X$, then we may define the set $\Upsilon := \{\zeta, \xi\}$. In this case, $diam(\Upsilon) = diam(T(\Upsilon)) = \|\xi - \zeta\| > 0$. Then, using (14.6), we get

$$\begin{aligned}&diam(T(\Upsilon)) > 0 \Longrightarrow \\ &F\Big(\int_0^{diam(\Upsilon)+\varphi(diam(\Upsilon))} f(s)ds\Big) \\ &= F\Big(\int_0^{diam(T(\Upsilon))+\varphi(diam(T(\Upsilon)))} f(s)ds\Big) \\ &\leq F\Big(\int_0^{diam((\Upsilon))+\varphi(diam((\Upsilon)))} f(s)ds\Big) + G\Big(\beta\Big(\int_0^{diam((\Upsilon))+\varphi(diam((\Upsilon)))} f(s)ds\Big)\Big).\end{aligned}$$

Therefore, we have

$$G\Big(\beta\Big(\int_0^{diam((\Upsilon))+\varphi(diam((\Upsilon)))} f(s)ds\Big)\Big) \geq 0$$

and hence

$$\beta\Big(\int_0^{diam((\Upsilon))+\varphi(diam((\Upsilon)))} f(s)ds\Big) \geq 1,$$

which is a contradiction and hence $\xi = \zeta$. This completes the proof.

If we consider $f(t) = 1$ in (14.6) of Proposition 14.1, then we obtain the following result.

Corollary 14.3 *Let X be a nonempty, bounded, closed and convex subset of a Banach space E and $T : X \to X$ be a continuous operator such that*

$$\begin{aligned} &\mu(TM) > 0 \Longrightarrow \\ &F(diam(TM) + \varphi(diam(TM))) \\ \leq\ &F(diam(M) + \varphi(diam(M))) + G(\beta(diam(M) + \varphi(diam(M)))) \end{aligned} \tag{14.7}$$

for all $M \subseteq X$, where $F \in \Delta_F$ and $(G, \beta) \in \Delta_{G,\beta}$ and a continuous mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. Then T has a unique fixed point in X.

Corollary 14.4 *Let X be a nonempty, bounded, closed and convex subset of a Banach space E and $T : X \to X$ be an operator such that*

$$\begin{aligned} &\|Tu - Tv\| > 0 \Longrightarrow \\ &F\Big(\int_0^{\|Tu-Tv\|+\varphi(\|Tu-Tv\|)} f(s)ds\Big) \\ \leq\ &F\Big(\int_0^{\|u-v\|+\varphi(\|u-v\|)} f(s)ds\Big) + G\Big(\beta\Big(\int_0^{\|u-v\|+\varphi(\|u-v\|)} f(s)ds\Big)\Big) \end{aligned} \tag{14.8}$$

for all $u, v \in X$, where $f \in \boldsymbol{I}$, $F \in \Delta_F$, $(G, \beta) \in \Delta_{G,\beta}$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and strictly increasing mapping. Then T has a unique fixed point.

Proof Let $\mu : \mathfrak{M}_E \to \mathbb{R}^+$ be a set quantity defined by the formula $\mu(X) = diam X$, where $diam X = \sup\{\|u - v\| : u, v \in X\}$ stands for the diameter of X. It is easily seen that μ is a MNC in a space E in the sense of Definition 14.1. Therefore, from (14.8), we have

$$\begin{aligned}
&\sup_{u,v\in X}\|Tu-Tv\|>0\Longrightarrow\\
&F\Big(\int_0^{\sup_{u,v\in X}\|Tu-Tv\|+\varphi(\sup_{u,v\in X}\|Tu-Tv\|)}f(s)ds\Big)\\
=&\sup_{u,v\in X}F\Big(\int_0^{\|Tu-Tv\|+\varphi(\|Tu-Tv\|)}f(s)ds\Big)\\
\le&\sup_{u,v\in X}\Big[F\Big(\int_0^{\|u-v\|+\varphi(\|u-v\|)}f(s)ds\Big)+G\Big(\beta\Big(\int_0^{\|u-v\|+\varphi(\|u-v\|)}f(s)ds\Big)\Big)\Big]\\
\le&F\Big(\int_0^{\sup_{u,v\in X}\|u-v\|+\varphi(\sup_{u,v\in X}\|u-v\|)}f(s)ds\Big)\\
&+G\Big(\beta\Big(\int_0^{\sup_{u,v\in X}\|u-v\|+\varphi(\sup_{u,v\in X}\|u-v\|)}f(s)ds\Big)\Big),
\end{aligned}$$

which implies that

$$\begin{aligned}
&diam(\mathscr{T}(\mathscr{X}))>0\Longrightarrow\\
&F\Big(\int_0^{diam(\mathscr{T}(\mathscr{X}))+\varphi(diam(\mathscr{T}(\mathscr{X})))}f(s)ds\Big)\\
\le&F\Big(\int_0^{diam(\mathscr{X})+\varphi(diam(\mathscr{X}))}f(s)ds\Big)+G\Big(\beta\Big(\int_0^{diam(\mathscr{X})+\varphi(diam(\mathscr{X}))}f(s)ds\Big)\Big).
\end{aligned}$$

Thus, following Proposition 14.1, $\mathscr{T}$ has a unique fixed point. This completes the proof.

If we consider $f(t)=1$ in (14.8) of Corollary 14.4, then we have the following result.

Corollary 14.5 *Let X be a nonempty, bounded, closed and convex subset of a Banach space E and $T:X\to X$ be an operator such that*

$$\begin{aligned}
&\|Tu-Tv\|>0\Longrightarrow\\
&F(\|Tu-Tv\|+\varphi(\|Tu-Tv\|))\\
\le&F(\|u-v\|+\varphi(\|u-v\|))+G(\beta(\|u-v\|+\varphi(\|u-v\|)))
\end{aligned}\tag{14.9}$$

for all $u,v\in X$, where $F\in\Delta_F$ and $(G,\beta)\in\Delta_{G,\beta}$ and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous and strictly increasing mapping. Then T has a unique fixed point.

Corollary 14.6 *Let $(E,\|\cdot\|)$ be a Banach space and X be a closed, convex subset of E. Let $T_1,T_2:X\to X$ be two operators satisfying the following conditions:*

(1) $(T_1 + T_2)(X) \subseteq X$;

(2) *there exist* $F \in \Delta_F$ *and* $(G, \beta) \in \Delta_{G,\beta}$ *and a continuous and increasing mapping* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\begin{aligned}
&\|T_1u - T_1v\| > 0 \Longrightarrow \\
&F\Big(\int_0^{\|T_1u-T_1v\|+\varphi(\|T_1u-T_1v\|)} f(s)ds\Big) \\
\leq\ &F\Big(\int_0^{\|u-v\|+\varphi(\|u-v\|)} f(s)ds\Big) + G\Big(\beta\Big(\int_0^{\|u-v\|+\varphi(\|u-v\|)} f(s)ds\Big)\Big); \quad (14.10)
\end{aligned}$$

(3) T_2 *is a continuous and compact operator.*

Then $\mathscr{J} := T_1 + T_2 : X \to X$ *has a fixed point* $u \in X$.

Proof Suppose that M is a subset of X with $\alpha(M) > 0$. By the notation of Kuratowski MNC, for each $n \in \mathbb{N}$, there exist $\mathscr{C}_1, \ldots, \mathscr{C}_{m(n)}$ bounded subsets such that $M \subseteq \bigcup_{i=1}^{m(n)} \mathscr{C}_i$ and $diam(\mathscr{C}_i) \leq \alpha(M) + \frac{1}{n}$. Suppose that $\alpha(T_1(M)) > 0$. Since $T_1(M) \subseteq \bigcup_{i=1}^{m(n)} T_1(\mathscr{C}_i)$, there exists $i_0 \in \{1, 2, \ldots, m(n)\}$ such that $\alpha(T_1(M)) \leq diam(T_1(\mathscr{C}_{i_0}))$. Using (14.10), we have

$$\begin{aligned}
&F\Big(\int_0^{\alpha(T_1(M))+\varphi(\alpha(T_1(M)))} f(s)ds\Big) \\
\leq\ &F\Big(\int_0^{diam(T_1(\mathscr{C}_{i_0}))+\varphi(diam(T_1(\mathscr{C}_{i_0})))} f(s)ds\Big) \\
\leq\ &F\Big(\int_0^{diam(\mathscr{C}_{i_0})+\varphi(diam(\mathscr{C}_{i_0}))} f(s)ds\Big) + G\Big(\beta\Big(\int_0^{diam(\mathscr{C}_{i_0})+\varphi(diam(\mathscr{C}_{i_0}))} f(s)ds\Big)\Big) \\
\leq\ &F\Big(\int_0^{\alpha(M)+\frac{1}{n}+\varphi(\alpha(M)+\frac{1}{n})} f(s)ds\Big) + G\Big(\beta\Big(\int_0^{\alpha(M)+\frac{1}{n}+\varphi(\alpha(M)+\frac{1}{n})} f(s)ds\Big)\Big).
\end{aligned}$$

Passing to the limit as $n \to \infty$, we get

$$\begin{aligned}
&F\Big(\int_0^{\alpha(T_1(M))+\varphi(\alpha(T_1(M)))} f(s)ds\Big) \\
\leq\ &F\Big(\int_0^{\alpha(M)+\varphi(\alpha(M))} f(s)ds\Big) + G\Big(\beta\Big(\int_0^{\alpha(M)+\varphi(\alpha(M))} f(s)ds\Big)\Big).
\end{aligned}$$

Using hypothesis (3), it follows from the notation of α that

$$
\begin{aligned}
& F\Big(\int_0^{\alpha(\mathscr{J}(M))+\varphi\big(\alpha(\mathscr{J}(M))\big)} f(s)ds\Big) \\
&= F\Big(\int_0^{\alpha(T_1(M)+T_2(M))+\varphi(\alpha(T_1(M)+T_2(M)))} f(s)ds\Big) \\
&\le F\Big(\int_0^{\alpha(T_1(M))+\alpha(\mathscr{T}_2(M))+\varphi(\alpha(T_1(M))+\alpha(T_2(M)))} f(s)ds\Big) \\
&= F\Big(\int_0^{\alpha(T_1(M))+\varphi(\alpha(T_1(M)))} f(s)ds\Big) \\
&\le F\Big(\int_0^{\alpha(M)+\varphi(\alpha(M)))} f(s)ds\Big) + G\Big(\beta\Big(\int_0^{\alpha(M)+\varphi(\alpha(M)))} f(s)ds\Big)\Big).
\end{aligned}
$$

Thus, by Theorem 14.3, $\mathscr{J}$ has a fixed point $u \in X$.

If we consider $f(t) = 1$ in (14.10) of Corollary 14.6, the we have the following result.

Corollary 14.7 *Let $(E, \|\cdot\|)$ be a Banach space and X be a closed, convex subset of E. Let $T_1, T_2 : X \to X$ be two operators satisfying the following conditions:*

(1) $(T_1 + T_2)(X) \subseteq X$*;*

(2) *there exist $F \in \Delta_F$, $(G, \beta) \in \Delta_{G,\beta}$ and a continuous and increasing mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$
\begin{aligned}
& \|T_1 u - T_1 v\| > 0 \Longrightarrow \\
& F(\|T_1 u - T_1 v\| + \varphi(\|T_1 u - T_1 v)\|)) \qquad (14.11)\\
\le\ & F(\|u - v\| + \varphi(\|u - v\|)) + G(\beta(\|u - v\| + \varphi(\|u - v\|)));
\end{aligned}
$$

(3) *T_2 is a continuous and compact operator.*

Then $\mathscr{J} := T_1 + T_2 : X \to X$ has a fixed point $u \in X$.

14.3 Darbo-Type $n-$Tupled Fixed Point Theorems

Definition 14.3 Let X be a nonempty set and $\mathscr{G} : X^n \to X$ be a given mapping with $n \ge 2$. An element $(x_1, x_2, \ldots, x_n) \in X^n$ is said to be an *n-tupled fixed point* of the mapping $\mathscr{G}$ if

$$
\begin{cases}
\mathscr{G}(x_1, x_2, \ldots, x_n) = x_1, \\
\mathscr{G}(x_2, x_3, \ldots, x_1) = x_2, \\
\vdots \\
\mathscr{G}(x_n, x_1, \ldots, x_{n-1}) = x_n.
\end{cases}
$$

In the following, we have denoted $\sum_J$ or $\prod_J$ as a summation or product in cyclic permutation over the product of X_i for each $i \in \{1, 2 \ldots, n\}$.

Theorem 14.5 *Let X be a nonempty, bounded, closed and convex subset of a Banach space E. Suppose that $\mathscr{G} : X^n \to X$ is a continuous operator satisfying the following condition: for each $\in \{1, 2 \ldots, n\}$,*

$$\begin{aligned} &\mu\Big(\mathscr{G}\Big(\prod_{i=1}^{n} X_i\Big)\Big) > 0 \Longrightarrow \\ &F\Big(\sum_J \mu\Big(\mathscr{G}\Big(\prod_{i=1}^{n} X_i\Big)\Big) + \varphi\Big(\ \mu\Big(\mathscr{G}\Big(\prod_{i=1}^{n} X_i\Big)\Big)\Big)\Big) \\ \le\ &F\Big(\sum_{i=1}^{n} \mu\big(X_i\big) + \varphi\Big(\sum_{i=1}^{n} \mu\big(X_i\big)\Big)\Big) + G\Big(\beta\Big(\sum_{i=1}^{n} \mu\big(X_i\big) + \varphi\Big(\sum_{i=1}^{n} \mu\big(X_i\big)\Big)\Big)\Big) \end{aligned} \tag{14.12}$$

for all $X_i \subseteq X$, where μ is an arbitrary MNC, $F \in \Delta_F$ and $(G, \beta) \in \Delta_{G,\beta}$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing and sub-additive mapping. Then $\mathscr{G}$ has at least an $n-$tupled fixed point in X^n.

Proof We start by considering a map $\widehat{\mathscr{G}} : X^n \to X^n$ defined by

$$\widehat{\mathscr{G}}(x_1, x_2, \ldots, x_n) = (\mathscr{G}(x_1, x_2, \ldots, x_n), \mathscr{G}(x_2, x_3, \ldots, x_1), \ldots, \mathscr{G}(x_n, x_1, \ldots, x_{n-1})).$$

With the virtue of continuity of $\mathscr{G}$, $\widehat{\mathscr{G}}$ is continuous. Define

$$\widehat{\mu}(M) = \sum_{i=1}^{n} \mu(X_1)$$

where X_i, $i = \{1, 2, \ldots, n\}$ denote the natural projections of X. Without loss of generality, let $\varnothing \neq M \subset X^n$. Hence, by the condition (14.13) and using (2^0) of Definition 14.1,

$$\begin{aligned} \widehat{\mu}(\widehat{\mathscr{G}}(M)) &\le \widehat{\mu}(\mathscr{G}(X_1 \times X_2 \times \cdots \times X_n) \times \mathscr{G}(X_2 \times X_3 \times \cdots \times X_1) \\ &\qquad \times \cdots \times \mathscr{G}(X_n \times X_1 \times \cdots \times X_{n-1})) \\ &= \sum_J \mu(\mathscr{G}(X_1 \times X_2 \times \cdots \times X_n)). \end{aligned}$$

Therefore, by the assumption, we have

$$\widehat{\mu}(\widehat{\mathscr{G}}(M)) > 0,$$

which implies

$$
\begin{aligned}
& F(\widehat{\mu}(\widehat{\mathscr{G}}(M)) + \varphi(\widehat{\mu}(\widehat{\mathscr{G}}(M)))) \\
& \leq F\Big(\widehat{\mu}\Big(\prod_{J} \mathscr{G}\Big(\prod_{i=1}^{n} X_i\Big)\Big) + \varphi\Big(\widehat{\mu}\Big(\prod_{J} \mathscr{G}\Big(\prod_{i=1}^{n} X_i\Big) \\
& \leq F\Big(\sum_{J} \mu(\mathscr{G}\Big(\prod_{i=1}^{n} X_i\Big)\Big) + \varphi\Big(\sum_{J} \mu\Big(\mathscr{G}\Big(\prod_{i=1}^{n} X_i\Big)\Big)\Big)\Big) \\
& \leq F\Big(\sum_{i=1}^{n} \mu(X_i) + \varphi\Big(\sum_{i=1}^{n} \mu(\mathscr{X}_i)\Big)\Big) + G\Big(\beta\Big(\sum_{i=1}^{n} \mu(X_i) + \varphi\Big(\sum_{i=1}^{n} \mu(\mathscr{X}_i)\Big)\Big)\Big) \\
& = F(\widehat{\mu}(M) + \varphi(\widehat{\mu}(M))) + G(\beta(\widehat{\mu}(M) + \varphi(\widehat{\mu}(M)))),
\end{aligned}
$$

that is,

$$
\begin{aligned}
& \widehat{\mu}(\widehat{\mathscr{G}}(M)) > 0 \Longrightarrow \\
& F(\widehat{\mu}(\widehat{\mathscr{G}}(M)) + \varphi(\widehat{\mu}(\widehat{\mathscr{G}}(M)))) \\
\leq\ & F(\widehat{\mu}(M) + \varphi(\widehat{\mu}(M))) + G(\beta(\widehat{\mu}(M) + \varphi(\widehat{\mu}(M)))).
\end{aligned}
$$

Therefore, from Theorem 14.4, we get that $\widehat{\mathscr{G}}$ has at least one fixed point in X^n and hence $\mathscr{G}$ has an $n-$tupled fixed point. This completes the proof.

Theorem 14.6 *Let X be a nonempty, bounded, closed and convex subset of a Banach space E. Suppose that $\mathscr{G} : X \times X \to X$ is a continuous operator. If there exist $F \in \Delta_F$, $(G, \beta) \in \Delta_{G,\beta}$ and a continuous and increasing mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for each $i \in \{1, 2 \ldots, n\}$,*

$$
\begin{aligned}
& \mu\Big(\mathscr{G}\Big(\prod_{i=1}^{n} X_i\Big)\Big) > 0 \Longrightarrow \\
& F\Big(\ \mu\Big(\mathscr{G}\Big(\prod_{i=1}^{n} X_i\Big)\Big) + \varphi\Big(\ \mu\Big(\mathscr{G}\Big(\prod_{i=1}^{n} X_i\Big)\Big)\Big)\Big) \\
\leq\ & F\left(\max\{\mu(X_1), \mu(X_2), \ldots, \mu(X_n)\} + \varphi\left(\max\{\mu(X_1), \mu(X_2), \ldots, \mu(X_n)\}\right)\right) \\
& + G\Big(\beta\Big(\max\{\mu(X_1), \mu(X_2), \ldots, \mu(X_n)\} \\
& + \varphi\Big(\max\{\mu(X_1), \mu(X_2), \ldots, \mu(X_n)\}\Big)\Big)\Big) \qquad (14.13)
\end{aligned}
$$

for all $X_i \subseteq X$, where μ is an arbitrary MNC, then $\mathscr{G}$ has at least an $n-$tupled fixed point in X^n.

Proof Consider the mapping $\widehat{\mathscr{G}} : X^n \to X^n$ defined by the formula

$$
\widehat{\mathscr{G}}(x_1, x_2, \ldots, x_n) = (\mathscr{G}(x_1, x_2, \ldots, x_n), \mathscr{G}(x_2, x_3, \ldots, x_1), \ldots, \mathscr{G}(x_n, x_1, \ldots, x_{n-1})).
$$

$\widehat{\mathscr{G}}$ is continuous due to the continuity of $\mathscr{G}$. Define

$$\widehat{\mu}(M) = \max\{\mu(X_1), \mu(X_2), \ldots, \mu(X_n)\},$$

where X_i for each $i = \{1, 2, \ldots, n\}$ denote the natural projections of X. Without loss of generality, let $\varnothing \neq M \subset X^n$. Following the previous theorem,

$$\begin{aligned}
&\widehat{\mu}(\widehat{\mathscr{G}}(M)) \\
&\leq \widehat{\mu}(\mathscr{G}(X_1 \times X_2 \times \cdots \times X_n) \times \mathscr{G}(X_2 \times X_3 \times \cdots \times X_1) \times \\
&\quad \ldots \times \mathscr{G}(X_n \times X_1 \times \cdots \times X_{n-1})) \\
&= \max \left\{ \begin{array}{c} \mu(\mathscr{G}(X_1 \times X_2 \times \cdots \times X_n), \\ \mu(\mathscr{G}(X_2 \times X_3 \times \cdots \times X_1)), \\ \cdots \\ \mu(\mathscr{G}(X_n \times X_1 \times \cdots \times X_{n-1})) \end{array} \right\},
\end{aligned}$$

which is, by the assumption,

$$\widehat{\mu}(\widehat{\mathscr{G}}(M)) > 0.$$

The condition (14.13) and (2^0) of Definition 14.1 imply that

$$\begin{aligned}
&F(\widehat{\mu}(\widehat{\mathscr{G}}(M)) + \varphi(\widehat{\mu}(\widehat{\mathscr{G}}(M)))) \\
&\leq F(\widehat{\mu}(\prod_J \mathscr{G}(\prod_{i=1}^n X_i)) \\
&\quad + \varphi(\widehat{\mu}(\prod_J \mathscr{G}(\prod_{i=1}^n X_i)) \\
&= F\left(\max\left\{ \begin{array}{c} \mu(\mathscr{G}(X_1 \times X_2 \times \cdots \times X_n)), \\ \mu(\mathscr{G}(X_2 \times X_3 \times \cdots \times X_1)), \\ \mu(\mathscr{G}(X_n \times X_1 \times \cdots \times X_{n-1})) \end{array} \right\} + \varphi\left(\max\left\{ \begin{array}{c} \mu(\mathscr{G}(X_1 \times X_2 \times \cdots \times X_n)), \\ \mu(\mathscr{G}(X_2 \times X_3 \times \cdots \times X_1)), \\ \mu(\mathscr{G}(X_n \times X_1 \times \cdots \times X_{n-1})) \end{array} \right\}\right)\right) \\
&= \max\left\{ \begin{array}{c} F(\mu(\mathscr{G}(X_1 \times X_2 \times \cdots \times X_n)) + \varphi(\mu(\mathscr{G}(X_1 \times X_2 \times \cdots \times X_1)))), \\ F(\mu(\mathscr{G}(X_2 \times X_3 \times \cdots \times X_1)) + \varphi(\mu(\mathscr{G}(X_2 \times X_3 \times \cdots \times X_1)))), \\ \vdots \\ F(\mu(\mathscr{G}(X_n \times X_1 \times \cdots \times X_{n-1})) + \varphi(\mu(\mathscr{G}(X_n \times X_1 \times \cdots \times X_{n-1})))), \end{array} \right\} \\
&\leq \max\left\{ \begin{array}{c} F(\max\{\mu(X_1), \mu(X_2), \ldots, \mu(X_n)\} + \varphi(\max\{\mu(X_1), \mu(X_2), \ldots, \mu(X_n)\})) \\ +G(\beta(\max\{\mu(X_1), \mu(X_2), \ldots, \mu(X_n)\} + \varphi(\max\{\mu(X_1), \mu(X_2), \ldots, \mu(X_n)\}))), \\ F(\max\{\mu(X_2), \mu(X_3, \ldots, \mu(X_1))\} + \varphi(\max\{\mu(X_2), \mu(X_3), \ldots, \mu(X_1)\})) \\ +G(\beta(\max\{\mu(X_2), \mu(X_3, \ldots, X_1)\} + \varphi(\max\{\mu(X_2), \mu(X_3), \ldots, \mu(X_1)\}))), \\ \vdots \\ F(\max\{\mu(X_n), \mu(X_1, \ldots, \mu(X_{n-1}))\} + \varphi(\max\{\mu(X_n), \mu(X_1), \ldots, \mu(X_{n-1})\})) \\ +G(\beta(\max\{\mu(X_n), \mu(X_1, \ldots, X_{n-1})\} + \varphi(\max\{\mu(X_n), \mu(X_1), \ldots, \mu(X_{n-1})\}))) \end{array} \right\} \\
&= F(\max\{\mu(X_1), \mu(X_2) \ldots, \mu(X_n)\} + \varphi(\max\{\mu(X_1), \mu(X_2) \ldots, \mu(X_n)\})) \\
&= F(\widehat{\mu}(M) + \varphi(\widehat{\mu}(M))) + G(\beta(\widehat{\mu}(M) + \varphi(\widehat{\mu}(M)))),
\end{aligned}$$

that is,

$$\begin{aligned}&\widehat{\mu}(\widehat{\mathscr{G}}(M) > 0 \Longrightarrow \\ &\tau + F(\widehat{\mu}(\widehat{\mathscr{G}}(M))) \le F(\widehat{\mu}(M) + \varphi(\widehat{\mu}(M))) + G(\beta(\widehat{\mu}(M) + \varphi(\widehat{\mu}(M)))).\end{aligned}$$

Hence, by Theorem 14.4, we reached that $\widehat{\mathscr{G}}$ has at least one fixed point in X^n and thus $\mathscr{G}$ has an $n-$tupled fixed point. This completes the proof.

Remark 14.2 In view of Corollaries 14.13–14.7, some new $n-$tupled fixed point results can be derived from Theorems 14.5 and 14.6.

14.4 Application I

Let $(X, \|.\|)$ be a real Banach algebra and let the symbol $C(I, X)$ stand for the space consisting of all continuous mappings $x : I = [0, 1] \to X$. We consider the existence of a solution $x \in C(I, X)$ to the following integral equation:

$$\begin{aligned}x(t) = f(t, x(t)) + Hx(t) \int_0^t \frac{(t^m - s^m)^{\alpha-1} m s^{m-1}}{\Gamma(\alpha)} k_1(g_1(t, s)) Q_1 x(s)\, ds \\ \times \int_0^t \frac{(t^n - s^n)^{\beta-1} n s^{n-1}}{\Gamma(\beta)} k_2(g_2(t, s)) Q_2 x(s)\, ds\end{aligned} \tag{14.14}$$

for all $t \in I = [0, 1]$, $0 < \alpha, \beta \le 1$ and $m, n > 0$.

We assume that the following conditions are satisfied:

(a_1) $f : I \times X \to X$ is a continuous mapping such that there exist a bijective, strictly increasing function $F : (0, \infty) \longrightarrow (-\infty, 0)$, $(G, \beta) \in \Delta_{G,\beta}$ and a nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\begin{aligned}&\|f(t, x) - f(t, y)\| > 0 \Longrightarrow \\ &F(\|f(t, x) - f(t, y)\| + \varphi(\|f(t, x) - f(t, y)\|)) \\ &\le F(\|x - y\| + \varphi(\|x - y\|)) + G(\beta((\|x - y\| + \varphi(\|x - y\|)))); \end{aligned} \tag{14.15}$$

(a_2) H, Q_1 and Q_2 are some operators acting continuously from the space $C(I, X)$ into itself and there are increasing functions $\psi_1, \psi_2, \psi_3 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\begin{aligned}\|H(x)\| &\le \psi_1(\|x\|), \\ \|Q_1(x)\| &\le \psi_2(\|x\|), \\ \|Q_2(x)\| &\le \psi_3(\|x\|);\end{aligned}$$

(a_3) $g_1, g_2 : I \times I \to \mathbb{R}$ are continuous and the functions $g_1(t,s)$ and $g_2(t,s)$ are nondecreasing for each variable t and s, separately;

(a_4) $k_1 : Img_1 \to \mathbb{R}_+$ is a continuous and nondecreasing function on the compact set Img_1;

(a_5) $k_2 : Img_2 \to \mathbb{R}_+$ is a continuous and nondecreasing function on the compact set Img_2;

(a_6) $\liminf_{\zeta\to\infty} \dfrac{\psi_1(\zeta).\psi_2(\zeta).\psi_3(\zeta)\|k_1\|\,\|k_2\|}{\zeta\Gamma(\alpha+1)\Gamma(\beta+1)} < 1.$

Theorem 14.7 *Under the assumptions (a_1)–(a_6), Eq. (14.14) has at least one solution in the space $x \in C(I, X)$.*

Proof Define an integral operator $T : C(I, X) \to C(I, X)$ by

$$Tx(t) = f(t, x(t)) + Hx(t)\,Fx(t)\,Gx(t),$$

where

$$Fx(t) = \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)}\, ms^{m-1}k_1(g_1(t,s))Q_1x(s)\,ds,$$
$$Gx(t) = \int_0^t \frac{(t^n - s^n)^{\beta-1}}{\Gamma(\beta)}\, ns^{n-1}k_2(g_2(t,s))Q_2x(s)\,ds.$$

Now, we show that the operator T has one fixed point. To this end, we define the following two mappings $T_1, T_2 : C(I, X) \to C(I, X)$ by:

$$T_1x(t) = f(t, x(t)),$$

$$T_2x(t) = Hx(t)\,Fx(t)\,Gx(t),$$

where $T = T_1 + T_2$. It is easy to see that T_1 is well-defined. Now, we show that T_2 is well-defined. Let $\varepsilon > 0$ arbitrarily and $x \in C(I, X)$ be given and fixed. Let $\epsilon > 0$ arbitrarily and $x \in C(I, X)$ be given and fixed. Since k_1 is uniformly continuous on the compact set Img_1, there exists $\delta_1(\epsilon) > 0$ such that, for all $t_1, t_2 \in I$ with $|t_2 - t_1| \le \delta_1(\epsilon)$, we have

$$|k_1(g_1(t_2,s)) - k_1(g_1(t_1,s))| < \frac{\Gamma(\alpha+1)\epsilon}{2(1+\|Q_1x\|)}.$$

Similarly, there exists $\delta_2(\epsilon) > 0$ such that, for all $t_1, t_2 \in I$ with $|t_2 - t_1| \le \delta_2(\epsilon)$, we have

$$|k_2(g_2(t_2,s)) - k_2(g_2(t_1,s))| < \frac{\Gamma(\beta+1)\epsilon}{2(1+\|Q_2x\|)}.$$

Put

$$\delta_3(\epsilon) = \min\left\{\delta_1(\varepsilon), \sqrt[\alpha]{\frac{\Gamma(\alpha+1)\varepsilon}{m^\alpha(1+2\,\|Q_1x\|\,\|k_1\|)}}\right\}$$

for all $t_1, t_2 \in I$ with $|t_2 - t_1| \le \delta_3(\epsilon)$, then we have

$$\begin{aligned}
&|(Fx)(t_2) - (Fx)(t_1)| \\
\le &\Big|\int_0^{t_2} \frac{(t_2^m - s^m)^{\alpha-1}}{\Gamma(\alpha)}\, ms^{m-1}k_1(g_1(t_2,s))Q_1x(s)ds \\
&- \int_0^{t_2} \frac{(t_2^m - s^m)^{\alpha-1}}{\Gamma(\alpha)}\, ms^{m-1}k_1(g_1(t_1,s))Q_1x(s)ds\Big| \\
&+ \Big|\int_0^{t_2} \frac{(t_2^m - s^m)^{\alpha-1}}{\Gamma(\alpha)}\, ms^{m-1}k_1(g_1(t_1,s))Q_1x(s)ds \\
&- \int_0^{t_1} \frac{(t_2^m - s^m)^{\alpha-1}}{\Gamma(\alpha)}\, ms^{m-1}k_1(g_1(t_1,s))Q_1x(s)ds\Big| \\
&+ \Big|\int_0^{t_1} \frac{(t_2^m - s^m)^{\alpha-1}}{\Gamma(\alpha)}\, ms^{m-1}k_1(g_1(t_1,s))Q_1x(s)ds \\
&- \int_0^{t_1} \frac{(t_1^m - s^m)^{\alpha-1}}{\Gamma(\alpha)}\, ms^{m-1}k_1(g_1(t_1,s))Q_1x(s)ds\Big| \\
\le &\int_0^{t_2} \frac{(t_2^m - s^m)^{\alpha-1}}{\Gamma(\alpha)}\, ms^{m-1}|k_1(g_1(t_2,s)) - k_1(g_1(t_1,s))||Q_1x(s)|ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2^m - s^m)^{\alpha-1}}{\Gamma(\alpha)}\, ms^{m-1}|k_1(g_1(t_1,s))||Q_1x(s)|ds \\
&+ \int_0^{t_1} \frac{|(t_2^m - s^m)^{\alpha-1} - (t_1^m - s^m)^{\alpha-1}|}{\Gamma(\alpha)}\, ms^{m-1}|k_1(g_1(t_1,s))||Q_1x(s)|ds.
\end{aligned}$$

Therefore, if we denote

$$\omega_{k_1og_1}(\delta_1, .) = \sup\{|k_1(g_1(t,s)) - k_1(g_1(t',s))| : t, t', s \in I \ and\ |t - t'| \le \delta_1\},$$

then we have

$$\begin{aligned}
\|(Fx)(t_2) - (Fx)(t_1)\| \le &\frac{\|Q_1x\|\omega_{k_1og_1}(\delta_1, .)}{\Gamma(\alpha)}\frac{t_2^{m\alpha}}{\alpha} + \frac{\|Q_1x\|\|k_1\|}{\Gamma(\alpha)}\frac{(t_2^m - t_1^m)^\alpha}{\alpha} \\
&+ \frac{\|Q_1x\|\|k_1\|}{\Gamma(\alpha)}\left[\frac{(t_2^m - t_1^m)^\alpha}{\alpha} + \frac{t_1^{m\alpha}}{\alpha} - \frac{t_2^{m\alpha}}{\alpha}\right] \\
\le &\frac{\|Q_1x\|\omega_{k_1og_1}(\delta_1, .)}{\Gamma(\alpha+1)} + \frac{2\|Q_1x\|\|k_1\|}{\Gamma(\alpha+1)}(t_2^m - t_1^m)^\alpha.
\end{aligned}$$

By applying the mean value theorem on $[t_1, t_2]$, we get

$$|t_2^m - t_1^m|^\alpha \le m^\alpha |t_2 - t_1|^\alpha.$$

Thus, from the last inequality, we get

$$\|(Fx)(t_2) - (Fx)(t_1)\| \leq \frac{\|Q_1x\|\omega_{k_1og_1}(\delta_1, .)}{\Gamma(\alpha+1)} + \frac{2\|Q_1x\|\|k_1\|}{\Gamma(\alpha+1)} m^\alpha |t_2 - t_1|^\alpha$$
$$\leq \frac{\varepsilon}{3(1 + \|Hx\|\,\|Gx\|)}. \tag{14.16}$$

Similarly, if we put $\delta_4(\epsilon) = \min\left\{\delta_2(\varepsilon), \sqrt[\beta]{\frac{\Gamma(\beta+1)\varepsilon}{n^\beta(1+2\,\|Q_2x\|\,\|k_2\|)}}\right\}$ for all $t_1, t_2 \in I$ with $|t_2 - t_1| \leq \delta_4(\epsilon)$, then we have

$$\|(Gx)(t_2) - (Gx)(t_1)\| \leq \frac{\|Q_2x\|\omega_{k_2og_2}(\delta_2, .)}{\Gamma(\beta+1)} + \frac{2\|Q_2x\|\|k_2\|}{\Gamma(\beta+1)} n^\beta |t_2 - t_1|^\beta$$
$$\leq \frac{\varepsilon}{3(1 + \|Hx\|\,\|Fx\|)},$$

where

$$\omega_{k_2og_2}(\delta_2, .) = \sup\{|k_2(g_2(t,s)) - k_2(g_2(t',s))| : t, t', s \in I \ and\ |t - t'| \leq \delta_2\}.$$

Also, for all $t \in I$, we have

$$\|(Fx)(t)\| \leq \frac{\|k_1\|\,\|Q_1x\|}{\Gamma(\alpha+1)}, \quad \|(Gx)(t)\| \leq \frac{\|k_2\|\,\|Q_2x\|}{\Gamma(\beta+1)}.$$

Also, since Hx is uniformly continuous on I, there exists $\delta_5(\varepsilon) > 0$ such that, for all $t_1, t_2 \in I$, with $|t_2 - t_1| < \delta_5(\varepsilon)$, we have

$$\|Hx(t_2) - Hx(t_1)\| < \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\varepsilon}{3\,(1 + \|k_1\|\,\|Q_1x\|)\,(1 + \|k_2\|\,\|Q_2x\|)}.$$

Put $\delta(\varepsilon) = \min\{\delta_3(\varepsilon), \delta_4(\varepsilon), \delta_5(\varepsilon)\}$ and $t_2 - t_1 \leq \delta(\varepsilon)$. Then we get

$$\begin{aligned}
\|T_2x(t_2) - T_2x(t_1)\| &= \|Hx(t_2)\,Fx(t_2)\,Gx(t_2) - Hx(t_1)\,Fx(t_1)\,Gx(t_1)\| \\
&\leq \|Hx(t_2) - Hx(t_1)\|\,\|Fx(t_2)\|\,\|Gx(t_2)\| \\
&\quad + \|Hx(t_1)\|\,\|Fx(t_2) - Fx(t_1)\|\,\|Gx(t_2)\| \\
&\quad + \|Hx(t_1)\|\,\|Fx(t_1)\|\,\|Gx(t_2) - Gx(t_1)\| \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Next, we show that T_2 is a continuous operator. Let $y \in C(I, X)$ and $\varepsilon > 0$. Since H, Q_1 and Q_2 are some operators acting continuously from the space $C(I, X)$ into itself, so there exist $\delta_1 > 0$, $\delta_2 > 0$ and $\delta_3 > 0$, such that

$$\begin{aligned}
&\forall x \in C(I,X),\ \ (\|x-y\|<\delta_1 \Longrightarrow \|Hx-Hy\|<\varepsilon_1(\varepsilon)),\\
&\forall x \in C(I,X),\ \ (\|x-y\|<\delta_2 \Longrightarrow \|Q_1x-Q_1y\|<\varepsilon_2(\varepsilon)),\\
&\forall x \in C(I,X),\ \ (\|x-y\|<\delta_3 \Longrightarrow \|Q_2x-Q_2y\|<\varepsilon_3(\varepsilon)),
\end{aligned}$$

for each $t \in I$, we have

$$\begin{aligned}
|Fx(t)-Fy(t)| =& \Big|\int_0^t \frac{(t^m-s^m)^{\alpha-1}}{\Gamma(\alpha)}\, ms^{m-1}k_1(g_1(t,s))Q_1x(s)ds\\
&- \int_0^t \frac{(t^m-s^m)^{\alpha-1}}{\Gamma(\alpha)}\, ms^{m-1}k_1(g_1(t,s))Q_1y(s)ds\Big|\\
&\le \int_0^t \frac{(t^m-s^m)^{\alpha-1}}{\Gamma(\alpha)}\, ms^{m-1}|k_1(g_1(t,s))||Q_1x(s)-Q_1y(s)|ds
\end{aligned}$$

and so

$$\|Fx-Fy\| \le \frac{\|k_1\|}{\Gamma(\alpha+1)}\,\|Q_1x-Q_1y\| \le \frac{\|k_1\|\,\varepsilon_2(\varepsilon)}{\Gamma(\alpha+1)}.$$

Similarly, we show that

$$\|Gx-Gy\| \le \frac{\|k_2\|}{\Gamma(\beta+1)}\,\|Q_2x-Q_2y\| \le \frac{\|k_2\|\,\varepsilon_3(\varepsilon)}{\Gamma(\beta+1)}.$$

Now, if we put $\delta = \min\{\delta_1,\delta_2,\delta_3\}$, then, for any $x \in C(I,X)$ that $\|x-y\|<\delta$, by the triangle inequality, we obtain

$$\begin{aligned}
&\|T_2x(t)-T_2y(t)\|\\
&= \|Hx(t)\,Fx(t)\,Gx(t)-Hy(t)\,Fy(t)\,Gy(t)\|\\
&\le \|Hx(t)-Hy(t)\|\,\|Fx(t)\|\,\|Gx(t)\| + \|Hy(t)\|\,\|Fx(t)-Fy(t)\|\,\|Gx(t)\|\\
&\quad + \|Hy(t)\|\,\|Fy(t)\|\,\|Gx(t)-Gy(t)\|\\
&\le \|Hx-Hy\|\,\|Fx\|\,\|Gx\| + \|Hy\|\,\|Fx-Fy\|\,\|Gx\| + \|Hy\|\,\|Fy\|\,\|Gx-Gy\|\\
&\le \varepsilon_1(\varepsilon)\frac{\|k_1\|\,\|Q_1x\|}{\Gamma(\alpha+1)}\,\frac{\|k_2\|\,\|Q_2x\|}{\Gamma(\beta+1)}\\
&\quad + \|Hy\|\,\frac{\|k_1\|\,\varepsilon_2(\varepsilon)}{\Gamma(\alpha+1)}\,\frac{\|k_2\|\,\|Q_2x\|}{\Gamma(\beta+1)} + \|Hy\|\,\frac{\|k_1\|\,\|Q_1y\|}{\Gamma(\alpha+1)}\,\frac{\|k_2\|\,\varepsilon_3(\varepsilon)}{\Gamma(\beta+1)}\\
&\le \varepsilon_1(\varepsilon)\frac{\|k_1\|\,\psi_2(\|x\|)}{\Gamma(\alpha+1)}\,\frac{\|k_2\|\,\psi_3(\|x\|)}{\Gamma(\beta+1)}\\
&\quad + \psi_1(\|y\|)\,\frac{\|k_1\|\,\varepsilon_2(\varepsilon)}{\Gamma(\alpha+1)}\,\frac{\|k_2\|\,\psi_2(\|x\|)}{\Gamma(\beta+1)} + \psi_1(\|y\|)\,\frac{\|k_1\|\,\psi_2(\|y\|)}{\Gamma(\alpha+1)}\,\frac{\|k_2\|\,\varepsilon_3(\varepsilon)}{\Gamma(\beta+1)}\\
&\le \varepsilon_1(\varepsilon)\frac{\|k_1\|\,\psi_2(\|y\|+\delta)}{\Gamma(\alpha+1)}\,\frac{\|k_2\|\,\psi_3(\|y\|+\delta)}{\Gamma(\beta+1)}\\
&\quad + \psi_1(\|y\|)\,\frac{\|k_1\|\,\varepsilon_2(\varepsilon)}{\Gamma(\alpha+1)}\,\frac{\|k_2\|\,\psi_3(\|y\|+\delta)}{\Gamma(\beta+1)} + \psi_1(\|y\|)\,\frac{\|k_1\|\,\psi_2(\|y\|)}{\Gamma(\alpha+1)}\,\frac{\|k_2\|\,\varepsilon_3(\varepsilon)}{\Gamma(\beta+1)}\\
&\le \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

where

$$\varepsilon_1 = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\varepsilon}{3[1+\psi_2(\|y\|+\delta)\|k_1\|][1+\psi_3(\|y\|+\delta)\|k_2\|]},$$
$$\varepsilon_2 = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\varepsilon}{3[1+\psi_1(\|y\|)\|k_1\|][1+\psi_3(\|y\|)\|k_2\|]},$$
$$\varepsilon_3 = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\varepsilon}{3[1+\psi_1(\|y\|)\|k_1\|][1+\psi_2(\|y\|)\|k_2\|]}.$$

Now, we show that T_2 is a compact operator. If $B = \{x \in C(I, X) : \|x\| < 1\}$ is the open unit ball of $C(I, X)$, then we claim that $\overline{T_2(B)}$ is a compact subset of $C(I, X)$. To see this, by the Arzelà–Ascoli theorem, we need only to show that $T_2(B)$ is an uniformly bounded and equi-continuous subset of $C(I, X)$.

First, we show that $T_2(B) = \{T_2 x : x \in B\}$ is uniformly bounded. By the conditions (a_2), for any $x \in B$, we have the following estimates:

$$\begin{aligned}\|T_2 x(t)\| &= \|Hx(t)\,Fx(t)\,Gx(t)\|\\ &\le \|Hx(t)\|\,\|Fx(t)\|\,\|Gx(t)\| \le \|Hx\|\,\|Fx\|\,\|Gx\|\\ &\le \psi_1(\|x\|)\frac{\|k_1\|\,\psi_2(\|x\|)}{\Gamma(\alpha+1)}\frac{\|k_2\|\,\psi_3(\|x\|)}{\Gamma(\beta+1)}\\ &\le \psi_1(1)\frac{\|k_1\|\,\psi_2(1)}{\Gamma(\alpha+1)}\frac{\|k_2\|\,\psi_3(1)}{\Gamma(\beta+1)}.\end{aligned}$$

Hence, putting $M := \psi_1(1)\frac{\|k_1\|\psi_2(1)}{\Gamma(\alpha+1)}\frac{\|k_2\|\psi_3(1)}{\Gamma(\beta+1)}$, we conclude that $T_2(B)$ is uniformly bounded. Now, we show that $T_2(B)$ is an uniformly equi-continuous subset of $C(I, X)$. To see this, let $x \in B$ be arbitrary, and let $\varepsilon > 0$. Since Hx, Fx and Gx are uniformly continuous, there exist some $\delta_1(\varepsilon), \delta_2(\varepsilon), \delta_3(\varepsilon) > 0$ such that

$$\begin{aligned}&\forall t_1, t_2 \in I, \quad (|t_2 - t_1| < \delta_1(\varepsilon) \Longrightarrow \|Hx(t_2) - Hx(t_1)\| < \varepsilon_1),\\ &\forall t_1, t_2 \in I, \quad (|t_2 - t_1| < \delta_2(\varepsilon) \Longrightarrow \|Fx(t_2) - Fx(t_1)\| < \varepsilon_2),\\ &\forall t_1, t_2 \in I, \quad (|t_2 - t_1| < \delta_3(\varepsilon) \Longrightarrow \|Gx(t_2) - Gx(t_1)\| < \varepsilon_3).\end{aligned}$$

Let $\delta(\varepsilon) = \min\{\delta_1(\varepsilon), \delta_2(\varepsilon), \delta_3(\varepsilon), \varepsilon_2\}$, where ε_2 and ε_3 depend on ε and will be given. Therefore, if $t_1, t_2 \in [0, T]$ satisfies $0 < t_2 - t_1 < \delta(\varepsilon)$ and $x \in B$, then we have the following estimates:

$$
\begin{aligned}
&\|T_2x\,(t_2) - T_2x\,(t_1)\| \\
&= \|Hx\,(t_2)\,Fx\,(t_2)\,Gx\,(t_2) - Hx\,(t_1)\,Fx\,(t_1)\,Gx\,(t_1)\| \\
&\le \|Hx\,(t_2) - Hx\,(t_1)\|\,\|Fx\,(t_2)\|\,\|Gx\,(t_2)\| + \|Hx\,(t_1)\|\,\|Fx\,(t_2) - Fx\,(t_1)\|\,\|Gx\,(t_2)\| \\
&\quad + \|Hx\,(t_1)\|\,\|Fx\,(t_1)\|\,\|Gx\,(t_2) - Gx\,(t_1)\| \\
&\le \varepsilon_1 \frac{\|k_1\|\,\psi_2(\|x\|)}{\Gamma(\alpha+1)} \frac{\|k_2\|\,\psi_3(\|x\|)}{\Gamma(\beta+1)} + \psi_1(\|x\|)\varepsilon_2 \frac{\|k_2\|\,\psi_3(\|x\|)}{\Gamma(\beta+1)} \\
&\quad + \psi_1(\|x\|) \frac{\|k_1\|\,\psi_2(\|x\|)}{\Gamma(\alpha+1)}\varepsilon_3 \\
&\le \varepsilon_1 \frac{\|k_1\|\,\psi_2(1)}{\Gamma(\alpha+1)} \frac{\|k_2\|\,\psi_3(1)}{\Gamma(\beta+1)} + \psi_1(1)\varepsilon_2 \frac{\|k_2\|\,\psi_3(1)}{\Gamma(\beta+1)} + \psi_1(1) \frac{\|k_1\|\,\psi_2(1)}{\Gamma(\alpha+1)}\varepsilon_3 \\
&\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon,
\end{aligned}
$$

where

$$
\begin{aligned}
\varepsilon_1 &= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\varepsilon}{3\,(1 + \|k_1\|\,\psi_2(1)\,\|k_2\|\,\psi_3(1))}, \\
\varepsilon_2 &= \frac{\Gamma(\beta+1)\varepsilon}{3\,(1 + \psi_1(1)\,\|k_2\|\,\psi_3(1))}, \\
\varepsilon_3 &= \frac{\Gamma(\alpha+1)\varepsilon}{3\,(1 + \psi_1(1)\,\|k_1\|\,\psi_2(1))}.
\end{aligned}
$$

Therefore, T_2 is a compact operator. Next, we show that T_1 satisfies in (14.11). Let $x, y \in C\,(I, X)$, and $\|T_1x - T_1y\| > 0$. By applying the fact that every continuous function attains its maximum on a compact set, there exists $t \in I$ such that $0 < \|T_1x - T_1y\| = \|f\,(t, x\,(t)) - f\,(t, y\,(t))\|$. By (a_1) and using the fact that F and φ are strictly increasing functions, we obtain

$$
\begin{aligned}
&F\,(\|T_1x - T_1y\| + \varphi\,(\|T_1x - T_1y\|)) \\
&= F\,(\|f\,(t, x\,(t)) - f\,(t, y\,(t))\| + \varphi\,(\|f\,(t, x\,(t)) - f\,(t, y\,(t))\|)) \\
&\le F\,(\|x - y\| + \varphi\,(\|x - y\|)) + G(\beta((\|x - y\| + \varphi\,(\|x - y\|)))).
\end{aligned}
$$

Hence T_1 satisfies in (14.11).

Now, we show that there exists some $M_1 > 0$ such that $\|T_1x\| \le M_1$ holds for each $x \in C\,(I, X)$. Since F is bijective and strictly increasing, we have

$$
\begin{aligned}
&\|T_1x - T_1y\| + \varphi\,(\|T_1x - T_1y\|) \\
&\le F^{-1}\left[F\,(\|x - y\| + \varphi\,(\|x - y\|)) + G(\beta((\|x - y\| + \varphi\,(\|x - y\|))))\right].
\end{aligned}
$$

Let $0 < \|x\| + \varphi\,(\|x\|)$, since $F\,(\|x\| + \varphi\,(\|x\|)) < 0$, the above inequality implies that

$$\begin{aligned}
\|T_1x\| &\le \|T_1x - T_10\| + \|T_10\| \\
&\le \|T_1x - T_10\| + \varphi(\|T_1x - T_10\|) + \|T_10\| \\
&\le F^{-1}[F(\|x\| + \varphi(\|x\|)) + G(\beta((\|x\| + \varphi(\|x\|))))] + \|T_10\| \\
&\le F^{-1}[G(\beta((\|x\| + \varphi(\|x\|))))] + \|T_10\| \le \|T_10\|.
\end{aligned}$$

Therefore, we have

$$\exists\, M_1 > 0 : \forall x \ (x \in C(I, X) \implies \|T_1x\| \le M_1),$$

where $M_1 := \|T_10\|$.

Finally, we claim that there exists some $r > 0$, such that $T(B_r(0)) \subseteq B_r(0)$ with $B_r(0) = \{x \in C(I, X) : \|x\| \le r\}$. On the contrary, for any $\zeta > 0$ there exists some $x_\zeta \in B_r(0)$ such that $\left\|T\left(x_\zeta\right)\right\| > \zeta$. This implies that $\liminf_{\zeta\to\infty} \frac{1}{\zeta}\left\|T\left(x_\zeta\right)\right\| \ge 1$. On the other hand, we have

$$\begin{aligned}
\left\|Tx_\zeta(t)\right\| &\le \left\|f\left(t, x_\zeta(t)\right)\right\| + \left\|Hx_\zeta(t)Fx_\zeta(t)Gx_\zeta(t)\right\| \\
&\le \left\|T_1x_\zeta\right\| + \left\|Hx_\zeta(t)\right\|\left\|Fx_\zeta(t)\right\|\left\|Gx_\zeta(t)\right\| \\
&\le M_1 + \left\|Hx_\zeta\right\| . \left\|Fx_\zeta\right\| . \left\|Gx_\zeta\right\| \\
&\le M_1 + \psi_1\left(\left\|x_\zeta\right\|\right) . \psi_2\left(\left\|x_\zeta\right\|\right) . \psi_3\left(\left\|x_\zeta\right\|\right) \frac{\|k_1\|}{\Gamma(\alpha+1)} \frac{\|k_2\|}{\Gamma(\beta+1)} \\
&\le M_1 + \psi_1(\zeta) . \psi_2(\zeta) . \psi_3(\zeta) \frac{\|k_1\|}{\Gamma(\alpha+1)} \frac{\|k_2\|}{\Gamma(\beta+1)}.
\end{aligned}$$

Hence, by the above estimate and the condition (a_6), we get

$$\liminf_{\zeta\to\infty} \frac{1}{\zeta}\left\|T\left(x_\zeta\right)\right\| \le \liminf_{\zeta\to\infty} \frac{\psi_1(\zeta) . \psi_2(\zeta) . \psi_3(\zeta) \|k_1\| \|k_2\|}{\zeta\Gamma(\alpha+1)\Gamma(\beta+1)} < 1,$$

which is a contradiction. Thus, in view of the above discussions and Corollary 14.7, we conclude that Eq. (14.14) has at least one solution in $B_r(0) \subseteq C(I, X)$. This completes the proof.

Corollary 14.8 *Let the assumptions of Theorem 14.7 be satisfied (with $m = n = 1$), then the fractional-order quadratic integral equation*

$$\begin{aligned}
x(t) = f(t, x(t)) + Hx(t) &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} k_1(g_1(t, s))Q_1x(s)\, ds \\
&\times \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} k_2(g_2(t, s))Q_2x(s)\, ds
\end{aligned}$$

has at least one solution $x \in C(I, X)$.

Corollary 14.9 *Let the assumptions of Corollary 14.8 be satisfied (with $Hx(t)) = 1$), then the fractional-order quadratic integral equation*

$$x(t) = f(t, x(t)) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} k_1(g_1(t,s)) Q_1 x(s)\, ds$$
$$\times \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} k_2(g_2(t,s)) Q_2 x(s)\, ds$$

has at least one solution $x \in C(I, X)$.

Corollary 14.10 *Let the assumptions of Corollary 14.9 be satisfied (with* $k_1 = k_2 = I, \alpha = \beta, g_1 = g_2 = g, f(t,x) = 0$*), then the fractional-order quadratic integral equation:*

$$x(t) = \left[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(t,s) x(s)\, ds \right]^2$$

has at least one solution $x \in C(I, X)$.

In what follows, we illustrate the above-obtained result by the following example.

Example 14.1 Consider the following functional integral equation of fractional order

$$\begin{aligned} x(t) = {} & \frac{1}{5}t^3 + \frac{2t^2 e^{-\lambda(t+1)}}{t^4+1}\cos(x(t)) \\ & + \frac{\sqrt[5]{|x(t)|}}{2\left(1+|x(t)|^2\right)} \int_0^t \frac{2s}{\Gamma(\frac{1}{2})\sqrt{t^2-s^2}} \left[\frac{1}{8}(t+s)+\frac{1}{4}\right] \ln\left(1+\frac{\sqrt[5]{|x(s)|}}{5}\right) ds \\ & \times \int_0^t \frac{3s^2}{\Gamma(\frac{1}{3})\sqrt[3]{(t^3-s^3)^2}} \frac{(t+\sqrt{s})^2}{12} \ln\left(1+\frac{\sqrt[3]{|x(s)|}}{3}\right) ds, \quad \forall \lambda > 0. \end{aligned} \tag{14.17}$$

In this example, we have $X = \mathbb{R}$, $g_1(t,s) = \frac{1}{4}\sqrt{t+s}$ and $g_2(t,s) = t + \sqrt{s}$, and these functions satisfy the assumption (a_3). Let $k_1 : [0, \frac{\sqrt{2}}{4}] \to \mathbb{R}_+$ and $k_2 : [0, 2] \to \mathbb{R}_+$ be given by $k_1(y) = 2y^2 + \frac{1}{4}$ and $k_2(y) = \frac{1}{12}y^2$, then k_1 and k_2 satisfy assumptions (a_4) and (a_5) with $||k_1|| = \frac{1}{2}$ and $||k_2|| = \frac{1}{3}$. Define the continuous operators $H, Q_1, Q_2 : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ given by

$$Hx = \frac{\sqrt[5]{|x|}}{2\left(1+|x|^2\right)},$$

$$Q_1 x = \ln\left(1+\frac{\sqrt[5]{|x|}}{5}\right), \quad Q_2 x = \ln\left(1+\frac{\sqrt[3]{|x|}}{3}\right),$$

respectively. Define the functions $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ given by $f(t,x) = \frac{1}{5}t^3 + \dfrac{2t^2 e^{-\lambda(t+1)}}{t^4+1}\cos(x)$ which is continuous and satisfies

$$|f(t,x)-f(t,y)| \leq \frac{2t^2 e^{-\lambda(t+1)}}{t^4+1} |\cos(x)-\cos(y)| \leq e^{-\lambda} |x-y| . \qquad (14.18)$$

So, we have

$$\begin{aligned} |f(t,x)-f(t,y)| + |f(t,x)-f(t,y)|^2 &\leq e^{-\lambda} |x-y| + e^{-2\lambda} |x-y|^2 \\ &\leq e^{-\lambda} \left(|x-y| + |x-y|^2\right). \end{aligned}$$

Now, by choosing the function $F : [0,\infty) \to (-\infty, 0)$ given by $F(t) = \ln(t)$, $G : \mathbb{R}^+ \to \mathbb{R}$ by $G(t) = \ln(t)$, $\beta : [0,\infty) \to [0,1)$ by $\beta(t) = e^{-\lambda}$ and the function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\varphi(t) = t^2$, it is easy to see that the inequality (14.18) implies that the condition (14.15) holds.

Indeed, if $|f(t,x)-f(t,y)| > 0$, then we have

$$\begin{aligned} & F[|f(t,x)-f(t,y)| + \varphi(|f(t,x)-f(t,y)|)] \\ &= F[|f(t,x)-f(t,y)| + |f(t,x)-f(t,y)|^2] \\ &= \ln[|f(t,x)-f(t,y)| + |f(t,x)-f(t,y)|^2] \\ &\leq \ln[e^{-\lambda}\left(|x-y| + |x-y|^2\right)] \\ &= \ln\left(|x-y| + |x-y|^2\right) + \ln(e^{-\lambda}) \\ &= F(|x-y| + \varphi(|x-y|)) + G(\beta(|x-y| + \varphi(|x-y|))). \end{aligned}$$

By choosing the strictly continuous functions $\psi_1, \psi_2, \psi_3 : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\psi_1(t) = \frac{\sqrt[5]{t}}{2}$, $\psi_2(t) = \frac{\sqrt[5]{t}}{5}$ and $\psi_3(t) = \frac{\sqrt[3]{t}}{3}$, we have

$$\|H(x)\| \leq \psi_1(\|x\|), \quad \|Q_1(x)\| \leq \psi_2(\|x\|), \quad \|Q_2(x)\| \leq \psi_3(\|x\|),$$

$$\liminf_{\zeta\to\infty} \frac{\psi_1(\zeta).\psi_2(\zeta).\psi_3(\zeta)\,\|k_1\|\,\|k_2\|}{\zeta\,\Gamma(\alpha+1)\Gamma(\beta+1)} = \liminf_{\zeta\to\infty} \frac{\sqrt[5]{\zeta} \times \sqrt[5]{\zeta} \times \sqrt[3]{\zeta}}{30\Gamma(\frac{1}{2})\Gamma(\frac{1}{3})\,\zeta} = 0 < 1$$

and this satisfies the assumption (a_6).

14.5 Combination of Some Effective Modified Methods to Solve Volterra Nonlinear Singular Mixed Integral Equations (14.14)

A singular integral equation occurs in some concepts of engineering mechanics, such as elasticity, plasticity and aerodynamics (see [18, 27]). The Cauchy integral equation is a kind of singular integral equation introduced in [20, 21], and this problem is solved with the help of some numerical methods as collocation points, Gaussian quadrature method and general quadrature collocation nodes in

[10, 17, 36], respectively. Also, in [32, 34], integral equations with singular logarithmic kernel are solved by Galerkin multi-wavelet and wavelet methods in turn. In [30, 38], the variational iteration method and the Adomian decomposition method are used to solve nonlinear mixed integral equations, respectively.

Now, we have nonlinear singular mixed integral equations which have more complexity with respect to the above problems. So we use a combination of some effective modified methods, in order to consider homotopy perturbation which is an important concept of topology and perturbations theory (see [14, 19]). For increasing the ability of this method, some modifications of the homotopy perturbation method were created by [28, 29] where the definition of homotopy perturbation is introduced by nonlinear operators. To relax the nonlinearity we use from linear combination of Adomian polynomials; to see some applications of Adomian decomposition method, refer to [1, 31, 35]. In this section, we use a combination of modified homotopy perturbation and Adomian decomposition method, where we convert a nonlinear problem to some easier linear or nonlinear problems and also to free of nonlinearity we use Adomian polynomials. In the following, we consider Volterra nonlinear singular mixed integral equations (14.14) in the following form:

$$x(t)-H(x(t))\int_0^t k_1(t,s)Q_1(x(s))\,ds \times \int_0^t k_2(t,s)Q_2(x(s))\,ds - f\,(t,x\,(t)) = 0 \tag{14.19}$$

for all $t \in [0,1]$, $0 < \alpha, \beta \leq 1$ and $m, n > 0$, where

$$k_1(t,s) = \frac{ms^{m-1}k_1(g_1(t,s))}{\Gamma(\alpha)(t^m - s^m)^{1-\alpha}}, \quad k_2(t,s) = \frac{ns^{n-1}k_2(g_2(t,s))}{\Gamma(\beta)(t^n - s^n)^{1-\beta}}. \tag{14.20}$$

The general operator form of (14.19) can be given to this form:

$$A(t,x(t)) - f(t,x(t)) = 0, \quad \forall t \in [0,1]. \tag{14.21}$$

Obviously, A is a nonlinear integral operator and f is a known analytic function. Similar to [33], we divide the general operator A to N_1 and N_2 operators and f function converts to simple functions $f_1(t)$ and $f_2(t,x(t))$, thus (14.21) can be expressed by $N_1(x) - f_1(t) + N_2(x) - f_2(t,x(t)) = 0$. Therefore, we define a modified homotopy perturbation as follows:

$$H(u,p) = N_1(u) - f_1(t) + p(N_2(u) - f_2(t,x(t))) = 0, \quad \forall p \in [0,1], \tag{14.22}$$

and

$$x(t) \simeq u(t) = u_0(t) + pu_1(t) + p^2u_2(t) + p^3u_3(t) + \ldots, \tag{14.23}$$

where p is an embedding parameter, with the help of variations of $p = 0$ to $p = 1$; then we obtain $N_1(u) = f_1(t)$ to $A(t,u(t)) = f(t,x(t))$. So, we can get the solution of (14.21) for $p = 1$ and $x(t) \simeq \lim_{p\to 1} u(t)$. By (14.19) and (14.22), we can give N_1 and N_2 operators in this form:

$$
\begin{aligned}
&N_1(u) = u(t),\\
&N_2(u) = -H(u(t)) \int_0^t k_1(t,s) Q_1(u(s))\, ds \times \int_0^t k_2(t,s) Q_2(u(s))\, ds.
\end{aligned}
\tag{14.24}
$$

By converting $f(t, x(t))$ to $f_1(t)$, $f_2(t, x(t))$ and replacing (14.23), (14.24) in (14.22), we conclude that

$$
\begin{aligned}
&\left(\sum_{i=0}^{\infty} p^i u_i(t) - f_1(t)\right) - p\Bigg(f_2\left(t, \sum_{i=0}^{\infty} p^i u_i(t)\right)\\
&\quad + H\left(\sum_{i=0}^{\infty} p^i u_i(t)\right) \int_0^t k_1(t,s) Q_1\left(\sum_{i=0}^{\infty} p^i u_i(s)\right) ds \int_0^t k_2(t,s) Q_2\left(\sum_{i=0}^{\infty} p^i u_i(s)\right) ds\Bigg)\\
&= 0.
\end{aligned}
$$

We approximate the above nonlinear functions by Adomian polynomials:

$$
\begin{aligned}
&\Big(u_0(t) + p u_1(t) + p^2 u_2(t) + \cdots - f_1(t)\Big)\\
&- p\Bigg(\sum_{i=0}^{\infty} p^i F_i(t) + \sum_{i=0}^{\infty} p^i H_i(t) \int_0^t k_1(t,s) \sum_{i=0}^{\infty} p^i Q_{1,i}(s) ds \int_0^t k_2(t,s) \sum_{i=0}^{\infty} p^i Q_{2,i}(s) ds \Bigg)\\
&= 0,
\end{aligned}
\tag{14.25}
$$

where Adomian polynomials are given in the suitable form:

$$
\begin{aligned}
F_n(t) &= \frac{1}{n!}\Big(\frac{d^n}{dp^n} f_2(t, \sum_{i=0}^{\infty} p^i u_i(t))\Big)_{p=0},\\
H_n(t) &= \frac{1}{n!}\Big(\frac{d^n}{dp^n} H(\sum_{i=0}^{\infty} p^i u_i(t))\Big)_{p=0},\\
Q_{1,n}(s) &= \frac{1}{n!}\Big(\frac{d^n}{dp^n} Q_1(\sum_{i=0}^{\infty} p^i u_i(s))\Big)_{p=0},\\
Q_{2,n}(s) &= \frac{1}{n!}\Big(\frac{d^n}{dp^n} Q_2(\sum_{i=0}^{\infty} p^i u_i(s))\Big)_{p=0}.
\end{aligned}
\tag{14.26}
$$

Rearranging (14.25) in terms of p powers concludes that

$$
\begin{aligned}
&p^0 : (u_0(t) - f_1(t)),\\
&p^j : (u_j(t) - F_{j-1}(t) - H_{j-1}(t) \int_0^t k_1(t,s) Q_{1,j-1}(s) ds \int_0^t k_2(t,s) Q_{2,j-1}(s) ds)
\end{aligned}
\tag{14.27}
$$

for each $j = 1, 2, 3, \ldots$. From the definition of the modified homotopy perturbation (14.22), the coefficients of p powers are equal to zero, so we approach an iterative algorithm to solve (14.19).

Table 14.1 Absolute errors

t	Absolute errors for (14.31)
0.0	0
0.1	2.9×10^{-4}
0.2	1.4×10^{-3}
0.3	3.7×10^{-3}
0.4	7.5×10^{-3}
0.5	1.3×10^{-2}
0.6	2.0×10^{-2}
0.7	3.0×10^{-2}
0.8	4.3×10^{-2}
0.9	5.9×10^{-2}
1.0	8.0×10^{-2}

Algorithm.

$$\begin{cases} u_0(t) = f_1(t), \\ u_j(t) = F_{j-1}(t) + H_{j-1}(t) \int_0^t k_1(t,s) Q_{1,j-1}(s) ds \int_0^t k_2(t,s) Q_{2,j-1}(s) ds) \end{cases} \tag{14.28}$$

for each $j = 1, 2, 3, \cdots$. According to algorithm (14.28), for Eq. (14.17) in Example (14.18), we have

$$\begin{cases} f_1(t) = 0, \quad f_2(t, x(t)) = \frac{1}{5}t^3 + \frac{2t^2 e^{-\lambda(t+1)}}{t^4+1} \cos(x(t)), \quad \forall \lambda > 0, \\ k_1(t,s) = \frac{s(t+s+2)}{4\Gamma(\frac{1}{2})\sqrt{t^2-s^2}}, \qquad k_2(t,s) = \frac{s^2(t+\sqrt{s})^2}{4\Gamma(\frac{1}{3})\sqrt[3]{(t^3-s^3)^2}}, \\ H(x(t)) = \frac{\sqrt[5]{|x(t)|}}{2\left(1+|x(t)|^2\right)}, \quad Q_1(x(s)) = \ln\left(1 + \frac{\sqrt[5]{|x(s)|}}{5}\right), \\ Q_2(x(s)) = \ln\left(1 + \frac{\sqrt[3]{|x(s)|}}{3}\right). \end{cases} \tag{14.29}$$

In the first stage of algorithm (14.28), we choose $\lambda = 1$, $u_0(t) = f_1(t) = 0$ and we compute Adomian polynomials $F_0(t)$, $H_0(t)$, $Q_{1,0}(s)$ and $Q_{2,0}(s)$ by (14.26) and replace them into the second stage of algorithm (14.28); then we obtain $u_0(t)$ and $u_1(t)$ as follows:

$$\begin{cases} u_0(t) = f_1(t) = 0, \\ u_1(t) = F_0(t) + H_0(t) \int_0^t k_1(t,s) Q_{1,0}(s) ds \int_0^t k_2(t,s) Q_{2,0}(s) ds = \frac{1}{5}t^3 + \frac{2t^2}{(t^4+1)e^{(t+1)}}. \end{cases} \tag{14.30}$$

Now, we can give an approximation of the solution of Volterra nonlinear singular mixed integral equations (14.17) by the first two terms of series (14.23),

$$x(t) \simeq u(t) = \sum_{i=0}^{1} u_i(t)) = \frac{1}{5}t^3 + \frac{2t^2}{(t^4+1)e^{(t+1)}}. \tag{14.31}$$

By substituting (14.31) into (14.17) and comparing both sides of it, absolute errors are shown in Table 14.1.

14.6 Application II

The utility function is a magnificence concept that measures preferences over a set of goods and services. The theory of estimated utility and the dual utility theory are two very standard and extensively recognized methods for the quantification of favorites and a basis of decisions under uncertainty. These classical topics in economics are involved in the plenitude of textbooks and monographs and characterize a benchmark for every other quantitative decision theory.

Recently, the utility function is represented by the Volterra integral operator of the first and second types [12]. The main problem in the utility theory is to maximize the utility function. The unique solution of the problem guarantee implies this maximization. Our aim is to apply Theorem 14.6 in terms of fractional calculus [25]. The uniqueness of the fixed point implies the maximum value of the utility function.

Let $C[a,b]$ be the space of all continuous functions that endow with the maximum norm. And let the weighted space $C_{\wp,\rho}[a,b]$ defined for the functions ϕ

$$C_{\wp,\rho}[a,b] = \left\{\phi : (a,b] \to \mathbb{R} : \left(\frac{\chi^\rho - a^\rho}{\rho}\right)^{\wp} \phi(\chi) \in C[a,b]\right\}, \quad \forall \wp \in [0,1),$$

and the norm

$$\left\|\phi(\chi)\right\|_{C_{\wp,\rho}[a,b]} = \left\| \left(\frac{\chi^\rho - a^\rho}{\rho}\right)^{\wp} \phi(\chi)\right\|_C.$$

We define the utility function by applying the fractional integral operator in Volterra style:

$$U(\chi) = U_0(\wp,\rho) + \frac{1}{\Gamma(\wp)} \int_a^\chi \left(\frac{\chi^\rho - \tau^\rho}{\rho}\right)^{\wp-1} \tau^{\rho-1} \Theta(\tau, U(\tau)) d\tau, \tag{14.32}$$

where $U_0 \geq 0$ is the initial utility value depending on the fractional powers $0<\wp<1$ and $\rho > 0$, and the integral

$$I^{\wp,\rho}\theta(\chi) := \frac{1}{\Gamma(\wp)} \int_a^\chi \left(\frac{\chi^\rho - \tau^\rho}{\rho}\right)^{\wp-1} \tau^{\rho-1}\theta(\tau)d\tau$$

is called the *generalized fractional integral* [16]. Clearly, $U(\chi) \in C_{\wp,\rho}[a,b]$ whenever $\Theta \in C_{\wp,\rho}[a,b]$ (see Lemma 5 in [25]).

In our discussion, we suppose that $\Theta(., U)$ is Lipschitz with a Lipschitz constant $\varepsilon > 0$. The optimal problem is to maximize the utility function U. We consider the problem:

$$\max_{\chi} U(\chi) := G(\beta(\chi)), \quad G \in \Delta_{G,\beta}. \tag{14.33}$$

Moreover, we assume that φ is the accumulation function of the utility function (the future value of U) satisfying a new combination $U + \varphi(U)$. Define the optimal problem of $U + \varphi(U)$ as follows:

$$\max_{\chi}[U + \varphi(U)] := F(U + \varphi(U)), \quad \forall F \in \Delta_F. \tag{14.34}$$

Now, it is ready to seek our result.

Theorem 14.8 *Consider the problems* (14.32)–(14.34)*. If, for all* $\wp \in [0, 1)$ *and* $\rho > 0$*,*

$$\varepsilon \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\wp} \frac{\Gamma(\rho)}{\Gamma(\rho + \wp)} < 1, \quad \forall \varepsilon > 0,$$

then there is a unique solution maximizing the problems (14.32)–(14.34)*.*

Proof Our aim is to show that there exists a unique solution to the Volterra integral equation, the Eq. (14.32). This equation can be translated into integral operator:

$$(TU)(\chi) = U(\chi),$$

where

$$(TU)(\chi) = [U_0(\wp, \rho) + I^{\wp,\rho}\Theta](\chi).$$

Now, we proceed to prove that $\|(TU_1)(\chi) - (TU_2)(\chi)\|_{C_{\wp,\rho}[a,b]} > 0$, where $U_1 \neq U_2$. We have

$$\begin{aligned}
&\|(TU_1)(\chi) - (TU_2)(\chi)\|_{C_{\wp,\rho}[a,b]} \\
&= \|I^{\wp,\rho}\Theta(\chi, U_1) - I^{\wp,\rho}\Theta(\chi, U_2)\|_{C_{\wp,\rho}[a,b]} \\
&= \| I^{\wp,\rho}[\Theta(\chi, U_1) - \Theta(\chi, U_2)] \|_{C_{\wp,\rho}[a,b]} \\
&\leq \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\wp} \frac{\Gamma(\rho)}{\Gamma(\rho + \wp)} \times \|\Theta(\chi, U_1) - \Theta(\chi, U_2)\|_{C_{\wp,\rho}[a,b]} \\
&\leq \varepsilon \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\wp} \frac{\Gamma(\rho)}{\Gamma(\rho + \wp)} \times \|U_1 - U_2\|_{C_{\wp,\rho}[a,b]}.
\end{aligned}$$

Thus, we obtain

$$\|(TU_1)(\chi) - (TU_2)(\chi)\|_{C_{\wp,\rho}[a,b]} > 0.$$

Now, we achieve the condition of Theorem 14.6.

$$
\begin{aligned}
& \sup_{U_1,U_2\in C_{\wp,\rho}[a,b]} \|(TU_1)(\chi)-(TU_2)(\chi)\|_{C_{\wp,\rho}[a,b]} 0 \Longrightarrow \\
& F(\sup_{U_1,U_2\in C_{\wp,\rho}[a,b]} \|(TU_1)(\chi)-(TU_2)(\chi)\|_{C_{\wp,\rho}[a,b]} \\
& +\varphi(\sup_{U_1,U_2\in C_{\wp,\rho}[a,b]} \|(TU_1)(\chi)-(TU_2)(\chi)\|_{C_{\wp,\rho}[a,b]})) \\
= & \sup_{U_1,U_2\in C_{\wp,\rho}[a,b]} F(\|(TU_1)(\chi)-(TU_2)(\chi)\|_{C_{\wp,\rho}[a,b]} \\
& +\varphi(\|(TU_1)(\chi)-(TU_2)(\chi)\|_{C_{\wp,\rho}[a,b]})) \\
\leq & \sup_{U_1,U_2\in C_{\wp,\rho}[a,b]} [F(\|U_1-U_2\|_{C_{\wp,\rho}[a,b]} + \varphi(\|U_1-U_2\|_{C_{\wp,\rho}[a,b]})) \\
& +G(\beta(\|U_1-U_2\|_{C_{\wp,\rho}[a,b]} + \varphi(\|U_1-U_2\|_{C_{\wp,\rho}[a,b]})))] \\
\leq & F(\sup_{U_1,U_2\in C_{\wp,\rho}[a,b]} \|U_1-U_2\|_{C_{\wp,\rho}[a,b]} + \varphi(\sup_{U_1,U_2\in C_{\wp,\rho}[a,b]} \|U_1-U_2\|_{C_{\wp,\rho}[a,b]})) \\
& +G(\beta(\sup_{U_1,U_2\in C_{\wp,\rho}[a,b]} \|U_1-U_2\| + \varphi(\sup_{U_1,U_2\in C_{\wp,\rho}[a,b]} \|U_1-U_2\|))).
\end{aligned}
$$

Hence this implies that T has a unique fixed point corresponding to the solution of the problems (14.32)–(14.34) and maximizing the utility function U.

Denoted by

$$
\omega := \varepsilon \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\wp} \frac{\Gamma(\rho)}{\Gamma(\rho+\wp)},
$$

we have Table 14.2.

Theorem 14.8 maximizes the utility function for one item. In other words, the set of goods contains one unit. The next result describes the set for n-items; in this case, we shall apply Theorem 14.6. The Stone–Geary utility function can be generated by using the integral operator $I^{\wp,\rho}$ as follows:

$$
U(\chi_1,\ldots,\chi_n) = U_0(\wp,\rho) + \frac{1}{\Gamma(\wp)} \prod_i^n \int_a^{\chi_i} \left(\frac{\chi_i^\rho - \tau_i^\rho}{\rho}\right)^{\wp-1} \tau_i^{\rho-1} \Theta_i(\tau,U)\, d\tau_i, \tag{14.35}
$$

Table 14.2 The correlation between the fractional parameters and ε in the interval [a, b] to achieve $\omega < 1$

$(\wp, \rho)$	ε	$\omega < 1$	[a, b]
(0.5, 0.5)	0.1	0.7	[0, 1]
(0.5, 0.5)	0.2	0.5	[0, 1]
(0.5, 0.5)	0.3	0.7	[0, 1]
(0.5, 1.0)	0.4	0.45	[0, 1]
(0.75, 1.5)	0.5	0.28	[0, 1]

where U is utility, $\chi_i \in [a, b]$ is consumption of good i, and $\wp$ and ρ are fractional parameters. Obviously, $U(\chi_1, \ldots, \chi_n) \in C_{\wp,\rho}[a, b]^n$ whenever $\Theta \in C_{\wp,\rho}[a, b]^n$. Assume that $\Theta_i(., U)$ is Lipschitz with a Lipschitz constant $\varepsilon_i > 0$. The optimal problem is to maximize the utility function U. We consider the problem

$$\max_{\chi_i} U(\chi_1, \ldots, \chi_n) := G(\beta(\chi_1, \ldots, \chi_n)), \quad \forall G \in \Delta_{G,\beta}. \tag{14.36}$$

Moreover, we assume that φ is the accumulation function of the utility function (the future value of U) satisfying a new combination $U + \varphi(U)$. Define the optimal problem of $U + \varphi(U)$ as follows:

$$\max_{\chi_i}[U(\chi_1, \ldots, \chi_n) + \varphi(U(\chi_1, \ldots, \chi_n))] := F(U(\chi_1, \ldots, \chi_n) + \varphi(U(\chi_1, \ldots, \chi_n))), \tag{14.37}$$

where $F \in \Delta_F$. We have the following result, which can be proved by applying Theorem 14.6 by letting

$$F(U(\chi_1, \ldots, \chi_n) + \varphi(U(\chi_1, \ldots, \chi_n))) = U(\chi_1, \ldots, \chi_n) + \varphi(U(\chi_1, \ldots, \chi_n)).$$

Theorem 14.9 *Consider the problems* (14.35)–(14.37). *If, for all* $\wp \in [0, 1)$ *and* $\rho > 0$,

$$\prod_{i=1}^{n} \varepsilon_i \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\wp} \frac{\Gamma(\rho)}{\Gamma(\rho + \wp)} < 1, \quad \forall \varepsilon_i > 0,$$

then there is a unique solution maximizing the problems (14.32)–(14.34).

14.7 Conclusion

In this chapter, we have proposed the notation of μ-set contractive mappings for two classes of functions involving a measure of noncompactness in Banach space, and proved Darbo-type fixed point and n-tupled fixed point results. Our work improved and generalized the results existing in the literature. In the end, we have applied our results to two different Volterra integral equations in Banach algebras, followed by an example.

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Chapter 15
Approximating Fixed Points of Suzuki (α, β)-Nonexpansive Mappings in Ordered Hyperbolic Metric Spaces

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Abstract In this chapter, we define the class of monotone (α, β)-nonexpansive mappings and prove that they have an approximate fixed point sequence in partially ordered hyperbolic metric spaces. We prove the Δ and strong convergence of the CR-iteration scheme.

15.1 Introduction and Preliminaries

In 2004, Kohlenbach [1] introduced hyperbolic metric spaces. Busemann spaces [2] are the well-known examples of hyperbolic metric spaces. Leauştean [3] showed that CAT(0) spaces are uniformly convex hyperbolic metric spaces. Recently, Bin

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Dehaish and Khamsi [4] obtained a fixed point theorem for a monotone nonexpansive mapping in the setting of partially ordered hyperbolic metric spaces.

On the other hand, to generalize nonexpansive mappings, Aoyama and Kohsaka [5] introduced a new class of nonexpansive mappings, namely, α-*nonexpansive mappings*, and obtained a fixed point theorem for such mappings.

Definition 15.1 ([5]) Let K be a nonempty subset of a Banach space M. A mapping $T : K \to K$ is said to be α-*nonexpansive* if, for all $u, v \in K$ and $\alpha \in [0, 1)$,

$$\|T(u) - T(v)\|^2 \leq \alpha\|T(u) - v\|^2 + \alpha\|u - T(v)\|^2 + (1 - 2\alpha)\|u - v\|^2. \quad (15.1)$$

Theorem 15.1 ([5]) *Let K be a nonempty closed and convex subset of a uniformly convex Banach space M and $T : K \to K$ be an α-nonexpansive mapping. Then $F(T)$ is nonempty if and only if there exists $u \in K$ such that $\{T^n(u)\}$ is bounded, where $F(T)$ denotes the set of fixed points of the mapping T.*

This class of mappings is recently extended to the class of (α, β)-nonexpansive mappings, which is defined by Amini-Harandi et al. [6].

Definition 15.2 ([6]) Let K be a nonempty subset of a Banach space M. A mapping $T : K \to K$ is said to be α-*nonexpansive* if, for all $u, v \in K$ and $\alpha, \beta \in [0, 1)$,

$$\begin{aligned} \|T(u) - T(v)\|^2 \leq\; & \alpha\|T(u) - v\|^2 + \alpha\|u - T(v)\|^2 \\ & + \beta\|T(u) - u\|^2 + \beta\|v - T(v)\|^2 + (1 - 2\alpha - 2\beta)\|u - v\|^2. \end{aligned} \quad (15.2)$$

Remark 15.1 We note that an (α, β)-nonexpansive mapping reduces to α-nonexpansive mapping when $\beta = 0$ and to a nonexpansive mapping when $\alpha = \beta = 0$.

On the other hand, to generalize nonexpansive mappings, Suzuki [7] introduced the following new class of mappings and obtained some existence and convergence results:

Definition 15.3 ([7]) Let E be a Banach space and K a nonempty subset of E. A mapping $T : K \to K$ is said to satisfy the *condition* (C) if, for all $u, v \in K$,

$$\frac{1}{2}\|u - T(u)\| \leq \|u - v\| \quad \text{implies} \quad \|T(u) - T(v)\| \leq \|u - v\|.$$

Let $(\mathscr{M}, d, \preceq)$ be a metric space with the metric d and the partial order $\preceq$. The following two definitions are due to Kohlenbach [1].

Definition 15.4 A triplet $(\mathscr{M}, d, W)$ is called a *hyperbolic metric space* if $(\mathscr{M}, d)$ is a metric space and $W : \mathscr{M} \times \mathscr{M} \times [0, 1] \to \mathscr{M}$ is a function satisfying the following conditions: for all $u, v, z, w \in \mathscr{M}$ and $\beta, \gamma \in [0, 1]$,

(H1) $d(z, W(u, v, \beta)) \leq (1 - \beta)d(z, u) + \beta d(z, v)$;
(H2) $d(W(u, v, \beta), W(u, v, \gamma)) = |\beta - \gamma|d(u, v)$;
(H3) $W(u, v, \beta) = W(v, u, 1 - \beta)$;
(H4) $d(W(u, z, \beta), W(v, w, \beta) \leq (1 - \beta)d(u, v) + \beta d(z, w)$.

Definition 15.5 Let $(\mathscr{M}, d, W)$ be a hyperbolic metric space. The set

$$\text{seg}[u, v] := \{W(u, v, \beta) : \beta \in [0, 1]\}$$

is called the *metric segment* with the endpoints u, v.

Remark 15.2 If only the condition (H1) is satisfied, then $(\mathscr{M}, d, W)$ is a *convex metric space* in the sense of Takahasi [8]. The conditions (H1)–(H3) are equivalent to $(\mathscr{M}, d, W)$ being a space of hyperbolic type in the sense of Goebel and Kirk [9]. The condition (H4) was considered by Itoh [10] and later used in [11] (with the restriction on β, i.e., $\beta = 1/2$) to define the class of hyperbolic metric spaces. The condition (H3) ensures that seg$[u, v]$ is an isometric image of the real line segment $[0, d(u, v)]$.

Throughout this paper, $W(u, v, \beta)$ is fixed as

$$W(u, v, \beta) := (1 - \beta)u \oplus \beta v.$$

We say that a subset $\mathscr{K}$ of $\mathscr{M}$ is *convex* if, for all $u, v \in \mathscr{K}$, $(1 - \beta)u \oplus \beta v \in \mathscr{K}$ for all $\beta \in [0, 1]$. We use $(\mathscr{M}, d)$ for $(\mathscr{M}, d, W)$ when there is no ambiguity. All normed linear spaces and Hilbert balls equipped with the hyperbolic metric are some examples of hyperbolic metric spaces [12].

Throughout, we assume that order intervals are closed and convex subsets of a hyperbolic metric space $(\mathscr{M}, d)$. We denote these as follows:

$$[a, \rightarrow) := \{u \in \mathscr{M}; a \preceq u\} \text{ and } (\leftarrow, b] := \{u \in \mathscr{M}; u \preceq b\}$$

for any $a, b \in \mathscr{M}$ (cf. [4]).

Definition 15.6 ([13, 14]) Let $(\mathscr{M}, d)$ be a hyperbolic metric space. For any $r > 0$ and $\varepsilon > 0$, set

$$\delta(r, \varepsilon) = \inf\left\{1 - \frac{1}{r}d\left(\frac{1}{2}u \oplus \frac{1}{2}v, a\right) : \ d(u, a) \le r, \ d(v, a) \le r, \ d(u, v) \ge r\varepsilon\right\}$$

for any $a \in \mathscr{M}$. We say that $\mathscr{M}$ is *uniformly convex* if $\delta(r, \varepsilon) > 0$ for any $r > 0$ and $\varepsilon > 0$.

Definition 15.7 ([15]) A metric space $(\mathscr{M}, d)$ is said to satisfy the *property* (R) if $\{C_n\}$ is a decreasing sequence of nonempty bounded convex and closed subsets of $\mathscr{M}$, then $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$.

Uniformly convex hyperbolic spaces enjoy the property (R) [4].

Let $\mathscr{K}$ be a nonempty subset of a hyperbolic metric space $(\mathscr{M}, d)$ and $\{u_n\}$ a bounded sequence in $\mathscr{M}$. For all $u \in \mathscr{M}$, define the following:

(1) The *asymptotic radius* of $\{u_n\}$ at u as $r(\{u_n\}, u) := \limsup_{n\to\infty} d(u_n, u)$.

(2) The *asymptotic radius* of $\{u_n\}$ relative to $\mathscr{K}$ as

$$r(\{u_n\}, \mathscr{K}) := \inf\{r(\{u_n\}, u) : u \in \mathscr{K}\}.$$

(3) The *asymptotic center* of $\{u_n\}$ relative to $\mathscr{K}$ by

$$A(\mathscr{K}, \{u_n\}) := \{u \in \mathscr{K} : r(u, \{u_n\}) = r(\{u_n\}, \mathscr{K})\}.$$

Lim [16] introduced the concept of Δ-convergence in metric spaces. Kirk and Panyanak [17] used Lim's concept to CAT(0) spaces and showed that many Banach spaces results involving weak convergence have precise analogs in this setting.

Definition 15.8 ([17]) A bounded sequence $\{u_n\}$ in $\mathscr{M}$ is said to Δ-converge to a point $u \in \mathscr{M}$ if u is the unique asymptotic center of every subsequence $\{u_{n_k}\}$ of $\{u_n\}$.

Definition 15.9 ([4]) Let $\mathscr{K}$ be a nonempty subset of a hyperbolic metric space $(\mathscr{M}, d)$. A function $\tau : \mathscr{K} \to [0, \infty)$ is called a *type function* if there exists a bounded sequence $\{u_n\}$ in $\mathscr{M}$ such that

$$\tau(u) = \limsup_{n\to\infty} d(u_n, u)$$

for any $u \in \mathscr{K}$.

Remark 15.3 We note that every bounded sequence generates an unique type function.

Now, we rephrase the concept of Δ-convergence in hyperbolic metric spaces.

Definition 15.10 A bounded sequence $\{u_n\}$ in $\mathscr{M}$ is said to Δ-*convergent* to a point $z \in \mathscr{M}$ if z is the unique point and a type function generated by every subsequence $\{u_{n_k}\}$ of $\{u_n\}$ attains its infimum at z.

Definition 15.11 ([18]) Let $\mathscr{K}$ be a subset of a metric space $(\mathscr{M}, d)$. A mapping $T : \mathscr{K} \to \mathscr{K}$ is said to satisfy the *condition* (I) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $d(u, T(u)) \geq f(D(u, F(T)))$ for all $u \in \mathscr{K}$, where $D(u, F(T))$ denotes the distance from u to $F(T)$.

15.2 Existence Results on Picard Iterations

First, we recall the following definitions and preliminary results:

Definition 15.12 ([4]) Let $(\mathscr{M}, d, \preceq)$ be a partially ordered metric space and $T : \mathscr{M} \to \mathscr{M}$ a mapping. The mapping T is said to be *monotone* if, for all $u, v \in \mathscr{M}$,

$$u \preceq v \text{ implies } T(u) \preceq T(v).$$

Definition 15.13 ([4]) Let $(\mathscr{M}, d, \preceq)$ be a partially ordered metric space and $T : \mathscr{M} \to \mathscr{M}$ be a mapping. The mapping T is said to be *monotone nonexpansive* if T is monotone and

$$d(T(u), T(v)) \leq d(u, v) \tag{15.3}$$

for all $u, v \in \mathscr{M}$ such that u and v are comparable.

We extend Definition 15.2 from Banach spaces to hyperbolic metric spaces as follows:

Definition 15.14 Let $(\mathscr{M}, d, \preceq)$ be a partially ordered metric space and $T : \mathscr{M} \to \mathscr{M}$ be a mapping. The mapping T is said to be *monotone* (α, β)-*nonexpansive* if T is monotone and there exist $\alpha, \beta \in [0, 1)$ such that

$$\begin{aligned} d(T(u), T(v))^2 \leq\ & \alpha d(T(u), v)^2 + \alpha d(u, T(v))^2 \\ & + \beta d(T(u), u)^2 + \beta d(v, T(v))^2 + (1 - 2\alpha - 2\beta) d(u, v)^2 \end{aligned} \tag{15.4}$$

for all $u, v \in \mathscr{M}$ such that u and v are comparable.

Moreover, we can introduce a new class by combining Definitions 15.4 and 15.2 as follows:

Definition 15.15 Let $(\mathscr{M}, d, \preceq)$ be a partially ordered metric space and $T : \mathscr{M} \to \mathscr{M}$ be a mapping. The mapping T is said to be *monotone Suzuki* (α, β)-*nonexpansive* if T is monotone and there exist $\alpha, \beta \in [0, 1)$ such that, if

$$\frac{1}{2} d(u, T(u)) \leq d(u, v),$$

then the condition (15.4) holds for all $u, v \in \mathscr{M}$ such that u and v are comparable.

If $\beta = 0$, then (α, β)-nonexpansive definition reduces to the concept of α-nonexpansive defined in [19, 20]. A $(0, 0)$-nonexpansive mapping is a monotone nonexpansive. The Suzuki case is introduced in [21]. An α-nonexpansive mapping T with a fixed point $w \in \mathscr{K}$ is quasi-nonexpansive, that is, $d(T(u), w) \leq d(u, w)$ for all $u \in \mathscr{K}$ and $w \in F(T)$ such that u and w are comparable. It may be completed following the proof of Proposition 2 [7].

We remark that the above proposition is not valid in general for $\beta \neq 0$. An example is presented in [6].

Lemma 15.1 ([4]) *Let $(\mathscr{M}, d)$ be a uniformly convex hyperbolic metric space and $\mathscr{K}$ be a nonempty closed and convex subset of $\mathscr{M}$. Let $\tau : \mathscr{K} \to [0, \infty)$ be a type function. Then τ is continuous. Moreover, there exists a unique minimum point $z \in \mathscr{K}$ such that*

$$\tau(z) = \inf\{\tau(u) : u \in \mathscr{K}\}.$$

Now, we present some existence results on a partially ordered hyperbolic metric space. For more details on ordered metric spaces and applications, one may refer to [22, 23].

Although α-nonexpansive mappings are defined for any real number $\alpha < 1$, as Ariza-Ruiz *et al.* [24] pointed out that this concept is trivial for $\alpha < 0$. From now on, we assume that $\alpha, \beta \in [0, 1)$.

Now, we present our first existence result which is a generalization of [4, Theorem 3.1] and [19, Theorem 3.5].

Theorem 15.2 *Let $(\mathscr{M}, d, \preceq)$ be a uniformly convex partially ordered hyperbolic metric space and $\mathscr{K}$ be a nonempty bounded closed and convex subset of $\mathscr{M}$ not reduced to one point. Let $T : \mathscr{K} \to \mathscr{K}$ be a monotone Suzuki (α, β)-nonexpansive mapping. Assume that there exists $u \in \mathscr{K}$ such that u and $T(u)$ are comparable. Then T has a fixed point.*

Proof Without loss of generality, we may assume that $u \preceq T(u)$. Since T is monotone, we get $T(u) \preceq T^2(u)$. Continuing in this way, we get

$$T(u) \preceq T^2(u) \preceq T^3(u) \preceq T^4(u) \preceq \cdots .$$

Define $u_n = T^n(u)$ for all $n \in \mathbb{N}$. Since $\mathscr{M}$ is uniformly convex, it satisfies the property (R) and, by the construction of $\{u_n\}$, we have

$$\mathscr{K}_\infty = \bigcap_{n=1}^{\infty}[u_n, \to) \bigcap \mathscr{K} = \bigcap_{n=1}^{\infty}\{u \in \mathscr{K}; u_n \preceq u\} \neq \varnothing.$$

Let $u \in \mathscr{K}_\infty$. Then $u_n \preceq u$. Since T is monotone, we have $u_n \preceq T(u_n) \preceq T(u)$ for all $n \in \mathbb{N}$. This implies that $T(\mathscr{K}_\infty) \subset \mathscr{K}_\infty$. Let $\tau : \mathscr{K}_\infty \to [0, \infty)$ be the type function generated by $\{u_n\}$, that is,

$$\tau(u) = \limsup_{n\to\infty} d(u_n, u).$$

From Lemma 15.1, it follows that there exists a unique element $w \in \mathscr{K}_\infty$ such that

$$\tau(w) = \inf\{\tau(u) : u \in \mathscr{K}_\infty\}.$$

Since $w \in \mathscr{K}_\infty$, $u_n \preceq w$ for all $n \in \mathbb{N}$. If $u_n = u_{n+1}$, then $d(u_n, u_{n+1}) \leq d(u_n, w)$ for all $n \in \mathbb{N}$. Again, if $u_n \prec u_{n+1}$, then $u_n \prec u_{n+1} \preceq w$. Thus, in the both cases, we have $d(u_n, u_{n+1}) \leq d(u_n, w)$ for all $n \in \mathbb{N}$ and so

$$\frac{1}{2}d(u_n, T(u_n)) \leq d(u_n, w).$$

In the generalized case, since Definition 15.15,

$$d(T(u_n), T(w))^2 \leq \alpha d(T(u_n), w)^2 + \alpha d(u_n, T(w))^2 \\ +\beta d(T(u_n), u_n)^2 + \beta d(w, T(w))^2 + (1 - 2\alpha - 2\beta) d(u_n, w)^2. \tag{15.5}$$

This equation is also valid in the not generalized case by the definition of the (α, β)-nonexpansive mapping.

Let $Tw \preceq w$. Since $d(u_n, u_{n+1}) \leq d(u_n, w)$ and $d(w, T(w)) \leq d(u_n, T(w))$, we have

$$\begin{aligned} d(u_{n+1}, T(w))^2 &= d(T(u_n), T(w))^2 \\ &\leq \alpha d(u_{n+1}, w)^2 + \alpha d(u_n, T(w))^2 + (1 - 2\alpha) d(u_n, w)^2. \end{aligned}$$

Taking $n \to \infty$, we have

$$\limsup_{n\to\infty} d(u_{n+1}, T(w))^2 \leq \alpha \limsup_{n\to\infty} d(u_{n+1}, w)^2 + \alpha \limsup_{n\to\infty} d(u_n, T(w))^2 \\ +(1 - 2\alpha) \limsup_{n\to\infty} d(u_n, w)^2$$

or

$$\limsup_{n\to\infty} d(u_n, T(w))^2 \leq \limsup_{n\to\infty} d(u_n, w)^2.$$

Thus we have

$$\limsup_{n\to\infty} d(u_n, T(w)) \leq \limsup_{n\to\infty} d(u_n, w).$$

Let $w \preceq Tw$. Since $d(u_n, u_{n+1}) \leq d(u_n, w)$ and $d(w, T(w)) \leq d(u_n, T(w))$, we have

$$d(u_{n+1}, T(w))^2 = d(T(u_n), T(w))^2 \leq \alpha d(u_{n+1}, w)^2 + (\alpha + \beta) d(u_n, T(w))^2 \\ +(1 - 2\alpha - \beta) d(u_n, w)^2.$$

Taking $n \to \infty$, we have

$$\limsup_{n\to\infty} d(u_{n+1}, T(w))^2 \leq \alpha \limsup_{n\to\infty} d(u_{n+1}, w)^2 + (\alpha + \beta) \limsup_{n\to\infty} d(u_n, T(w))^2 \\ +(1 - 2\alpha - \beta) \limsup_{n\to\infty} d(u_n, w)^2$$

or

$$\limsup_{n\to\infty} d(u_n, T(w))^2 \leq \limsup_{n\to\infty} d(u_n, w)^2.$$

Thus we have

$$\limsup_{n\to\infty} d(u_n, T(w)) \leq \limsup_{n\to\infty} d(u_n, w).$$

Since $\tau(w) = \inf\{\tau(u); u \in \mathscr{K}_\infty\}$, the uniqueness of minimum point, it follows that $T(w) = w$, that is, w is a fixed point of T. This completes the proof.

The following result is slightly different than Theorem 15.2. In this result instead of taking the domain of T a bounded set, the sequence of iterates at a point is considered as bounded and the proof is similar to Theorem 3.7 in [19].

Theorem 15.3 *Let $(\mathscr{M}, d, \preceq)$ be a complete uniformly convex partially ordered hyperbolic metric space and $\mathscr{K}$ be a nonempty bounded convex and closed subset of $\mathscr{M}$ not reduced to one point. Let $T : \mathscr{K} \to \mathscr{K}$ be a monotone Suzuki (α, β)-nonexpansive mapping. Assume that there exists $u \in \mathscr{K}$ such that u and $T(u)$ are comparable. Then $F(T)$ is nonempty if and only if $\{T^n(u)\}$ is a bounded sequence and there exists a point $v \in \mathscr{K}$ such that every point of sequence $\{u_n\}$ are comparable with v.*

15.3 Existence Results on the CR-Iteration

In 2012, Chugh et al. [25] introduced the following iterative process:

$$\begin{cases} u_1 \in \mathscr{K}, \\ v_n = \gamma_n T(u_n) \oplus (1 - \gamma_n) u_n, \\ w_n = \beta_n T(v_n) \oplus (1 - \beta_n) T(u_n), \\ u_{n+1} = \alpha_n T(w_n) \oplus (1 - \alpha_n) w_n \end{cases} \tag{15.6}$$

for each $n \in \mathbb{N}$, where $\{\alpha\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the real sequences in $[0, 1]$. It is called the *CR-iteration*. If we take $\alpha_n = 0$, the CR-iterative process reduces to the S-iteration [26].

Lemma 15.2 *Let $(\mathscr{M}, d, \preceq)$ be a partially ordered hyperbolic metric space and $\mathscr{K}$ be a nonempty closed and convex subset of $\mathscr{M}$. Let $T : \mathscr{K} \to \mathscr{K}$ be a monotone mapping. Let $u_1 \in \mathscr{K}$ such that $u_1 \preceq T(u_1)$ (or $T(u_1) \preceq u_1$). Then the sequence $\{u_n\}$ defined by (15.6), we have the following:*

(1) $u_n \preceq T(u_n) \preceq u_{n+1}$ (or $u_{n+1} \preceq T(u_n) \preceq u_n$) for each $n \in \mathbb{N}$.

(2) $u_n \preceq p$ (or $p \preceq u_n$) provided $\{u_n\}$ Δ-converges to a point $p \in \mathscr{K}$ for each $n \in \mathbb{N}$.

Proof (1) By induction, we prove our first result. By the assumption, we have $u_1 \preceq T(u_1)$ and, by the convexity of the ordered interval $[u_1, T(u_1)]$ and (15.6), we have

$$u_1 \preceq v_1 \preceq T(u_1). \tag{15.7}$$

Since T is monotone, we have $T(u_1) \preceq T(v_1)$ and, by the convexity of ordered interval $[T(u_1), T(v_1)]$ and (15.6), we have

$$T(u_1) \preceq w_1 \preceq T(v_1). \tag{15.8}$$

Combining (15.7) and (15.8), we get

$$u_1 \preceq v_1 \preceq T(u_1) \preceq w_1.$$

Since T is monotone, we have $w_1 \preceq T(v_1) \preceq T(w_1)$ and, by the convexity of ordered interval $[w_1, T(w_1)]$ and (15.6), we have

$$w_1 \preceq u_2 \preceq T(w_1). \tag{15.9}$$

Combining (15.7), (15.8) and (15.9), we get

$$u_1 \preceq v_1 \preceq T(u_1) \preceq w_1 \preceq u_2 \preceq T(w_1).$$

Thus the result is true for $n = 1$. Similarly, for $n = k - 1$, we have

$$u_{k-1} \preceq T(u_{k-1}) \preceq u_k$$

and so, by induction, for each $n \in \mathbb{N}$,

$$u_n \preceq v_n \preceq T(u_n) \preceq w_n \preceq u_{n+1} \preceq T(w_n), \quad w_n \preceq T(v_n) \preceq T(w_n). \tag{15.10}$$

(2) Suppose tht p is a Δ-limit of $\{u_n\}$. Here the sequence $\{u_n\}$ is monotone increasing and the order interval $[u_m, \rightarrow)$ is closed and convex. Now, we claim that $p \in [u_m, \rightarrow)$ for a fixed $m \in \mathbb{N}$. If $p \notin [u_m, \rightarrow)$, then the type function generated by the subsequence $\{u_r\}$ of $\{u_n\}$ defined by leaving first $m - 1$ terms of sequence $\{u_n\}$ will not attain an infimum at p, which is a contradiction to the assumption that p is a Δ-limit of the sequence $\{u_n\}$. This completes the proof.

Now, we give a main result in this section

Theorem 15.4 *Let $(\mathscr{M}, d, \preceq)$ be a uniformly convex partially ordered hyperbolic metric space and $\mathscr{K}$ be a nonempty convex and closed subset of $\mathscr{M}$. Let $T : \mathscr{K} \to \mathscr{K}$ be a monotone Suzuki (α, β)-nonexpansive mapping. Assume that there exists $u_1 \in \mathscr{K}$ such that u_1 and $T(u_1)$ are comparable. Let a sequence $\{u_n\}$ generated by (15.6) is bounded and suppose that there exists a point $v \in \mathscr{K}$ such that every point of the sequence $\{u_n\}$ is comparable with v and*

$$\lim_{n\to\infty} \inf d(T(u_n), u_n) = 0.$$

Then T has a fixed point.

Proof Suppose that $\{u_n\}$ is a bounded sequence and $\lim\limits_{n\to\infty} \inf d(T(u_n), u_n) = 0$. Then there exist a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that

$$\lim_{n\to\infty} d(T(u_{n_j}), u_{n_j}) = 0. \tag{15.11}$$

Since u_{n_j} is bounded, there exist $u' \in \mathscr{K}$ and $U \in \mathbb{R}$ such that $d(u_{n_j}, u') \leq U$. Therefore, we have

$$d(T(u_{n_j}), u') \leq d(T(u_{n_j}), u_{n_j}) + d(u_{n_j}, u').$$

Thus $\{T(u_{n_j})\}$ is bounded. By Lemma 15.2, we have $u_1 \preceq u_{n_j} \preceq u_{n_{j+1}}$. Define

$$\mathscr{K}_j = \{u \in \mathscr{K} : u_{n_j} \preceq u\}$$

for all $j \in \mathbb{N}$. Clearly, for each $j \in \mathbb{N}$, $\mathscr{K}_j$ is a closed and convex. Since $v \in \mathscr{K}_j$, it follows that $\mathscr{K}_j$ is nonempty. Let

$$\mathscr{K}_\infty = \bigcap_{j=1}^{\infty} \{u \in \mathscr{K} : u_{n_j} \preceq u\} \neq \varnothing$$

be a closed convex subset of $\mathscr{K}$. Let $u \in \mathscr{K}_\infty$. Then $u_{n_j} \preceq u$ for each $j \in \mathbb{N}$. Since T is monotone, it follows that, for each $j \in \mathbb{N}$,

$$u_{n_j} \preceq T(u_{n_j}) \preceq T(u).$$

This implies that $T(\mathscr{K}_\infty) \subset \mathscr{K}_\infty$. Let $\tau : \mathscr{K}_\infty \to [0, \infty)$ be the type function generated by $\{T(u_{n_j})\}$, that is,

$$\tau(u) = \limsup_{j\to\infty} d(T(u_{n_j}), u).$$

From Lemma 15.1, it follows that there exists a unique element $w \in \mathscr{K}_\infty$ such that

$$\tau(w) = \inf\{\tau(u); u \in \mathscr{K}_\infty\}.$$

By the definition of the type function, we have

$$\tau(T(w)) = \limsup_{j\to\infty} d(T(u_{n_j}), T(w)).$$

By the triangle inequality and (15.11), we have

$$\begin{aligned}
\limsup_{j\to\infty} d(T(u_{n_j}), u) &\leq \limsup_{j\to\infty} d(T(u_{n_j}), u_{n_j}) + \limsup_{j\to\infty} d(u_{n_j}, u) \\
&= \limsup_{j\to\infty} d(u_{n_j}, u).
\end{aligned}$$

Similarly, we have

$$\limsup_{j\to\infty} d(u_{n_j}, u) \leq \limsup_{j\to\infty} d(T(u_{n_j}), u).$$

Therefore, we have

$$\limsup_{j\to\infty} d(u_{n_j}, u) = \limsup_{j\to\infty} d(T(u_{n_j}), u). \tag{15.12}$$

Since $w \in \mathscr{K}_\infty$, we have $u_{n_j} \preceq w$ for each $j \in \mathbb{N}$. It follows from the monotonicity of T and Lemma 15.2 that $u_{n_j} \preceq T(u_{n_j}) \preceq w$ for each $j \in \mathbb{N}$. Then we have

$$d(u_{n_j}, T(u_{n_j})) \leq d(u_{n_j}, w)$$

for all $j \in \mathbb{N}$. By (15.3), for each $j \in \mathbb{N}$, we have

$$\begin{aligned} d(T(u_{n_j}), T(w))^2 \leq\ & \alpha d(T(u_{n_j}), w)^2 + \alpha d(u_{n_j}, T(w))^2 \\ & + \beta d(T(u_{n_j}), u_{n_j})^2 + \beta d(w, T(w))^2 \\ & + (1 - 2\alpha - 2\beta) d(u_{n_j}, w)^2. \end{aligned} \tag{15.13}$$

Letting $a_j := d(T(u_{n_j}), T(w))$, $b_j := d(u_{n_j}, T(u_{n_j}))$ and $c_j := d(u_{n_j}, w)$ for each $j \in \mathbb{N}$. Then, for all $j \in \mathbb{N}$, we have

$$d(u_{n_j}, T(w))^2 = \left(a_j + b_j\right)^2 = a_j^2 + b_j^2 + 2a_j b_j$$

and

$$d(w, T(w))^2 = \left(a_j + b_j + c_j\right)^2 = a_j^2 + b_j^2 + c_j^2 + 2a_j b_j + 2a_j c_j + 2b_j c_j.$$

Thus, by the triangle inequality, it follows that (15.13) reduces to

$$\begin{aligned} (1 - \alpha - \beta) a_j^2 \leq\ & \alpha d(T(u_{n_j}), w)^2 + (\alpha + 2\beta) b_j^2 + (1 - 2\alpha - \beta) c_j^2 \\ & + 2(\alpha + \beta) a_j b_j + 2\beta a_j c_j + 2\beta b_j c_j. \end{aligned}$$

Using (15.11), we get

$$\begin{aligned} (1 - \alpha - \beta) \limsup_{j\to\infty} a_j^2 \leq\ & \alpha \limsup_{j\to\infty} d(T(u_{n_j}), w)^2 + (1 - 2\alpha - \beta) \limsup_{j\to\infty} c_j^2 \\ & + 2\beta \limsup_{j\to\infty} a_j c_j. \end{aligned}$$

By (15.12), we have

$$\limsup_{j\to\infty} d(T(u_{n_j}), T(w))^2 \leq \limsup_{j\to\infty} d(T(u_{n_j}), w)^2.$$

This implies that

$$\limsup_{j\to\infty} d(T(u_{n_j}), T(w)) \leq \limsup_{j\to\infty} d(T(u_{n_j}), w).$$

Therefore, we have $\tau(T(w)) \leq \tau(w)$ and, further, by the uniqueness of the minimum point, $T(w) = w$. This completes the proof.

15.4 Convergence Results

In this section, we discuss some convergence results for CR-iteration process in partially ordered hyperbolic metric spaces.

Lemma 15.3 ([27]) *Let $(\mathscr{M}, d)$ be a uniformly convex hyperbolic metric space with monotone modulus of the uniform convexity δ. Let $w \in \mathscr{M}$ and $\{\alpha_n\}$ be a sequence such that $0 < a \leq \alpha_n \leq b < 1$ for each $j \in \mathbb{N}$. If $\{u_n\}$ and $\{v_n\}$ are the sequence in $\mathscr{M}$ such that*

$$\limsup_{n\to\infty} d(u_n, w) \leq r, \quad \limsup_{n\to\infty} d(v_n, w) \leq r$$

and

$$\lim_{n\to\infty} d(\alpha_n v_n \oplus (1-\alpha_n)u_n, w) = r$$

for some $r \geq 0$, then we have $\lim_{n\to\infty} d(v_n, u_n) = 0$.

Theorem 15.5 *Let $(\mathscr{M}, d, \preceq)$ be a uniformly convex partially ordered hyperbolic metric space and $\mathscr{K}$ a nonempty convex and closed subset of $\mathscr{M}$. Let $T : \mathscr{K} \to \mathscr{K}$ be a monotone Suzuki (α, β)-nonexpansive mapping. Assume that there exists $u_1 \in \mathscr{K}$ such that u_1 and $T(u_1)$ are comparable. Suppose that $F(T)$ is nonempty and u_1 and w are comparable for every $w \in F(T)$. Let $\{u_n\}$ be the sequence defined by (15.6). Then following assertions hold:*

(1) *the sequence $\{u_n\}$ is bounded.*
(2) $\max\{d(u_{n+1}, w), d(v_n, w), d(w_n, w)\} \leq d(u_n, w)$ *for each $n \in \mathbb{N}$.*
(3) $\lim_{n\to\infty} d(u_n, w)$ *and* $\lim_{n\to\infty} D(u_n, F(T))$ *exist.*
(4) $\lim_{n\to\infty} d(T(u_n), u_n) = 0$.

Proof Without loss of generality, we may assume that $u_1 \preceq w$. Then, by the monotonicity of T, $T(u_1) \preceq T(w) = w$. By (15.10), we have $v_1 \preceq T(u_1) \preceq T(w) = w$. By the monotonicity of T and (15.10), $w_1 \preceq T(v_1) \preceq T(w) = w$. Since T is monotone, we have $T(w_1) \preceq T(w) = w$. Then, again from (15.10), we have

$$u_2 \preceq T(w_1) \preceq w.$$

Continuing in this way, we get

$$u_n \preceq w$$

for each $n \in \mathbb{N}$. By (15.10), we have $d(T(u_n), u_n) \leq d(u_n, w)$ and, by Definition 15.15, we have

$$\begin{aligned} d(T(u_n), w)^2 &\leq \alpha d(T(u_n), w)^2 + \alpha d(u_n, w)^2 + \beta d(T(u_n), u_n)^2 \\ &\quad +(1-2\alpha-2\beta)d(u_n, w)^2 \\ &\leq \alpha d(T(u_n), w)^2 + (1-\alpha-\beta)d(u_n, w)^2. \end{aligned}$$

So, $d(T(u_n), w) \leq d(u_n, w)$. Similarly, we have

$$d(T(v_n), w) \leq d(v_n, w), \quad d(T(w_n), w) \leq d(w_n, w).$$

By (15.6), we have

$$\begin{aligned} d(v_n, w) &= d(\gamma_n T(u_n) \oplus (1-\gamma_n)u_n, w) \\ &\leq \gamma_n d(T(u_n), w) + (1-\gamma_n)d(u_n, w) \\ &\leq \gamma_n d(u_n, w) + (1-\gamma_n)d(u_n, w) \\ &= d(u_n, w). \end{aligned}$$

Further, by (15.6), we have

$$\begin{aligned} d(w_n, w) &= d(\beta_n T(v_n) \oplus (1-\beta_n)T(u_n), w) \\ &\leq \beta_n d(T(v_n), w) + (1-\beta_n)d(T(u_n), w) \\ &\leq \beta_n d(v_n, w) + (1-\beta_n)d(u_n, w) \\ &\leq \beta_n d(u_n, w) + (1-\beta_n)d(u_n, w) \\ &= d(u_n, w). \end{aligned} \tag{15.14}$$

Finally, by (15.6), we have

$$\begin{aligned} d(u_{n+1}, w) &= d(\alpha_n T(w_n) \oplus (1-\alpha_n)w_n, w) \\ &\leq \alpha_n d(T(w_n), w) + (1-\alpha_n)d(w_n, w) \\ &\leq \beta_n d(w_n, w) + (1-\beta_n)d(u_n, w) \\ &\leq \beta_n d(u_n, w) + (1-\beta_n)d(u_n, w) \\ &= d(u_n, w). \end{aligned}$$

Thus the sequence $\{d(u_n, w)\}$ is bounded and monotonic decreasing so $\lim_{n\to\infty} d(u_n, w)$ exists. For each $w \in F(T)$, since we have $d(u_{n+1}, w) \leq d(u_n, w)$ for each $n \in \mathbb{N}$, taking the infimum over all $w \in F(T)$, we get $D(u_{n+1}, F(T)) \leq D(u_n, F(T))$ for all $n \in \mathbb{N}$. So, the sequence $\{D(u_n, F(T))\}$ is bounded and monotone decreasing. Therefore, it follows that $\lim_{n\to\infty} D(u_n, F(T))$ exists. Suppose that

$$\lim_{n\to\infty} d(u_n, w) = r. \tag{15.15}$$

By (15.15), we have

$$\limsup_{n\to\infty} d(T(u_n), w), \limsup_{n\to\infty} d(T(v_n), w), \limsup_{n\to\infty} d(T(w_n), w) \le r. \tag{15.16}$$

By (15.14) and (15.15), we have

$$\limsup_{n\to\infty} d(v_n, w), \limsup_{n\to\infty} d(w_n, w) \le r. \tag{15.17}$$

By (15.6), we have

$$r = \lim_{n\to\infty} d(u_{n+1}, w) = \lim_{n\to\infty} d((1-\alpha_n)w_n \oplus \alpha_n T(w_n), w). \tag{15.18}$$

In view of (15.16), (15.18) and Lemma 15.3, we get

$$\lim_{n\to\infty} d(w_n, T(w_n)) = 0. \tag{15.19}$$

Again, by (15.6), we have

$$\begin{aligned} d(u_{n+1}, T(w_n)) &= d((1-\alpha_n)w_n \oplus \alpha_n T(w_n), T(w_n)) \\ &\le (1-\alpha_n) d(w_n, T(w_n)). \end{aligned}$$

Letting $n \to \infty$ and using (15.19), we get

$$\lim_{n\to\infty} d(u_{n+1}, T(w_n)) = 0. \tag{15.20}$$

Now, observe that

$$\begin{aligned} d(u_{n+1}, w) &\le d(u_{n+1}, T(w_n)) + d(T(w_n), w) \\ &\le d(u_{n+1}, T(w_n)) + d(w_n, w), \end{aligned}$$

which yields that

$$r \le \liminf_{n\to\infty} d(w_n, w). \tag{15.21}$$

From (15.17) and (15.21), we get

$$r = \lim_{n\to\infty} d(w_n, w) = \liminf_{n\to\infty} d((1-\beta_n)T(u_n) \oplus \beta_n T(v_n), w). \tag{15.22}$$

Finally, from (15.15) and Lemma 15.3, we conclude that $\lim_{n\to\infty} d(T(u_n), T(v_n)) = 0$.

Again, by (15.6), we have

$$\begin{aligned} d(w_n, T(v_n)) &= d((1-\beta_n)T(u_n) \oplus \beta_n T(v_n), T(v_n)) \\ &\leq (1-\beta_n) d(T(u_n), T(v_n)). \end{aligned}$$

Letting $n \to \infty$ and using (15.19), we get

$$\lim_{n\to\infty} d(w_n, T(v_n)) = 0. \tag{15.23}$$

Now, observe that

$$\begin{aligned} d(w_n, w) &\leq d(w_n, T(v_n)) + d(T(v_n), w) \\ &\leq d(w_n, T(v_n)) + d(v_n, w), \end{aligned}$$

which yields that

$$r \leq \liminf_{n\to\infty} d(v_n, w). \tag{15.24}$$

From (15.17) and (15.21), we get

$$r = \lim_{n\to\infty} d(v_n, w) = \liminf_{n\to\infty} d((1-\gamma_n)u_n \oplus \gamma_n T(u_n), w). \tag{15.25}$$

Finally, from (15.17), (15.22) and Lemma (15.3), we conclude that $\lim\limits_{n\to\infty} d(T(u_n), u_n) = 0$. This completes the proof.

Now, we present a result for the Δ-convergence.

Theorem 15.6 *Let $(\mathscr{M}, d, \preceq)$ be a uniformly convex partially ordered hyperbolic metric space. Let $\mathscr{K}$, T and $\{u_n\}$ be the same as in Theorem 15.5. If $F(T)$ is nonempty and totally ordered, then $\{u_n\}$ Δ-converges to a fixed point of T.*

Proof By Theorem 15.5, $\{u_n\}$ is a bounded sequence. Therefore, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $\{u_{n_j}\}$ Δ-converges to some $p \in \mathscr{K}$. By using Lemma 15.2, we have

$$u_1 \preceq u_{n_j} \preceq p \quad (\text{or } p \preceq u_{n_j} \preceq u_1)$$

for each $j \in \mathbb{N}$.

Now, we show that every Δ-convergent subsequence of $\{u_n\}$ has a unique Δ-limit in $F(T)$. Arguing by contradiction, suppose that $\{u_n\}$ has two subsequence $\{u_{n_j}\}$ and $\{u_{n_k}\}$ Δ-converging to p and q, respectively. By Theorem 15.5, $\{u_{n_j}\}$ is bounded and $d(T(u_{n_j}), u_{n_j}) = 0$. We claim that $p \in F(T)$. By following the proof of Theorem 15.4, we have $T(p) = p$. By the similar argument, $T(q) = q$. Since $\lim\limits_{n\to\infty} d(u_n, w)$ exists for all $w \in F(T)$, by the definition of the Δ-convergence and Lemma 15.1, we have

$$\begin{aligned}\limsup_{n\to\infty} d(u_n, p) &= \limsup_{j\to\infty} d(u_{n_j}, p) < \limsup_{j\to\infty} d(u_{n_j}, q)\\ &= \limsup_{n\to\infty} d(u_n, q) = \limsup_{k\to\infty} d(u_{n_k}, q)\\ &< \limsup_{k\to\infty} d(u_{n_k}, p)\\ &= \limsup_{n\to\infty} d(u_n, p),\end{aligned}$$

which is a contradiction, unless $p = q$. This completes the proof.

Next we present a strong convergence theorem.

Theorem 15.7 *Let $(\mathscr{M}, d, \preceq)$ be a uniformly convex partially ordered hyperbolic metric space and let $\mathscr{K}$, T and $\{u_n\}$ be the same as in Theorem 15.5. Suppose that $F(T)$ is nonempty and totally ordered. Then the sequence $\{u_n\}$ converges strongly to a fixed point of T if and only if*

$$\liminf_{n\to\infty} D(u_n, F(T)) = 0.$$

Proof Suppose that $\liminf_{n\to\infty} D(u_n, F(T))=0$. From Theorem 15.5, $\lim_{n\to\infty} D(u_n, F(T))$ exists and so

$$\lim_{n\to\infty} D(u_n, F(T)) = 0. \tag{15.26}$$

First, we show that set $F(T)$ is closed. For this, let $\{z_n\}$ be a sequence in $F(T)$ converging strongly to a point $z \in \mathscr{K}$. By Definition 15.15, we have

$$\begin{aligned}\limsup_{n\to\infty} d(T(z_n), T(z))^2 \leq{}& \alpha \limsup_{n\to\infty} d(T(z_n), z)^2 + \alpha \limsup_{n\to\infty} d(z_n, T(z))^2\\ &+\beta \limsup_{n\to\infty} d(T(z_n), z_n)^2 + \beta \limsup_{n\to\infty} d(z, T(z))^2\\ &+(1-2\alpha-2\beta) \limsup_{n\to\infty} d(z_n, z)^2.\end{aligned}$$

Since $d(z, T(z)) \leq d(z, z_n) + d(z_n, T(z))$, it follows that

$$\begin{aligned}\limsup_{n\to\infty} d(z_n, T(z)) &= \limsup_{n\to\infty} d(T(z_n), T(z))\\ &\leq \limsup_{n\to\infty} \Big(1 + \frac{2\beta}{1-\alpha-\beta} d(z_n, T(z))\Big) d(z_n, z)\\ &= 0.\end{aligned}$$

Thus $\{z_n\}$ converges strongly to $T(z)$. This implies that $T(z) = z$. Therefore, $F(T)$ is closed. In view of (15.26), let $\{u_{n_j}\}$ be a subsequence of sequence $\{u_n\}$ such that

$$d(u_{n_j}, z_j) \leq \frac{1}{2^j}$$

for each $j \geq 1$, where $\{w_j\}$ is a sequence in $F(T)$. By Theorem 15.5, we have

$$d(u_{n_{j+1}}, z_j) \leq d(u_{n_j}, z_j) \leq \frac{1}{2^j}. \tag{15.27}$$

Now, by the triangle inequality and (15.27), we have

$$\begin{aligned} d(z_{j+1}, z_j) &\leq d(z_{j+1}, u_{n_{j+1}}) + d(u_{n_{j+1}}, z_j) \\ &\leq \frac{1}{2^{j+1}} + \frac{1}{2^j} \\ &< \frac{1}{2^{j-1}}. \end{aligned}$$

A standard argument shows that $\{z_j\}$ is a Cauchy sequence. Since $F(T)$ is closed, it follows that $\{z_j\}$ converges to some point $z \in F(T)$. Now, we have

$$d(u_{n_j}, z) \leq d(u_{n_j}, z_j) + d(z_j, z).$$

Letting $j \to \infty$ implies that $\{u_{n_j}\}$ converges strongly to z. By Lemma 15.5, $\lim_{n\to\infty} d(u_n, z)$ exists. Hence $\{u_n\}$ converges strongly to z.

The converse part is obvious. This completes the proof.

Theorem 15.8 *Let $(\mathscr{M}, d, \preceq)$ be a uniformly convex partially ordered hyperbolic metric space and $\mathscr{K}$, T and $\{u_n\}$ be same as in Theorem 15.5. Let T satisfy the condition (I) and $F(T)$ be nonempty. Then $\{u_n\}$ converges strongly to a fixed point of T.*

Proof From Theorem 15.5, it follows that

$$\liminf_{n\to\infty} d(T(u_n), u_n) = 0. \tag{15.28}$$

Since T satisfies the condition (I), we have

$$d(T(u_n), u_n) \geq f(D(u_n, F(T))).$$

From (15.28), we get

$$\liminf_{n\to\infty} f(D(u_n, F(T))) = 0.$$

Since $f : [0, \infty) \to \mathbb{R}$ is a nondecreasing function with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, we have

$$\liminf_{n\to\infty} D(u_n, F(T)) = 0.$$

Therefore, all the assumptions of Theorem 15.7 are satisfied and so $\{u_n\}$ converges strongly to a fixed point of T. This completes the proof.

15.5 Conclusions

Iterative methods of finding fixed points of nonexpansive mappings are a very challenging problem. There are several methods that have been studied to approximate them. In this paper, we have proposed a general definition of monotone Suzuki (α, β)-nonexpansive mapping and we have proposed the CR-iteration method in partially ordered hyperbolic metric space for finding a fixed point of the proposed mapping.

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Chapter 16
Generalized JS-Contractions in b-Metric Spaces with Application to Urysohn Integral Equations

Hemant Kumar Nashine and Zoran Kadelburg

Abstract We introduce the notion of α-G-JS-type contractions for two pairs of self-mappings in b-metric spaces. Coincidence points, common fixed points, their uniqueness, as well as periodic points are studied for these mappings under α-compatible and relatively partially α-weakly increasing conditions on α-complete b-metric spaces. The results are verified through an example in order to check their effectiveness and applicability. We apply the results to obtain the existence of solutions for a system of Urysohn integral equations.

Keywords b-metric space · F-contraction · α-admissible mapping · Common fixed point · Urysohn integral equation

16.1 Introduction

For recent development of metric fixed point theory and its contributions in various disciplines from application point of view, we refer to [1] and the references therein.

The notion of b-metric space as an extension of metric space was introduced by Bakhtin in [3] and then extensively used by Czerwik in [5, 6]. Since then, a lot of papers on the fixed point theory for a range of classes of single-valued and multi-valued operators in such spaces have become available.

In 2012, Wardowski [16] and Samet et al. [14] introduced two different notions, named as F-contraction and α-admissible mappings, respectively, and investigated

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the existence of fixed points for such mappings, thus generalizing Banach Contraction Principle (BCP). Thereafter a lot of work has been done in fixed point theory using these concepts and their modified forms. Subsequently, Hussain et al. [7] introduced the concept of α-completeness in metric spaces and, further, Yamaod et al. [17] extended this notion to b-metric spaces. They introduced also notions of α-compatible, α-weakly increasing and relatively partially α-weakly increasing mappings and derived fixed point results using these notions for four maps in b-metric spaces. In 2014, Jleli and Samet [12] introduced a new type of control functions and generalized the BCP.

In the present chapter, we give an improved version of the common fixed point result given in [17] by considering a new contraction condition for two pairs of mappings, named as α-G-JS-contraction in α-complete b-metric spaces. We also present some criteria for the uniqueness of a common fixed point and discuss periodic points. In order to illustrate the results, we present an example. The considered α-G-JS-contraction condition not only generalizes the known ones but also includes the contraction conditions considered in [12, 13, 17] and many others as special cases.

Finally, we utilize our results to prove the existence and uniqueness of the following system of Urysohn integral equations:

$$u(t) = \hbar_j(t) + \int_0^T \Upsilon_j(t, s, u(s))\, ds, \quad \forall t \in [0, T], \ \ j \in \{1, 2, 3, 4\},$$

where $T > 0$, $t \in [0, T]$, $\hbar_j \colon [0, T] \to \mathbb{R}$ and $\Upsilon_j \colon [0, T]^2 \times \mathbb{R} \to \mathbb{R}$ ($j \in \{1, 2, 3, 4\}$) are given mappings.

16.2 Preliminaries

Throughout this chapter, we denote by $\mathbb{N}, \mathbb{R}^+, \mathbb{R}_0^+$ and $\mathbb{R}$ the sets of positive integers, positive real numbers, nonnegative real numbers, and real numbers, respectively.

Recall (see, e.g., [5]) that, for a nonempty set $\mathscr{X}$ and a given real number $s \geq 1$, a function $d_b \colon \mathscr{X} \times \mathscr{X} \to \mathbb{R}_0^+$ satisfying the following conditions:

(B_1) $d_b(x, y) = 0$ if and only if $x = y$;
(B_2) $d_b(x, y) = d(y, x)$;
(B_3) $d_b(x, y) \leq s[d_b(x, z) + d_b(z, y)]$ for all $x, y, z \in \mathscr{X}$

is called a *b -metric* on $\mathscr{X}$. The pair $(\mathscr{X}, d_b)$ is called a *b -metric space* with coefficient $s \geq 1$.

Any metric space is a b-metric space with $s = 1$, but the class of b-metric spaces is effectively larger than that of metric spaces. A typical example is the following.

Example 16.1 Let $(\mathscr{X}, d)$ be a metric space and the mapping $d_b \colon \mathscr{X} \times \mathscr{X} \to \mathbb{R}^+$ be defined by

$$d_b(x, y) = [d(x, y)]^p, \quad \forall x, y \in \mathscr{X},$$

where $p \geq 1$ is a fixed real number. Then $(\mathscr{X}, d_b)$ is a b-metric space with coefficient $s = 2^{p-1}$ (not being a metric space if $p > 1$). The triangular inequality (B_3) can easily be checked using the convexity of function $\mathbb{R}_0^+ \ni t \mapsto t^p$.

The concepts of b-convergent sequence, b-Cauchy sequence, b-continuity and b-completeness in b-metric spaces are introduced in the same way as in metric spaces (see, e.g., [4]). In particular, a function $f : \mathscr{X} \to \mathscr{Y}$ between two b-metric spaces is called b-continuous at a point $x \in \mathscr{X}$ if it is b-sequentially continuous at x, that is, if $\{fx_n\}$ is b-convergent to fx in $\mathscr{Y}$ for each sequence $\{x_n\}$ which is b-convergent to x in $\mathscr{X}$.

Each b-convergent sequence in a b-metric space has a unique limit and it is also a b-Cauchy sequence. However, a b-metric itself might not be continuous. Hence, the following lemma about b-convergent sequences is required in the proof of our main results.

Lemma 16.1 (see [2]) *Let $(\mathscr{X}, d_b)$ be a b-metric space with coefficient $s \geq 1$ and let $\{x_n\}$ and $\{y_n\}$ be b-convergent to points $x, y \in \mathscr{X}$, respectively. Then*

$$\frac{1}{s^2} d_b(x, y) \leq \liminf_{n\to\infty} d_b(x_n, y_n) \leq \limsup_{n\to\infty} d_b(x_n, y_n) \leq s^2 d_b(x, y).$$

If $x = y$, then $\lim_{n\to\infty} d_b(x_n, y_n) = 0$. Moreover, for each $z \in \mathscr{X}$, we have

$$\frac{1}{s} d_b(x, z) \leq \liminf_{n\to\infty} d_b(x_n, z) \leq \limsup_{n\to\infty} d_b(x_n, z) \leq s d_b(x, z).$$

For a self-mapping $\mathscr{J}$ on a nonempty set $\mathscr{X}$ and a point $x \in \mathscr{X}$, we use the following notation: $\mathscr{J}^{-1}(x) = \{u \in \mathscr{X} : \mathscr{J}u = x\}$.

Definition 16.1 Let $\mathscr{X}$ be a nonempty set, $\alpha : \mathscr{X} \times \mathscr{X} \to [0, +\infty)$ and $\mathscr{J}, \mathscr{K}, \mathscr{T} : \mathscr{X} \to \mathscr{X}$ be four mappings such that $\mathscr{J}(\mathscr{X}) \subseteq \mathscr{T}(\mathscr{X})$ and $\mathscr{K}(\mathscr{X}) \subseteq \mathscr{T}(\mathscr{X})$. The ordered pair $(\mathscr{J}, \mathscr{K})$ is said to be

(1) *α-weakly increasing* with respect to $\mathscr{T}$ if, for all $x \in \mathscr{X}$, we have $\alpha(\mathscr{J}x, \mathscr{K}y) \geq 1$ for all $y \in \mathscr{T}^{-1}(\mathscr{J}x)$ and $\alpha(\mathscr{K}x, \mathscr{J}y) \geq 1$ for all $y \in \mathscr{T}^{-1}(\mathscr{K}x)$.

(2) *partially α-weakly increasing* with respect to $\mathscr{T}$ if $\alpha(\mathscr{J}x, \mathscr{K}y) \geq 1$ for all $y \in \mathscr{T}^{-1}(\mathscr{J}x)$.

In particular,

(3) If $\mathscr{T}$ =the identity mapping on $\mathscr{X}$, then the pair $(\mathscr{J}, \mathscr{K})$ is called $(partially)$ *α-weakly increasing* [17].

(4) If $\mathscr{K} = \mathscr{J}$, then we say that $\mathscr{J}$ is $(partially)$ *α-weakly increasing* with respect to $\mathscr{T}$. If, moreover, $\mathscr{T}$ = the identity mapping on $\mathscr{X}$, then we say that $\mathscr{J}$ is $(partially)$ *α-weakly increasing*.

Definition 16.2 ([17]) Let $(\mathscr{X}, d_b)$ be a b-metric space, $\alpha : \mathscr{X} \times \mathscr{X} \to [0, +\infty)$ and $\mathscr{J}, \mathscr{K} : \mathscr{X} \to \mathscr{X}$ be three mappings. The pair $(\mathscr{J}, \mathscr{K})$ is said to be α *-compatible* if

$$\lim_{n\to\infty} d_b(\mathscr{J}\mathscr{K}x_n, \mathscr{K}\mathscr{J}x_n) = 0$$

whenever $\{x_n\}$ is a sequence in $\mathscr{X}$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \mathscr{J}x_n = \lim_{n\to\infty} \mathscr{K}x_n = t$$

for some $t \in \mathscr{X}$.

Definition 16.3 ([17]) Let $(\mathscr{X}, d_b)$ be a b-metric space, $\alpha\colon \mathscr{X} \times \mathscr{X} \to [0, +\infty)$ and $\mathscr{J}: \mathscr{X} \to \mathscr{X}$ be two mappings. We say that $\mathscr{J}$ is *α-continuous* at a point $x \in \mathscr{X}$ if, for each sequence $\{x_n\}$ in $\mathscr{X}$ with $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \mathscr{J}x_n = \mathscr{J}x.$$

The following notion is a modified version (in b-metric spaces) of the one from the paper [8].

Definition 16.4 Let $(\mathscr{X}, d_b)$ be a b-metric space, and $\alpha\colon \mathscr{X} \times \mathscr{X} \to [0, +\infty)$. A mapping $\mathscr{J}: \mathscr{X} \to \mathscr{X}$ is said to be α *-dominating* on $\mathscr{X}$ if $\alpha(x, \mathscr{J}x) \geq 1$ for each x in $\mathscr{X}$.

Following Jleli and Samet [12], denote by Θ the family of all functions $\theta\colon (0, \infty) \to [1, \infty)$ with the following properties:

(θ1) θ is strictly increasing;

(θ2) for all sequences $\{\alpha_n\} \subseteq (0, \infty)$,

$$\lim_{n\to\infty} \alpha_n = 0 \Leftrightarrow \lim_{n\to\infty} \theta(\alpha_n) = 1;$$

(θ3) there exist $0 < r < 1$ and $\ell \in (0, +\infty]$ such that

$$\lim_{t\to 0^+} \frac{\theta(t) - 1}{t^r} = \ell.$$

Note that the Θ is a rich class of functions. Some examples of functions belonging to Θ are $\theta_1(t) = e^{\sqrt{te^t}}$, $\theta_2(t) = 2 - \frac{2}{\pi}\arctan\left(\frac{1}{t^\alpha}\right)$ for $0 < \alpha < 1$, etc.

Hussain and Salimi [9] introduced α-GF-contractions with respect to a general family of functions G. We will use the following slightly modified family Δ_G of all functions $G\colon (\mathbb{R}_0^+)^4 \to \mathbb{R}_0^+$ satisfying

($G1$) there exists $\tau > 0$ such that $G(t_1, t_2, t_3, t_4) = \tau$ for all $t_1, t_2, t_3, t_4 \in \mathbb{R}_0^+$ with $t_1t_2t_3t_4 = 0$;

($G2$) there exists $\tau > 0$ such that $\lim_{n\to\infty} G(t_1^n, t_2^n, t_3^n, t_4^n) = \tau$ for all sequences $\{t_i^n\}_{n\in\mathbb{N}}$ of nonnegative real numbers ($i \in \{1, 2, 3, 4\}$) such that $\lim_{n\to\infty} t_i^n = 0$ for some $i \in \{1, 2, 3, 4\}$.

Example 16.2 The following are some examples of functions belonging to Δ_G:

(1) $G_1(a, b, c, d) = L \min\{a, b, c, d\} + \tau$, where $L \in \mathbb{R}_0^+$ and $\tau > 0$.
(2) $G_2(a, b, c, d) = \tau e^{L \min\{a,b,c,d\}}$, where $L \in \mathbb{R}_0^+$ and $\tau > 0$.
(3) $G_3(a, b, c, d) = L \ln(\min\{a, b, c, d\} + 1) + \tau$, where $L \in \mathbb{R}_0^+$ and $\tau > 0$.
(4) $G_4(a, b, c, d) = \tau - \dfrac{\tau d}{L + d}$, where $L \in \mathbb{R}_0^+$ and $\tau > 0$.

Throughout this chapter, the set of all fixed points (coincidence points, common fixed points) of a self-mapping $\mathscr{J}$ (and self-mapping $\mathscr{K}$) on a nonempty set $\mathscr{X}$ is denoted by $Fix(\mathscr{J})$ ($C(\mathscr{J}, \mathscr{K})$, $CF(\mathscr{J}, \mathscr{K})$), i.e.,

$$\begin{aligned} Fix(\mathscr{J}) &:= \{x \in \mathscr{X} : \mathscr{J}x = x\}, \\ C(\mathscr{J}, \mathscr{K}) &:= \{x \in \mathscr{X} : \mathscr{J}x = \mathscr{K}x\} \\ CF(\mathscr{J}, \mathscr{K}) &:= \{x \in \mathscr{X} : x = \mathscr{J}x = \mathscr{K}x\}. \end{aligned}$$

16.3 Main Results

Throughout this chapter, unless otherwise stated, for all elements x and y in a b-metric space $(\mathscr{X}, d_b)$ with coefficient $s \geq 1$ and four mappings $\mathscr{J}, \mathscr{K}, \mathscr{S}, \mathscr{T} \colon \mathscr{X} \to \mathscr{X}$, we denote

$$\Delta_b(x, y) = \max\left\{ d_b(\mathscr{S}x, \mathscr{T}y), d_b(\mathscr{S}x, \mathscr{J}x), d_b(\mathscr{T}y, \mathscr{K}y), \frac{d_b(\mathscr{S}x, \mathscr{K}y) + d_b(\mathscr{T}y, \mathscr{J}x)}{2s} \right\}.$$

Combining approaches from the papers [9, 12], adapted to the ambient of b-metric spaces, we introduce the following concept.

Definition 16.5 Let $(\mathscr{X}, d_b)$ be a b-metric space with coefficient $s \geq 1$, $\mathscr{J}, \mathscr{K}, \mathscr{S}, \mathscr{T} \colon \mathscr{X} \to \mathscr{X}$ and $\alpha \colon \mathscr{X} \times \mathscr{X} \to [0, \infty)$ be given mappings. Then $(\mathscr{J}, \mathscr{K}, \mathscr{S}, \mathscr{T})$ is called an *α-G-JS-contraction* if the following condition holds: there exist $\theta \in \Theta$ and $G \in \Delta_G$, such that, for all $x, y \in \mathscr{X}$,

$$\begin{cases} (\alpha(\mathscr{S}x, \mathscr{T}y) \geq 1 \text{ or } \alpha(\mathscr{T}x, \mathscr{S}y) \geq 1) \text{ with } d_b(\mathscr{J}x, \mathscr{K}y) > 0 \implies \\ \theta(s d_b(\mathscr{J}x, \mathscr{K}y)) \leq \theta(\Delta_b(x, y))^{G\left(d_b(\mathscr{S}x, \mathscr{J}x), d_b(\mathscr{T}y, \mathscr{K}y), d_b(\mathscr{S}x, \mathscr{K}y), d_b(\mathscr{T}y, \mathscr{J}x)\right)}. \end{cases} \tag{16.1}$$

We denote by $\Xi_b(\mathscr{X}, \alpha, \Theta, \Delta_G)$ the collection of all α-G-JS-contractions on a b-metric space $(\mathscr{X}, d_b)$ with coefficient $s \geq 1$.

16.3.1 Discussion on Coincidence Point Results

Theorem 16.1 *Let $(\mathscr{X}, d_b)$ be an α-complete b-metric space with coefficient $s \geq 1$, where $\alpha\colon \mathscr{X} \times \mathscr{X} \to [0, +\infty)$, and let $\mathscr{J}, \mathscr{K}, \mathscr{S}, \mathscr{T}\colon \mathscr{X} \to \mathscr{X}$ be given mappings. Suppose that the following conditions hold:*

$(H1)$ $(\mathscr{J}, \mathscr{S}, \mathscr{K}, \mathscr{T}) \in \Xi_b(\mathscr{X}, \alpha, \Theta, \Delta_G)$;

$(H2)$ $\mathscr{J}(\mathscr{X}) \subseteq \mathscr{T}(\mathscr{X})$ and $\mathscr{K}(\mathscr{X}) \subseteq \mathscr{S}(\mathscr{X})$;

$(H3)$ the pairs $(\mathscr{J}, \mathscr{K})$ and $(\mathscr{K}, \mathscr{J})$ are partially α-weakly increasing with respect to $\mathscr{T}$ and $\mathscr{S}$, respectively;

$(H4)$ α is a transitive mapping, that is, for all $x, y, z \in \mathscr{X}$,

$$\alpha(x, y) \geq 1 \;\; \text{and} \;\; \alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1;$$

$(H5)$ $\mathscr{J}, \mathscr{K}, \mathscr{S}$ and $\mathscr{T}$ are α-continuous;

$(H6)$ the pairs $(\mathscr{J}, \mathscr{S})$ and $(\mathscr{K}, \mathscr{T})$ are α-compatible.

Then there exists $x^* \in \mathscr{X}$ such that $x^* \in C(\mathscr{J}, \mathscr{S}) \cap C(\mathscr{K}, \mathscr{T})$. Moreover, if $\alpha(\mathscr{S}x^*, \mathscr{T}x^*) \geq 1$ or $\alpha(\mathscr{T}x^*, \mathscr{S}x^*) \geq 1$, then $x^* \in C(\mathscr{J}, \mathscr{S}, \mathscr{K}, \mathscr{T})$.

Proof Starting from an arbitrary point $x_0 \in \mathscr{X}$ and using the condition (H2), we can consider sequences $\{x_n\}$ and $\{z_n\}$ in $\mathscr{X}$ defined by

$$z_{2n+1} = \mathscr{T}x_{2n+1} = \mathscr{J}x_{2n}, \quad z_{2n+2} = \mathscr{S}x_{2n+2} = \mathscr{K}x_{2n+1}$$

for $n \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$. Since $x_1 \in \mathscr{T}^{-1}(\mathscr{J}x_0)$, $x_2 \in \mathscr{S}^{-1}(\mathscr{K}x_1)$ and the pairs $(\mathscr{J}, \mathscr{K})$ and $(\mathscr{K}, \mathscr{J})$ satisfy (H3), therefore we have

$$\alpha(z_1, z_2) = \alpha(\mathscr{J}x_0, \mathscr{K}x_1) \geq 1, \quad \alpha(z_2, z_3) = \alpha(\mathscr{K}x_1, \mathscr{J}x_2) \geq 1.$$

Repeating this process, we obtain

$$\alpha(z_n, z_{n+1}) \geq 1, \quad \forall n \in \mathbb{N}^*. \tag{16.2}$$

First, we need to prove that

$$\lim_{n\to\infty} d_b(z_n, z_{n+1}) = 0.$$

For all $k \in \mathbb{N}^*$, we define $\rho_k = d_b(z_k, z_{k+1})$. If we assume that $\rho_{k_0} = 0$ for some $k_0 \in \mathbb{N}^*$, then $z_{k_0} = z_{k_0+1}$, and the proof is finished. So assume $z_n \neq z_{n+1}$ for all $n \geq 0$. Then $\rho_n > 0$ for all $n \in \mathbb{N}^*$.

Suppose that n is an odd number. Since $\alpha(z_n, z_{n+1}) \geq 1$, from $(\mathscr{J}, \mathscr{K}, \mathscr{S}, \mathscr{T}) \in \Xi_b(\mathscr{X}, \alpha, \Theta, \Delta_G)$, it follows that the condition (16.1) implies that

$$\begin{aligned}
&\theta(s\rho_{n+1})\\
&= \theta(sd_b(z_{n+1}, z_{n+2}))\\
&= \theta(sd_b(\mathscr{J}x_n, \mathscr{K}x_{n+1}))\\
&\leq \theta(\Delta_b(x_n, x_{n+1}))^{G(d_b(\mathscr{S}x_n, \mathscr{J}x_n), d_b(\mathscr{T}x_{n+1}, \mathscr{K}x_{n+1}), d_b(\mathscr{S}x_n, \mathscr{K}x_{n+1}), d_b(\mathscr{T}x_{n+1}, \mathscr{J}x_n))}\\
&= \theta(\Delta_b(x_n, x_{n+1}))^{G(d_b(z_n, z_{n+1}), d_b(z_{n+1}, z_{n+1}), d_b(z_n, z_{n+1}), d_b(z_{n+1}, z_{n+1}))}\\
&= \theta(\Delta_b(x_n, x_{n+1}))^{G(\rho_n, 0, \rho_n, 0)},
\end{aligned} \tag{16.3}$$

where

$$\begin{aligned}
&\Delta_b(x_n, x_{n+1})\\
&= \max\left\{\begin{array}{c} d_b(\mathscr{S}x_n, \mathscr{T}x_{n+1}), d_b(\mathscr{S}x_n, \mathscr{J}x_n), d_b(\mathscr{T}x_{n+1}, \mathscr{K}x_{n+1}),\\ \dfrac{d_b(\mathscr{S}x_n, \mathscr{K}x_{n+1}) + d_b(\mathscr{T}x_{n+1}, \mathscr{J}x_n)}{2s} \end{array}\right\}\\
&= \max\left\{\begin{array}{c} d_b(z_n, z_{n+1}), d_b(z_n, z_{n+1}), d_b(z_{n+1}, z_{n+2}),\\ \dfrac{d_b(z_n, z_{n+2}) + d_b(z_{n+1}, z_{n+1})}{2s} \end{array}\right\}\\
&= \max\left\{d_b(z_n, z_{n+1}), d_b(z_{n+1}, z_{n+2}), \frac{d_b(z_n, z_{n+1}) + d_b(z_{n+1}, z_{n+2})}{2s}\right\}\\
&= \max\{d_b(z_n, z_{n+1}), d_b(z_{n+1}, z_{n+2})\}\\
&= \max\{\rho_n, \rho_{n+1}\}.
\end{aligned}$$

Also, by the property (G1) of $G \in \Delta_G$, there exists $\tau > 0$ such that

$$G(\rho_n, 0, \rho_n, 0) = \tau.$$

Therefore the above inequalities with (16.3) yield

$$\theta(s\rho_{n+1}) \leq \theta(\max\{\rho_n, \rho_{n+1}\})^{\tau}. \tag{16.4}$$

If $\Delta_b(x_n, x_{n+1})) = \rho_{n+1}$ for some $n \in \mathbb{N}$, then the inequality (16.4) implies that

$$\theta(s\rho_{n+1}) \leq \theta(\rho_{n+1})^{\tau},$$

which is a contradiction since $\tau > 0$. Therefore, $\Delta_b(x_n, x_{n+1}) = \rho_n$ for all $n \in \mathbb{N}$ and so, from (16.3), we have

$$\theta(s\rho_n) \leq \theta(\rho_{n-1})^{\tau}. \tag{16.5}$$

In a similar way, we can establish the inequality (16.5) when n is an even number. Therefore, we have

$$1 \leq \theta(s^n \rho_n) \leq \theta(s^{n-1}\rho_{n-1}))^{\tau} \leq \theta(s^{n-2}\rho_{n-2})^{\tau^2} \leq \ldots \leq \theta(\rho_0)^{\tau^n} \tag{16.6}$$

for all $n \in \mathbb{N}$, that is,

$$1 \leq \theta(\rho_0)^{\tau^n}. \tag{16.7}$$

From (16.7), we get $\theta(s^n \rho_n) \to 1$ as $n \to \infty$. Thus, from (F2), we have

$$\lim_{n\to\infty} s^n \rho_n = 0. \tag{16.8}$$

Now, by the property (θ2), we have

$$\lim_{n\to\infty} \rho_n = 0, \tag{16.9}$$

and, by the property (θ3), there exist $k \in (0, 1)$ and $0 < \ell \leq \infty$ such that

$$\lim_{n\to\infty} \frac{\theta(s^n \rho_n) - 1}{(s^n \rho_n)^k} = \ell. \tag{16.10}$$

Assume that $\ell < \infty$ and let $B = \ell/2$. From the definition of the limit there exists $n_0 \in N$ such that

$$\left|\frac{\theta(s^n \rho_n) - 1}{(s^n \rho_n)^k} - \ell\right| \leq B, \quad \forall n \geq n_0$$

which implies that

$$\frac{\theta(s^n \rho_n) - 1}{(s^n \rho_n)^k} \geq \ell - B = B, \quad \forall n \geq n_0$$

and so

$$n(s^n \rho_n)^k \leq nA[\theta(s^n \rho_n) - 1], \quad \forall n \geq n_0,$$

where $A = 1/B$. Now, assume that $\ell = \infty$. Let $B > 0$ be a given real number. From the definition of the limit, there exists $n_0 \in N$ such that

$$\left|\frac{\theta(s^n \rho_n) - 1}{(s^n \rho_n)^k} - \ell\right| \geq B, \quad \forall n \geq n_0$$

which implies that

$$n(s^n \rho_n)^k \leq nA[\theta(s^n \rho_n) - 1], \quad \forall n \geq n_0,$$

where $A = 1/B$. Hence, in all cases, there exist $A > 0$ and $n_0 \in N$ such that

$$n(s^n \rho_n)^k \leq nA[\theta(s^n \rho_n) - 1], \quad \forall n \geq n_0.$$

From (16.6) we have

$$n(s^n\rho_n)^k \le nA[\theta(\rho_n)^{\tau^n} - 1], \quad \forall n \ge n_0. \tag{16.11}$$

Passing to the limit as $n \to \infty$ in (16.11), we obtain

$$\lim_{n\to\infty} n(s^n\rho_n)^k = 0.$$

Now the last limit implies that the series $\sum_{n=1}^{\infty} s^n\rho_n$ is convergent and hence $\{z_n\}$ is a b-Cauchy sequence in $\mathscr{X}$. Since the inequality (16.2) holds, by the α-completeness of b-metric space $(\mathscr{X}, d_b)$, there exists $x^* \in \mathscr{X}$ such that

$$\lim_{n\to\infty} d_b(z_n, x^*) = 0$$

and so

$$\lim_{n\to\infty} d_b(z_{2n+1}, x^*) = \lim_{n\to\infty} d_b(\mathscr{T}x_{2n+1}, x^*) = \lim_{n\to\infty} d_b(\mathscr{J}x_{2n}, x^*) = 0 \tag{16.12}$$

and

$$\lim_{n\to\infty} d_b(z_{2n+2}, x^*) = \lim_{n\to\infty} d_b(\mathscr{S}x_{2n+2}, x^*) = \lim_{n\to\infty} d_b(\mathscr{K}x_{2n+1}, x^*) = 0. \tag{16.13}$$

From (16.12) and (16.13), we have $\mathscr{J}x_{2n} \to x^*$ and $\mathscr{S}x_{2n} \to x^*$ as $n \to \infty$. Since $(\mathscr{J}, \mathscr{S})$ is α-compatible, by (16.2), we have

$$\lim_{n\to\infty} d_b(\mathscr{S}\mathscr{J}x_{2n}, \mathscr{J}\mathscr{S}x_{2n}) = 0. \tag{16.14}$$

By (16.2), the α-continuity of $\mathscr{S}$, $\mathscr{J}$ and Lemma 16.1, we obtain

$$\lim_{n\to\infty} d_b(\mathscr{S}\mathscr{J}x_{2n}, \mathscr{S}x^*) = 0 = \lim_{n\to\infty} d_b(\mathscr{J}\mathscr{S}x_{2n}, \mathscr{J}x^*). \tag{16.15}$$

By the (B3) property, we have

$$\begin{aligned}
&d_b(\mathscr{S}x^*, \mathscr{J}x^*)\\
&\le s[d_b(\mathscr{S}x^*, \mathscr{S}\mathscr{J}x_{2n}) + d_b(\mathscr{S}\mathscr{J}x_{2n}, \mathscr{J}x^*)]\\
&\le sd_b(\mathscr{S}x^*, \mathscr{S}\mathscr{J}x_{2n}) + s^2[d_b(\mathscr{S}\mathscr{J}x_{2n}, \mathscr{J}\mathscr{S}x_{2n}) + d_b(\mathscr{J}\mathscr{S}x_{2n}, \mathscr{J}x^*)]
\end{aligned}$$

for all $n \in \mathbb{N}^*$. Passing to the limit as $n \to \infty$ in the above inequality and using (16.14)–(16.15), we obtain $d_b(\mathscr{S}x^*, \mathscr{J}x^*) \le 0$. This implies that $d_b(\mathscr{S}x^*, \mathscr{J}x^*) = 0$ and so $x^* \in C(\mathscr{J}, \mathscr{S})$, that is, x^* is a coincidence point of $\mathscr{J}$ and $\mathscr{S}$. Similarly, we can prove that x^* is also a coincidence point of $\mathscr{K}$ and $\mathscr{T}$.

Finally, we prove that x^* is a coincidence point of $\mathscr{J}$, $\mathscr{K}$, $\mathscr{S}$ and $\mathscr{T}$ provided that

$$\alpha(\mathscr{T}x^*, \mathscr{S}x^*) \geq 1 \text{ or } \alpha(\mathscr{S}x^*, \mathscr{T}x^*) \geq 1.$$

Suppose, to the contrary, that $\mathscr{J}x^* \neq \mathscr{K}x^*$. Then, from (16.1), we have

$$\begin{aligned} &\theta(sd_b(\mathscr{J}x^*, \mathscr{K}x^*)) \\ &\leq \theta(\Delta_b(x^*, x^*))^{G(d_b(\mathscr{S}x^*, \mathscr{J}x^*), d_b(\mathscr{T}x^*, \mathscr{K}x^*), d_b(\mathscr{S}x^*, \mathscr{K}y), d_b(\mathscr{T}y, \mathscr{J}x^*))}, \end{aligned} \tag{16.16}$$

where

$$\begin{aligned} \Delta_b(x^*, x^*) &= \max\left\{ \begin{array}{c} d_b(\mathscr{S}x^*, \mathscr{T}x^*), d_b(\mathscr{S}x^*, \mathscr{J}x^*), d_b(\mathscr{T}x^*, \mathscr{K}x^*), \\ \dfrac{d_b(\mathscr{S}x^*, \mathscr{K}x^*) + d_b(\mathscr{T}x^*, \mathscr{J}x^*)}{2s} \end{array} \right\} \\ &= \max\left\{ d_b(\mathscr{J}x^*, \mathscr{K}x^*), 0, 0, \frac{d_b(\mathscr{J}x^*, \mathscr{K}x^*)}{s} \right\} \\ &= d_b(\mathscr{J}x^*, \mathscr{K}x^*). \end{aligned}$$

Also, by the property (G1) of $G \in \Delta_G$, there exists $\tau > 0$ such that

$$\begin{aligned} &G(d_b(\mathscr{S}x^*, \mathscr{J}x^*), d_b(\mathscr{T}x^*, \mathscr{K}x^*), d_b(\mathscr{S}x^*, \mathscr{K}x^*), d_b(\mathscr{T}x^*, \mathscr{J}x^*)) \\ &\quad = G(0, 0, d_b(\mathscr{J}x^*, \mathscr{K}x^*), d_b(\mathscr{K}x^*, \mathscr{J}x^*)) = \tau. \end{aligned}$$

It follows from (16.16) that

$$\theta(sd_b(\mathscr{J}x^*, \mathscr{K}x^*)) \leq \theta(d_b(\mathscr{J}x^*, \mathscr{K}x^*))^{\tau}. \tag{16.17}$$

Now, by the property (θ1) with $\tau > 0$, it follows from (16.17) that

$$sd_b(\mathscr{J}x^*, \mathscr{K}x^*) < d_b(\mathscr{J}x^*, \mathscr{K}x^*),$$

a contradiction, except when $d_b(\mathscr{J}x^*, \mathscr{K}x^*) = 0$. Thus $\mathscr{J}x^* = \mathscr{K}x^*$ and hence $x^* \in C(\mathscr{J}, \mathscr{S}, \mathscr{K}, \mathscr{T})$. This completes the proof.

We note that the previous result can still be valid, under some additional assumptions, for $\mathscr{J}$, $\mathscr{K}$, $\mathscr{S}$, $\mathscr{T}$ not necessarily α-continuous. We have the following result.

Theorem 16.2 *Let $(\mathscr{X}, d_b)$ be an α-complete b-metric space with coefficient $s \geq 1$, $\alpha\colon \mathscr{X} \times \mathscr{X} \to [0, +\infty)$ and $\mathscr{J}, \mathscr{K}, \mathscr{S}, \mathscr{T}\colon \mathscr{X} \to \mathscr{X}$ be given mappings. Suppose that the assumptions $(H1)$–$(H4)$ of Theorem 16.1 hold as well as:*

($\widehat{H5}$) *$\mathscr{T}(\mathscr{X})$ and $\mathscr{S}(\mathscr{X})$ are b-closed subsets of $\mathscr{X}$;*

($\widehat{H6}$) *the pairs $(\mathscr{J}, \mathscr{S})$ and $(\mathscr{K}, \mathscr{T})$ are weakly compatible;*

$(H7)$ *$\mathscr{X}$ is α-regular, i.e., if $\{u_n\}$ is a sequence in $\mathscr{X}$ with $\alpha(u_n, u_{n+1}) \geq 1$ for $n \in \mathbb{N}$ and $u_n \to u^*$ as $n \to \infty$, then $\alpha(u_n, u^*) \geq 1$ for all $n \in \mathbb{N}$.*

Then there exists $x^* \in \mathscr{X}$ such that $x^* \in C(\mathscr{J}, \mathscr{S}) \cap C(\mathscr{K}, \mathscr{T})$. Moreover, if $\alpha(\mathscr{S}x^*, \mathscr{T}x^*) \geq 1$ or $\alpha(\mathscr{T}x^*, \mathscr{S}x^*) \geq 1$, then $x^* \in C(\mathscr{J}, \mathscr{S}, \mathscr{K}, \mathscr{T})$.

Proof Following the proof of Theorem 16.1, we obtain a b-Cauchy sequence $\{z_n\}$ in the α-complete b-metric space $(\mathscr{X}, d_b)$. Hence there exists $x^* \in \mathscr{X}$ such that

$$\lim_{n\to\infty} d_b(z_n, x^*) = 0.$$

Combining the hypothesis $(\widehat{H}5)$ for $\mathscr{T}(\mathscr{X})$ and $\{z_{2n+1}\} \subseteq \mathscr{T}(\mathscr{X})$, we have $x^* \in \mathscr{T}(\mathscr{X})$. Hence there exists $\xi \in \mathscr{X}$ such that $x^* = \mathscr{T}\xi$ and

$$\lim_{n\to\infty} d(z_{2n+1}, \mathscr{T}\xi) = \lim_{n\to\infty} d(\mathscr{T}x_{2n+1}, \mathscr{T}\xi) = 0.$$

Similarly, using the hypothesis $(\widehat{H}5)$ for $\mathscr{S}(\mathscr{X})$ and $\{z_{2n}\} \subseteq \mathscr{S}(\mathscr{X})$, we have $x^* \in \mathscr{S}(\mathscr{X})$. Hence there exists $\zeta \in \mathscr{X}$ such that $x^* = \mathscr{T}\xi = \mathscr{S}\zeta$ and

$$\lim_{n\to\infty} d(z_{2n}, \mathscr{S}\zeta) = \lim_{n\to\infty} d(\mathscr{S}x_{2n}, \mathscr{T}\zeta) = 0.$$

Further, we prove that x^* is a coincidence point of $\mathscr{J}$ and $\mathscr{S}$. Since $\mathscr{T}x_{2n+1} \to x^* = \mathscr{S}\zeta$ as $n \to \infty$, it follows from the hypothesis (H7), that is, α-regularity of $\mathscr{X}$ that $\alpha(\mathscr{T}x_{2n+1}, \mathscr{S}\zeta) \geq 1$. Suppose, to the contrary, that $\mathscr{J}\zeta \neq x^*$. Then we have from (16.1),

$$\begin{aligned}
&\theta(sd_b(\mathscr{J}\zeta, \mathscr{K}x_{2n+1}))\\
&\leq \theta(\Delta_b(\zeta, x_{2n+1}))^{Gd_b(\mathscr{S}\zeta,\mathscr{J}\zeta),d_b(\mathscr{T}x_{2n+1},\mathscr{K}x_{2n+1}),d_b(\mathscr{S}\zeta,\mathscr{K}x_{2n+1}),d_b(\mathscr{T}x_{2n+1},\mathscr{J}\zeta)},
\end{aligned} \tag{16.18}$$

where

$$\begin{aligned}
&\Delta_b(\zeta, x_{2n+1})\\
&= \max\left\{\begin{array}{c} d_b(\mathscr{S}\zeta, \mathscr{T}x_{2n+1}), d_b(\mathscr{S}\zeta, \mathscr{J}\zeta), d_b(\mathscr{T}x_{2n+1}, \mathscr{K}x_{2n+1}),\\ \dfrac{d_b(\mathscr{S}\zeta, \mathscr{K}x_{2n+1}) + d_b(\mathscr{T}x_{2n+1}, \mathscr{J}\zeta)}{2s} \end{array}\right\}
\end{aligned}$$

which implies that

$$\begin{aligned}
&\lim_{n\to\infty} \Delta_b(\zeta, x_{2n+1})\\
&= \lim_{n\to\infty} \max\left\{\begin{array}{c} d_b(\mathscr{S}\zeta, \mathscr{T}x_{2n+1}), d_b(\mathscr{S}\zeta, \mathscr{J}\zeta), d_b(\mathscr{T}x_{2n+1}, \mathscr{K}x_{2n+1}),\\ \dfrac{d_b(\mathscr{S}\zeta, \mathscr{K}x_{2n+1}) + d_b(\mathscr{T}x_{2n+1}, \mathscr{J}\zeta)}{2s} \end{array}\right\}\\
&= \max\left\{0, d_b(x^*, \mathscr{J}\zeta), 0, \frac{0 + d_b(x^*, \mathscr{J}\zeta)}{2s}\right\}\\
&= d_b(x^*, \mathscr{J}\zeta).
\end{aligned} \tag{16.19}$$

Since $\lim_{n\to\infty} d_b(\mathscr{T}x_{2n+1}, \mathscr{K}x_{2n+1}) = 0$, from the property (G2) of $G \in \Delta_G$, there exists $\tau > 0$ such that

$$\lim_{n\to\infty} G\begin{pmatrix} d_b(\mathscr{S}\zeta, \mathscr{J}\zeta), d_b(\mathscr{T}x_{2n+1}, \mathscr{K}x_{2n+1}), \\ d_b(\mathscr{S}\zeta, \mathscr{K}x_{2n+1}), d_b(\mathscr{T}x_{2n+1}, \mathscr{J}\zeta) \end{pmatrix} = \tau.$$

Therefore, it follows from the continuity of F, applying the limit as $n \to \infty$ in (16.18) and using (16.19), that

$$\theta(sd_b(\mathscr{J}\zeta, x^*)) \le \theta(d_b(x^*, \mathscr{J}\zeta))^{\tau}. \tag{16.20}$$

Now, by the property (θ1) with $\tau > 0$, it follows from (16.20) that

$$sd_b(\mathscr{J}\zeta, x^*) < d_b(\mathscr{J}\zeta, x^*),$$

which is a contradiction, except when $d_b(\mathscr{J}\zeta, x^*) = 0$. Hence $x^* = \mathscr{J}\zeta$ and so $\mathscr{S}\zeta = x^* = \mathscr{J}\zeta$. By the hypothesis ($\widehat{H}6$) for the pair $(\mathscr{J}, \mathscr{S})$, we have

$$\mathscr{J}x^* = \mathscr{J}\mathscr{S}\zeta = \mathscr{S}\mathscr{J}\zeta = \mathscr{S}x^*,$$

which shows that x^* is a coincidence point of $\mathscr{J}$ and $\mathscr{S}$. Likewise, we can obtain that x^* is a coincidence point of the pair $(\mathscr{K}, \mathscr{T})$. Using similar arguments as in the previous theorem, we can show that $x^* \in C(\mathscr{J}, \mathscr{S}, \mathscr{K}, \mathscr{T})$. This completes the proof.

16.3.2 Discussion on Common Fixed Point Results

Theorem 16.3 *Under the hypotheses of Theorem 16.1 (or Theorem 16.2), $\mathscr{J}, \mathscr{K}, \mathscr{S}, \mathscr{T}$ have a common fixed point in $\mathscr{X}$ provided the following condition holds:*

($H8$) $\mathscr{S}$ or $\mathscr{T}$ is an α-dominating map.

Proof From Theorem 16.1 (or Theorem 16.2), there exists an $x^* \in \mathscr{X}$ such that $x^* \in C(\mathscr{J}, \mathscr{S}, \mathscr{K}, \mathscr{T})$. Since the pair $(\mathscr{J}, \mathscr{S})$ is weakly compatible, we have $\mathscr{J}\mathscr{S}x^* = \mathscr{S}\mathscr{J}x^*$. Let $u^* = \mathscr{J}x^* = \mathscr{S}x^*$. Therefore, we have $\mathscr{J}u^* = \mathscr{S}u^*$. Similarly, since the pair $(\mathscr{K}, \mathscr{T})$ is weakly compatible, we have $\mathscr{K}\mathscr{T}x^* = \mathscr{T}\mathscr{K}x^*$. Let $u^* = \mathscr{K}x^* = \mathscr{T}x^*$. Therefore, we have $\mathscr{K}u^* = \mathscr{T}u^*$. Since $\mathscr{S}$ (or $\mathscr{T}$) is an α-dominating map,

$$\alpha(u^*, \mathscr{S}u^*) = \alpha(\mathscr{T}x^*, \mathscr{S}u^*) \ge 1.$$

If $u^* = x^*$, then x^* is a common fixed point of $\mathscr{J}, \mathscr{K}, \mathscr{S}$ and $\mathscr{T}$. If $u^* \ne x^*$, then, using $\alpha(\mathscr{T}x^*, \mathscr{S}u^*) \ge 1$, from (16.1), we have

$$\begin{aligned}&\theta(sd_b(\mathcal{J}x^*,\mathcal{K}u^*))\\&\quad\leq\theta(\Delta_b(x^*,u^*))^{G(d_b(\mathcal{S}x^*,\mathcal{J}x^*),d_b(\mathcal{T}u^*,\mathcal{K}u^*),d_b(\mathcal{S}x^*,\mathcal{K}u^*),d_b(\mathcal{T}u^*,\mathcal{J}x^*)},\end{aligned}\tag{16.21}$$

where

$$\begin{aligned}\Delta_b(x^*,u^*)&=\max\left\{\begin{array}{c}d_b(\mathcal{S}x^*,\mathcal{T}u^*),d_b(\mathcal{S}x^*,\mathcal{J}x^*),d_b(\mathcal{T}u^*,\mathcal{K}u^*),\\ \dfrac{d_b(\mathcal{S}x^*,\mathcal{K}u^*)+d_b(\mathcal{T}u^*,\mathcal{J}x^*)}{2s}\end{array}\right\}\\&=\max\left\{d_b(u^*,\mathcal{K}u^*),0,0,\frac{d_b(u^*,\mathcal{K}u^*)}{s}\right\}\\&=d_b(u^*,\mathcal{K}u^*).\end{aligned}\tag{16.22}$$

Also, since $G\in\Delta_G$, there exist $\tau>0$ such that

$$G\begin{pmatrix}d_b(\mathcal{S}x^*,\mathcal{J}x^*),d_b(\mathcal{T}u^*,\mathcal{K}u^*),\\ d_b(\mathcal{S}x^*,\mathcal{K}u^*),d_b(\mathcal{T}u^*,\mathcal{J}x^*)\end{pmatrix}=G\left(0,0,d_b(u^*,\mathcal{K}u^*),d_b(\mathcal{K}u^*,u^*)\right)=\tau.\tag{16.23}$$

Therefore, from (16.21)–(16.23), we have

$$\theta(sd_b(u^*,\mathcal{K}u^*))\leq\theta(d_b(u^*,\mathcal{K}u^*))^{\tau}.\tag{16.24}$$

Now by the property $(\theta1)$ with $\tau>0$, it follows from (16.24) that

$$sd_b(u^*,\mathcal{K}u^*)<d_b(u^*,\mathcal{K}u^*),$$

which is a contradiction, except when $d_b(u^*,\mathcal{K}u^*)=0$. Hence $u^*=\mathcal{K}u^*$, which implies that u^* is a common fixed point of $\mathcal{J},\mathcal{K},\mathcal{S}$ and $\mathcal{T}$. This completes the proof.

16.3.3 Uniqueness of Common Fixed Point

To ensure the uniqueness of the common fixed point for the pair $(\mathcal{T},\mathcal{S})$ of mappings, we will consider the following hypothesis:

$(H9)$ for all $x,y\in CF(\mathcal{S},\mathcal{T})$, $\alpha(x,y)\geq1$ or $\alpha(y,x)\geq1$.

Theorem 16.4 *Adding condition $(H9)$ for the pair $(\mathcal{T},\mathcal{S})$ to the hypotheses of Theorem 16.3, the uniqueness of the common fixed point x^* of $\mathcal{J},\mathcal{K},\mathcal{S}$ and $\mathcal{T}$ is obtained.*

Proof Suppose that $\hat{x}$ is another common fixed point of $\mathcal{J},\mathcal{K},\mathcal{S}$ and $\mathcal{T}$ and, contrary to what is going to be proved, $d_b(\mathcal{J}x^*,\mathcal{K}\hat{x})=d_b(x^*,\hat{x})>0$. Using $(H9)$ for all $x^*,\hat{x}\in CF(\mathcal{T},\mathcal{S})$, we have

$$\alpha(\mathscr{T}x^*, \mathscr{S}\hat{x}) = \alpha(x^*, \hat{x}) \geq 1. \tag{16.25}$$

Now, we can replace x by x^* and y by $\hat{x}$ in the condition (16.1) and we get easily with (16.25):

$$\begin{aligned}\theta(\Delta_b(x^*, \hat{x})) &\leq \theta(sd_b(\mathscr{J}x^*, \mathscr{K}\hat{x})) \\ &\leq \theta(d_b(x^*, \hat{x}))^{G(d_b(\mathscr{S}x^*, \mathscr{J}x^*), d_b(\mathscr{T}\hat{x}, \mathscr{K}\hat{x}), d_b(\mathscr{S}x^*, \mathscr{K}\hat{x}), d_b(\mathscr{T}\hat{x}, \mathscr{J}x^*))} \\ &= \theta(d_b(x^*, \hat{x}))^{G(0,0,d_b(x^*,\hat{x}),d_b(\hat{x},x^*))},\end{aligned} \tag{16.26}$$

where

$$\begin{aligned}\Delta_b(x^*, \hat{x}) &= \max\left\{\begin{array}{c} d_b(\mathscr{S}x^*, \mathscr{T}\hat{x}), d_b(\mathscr{S}x^*, \mathscr{J}x^*), d_b(\mathscr{T}\hat{x}, \mathscr{K}\hat{x}), \\ \dfrac{d_b(\mathscr{S}x^*, \mathscr{K}\hat{x}) + d_b(\mathscr{T}\hat{x}, \mathscr{J}x^*)}{2s} \end{array}\right\} \\ &= \max\left\{d_b(x^*, \hat{x}), 0, 0, \frac{d_b(x^*, \hat{x})}{s}\right\} \\ &= d_b(x^*, \hat{x}).\end{aligned} \tag{16.27}$$

Also, since $G \in \Delta_G$, there exist $\tau > 0$ such that

$$G\left(0, 0, d_b(x^*, \hat{x}), d_b(\hat{x}, x^*)\right) = \tau. \tag{16.28}$$

Therefore, from (16.26)–(16.28), we get

$$\theta(d_b(x^*, \hat{x})) \leq \theta(d_b(x^*, \hat{x}))^{\tau}.$$

Now, by the property $(\theta 1)$ with $\tau > 0$, it follows that

$$d_b(x^*, \hat{x})) < \theta(d_b(x^*, \hat{x}),$$

a contradiction, which implies that $x^* = \hat{x}$. This completes the proof.

If, in Theorems 16.3 and 16.4, the mappings $\mathscr{T}$ and $\mathscr{S}$ are identities, then they can be formulated as results for obtaining the existence and uniqueness of a common fixed point for two mappings $\mathscr{J}, \mathscr{K}$.

Corollary 16.1 *Let $(\mathscr{X}, d_b)$ be an α-complete b-metric space with coefficient $s \geq 1$, $\alpha\colon \mathscr{X} \times \mathscr{X} \to [0, +\infty)$, and $\mathscr{J}, \mathscr{K}\colon \mathscr{X} \to \mathscr{X}$ be given mappings. Suppose that the following conditions hold:*

$(C1)$ there exist $G \in \Delta_G$ and $F \in \mathfrak{F}_s$ such that

$$\begin{cases} (\alpha(x, y) \geq 1 \text{ or } \alpha(y, x) \geq 1) \text{ with } d_b(\mathscr{J}x, \mathscr{K}y) > 0 \\ \Rightarrow \theta(sd_b(\mathscr{J}x, \mathscr{K}y)) \leq \theta(\Delta_b'(x, y))^{G(d_b(x, \mathscr{J}x), d_b(y, \mathscr{K}y), d_b(x, \mathscr{K}y), d_b(y, \mathscr{J}x))}, \end{cases}$$

where

$$\Delta'_b(x, y) = \max\left\{d_b(x, y), d_b(x, \mathscr{J}x), d_b(y, \mathscr{K}y), \frac{d_b(x, \mathscr{K}y) + d_b(y, \mathscr{J}x)}{2s}\right\}$$

for all $x, y \in \mathscr{X}$*;*

$(C2)$ *the pair* $(\mathscr{J}, \mathscr{K})$ *is* α*-weakly increasing;*

$(C3)$ α *is a transitive mapping;*

$(C4)$ $\mathscr{J}, \mathscr{K}$ *are* α*-continuous.*

Then there exists a common fixed point $x^* \in \mathscr{X}$ *of* $\mathscr{J}$ *and* $\mathscr{K}$*. Moreover, if*

$(C5)$ *for all* $x, y \in \mathscr{X}$, $\alpha(x, y) \geq 1$ *or* $\alpha(y, x) \geq 1$,

then the common fixed point of $\mathscr{J}, \mathscr{K}$ *is unique.*

16.3.4 Example

We present an example which illustrates a possible usage of our results.

Example 16.3 (*inspired by* [2]) Let $\mathscr{X} = [0, 1]$ be equipped by the b-metric $d_b(x, y) = (x - y)^2$ and $\alpha\colon \mathscr{X} \times \mathscr{X} \to \mathbb{R}_0^+$ be given as

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(\mathscr{X}, d_b)$ is an α-complete b-metric space with $s = 2$. Consider the mappings $\mathscr{J}, \mathscr{K}, \mathscr{S}, \mathscr{T}\colon \mathscr{X} \to \mathscr{X}$ defined by

$$\mathscr{J}x = \begin{cases} 0, & \text{if } 0 \leq x \leq 1/4, \\ 1/16, & \text{if } 1/4 < x \leq 1; \end{cases} \qquad \mathscr{K}x = 0 \text{ for } 0 \leq x \leq 1;$$

$$\mathscr{T}x = \begin{cases} x, & \text{if } 0 \leq x \leq 1/4, \\ 1, & \text{if } 1/4 < x \leq 1; \end{cases} \qquad \mathscr{S}x = \begin{cases} 0, & \text{if } x = 0, \\ 1/4, & \text{if } 0 < x \leq 1/4, \\ 1, & \text{if } 1/4 < x \leq 1. \end{cases}$$

The only condition of Theorem 16.4 that has to be checked is (H1)—all others are easily seen to hold true.

Take $\theta \in \Theta$ defined by $\theta(t) = e^{\sqrt{t}}$, $t > 0$, and $G \in \Delta_G$ given as $G(t_1, t_2, t_3, t_4) = \min\{t_1, t_2, t_3, t_4\} + \tau$, where $\tau = 1/(8\sqrt{2})$. We will check the contractive condition (16.1). Consider the following cases:

1° $0 \leq x \leq 1/4$, $0 \leq y \leq 1$. Then $d_b(\mathscr{J}x, \mathscr{K}y) = 0$ and there is nothing to prove.

2° $1/4 < x \le 1$, $y = 0$. Then $d_b(\mathscr{J}x, \mathscr{K}y) = (1/16)^2$, $\Delta_b(x,y) \ge d_b(\mathscr{S}x, \mathscr{T}y) = 1$ and $G\begin{pmatrix} d_b(\mathscr{S}x, \mathscr{J}x), d_b(\mathscr{T}y, \mathscr{K}y), \\ d_b(\mathscr{S}x, \mathscr{K}y), d_b(\mathscr{T}y, \mathscr{J}x) \end{pmatrix} = \tau$ and then (16.1) holds true as

$$\begin{aligned}\theta(sd_b(\mathscr{J}x, \mathscr{K}y)) &= e^{\sqrt{2}/16} = e^{\tau\sqrt{d_b(\mathscr{S}x,\mathscr{T}y)}} \le e^{\tau\sqrt{\Delta_b(x,y)}} \\ &\le \theta(\Delta_b(x,y))^{G\begin{pmatrix} d_b(\mathscr{S}x, \mathscr{J}x), d_b(\mathscr{T}y, \mathscr{K}y), \\ d_b(\mathscr{S}x, \mathscr{K}y), d_b(\mathscr{T}y, \mathscr{J}x) \end{pmatrix}}.\end{aligned}$$

3° $1/4 < x \le 1$, $0 < y \le 1/4$. Then we have

$$d_b(\mathscr{J}x, \mathscr{K}y) = (1/16)^2, \quad \Delta_b(x,y) \ge d_b(\mathscr{S}x, \mathscr{T}y) = (1-y)^2 \ge 9/16$$

and

$$G\begin{pmatrix} d_b(\mathscr{S}x, \mathscr{J}x), d_b(\mathscr{T}y, \mathscr{K}y), \\ d_b(\mathscr{S}x, \mathscr{K}y), d_b(\mathscr{T}y, \mathscr{J}x) \end{pmatrix} = y^2 + \tau \le (1/16) + \tau.$$

In this case, (16.1) reduces to

$$\begin{aligned}\theta(sd_b(\mathscr{J}x, \mathscr{K}y)) &= e^{\sqrt{2}/16} \le [e^{\sqrt{\frac{9}{16}}}]^{(y^2+\tau)} \le [e^{1-y}]^{(y^2+\tau)} \\ &\le [e^{\sqrt{d_b(\mathscr{S}x,\mathscr{T}y)}}]^{(y^2+\tau)} \le [e^{\sqrt{\Delta_b(x,y)}}]^{(y^2+\tau)} \\ &\le \theta(\Delta_b(x,y))^{G\begin{pmatrix} d_b(\mathscr{S}x, \mathscr{J}x), d_b(\mathscr{T}y, \mathscr{K}y), \\ d_b(\mathscr{S}x, \mathscr{K}y), d_b(\mathscr{T}y, \mathscr{J}x) \end{pmatrix}}\end{aligned}$$

and holds true for the chosen value of τ.

4° $1/4 < x \le 1$, $1/4 < y \le 1$. Then we have

$$d_b(\mathscr{J}x, \mathscr{K}y) = (1/16)^2, \quad \Delta_b(x,y) \ge d_b(\mathscr{T}y, \mathscr{K}y) = 1$$

and

$$G\begin{pmatrix} d_b(\mathscr{S}x, \mathscr{J}x), d_b(\mathscr{T}y, \mathscr{K}y), \\ d_b(\mathscr{S}x, \mathscr{K}y), d_b(\mathscr{T}y, \mathscr{J}x) \end{pmatrix} = (15/16)^2 + \tau.$$

In this case, (16.1) reduces to

$$\begin{aligned}\theta(sd_b(\mathscr{J}x, \mathscr{K}y)) &= e^{\sqrt{2}/16} \le e^{(\frac{15}{16})^2+\tau} \\ &= [e^{\sqrt{d_b(\mathscr{T}y,\mathscr{K}y)}}]^{((\frac{15}{16})^2+\tau)} \le [e^{\sqrt{\Delta_b(x,y)}}]^{((\frac{15}{16})^2+\tau)} \\ &\le \theta(\Delta_b(x,y))^{G\begin{pmatrix} d_b(\mathscr{S}x, \mathscr{J}x), d_b(\mathscr{T}y, \mathscr{K}y), \\ d_b(\mathscr{S}x, \mathscr{K}y), d_b(\mathscr{T}y, \mathscr{J}x) \end{pmatrix}}\end{aligned}$$

and also holds true for the chosen value of τ.

Thus all the conditions of Theorem 16.4 are fulfilled and we conclude that the mappings $\mathscr{J}, \mathscr{K}, \mathscr{S}, \mathscr{T}$ have a unique common fixed point (which is $x^* = 0$).

16.3.5 Periodic Point Results

It is an obvious fact that, if $\mathscr{J}$ is a self-map which has a fixed point u, then u is also a fixed point of $\mathscr{J}^n$ for arbitrary $n \in \mathbb{N}$. However, the converse is false, i.e., a self-map can have a "periodic" point (a point u satisfying $\mathscr{J}^n u = u$ for some $n \in \mathbb{N}$) which is not its fixed point. In this subsection, we prove some periodic point results for self-mappings on an α-complete b-metric space.

Definition 16.6 ([11]) (1) A mapping $\mathscr{J} : \mathscr{X} \to \mathscr{X}$ is said to have the *property* (P) if it has no periodic points, i.e., if $Fix(\mathscr{J}^n) = Fix(\mathscr{J})$ for each $n \in \mathbb{N}$.

(2) Two mappings $\mathscr{J}, \mathscr{K} : \mathscr{X} \to \mathscr{X}$ are said to have the *property* (Q) if $Fix(\mathscr{J}^n) \cap Fix(\mathscr{K}^n) = Fix(\mathscr{J}) \cap Fix(\mathscr{K})$ for each $n \in \mathbb{N}$.

Theorem 16.5 *In addition to the hypotheses of Corollary 16.1, let the following condition hold:*

$(H10)$ *If $w \in Fix(\mathscr{J}^n) \cap Fix(\mathscr{K}^n)$ and $w \notin Fix(\mathscr{J}) \cap Fix(\mathscr{K})$, then we have*

$$\alpha(\mathscr{J}^{n-1}w, \mathscr{K}^n w) \geq 1 \ \text{ or } \ \alpha(\mathscr{J}^n w, \mathscr{J}^{n-1}w) \geq 1.$$

Then $\mathscr{J}$ and $\mathscr{K}$ have the property (Q).

Proof By Corollary 16.1, $\mathscr{J}$ and $\mathscr{K}$ have a unique common fixed point in $\mathscr{X}$. Suppose $w \in Fix(\mathscr{J}^n) \cap Fix(\mathscr{K}^n)$ and $w \notin Fix(\mathscr{J}) \cap Fix(\mathscr{K})$; then $d_b(w, \mathscr{J}w) > 0$ or $d_b(w, \mathscr{K}w) > 0$ (for example, let the latter condition hold). Applying $(H10)$ and $(H1)$, we get

$$\begin{aligned} \theta(sd_b(w, \mathscr{K}w)) &= \theta(sd_b(\mathscr{J}(\mathscr{J}^{n-1}w), \mathscr{K}(\mathscr{K}^n w))) \\ &\leq \theta(\Delta_b(\mathscr{J}^{n-1}w, \mathscr{K}^n w))^{\Lambda}, \end{aligned} \tag{16.29}$$

where

$$\begin{aligned} \Lambda &= G\begin{pmatrix} d_b(\mathscr{J}^{n-1}w, \mathscr{J}\mathscr{J}^{n-1}w), d_b(\mathscr{K}^n w, \mathscr{K}\mathscr{K}^n w), \\ d_b(\mathscr{J}^{n-1}w, \mathscr{K}\mathscr{K}^n w), d_b(\mathscr{K}^n w, \mathscr{J}\mathscr{J}^{n-1}w) \end{pmatrix} \\ &= G(d_b(\mathscr{J}^{n-1}w, w), d_b(w, \mathscr{K}w), d_b(\mathscr{J}^{n-1}w, w), 0). \end{aligned} \tag{16.30}$$

Since $G \in \Delta_G$, there exists $\tau > 0$, such that

$$\Lambda = G\left(d_b(\mathscr{J}^{n-1}w, w), d_b(w, \mathscr{K}w), d_b(\mathscr{J}^{n-1}w, w), 0\right) = \tau. \tag{16.31}$$

Also, we have

$$
\begin{aligned}
&\Delta_b(\mathscr{J}^{n-1}w, \mathscr{K}^n w) \\
&= \max\left\{\begin{array}{c} d_b(\mathscr{J}^{n-1}w, \mathscr{K}^n w), d_b(\mathscr{J}^{n-1}w, \mathscr{J}\mathscr{J}^{n-1}w), d_b(\mathscr{K}^n w, \mathscr{K}\mathscr{K}^n w), \\ \dfrac{d_b(\mathscr{J}^{n-1}w, \mathscr{K}\mathscr{K}^n w) + d_b(\mathscr{K}^n w, \mathscr{J}\mathscr{J}^{n-1}w)}{2s} \end{array}\right\} \\
&= \max\Big\{d_b(\mathscr{J}^{n-1}w, w), d_b(\mathscr{J}^{n-1}w, w), d_b(w, \mathscr{K}w), \\
&\qquad \frac{d_b(\mathscr{J}^{n-1}w, \mathscr{K}w) + d_b(w, w)}{2s}\Big\} \\
&\leq \max\left\{d_b(\mathscr{J}^{n-1}w, w), d_b(w, \mathscr{K}w), \frac{d_b(\mathscr{J}^{n-1}w, w) + d_b(w, \mathscr{K}w)}{2}\right\} \\
&= \max\{d_b(\mathscr{J}^{n-1}w, w), d_b(w, \mathscr{K}w)\} \\
&= d_b(\mathscr{J}^{n-1}w, w). \qquad (16.32)
\end{aligned}
$$

Consequently, from (16.29)–(16.32), we can write

$$
\begin{aligned}
1 \leq \theta(s d_b(w, \mathscr{K}w)) &\leq \theta(d_b(\mathscr{J}^{n-1}w, \mathscr{J}^n w))^{\tau} \\
&\leq \theta(d_b(\mathscr{J}^{n-2}w, \mathscr{J}^{n-1}w))^{\tau^2} \\
&\leq \cdots \\
&\leq \theta(d_b(w, \mathscr{J}w))^{\tau^n}.
\end{aligned}
$$

By taking the limit as $n \to \infty$ in the above inequality, we have $\theta(s d_b(w, \mathscr{K}w)) = 1$, which is a contradiction and hence we deduce that $d_b(w, \mathscr{K}w) = 0$, that is, $\mathscr{K}w = w$. From the conclusion of Corollary 16.1, we also have $\mathscr{J}w = w$. Therefore, $Fix(\mathscr{J}^n) \cap Fix(\mathscr{K}^n) = Fix(\mathscr{J}) \cap Fix(\mathscr{K})$ for all $n \in \mathbb{N}$. This completes the proof.

16.4 Application

Consider the following system of Urysohn integral equations:

$$
\begin{cases}
u(t) = \hbar_1(t) + \int_0^T \Upsilon_1(t, s, u(s))\, ds, & t \in [0, T], \\
u(t) = \hbar_2(t) + \int_0^T \Upsilon_2(t, s, u(s))\, ds, & t \in [0, T], \\
u(t) = \hbar_3(t) + \int_0^T \Upsilon_3(t, s, u(s))\, ds, & t \in [0, T], \\
u(t) = \hbar_4(t) + \int_0^T \Upsilon_4(t, s, u(s))\, ds, & t \in [0, T],
\end{cases}
\qquad (16.33)
$$

where $T > 0, t \in [0, T], \hbar_i : [0, T] \to \mathbb{R}$ and $\Upsilon_i : [0, T]^2 \times \mathbb{R} \to \mathbb{R}$ $(i \in \{1, 2, 3, 4\})$ are given mappings.

The purpose of this section is to apply Theorems 16.3 and 16.4 in order to prove the existence and uniqueness of common solution of (16.33). For more detailed study, one can refer to [15, 17] and some other related papers.

Let $I = [0, T]$ and $\mathscr{X} := C(I, \mathbb{R})$ be equipped with the usual maximum norm, i.e., $\|u\|_{\mathscr{X}} = \max_{t\in I} |u(t)|$ for all $u \in C(I, \mathbb{R})$. Then $(\mathscr{X}, \|\cdot\|_{\mathscr{X}})$ is a complete metric space. The distance in $\mathscr{X}$ is given by

$$d_\infty(u, v) = \max_{t\in I} |u(t) - v(t)|, \quad \forall u, v \in \mathscr{X}.$$

Moreover, we can define a b-metric d_b on $\mathscr{X}$ by $d_b(u, v) = [d_\infty(u, v)]^p$ for some $p > 1$ and all $u, v \in \mathscr{X}$. Since $(\mathscr{X}, d_\infty)$ is complete, we deduce that $(\mathscr{X}, d_b)$ is a b-complete b-metric space with $s = 2^{p-1}$. Throughout this section, for each $i \in \{1, 2, 3, 4\}$ and Υ_i in (16.33), we will denote by $\Psi_i : \mathscr{X} \to \mathscr{X}$ the operator defined by

$$\Psi_i u(t) := \int_0^T \Upsilon_i(t, s, u(s))\, ds, \quad \forall u \in \mathscr{X},\ t \in I.$$

We will also use the following partial order on $\mathscr{X}$:

$$u \preceq v \iff u(t) \le v(t), \quad \forall t \in [0, T].$$

Theorem 16.6 *Suppose that the following hypotheses hold:*
(U1) *There exist* $\lambda \in (0, 1)$ *and* $p > 1$ *such that, for all* $u, v \in \mathscr{X}$,

$$\begin{cases} 2u - \Psi_3 u - \hbar_3 \preceq 2v - \Psi_4 v - \hbar_4 \ \ or \\ 2u - \Psi_4 u - \hbar_4 \preceq 2v - \Psi_3 v - \hbar_3 \end{cases} \tag{16.34}$$

implies that

$$\begin{aligned} 2^{p-1} \max_{t\in I} \mathscr{A}(u, v)(t)\ \exp\{2^{p-1} \max_{t\in I} \mathscr{A}(u, v)(t)\} \\ \le \lambda \max_{t\in I} \Delta_b(u, v)(t)\ \exp\{\max_{t\in I} \Delta_b(u, v)(t)\}, \end{aligned} \tag{16.35}$$

where

$$\begin{aligned} &\Delta_b(u, v)(t) \\ &= \max\left\{\mathscr{B}(u, v)(t), \mathscr{C}(u, v)(t), \mathscr{D}(u, v)(t), \frac{1}{2^p}[\mathscr{E}(u, v)(t) + \mathscr{F}(u, v)(t)]\right\} \end{aligned}$$

and

$$\begin{aligned}
\mathscr{A}(u,v)(t) &= |\Psi_1 u(t) + \hbar_1(t) - \Psi_2 v(t) - \hbar_2(t)|^p,\\
\mathscr{B}(u,v)(t) &= |2u(t) - \Psi_3 u(t) - \hbar_3(t) - 2v(t) + \Psi_4 v(t) + \hbar_4(t)|^p,\\
\mathscr{C}(u,v)(t) &= |\Psi_1 u(t) + \hbar_1(t) - 2u(t) + \Psi_3 u(t) + \hbar_3(t)|^p,\\
\mathscr{D}(u,v)(t) &= |\Psi_2 v(t) + \hbar_2(t) - 2v(t) + \Psi_4 v(t) + \hbar_4(t)|^p,\\
\mathscr{E}(u,v)(t) &= |\Psi_2 v(t) + \hbar_2(t) - 2u(t) + \Psi_3 u(t) + \hbar_3(t)|^p,\\
\mathscr{F}(u,v)(t) &= |\Psi_1 u(t) + \hbar_1(t) - 2v(t) + \Psi_4 v(t) + \hbar_4(t)|^p.
\end{aligned}$$

(U2) *For each $u \in \mathscr{X}$, there is some $v \in \mathscr{X}$ such that*

$$\Psi_1 u + \hbar_1 = 2v - \Psi_4 v - \hbar_4$$

and, for each $u \in \mathscr{X}$, there is some $v \in \mathscr{X}$ such that

$$\Psi_2 u + \hbar_2 = 2v - \Psi_3 v - \hbar_3.$$

(U3) *For all $u, v \in \mathscr{X}$, we have*

$$2v - \Psi_4 v - \hbar_4 = \Psi_1 u + \hbar_1 \implies \Psi_1 u + \hbar_1 \preceq \Psi_2 v + \hbar_2,$$

and

$$2v - \Psi_3 v - \hbar_3 = \Psi_2 u + \hbar_2 \implies \Psi_2 u + \hbar_2 \preceq \Psi_1 v + \hbar_1.$$

(U4) *The mappings $\hbar_i\colon I \to \mathbb{R}$ and $\Upsilon_i\colon [0,T]^2 \times \mathbb{R} \to \mathbb{R}$ $(i \in \{1,2,3,4\})$ are continuous.*

(U5$_1$) *If $\{u_n\}$ is a sequence in $\mathscr{X}$ such that $u_n \preceq u_{n+1}$ for all $n \in \mathbb{N}$ and, for all $y \in \mathscr{X}$,*

$$\begin{aligned}
\max_{t\in I} |\Psi_1 u_n(t) + \hbar_1(t) - y(t)|^p &\to 0 \ \text{ as } n \to \infty,\\
\max_{t\in I} |2u_n(t) - \Psi_3 u_n(t) - \hbar_3(t) - y(t)|^p &\to 0 \ \text{ as } n \to \infty,
\end{aligned}$$

then

$$\begin{aligned}
&\max_{t\in I} |[\hbar_1(t) + \Psi_1(2u_n(t) - \Psi_3 u_n(t) - \hbar_3(t))]\\
&\qquad - [2(\Psi_1 u_n(t) + \hbar_1(t)) - \Psi_3(\Psi_1 u_n(t) + \hbar_1(t)) - \hbar_3(t)]|^p \to 0 \ \text{ as } n \to \infty.
\end{aligned}$$

(U5$_2$) *If $\{u_n\}$ is a sequence in $\mathscr{X}$ such that $u_n \preceq u_{n+1}$ for all $n \in \mathbb{N}$ and, for all $y \in \mathscr{X}$,*

$$\max_{t\in I} |\Psi_2 u_n(t) + \hbar_2(t) - y(t)|^p \to 0 \ \ as\ n \to \infty,$$
$$\max_{t\in I} |2u_n(t) - \Psi_4 u_n(t) - \hbar_4(t) - y(t)|^p \to 0 \ \ as\ n \to \infty,$$

then

$$\max_{t\in I} |[\hbar_2(t) + \Psi_2(2u_n(t) - \Psi_4 u_n(t) - \hbar_4(t))] \\ - [2(\Psi_2 u_n(t) + \hbar_2(t)) - \Psi_4(\Psi_2 u_n(t) + \hbar_2(t)) - \hbar_4(t)]|^p \to 0\ as\ n \to \infty.$$

(U6) $u \preceq 2u - \Psi_3 u - \hbar_3$ *for all* $u \in \mathscr{X}$ *or* $u \preceq 2u - \Psi_4 u - \hbar_4$ *for all* $u \in \mathscr{X}$.

Then the system (16.33) *has a solution. Moreover, if*

(U7) *for any two solutions* u^*, v^* *of the system* (16.33), $u^* \preceq v^*$ *or* $v^* \preceq u^*$ *holds, then the solution of* (16.33) *is unique.*

Proof Define the mappings $\mathscr{J}, \mathscr{K}, \mathscr{S}, \mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$\begin{aligned} \mathscr{J}u(t) &= \Psi_1 u(t) + \hbar_1(t) = \int_0^T \Upsilon_1(t, s, u(s))\, ds + \hbar_1(t), \\ \mathscr{K}u(t) &= \Psi_2 u(t) + \hbar_2(t) = \int_0^T \Upsilon_2(t, s, u(s))\, ds + \hbar_2(t), \\ \mathscr{S}u(t) &= 2u(t) - \Psi_3 u(t) - \hbar_3(t) = 2u(t) - \int_0^T \Upsilon_3(t, s, u(s))\, ds - \hbar_3(t), \\ \mathscr{T}u(t) &= 2u(t) - \Psi_4 u(t) - \hbar_4(t) = 2u(t) - \int_0^T \Upsilon_4(t, s, u(s))\, ds - \hbar_4(t), \end{aligned} \tag{16.36}$$

respectively. Define also a function $\alpha : \mathscr{X}^2 \to [0, \infty)$ by

$$\alpha(u, v) = \begin{cases} 1, & \text{if } u(t) \le v(t) \text{ for all } t \in I, \\ 0, & \text{otherwise.} \end{cases}$$

Now, we check the validity of the conditions (H1)–(H6) of Theorem 16.1 and (H8) of Theorem 16.3 as well as (under the assumption (U7)), (H9) of Theorem 16.4.

(H1) By the definition (16.36) of the mappings $\mathscr{J}, \mathscr{K}, \mathscr{S}, \mathscr{T}$ and the definition of b-metric d_b, we have that, for all $u, v \in \mathscr{X}$,

$$\begin{cases} d_b(\mathcal{J}u, \mathcal{K}v) = \max\limits_{t\in[0,T]} |\Psi_1 u(t) + \hbar_1(t) - \Psi_2 v(t) - \hbar_2(t)|^p, \\ d_b(\mathcal{S}u, \mathcal{T}v) = \max\limits_{t\in[0,T]} |2u(t) - \Psi_3 u(t) - \hbar_3(t) - 2v(t) + \Psi_4 v(t) + \hbar_4(t)|^p, \\ d_b(\mathcal{J}v, \mathcal{S}u) = \max\limits_{t\in[0,T]} |\Psi_1 u(t) + \hbar_1(t) - 2u(t) + \Psi_3 u(t) + \hbar_3(t)|^p, \\ d_b(\mathcal{K}v, \mathcal{T}v) = \max\limits_{t\in[0,T]} |\Psi_2 v(t) + \hbar_2(t) - 2v(t) + \Psi_4 v(t) + \hbar_4(t)|^p, \\ d_b(\mathcal{J}u, \mathcal{T}v) = \max\limits_{t\in[0,T]} |\Psi_1 u(t) + \hbar_1(t) - 2v(t) + \Psi_4 v(t) + \hbar_4(t)|^p, \\ d_b(\mathcal{S}u, \mathcal{K}v) = \max\limits_{t\in[0,T]} |\Psi_2 v(t) + \hbar_2(t) - 2u(t) + \Psi_3 u(t) + \hbar_3(t)|^p, \end{cases}$$

respectively. Suppose that $\alpha(\mathcal{S}u, \mathcal{T}v) \geq 1$. Then we have $\mathcal{S}u \preceq \mathcal{T}v$, i.e., the assumption (16.34) of (U1) holds and hence also its conclusion (16.35) holds true. But this means that the implication (16.1) is valid for the function $\theta \in \Theta$ given as $\theta(t) = \exp\{\sqrt{t\ \exp(t)}\}$ and $G \in \Delta_G$ given as $G(t_1, t_2, t_3, t_4) \equiv \tau$ $(\tau^2 = \lambda)$, $\tau > 0$. Hence (H1) is proved.

(H2) is a direct consequence of the assumption (U2).

(H3) Let $u \in \mathcal{X}$ and $v \in \mathcal{T}^{-1}(\mathcal{J}u)$. Then $2v - \Psi_4 v - \hbar_4 = \Psi_1 u + \hbar_1$ and, by the assumption (U3), $\Psi_1 u + \hbar_1 \preceq \Psi_2 v + \hbar_2$ holds. That is, $\mathcal{J}u \preceq \mathcal{K}v$ and so $\alpha(\mathcal{J}u, \mathcal{K}v) \geq 1$. Hence the pair $(\mathcal{J}, \mathcal{K})$ is partially α-weakly increasing w.r.t. $\mathcal{T}$. Similarly, the pair $(\mathcal{K}, \mathcal{J})$ is partially α-weakly increasing w.r.t. $\mathcal{S}$.

(H4) follows easily from the definition of mapping α and (H5) follows from the assumption (U4).

(H6) Let $\{u_n\}$ be a sequence in $\mathcal{X}$ such that $\alpha(u_n, u_{n+1}) \geq 1$, i.e., $u_n \preceq u_{n+1}$ for $n \in \mathbb{N}$, and let $\lim_{n\to\infty} \mathcal{J}u_n = \lim_{n\to\infty} \mathcal{S}u_n = y$ in $(\mathcal{X}, d_b)$, i.e.,

$$\max_{t\in I} |\Psi_1 u_n(t) + \hbar_1(t) - y(t)|^p \to 0 \text{ as } n \to \infty,$$
$$\max_{t\in I} |2u_n(t) - \Psi_3 u_n(t) - \hbar_3(t) - y(t)|^p \to 0 \text{ as } n \to \infty.$$

By the assumption ($U5_1$), it follows that

$$\max_{t\in I} |[\hbar_1(t) + \Psi_1(2u_n(t) - \Psi_3 u_n(t) - \hbar_3(t))] \\ - [2(\Psi_1 u_n(t) + \hbar_1(t)) - \Psi_3(\Psi_1 u_n(t) + \hbar_1(t)) - \hbar_3(t)]|^p \to 0 \text{ as } n \to \infty,$$

i.e., $\lim_{n\to\infty} d_b(\mathcal{J}\mathcal{S}u_n, \mathcal{S}\mathcal{J}u_n) = 0$. Hence the pair $(\mathcal{J}, \mathcal{S})$ is α-compatible. Similarly, it follows from ($U5_2$) that the pair $(\mathcal{K}, \mathcal{T})$ is α-compatible.

The condition (H8) (that $\mathcal{S}$ or $\mathcal{T}$ is an α-dominating map) follows directly from the assumption (U6).

Thus all the conditions of Theorem 16.3 are fulfilled and it follows that the mappings $\mathcal{J}, \mathcal{K}, \mathcal{S}, \mathcal{T}$ have a common fixed point $u^* \in \mathcal{X}$. It is easy to see that u^* is then a solution of the system (16.33).

Finally, if the assumption (U7) is fulfilled, then it follows that the condition (H9) of Theorem 16.4 holds and hence the solution of (16.33) is unique. This completes the proof.

16.5 Conclusion

In this chapter, the notion of α-G-JS-type contraction for four mappings in the setup of b-metric spaces has been introduced, and coincidence points, common fixed points, their uniqueness, as well as periodic points have been discussed under α-compatible and relatively partially α-weakly increasing conditions on α-complete b-metric spaces. The given notions and results are illustrated by a suitable example, followed by application to the proof of existence of solutions for a system of Urysohn integral equations.

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Chapter 17
Unified Multi-tupled Fixed Point Theorems Involving Monotone Property in Ordered Metric Spaces

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Abstract In this chapter, we introduce a generalized notion of monotone property and prove some results regarding existence and uniqueness of multi-tupled fixed points for nonlinear contraction mappings satisfying monotone property in ordered complete metric spaces. Our results unify several classical and well-known n-tupled (including coupled, tripled and quadruple ones) fixed point results in the existing literature.

Keywords $*$-fixed point · Ordered metric spaces · Monotone property · φ-contractions

17.1 Introduction

Throughout the chapter, the following symbols and notations are involved.

(1) As usual, (X, d), $(X, \preceq)$ and $(X, d, \preceq)$ are termed as metric space, ordered set and ordered metric space, wherein X stands for a nonempty set, d for a metric on

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X and $\preceq$ for a partial order on X. Moreover, if the metric space (X, d) is complete, then $(X, d, \preceq)$ is termed as ordered complete metric space.

(2) $\succeq$ denotes dual partial order of $\preceq$ (i.e., $x \succeq y$ means $y \preceq x$).

(3) $\mathbb{N}$ and $\mathbb{N}_0$ stands for the sets of positive and non-negative integers respectively (i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$).

(4) n stand for a fixed natural number greater than 1, while $m, l \in \mathbb{N}_0$.

(5) I_n denotes the set $\{1, 2, \ldots, n\}$ and we use $i, j, k \in I_n$.

(6) For a nonempty set X, X^n denotes the Cartesian product of n identical copies of X, i.e., $X^n := X \times X \times \overset{(n)}{\cdots} \times X$. We call X^n the *n-dimensional product set* induced by X.

(7) A sequence in X is denoted by $\{x^{(m)}\}$ and a sequence in X^n is denoted by $\{\mathrm{U}^{(m)}\}$ where $\mathrm{U}^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})$ such that for each $i \in I_n$, $\{x_i^{(m)}\}$ is a sequence in X.

Starting from the Bhaskar-Lakshmikantham coupled fixed point theorem [1], the branch of multi-tupled fixed point theory in ordered metric spaces is progressed in high speed during only one decade. Then, coupled fixed point theorems are extended up to higher dimensional product set by appearing tripled (in [2]), quadrupled (in [3]) and n-tupled (in [4]) fixed point theorems. Here it can be highlighted that extension of coupled fixed point up to higher dimensional product set is not unique. It is defined by various authors in different ways. In recent years, some authors paid attention to unify the different types of multi-tupled fixed points. A first attempt of this kind was given by Berzig and Samet [5], wherein the authors defined a unified notion of n-tupled fixed point by using $2n$ mappings from I_n to I_n. Later, Roldán et al. [6] extended the notion of n-tupled fixed point of Berzig and Samet [5] by introducing the notion of Υ-fixed point based on n mappings from I_n to I_n. In 2016, Alam et al. [7] modified the notion of Υ-fixed point by introducing the notion of $*$-fixed point depending on a binary operation $*$ on I_n. Although the notion of $*$-fixed point is equivalent to that of Υ-fixed point (see [7]) but it is relatively more natural and effective as compared to Υ-fixed point due to its matrix representation. Here it can be pointed out that Choban and Berinde [8] also proved some multidimensional fixed point results in certain distance spaces for λ-contractions.

One of the common properties of multi-tupled fixed point theory in the context of ordered metric spaces is that the mapping $F : X^n \to X$ satisfies mixed monotone property (for instance, see [9–12]). In order to avoid the mixed monotone property in such results, authors in [13–21] utilized the notion of monotone property.

The aim of this chapter is to extend the notion of monotone property for the mapping $F : X^n \to X$ and utilizing this and to prove some existence and uniqueness results on $*$-coincidence points under φ-contractions due to Boyd and Wong [22].

17.2 Extended Notions Upto Product Sets

With a view to extend the domain of the mapping $f : X \longrightarrow X$ to n-dimensional product set X^n, we introduce the variants of the notions of monotonicity, fixed and coincidence points, continuity, g-continuity, compatibility, and weak compatibility for the mapping $F : X^n \to X$. Recall that a binary operation $*$ on a set S is a mapping from $S \times S$ to S and a permutation π on a set S is a one-one mapping from S onto itself ($cf.$ Herstein [23]). Throughout this manuscript, we adopt the following notations:

(1) In order to understand a binary operation $*$ on I_n, we denote the image of any element $(i, k) \in I_n \times I_n$ under $*$ by i_k rather than $*(i, k)$.

(2) A binary operation $*$ on I_n can be identically represented by an $n \times n$ matrix throughout its ordered image such that the first and second components run over rows and columns, respectively, i.e.,

$$* = [m_{ik}]_{n\times n} \text{ where } m_{ik} = i_k \text{ for each } i, k \in I_n.$$

(3) A permutation π on I_n can be identically represented by an n-tuple throughout its ordered image, i.e.,

$$\pi = (\pi(1), \pi(2), \ldots, \pi(n)).$$

(4) $\mathfrak{B}_\mathfrak{n}$ denotes the family of all binary operations $*$ on I_n, i.e.,

$$\mathfrak{B}_\mathfrak{n} = \{* : * : I_n \times I_n \to I_n\}.$$

Remark 17.1 It is clear, for each $i \in I_n$, that

$$\{i_1, i_2, \ldots, i_n\} \subseteq I_n.$$

We define generalized notions of monotone property as follows.

Definition 17.1 Let $(X, \preceq)$ be an ordered set and $F : X^n \to X$ and $g : X \to X$ two mappings. We say that F has the *argumentwise g-monotone property* if F is g-increasing in each of its arguments, i.e., for any $x_1, x_2, \ldots, x_n \in X$ and $i \in I_n$,

$$\begin{aligned}&\underline{x}_i, \overline{x}_i \in X, \quad g(\underline{x}_i) \preceq g(\overline{x}_i)\\ &\Longrightarrow F(x_1, x_2, \ldots, x_{i-1}, \underline{x}_i, x_{i+1}, \ldots, x_n) \preceq F(x_1, x_2, \ldots, x_{i-1}, \overline{x}_i, x_{i+1}, \ldots, x_n).\end{aligned}$$

Definition 17.2 Let $(X, \preceq)$ be an ordered set and $F : X^n \to X$ and $g : X \to X$ two mappings. We say that F has the *g-monotone property* if, for any $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$,

$$\begin{aligned}&g(x_1) \preceq g(y_1),\ g(x_2) \preceq g(y_2),\ \ldots,\ g(x_n) \preceq g(y_n)\\ &\Longrightarrow F(x_1, x_2, \ldots, x_n) \preceq F(y_1, y_2, \ldots, y_n).\end{aligned}$$

On particularizing with $g = I$, the identity mapping on X, the notions employed in Definitions 17.1 and 17.2 are, respectively, called argumentwise monotone property and monotone property.

Notice that the notion of 'monotone mappings' introduced by Borcut [13] is the same as the notion of 'argumentwise monotone property' presented in Definition 17.1 but different from 'monotone property' embodied in Definition 17.2. Henceforth, coherently with Definition 17.1, we prefer employing the term 'argumentwise monotone property' instead of 'monotone mappings'.

It is clear that if F has argumentwise monotone property (resp. argumentwise g-monotone property) then it also has monotone property (resp. g-monotone property).

Definition 17.3 ([7]) Let X be a nonempty set, $* \in \mathfrak{B}_\mathfrak{n}$ and $F : X^n \to X$ and $g : X \to X$ two mappings. An element $(x_1, x_2, \ldots, x_n) \in X^n$ is called an *n-tupled coincidence point* of F and g w.r.t. $*$ (or, in short, *$*$-coincidence point* of F and g) if

$$F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) = g(x_i) \quad \text{for each } i \in I_n.$$

In this case, $(gx_1, gx_2, \ldots, gx_n)$ is called a *point of $*$-coincidence* of F and g

Notice that if g is an identity mapping on I_n then the notion employed in Definition 17.3 is called an n-tupled fixed point of F w.r.t. $*$ (or, in short, $*$-fixed point of F).

Definition 17.4 ([7]) Let X be a nonempty set, $* \in \mathfrak{B}_\mathfrak{n}$ and $F : X^n \to X$ and $g : X \to X$ two mappings. An element $(x_1, x_2, \ldots, x_n) \in X^n$ is called a *common n-tupled fixed point* of F and g w.r.t. $*$ (or, in short, *common $*$-fixed point* of F and g) if

$$F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) = g(x_i) = x_i \quad \text{for each } i \in I_n.$$

Definition 17.5 ([7]) A binary operation $*$ on I_n is called *permuted* if each row of matrix representation of $*$ forms a permutation on I_n.

Example 17.1 ([7]) On I_3, consider two binary operations:

$$* = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \quad \circ = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 2 \end{bmatrix}$$

$*$ is permuted as each of rows $(1, 2, 3)$, $(2, 1, 3)$, $(3, 2, 1)$ is a permutation on I_3, while $\circ$ is not permuted as last row $(3, 3, 2)$ is not permutation on I_3.

Proposition 17.1 ([7]) *A permutation $*$ on I_n is permuted if and only if, for each $i \in I_n$,*

$$\{i_1, i_2, \ldots, i_n\} = I_n.$$

Definition 17.6 ([7]) Let (X, d) be a metric space, $F : X^n \to X$ be a mapping and let $(x_1, x_2, \ldots, x_n) \in X^n$. We say that F is *continuous* at $(x_1, x_2, \ldots, x_n)$ if, for any sequences $\{x_1^{(m)}\}, \{x_2^{(m)}\}, \ldots, \{x_n^{(m)}\} \subset X$,

$$x_1^{(m)} \xrightarrow{d} x_1,\ x_2^{(m)} \xrightarrow{d} x_2,\ \ldots,\ x_n^{(m)} \xrightarrow{d} x_n$$
$$\Longrightarrow F(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \xrightarrow{d} F(x_1, x_2, \ldots, x_n).$$

Moreover, F is called *continuous* if it is continuous at each point of X^n.

Definition 17.7 ([7]) Let (X, d) be a metric space, $F : X^n \to X$, $g : X \to X$ be two mappings and let $(x_1, x_2, \ldots, x_n) \in X^n$. We say that F is *g-continuous* at $(x_1, x_2, \ldots, x_n)$ if for any sequences $\{x_1^{(m)}\}, \{x_2^{(m)}\}, \ldots, \{x_n^{(m)}\} \subset X$,

$$g(x_1^{(m)}) \xrightarrow{d} g(x_1),\ g(x_2^{(m)}) \xrightarrow{d} g(x_2), \ldots,\ g(x_n^{(m)}) \xrightarrow{d} g(x_n)$$

$$\Longrightarrow F(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \xrightarrow{d} F(x_1, x_2, \ldots, x_n).$$

Moreover, F is called *g-continuous* if it is g-continuous at each point of X^n.

Notice that, setting $g = I$ (: the identity mapping on X), Definition 17.7 reduces to Definition 17.6.

Let $(X, d, \preceq)$ be an ordered metric space and $\{x_n\}$ be a sequence in X. We adopt the following notations:

(1) If $\{x_n\}$ is increasing and $x_n \xrightarrow{d} x$, then we denote it symbolically by $x_n \uparrow x$.
(2) If $\{x_n\}$ is decreasing and $x_n \xrightarrow{d} x$, then we denote it symbolically by $x_n \downarrow x$.
(3) If $\{x_n\}$ is monotone and $x_n \xrightarrow{d} x$, then we denote it symbolically by $x_n \uparrow\downarrow x$.

Definition 17.8 Let $(X, d, \preceq)$ be an ordered metric space, $F : X^n \to X$ be a mapping and let $(x_1, x_2, \ldots, x_n) \in X^n$. We say that F is

(1) $\overline{\mathrm{O}}$-*continuous* at $(x_1, x_2, \ldots, x_n)$ if, for any sequences $\{x_1^{(m)}\}, \{x_2^{(m)}\}, \ldots, \{x_n^{(m)}\} \subset X$,

$$x_1^{(m)} \uparrow x_1,\ x_2^{(m)} \uparrow x_2, \ldots,\ x_n^{(m)} \uparrow x_n$$

$$\Longrightarrow F(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \xrightarrow{d} F(x_1, x_2, \ldots, x_n);$$

(2) $\underline{\mathrm{O}}$-*continuous* at $(x_1, x_2, \ldots, x_n)$ if, for any sequences $\{x_1^{(m)}\}, \{x_2^{(m)}\}, \ldots, \{x_n^{(m)}\} \subset X$,

$$x_1^{(m)} \downarrow x_1,\ x_2^{(m)} \downarrow x_2, \ldots,\ x_n^{(m)} \downarrow x_n$$

$$\Longrightarrow F(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \xrightarrow{d} F(x_1, x_2, \ldots, x_n);$$

(3) O-*continuous* at $(x_1, x_2, \ldots, x_n)$ if, for any sequences $\{x_1^{(m)}\}, \{x_2^{(m)}\}, \ldots, \{x_n^{(m)}\} \subset X$,

$$x_1^{(m)} \uparrow\downarrow x_1,\ x_2^{(m)} \uparrow\downarrow x_2, \ldots,\ x_n^{(m)} \uparrow\downarrow x_n$$

$$\Longrightarrow F(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \xrightarrow{d} F(x_1, x_2, \ldots, x_n).$$

Moreover, F is called *O-continuous* (resp., *$\overline{\mathrm{O}}$-continuous*, *$\underline{\mathrm{O}}$-continuous*) if it is O-continuous (resp., $\overline{\mathrm{O}}$-continuous, $\underline{\mathrm{O}}$-continuous) at each point of X^n.

Remark 17.2 In an ordered metric space, the continuity $\Longrightarrow$ the O-continuity $\Longrightarrow$ the $\overline{\mathrm{O}}$-continuity as well as the $\underline{\mathrm{O}}$-continuity.

Definition 17.9 Let $(X, d, \preceq)$ be an ordered metric space, $F : X^n \to X, g : X \to X$ two mappings and let $(x_1, x_2, \ldots, x_n) \in X^n$. We say that F is

(1) $(g, \overline{\mathrm{O}})$*-continuous* at $(x_1, x_2, \ldots, x_n)$ if, for any sequences $\{x_1^{(m)}\}$, $\{x_2^{(m)}\}$, …, $\{x_n^{(m)}\} \subset X$,

$$g(x_1^{(m)}) \uparrow g(x_1),\ g(x_2^{(m)}) \uparrow g(x_2), \ldots,\ g(x_n^{(m)}) \uparrow g(x_n)$$

$$\Longrightarrow F(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \xrightarrow{d} F(x_1, x_2, \ldots, x_n);$$

(2) $(g, \underline{\mathrm{O}})$*-continuous* at $(x_1, x_2, \ldots, x_n)$ if for any sequences $\{x_1^{(m)}\}$, $\{x_2^{(m)}\}$, …, $\{x_n^{(m)}\} \subset X$,

$$g(x_1^{(m)}) \downarrow g(x_1),\ g(x_2^{(m)}) \downarrow g(x_2), \ldots,\ g(x_n^{(m)}) \downarrow g(x_n)$$

$$\Longrightarrow F(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \xrightarrow{d} F(x_1, x_2, \ldots, x_n);$$

(3) (g, O) *-continuous* at $(x_1, x_2, \ldots, x_n)$ if, for any sequences $\{x_1^{(m)}\}$, $\{x_2^{(m)}\}$, …, $\{x_n^{(m)}\} \subset X$,

$$g(x_1^{(m)}) \uparrow\downarrow g(x_1),\ g(x_2^{(m)}) \uparrow\downarrow g(x_2), \ldots,\ g(x_n^{(m)}) \uparrow\downarrow g(x_n)$$

$$\Longrightarrow F(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \xrightarrow{d} F(x_1, x_2, \ldots, x_n).$$

Notice that, setting $g = I$ (: the identity mapping on X), Definition 17.9 reduces to Definition 17.8.

Remark 17.3 In an ordered metric space, the g-continuity $\Longrightarrow$ the (g, O)-continuity $\Longrightarrow$ the $(g, \overline{\mathrm{O}})$-continuity as well as the $(g, \underline{\mathrm{O}})$-continuity.

Definition 17.10 ([7]) Let X be a nonempty set and $F : X^n \to X$, $g : X \to X$ two mappings. We say that F and g are *commuting* if, for all $x_1, x_2, \ldots, x_n \in X$,

$$g(F(x_1, x_2, \ldots, x_n)) = F(gx_1, gx_2, \ldots, gx_n).$$

Definition 17.11 ([7]) Let (X, d) be a metric space and $F : X^n \to X$, $g : X \to X$ be two mappings. We say that F and g are $*$-*compatible* if, for any sequences $\{x_1^{(m)}\}, \{x_2^{(m)}\}, \ldots, \{x_n^{(m)}\} \subset X$ and $z_1, z_2, \ldots, z_n \in X$,

$$g(x_i^{(m)}) \xrightarrow{d} z_i \text{ and } F(x_{i_1}^{(m)}, x_{i_2}^{(m)}, \ldots, x_{i_n}^{(m)}) \xrightarrow{d} z_i \ \text{ for each } i \in I_n$$

$$\Longrightarrow \lim_{m\to\infty} d(gF(x_{i_1}^{(m)}, x_{i_2}^{(m)}, \ldots, x_{i_n}^{(m)}), F(gx_{i_1}^{(m)}, gx_{i_2}^{(m)}, \ldots, gx_{i_n}^{(m)})) = 0 \ \text{ for each } i \in I_n.$$

Definition 17.12 Let $(X, d, \preceq)$ be an ordered metric space and $F : X^n \to X$, $g : X \to X$ be two mappings. We say that F and g are

(1) $(*, \overline{\mathrm{O}})$-*compatible* if, for any sequences $\{x_1^{(m)}\}, \{x_2^{(m)}\}, \ldots, \{x_n^{(m)}\} \subset X$ and $z_1, z_2, \ldots, z_n \in X$,

$$g(x_i^{(m)}) \uparrow z_i \text{ and } F(x_{i_1}^{(m)}, x_{i_2}^{(m)}, \ldots, x_{i_n}^{(m)}) \uparrow z_i \ \text{ for each } i \in I_n$$

$$\Longrightarrow \lim_{m\to\infty} d(gF(x_{i_1}^{(m)}, x_{i_2}^{(m)}, \ldots, x_{i_n}^{(m)}), F(gx_{i_1}^{(m)}, gx_{i_2}^{(m)}, \ldots, gx_{i_n}^{(m)})) = 0;$$

(2) $(*, \underline{\mathrm{O}})$-*compatible* if, for any sequences $\{x_1^{(m)}\}, \{x_2^{(m)}\}, \ldots, \{x_n^{(m)}\} \subset X$ and $z_1, z_2, \ldots, z_n \in X$,

$$g(x_i^{(m)}) \downarrow z_i \text{ and } F(x_{i_1}^{(m)}, x_{i_2}^{(m)}, \ldots, x_{i_n}^{(m)}) \downarrow z_i \ \text{ for each } i \in I_n$$

$$\Longrightarrow \lim_{m\to\infty} d(gF(x_{i_1}^{(m)}, x_{i_2}^{(m)}, \ldots, x_{i_n}^{(m)}), F(gx_{i_1}^{(m)}, gx_{i_2}^{(m)}, \ldots, gx_{i_n}^{(m)})) = 0 \ \text{ for each } i \in I_n;$$

(3) $(*, \mathrm{O})$-*compatible* if, for any sequences $\{x_1^{(m)}\}, \{x_2^{(m)}\}, \ldots, \{x_n^{(m)}\} \subset X$ and $z_1, z_2, \ldots, z_n \in X$,

$$g(x_i^{(m)}) \uparrow\downarrow z_i \text{ and } F(x_{i_1}^{(m)}, x_{i_2}^{(m)}, \ldots, x_{i_n}^{(m)}) \uparrow\downarrow z_i \ \text{ for each } i \in I_n$$

$$\Longrightarrow \lim_{m\to\infty} d(gF(x_{i_1}^{(m)}, x_{i_2}^{(m)}, \ldots, x_{i_n}^{(m)}), F(gx_{i_1}^{(m)}, gx_{i_2}^{(m)}, \ldots, gx_{i_n}^{(m)})) = 0 \ \text{ for each } i \in I_n.$$

Definition 17.13 ([7]) Let X be a nonempty set and $F : X^n \to X$, $g : X \to X$ be two mappings. We say that F and g are $(*, w)$-*compatible* if, for any $x_1, x_2, \ldots, x_n \in X$,

$$g(x_i) = F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \ \text{ for each } i \in I_n$$

$$\Longrightarrow g(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n})) = F(gx_{i_1}, gx_{i_2}, \ldots, gx_{i_n}) \ \text{ for each } i \in I_n.$$

Remark 17.4 Evidently, in an ordered metric space, the commutativity $\Longrightarrow$ the $*$-compatibility $\Longrightarrow$ the $(*, \mathrm{O})$-compatibility $\Longrightarrow$ the $(*, \overline{\mathrm{O}})$-compatibility as well as the $(*, \underline{O})$-compatibility $\Longrightarrow$ the $(*, w)$-compatibility for a pair of mappings $F : X^n \to X$ and $g : X \to X$.

Proposition 17.2 ([7]) *If F and g are $(*, w)$-compatible, then every point of $*$-coincidence of F and g is also an $*$-coincidence point of F and g.*

17.3 Auxiliary Results

In this section, we discuss some basic results, which provide the tools for reduction of the multi-tupled fixed point results from the corresponding fixed point results.

Before doing this, we consider the following induced notations:

(1) For any $\mathrm{U} = (x_1, x_2, \ldots, x_n) \in X^n$, $* \in \mathfrak{B}_\mathfrak{n}$ and $i \in I_n$, U_i^* denotes the ordered element $(x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ of X^n.

(2) For each $* \in \mathfrak{B}_\mathfrak{n}$, a mapping $F : X^n \to X$ induces an associated mapping $F_* : X^n \to X^n$ defined by

$$F_*(\mathrm{U}) = (F\mathrm{U}_1^*, F\mathrm{U}_2^*, \ldots, F\mathrm{U}_n^*), \quad \forall \mathrm{U} \in X^n.$$

(3) A mapping $g : X \to X$ induces an associated mapping $G : X^n \to X^n$ defined by

$$G(\mathrm{U}) = (gx_1, gx_2, \ldots, gx_n), \quad \forall \mathrm{U} = (x_1, x_2, \ldots, x_n) \in X^n.$$

(4) For a metric space (X, d), Δ_n and ∇_n denote two metrics on product set X^n defined by: for all $\mathrm{U} = (x_1, x_2, \ldots, x_n)$, $\mathrm{V} = (y_1, y_2, \ldots, y_n) \in X^n$,

$$\Delta_n(\mathrm{U}, \mathrm{V}) = \frac{1}{n} \sum_{i=1}^{n} d(x_i, y_i),$$

$$\nabla_n(\mathrm{U}, \mathrm{V}) = \max_{i \in I_n} d(x_i, y_i).$$

(5) For any ordered set $(X, \preceq)$, $\sqsubseteq_n$ denotes a partial order on X^n defined by for all $\mathrm{U} = (x_1, x_2, \ldots, x_n)$, $\mathrm{V} = (y_1, y_2, \ldots, y_n) \in X^n$,

$$\mathrm{U} \sqsubseteq_n \mathrm{V} \Longleftrightarrow x_i \preceq y_i \ \text{ for each } i \in I_n.$$

Remark 17.5 The following facts are straightforward:

(1) $F_*(X^n) \subseteq (FX^n)^n$.
(2) $G(X^n) = (gX)^n$.
(3) $(G\mathrm{U})_i^* = G(\mathrm{U}_i^*)$ for all $\mathrm{U} \in X^n$.
(4) $\frac{1}{n}\nabla_n \leq \Delta_n \leq \nabla_n$ (i.e., both the metrics Δ_n and ∇_n are equivalent).

In what follows, we use order-theoretic analogues (namely, O, $\overline{\text{O}}$, $\underline{\text{O}}$ analogues) of some frequently used metrical notions (such as completeness, closedness, continuity, g-continuity and compatibility) introduced by Alam et al. [24, 25]. For the sake of brevity, we skip to record these notions.

Alam et al. [26] formulated the following notions by using certain properties on ordered metric space (in order to avoid the necessity of the continuity requirement on underlying mapping) utilized by earlier authors especially from [1, 9, 27, 28] besides some other ones.

Definition 17.14 ([26]) Let $(X, d, \preceq)$ be an ordered metric space and g a self-mapping on X. We say that

(1) $(X, d, \preceq)$ has *g-ICU* $(increasing - convergence - upperbound)$ *property* if g-image of every increasing convergent sequence $\{x_n\}$ in X is bounded above by g-image of its limit (as an upper bound), i.e.,

$$x_n \uparrow x \Longrightarrow g(x_n) \preceq g(x), \quad \forall n \in \mathbb{N}_0;$$

(2) $(X, d, \preceq)$ has *g-DCL* $(decreasing - convergence - lowerbound)$ *property* if g-image of every decreasing convergent sequence $\{x_n\}$ in X is bounded below by g-image of its limit (as a lower bound), i.e.,

$$x_n \downarrow x \Longrightarrow g(x_n) \succeq g(x), \quad \forall n \in \mathbb{N}_0;$$

(3) $(X, d, \preceq)$ has the *g-MCB* (monotone-convergence-boundedness) *property* if X has both *g-ICU* as well as the *g-DCL* property.

Notice that under the restriction $g = I$, the identity mapping on X, the notions of the *g-ICU* property, the *g-DCL* property and the *g-MCB* property are, respectively, called the *ICU* property, the *DCL* property, and the *MCB* property.

Definition 17.15 ([25]) Let $(X, d, \preceq)$ be an ordered metric space and Y a nonempty subset of X. Then d and $\preceq$, respectively, induce a metric d_Y and a partial order $\preceq_Y$ on Y so that

$$d_Y(x, y) = d(x, y), \quad \forall x, y \in Y,$$

$$x \preceq_Y y \Longleftrightarrow x \preceq y, \quad \forall x, y \in Y.$$

Thus $(Y, d_Y, \preceq_Y)$ is an ordered metric space, which is called a *subspace* of $(X, d, \preceq)$.

Conventionally, we opt to refer Y as a subspace of X rather than saying $(Y, d_Y, \preceq_Y)$ a subspace of $(X, d, \preceq)$ and continue to write d and $\preceq$ instead of d_Y and $\preceq_Y$, respectively.

The following family of control functions is indicated in Boyd and Wong [22], but was later used in Jotic [29].

$$\Omega = \left\{\varphi : [0, \infty) \to [0, \infty) : \varphi(t) < t, \quad \limsup_{r \to t^+} \varphi(r) < t \text{ for each t} > 0\right\}.$$

The following coincidence theorems are crucial results to prove our main results:

Lemma 17.1 *Let $(X, d, \preceq)$ be an ordered metric space and Y an $\overline{\mathrm{O}}$-complete (resp., $\underline{\mathrm{O}}$-complete) subspace of X. Let f and g be two self-mappings on X. Suppose that the following conditions hold:*

(a) *$f(X) \subseteq g(X) \cap Y$;*
(b) *f is g-increasing;*
(c) *f and g are $\overline{\mathrm{O}}$-compatible (resp., $\underline{\mathrm{O}}$-compatible);*
(d) *g is $\overline{\mathrm{O}}$-continuous (resp., $\underline{\mathrm{O}}$-continuous);*
(e) *either f is $\overline{\mathrm{O}}$-continuous (resp., $\underline{\mathrm{O}}$-continuous) or $(Y, d, \preceq)$ has the g-ICU property (resp., the g-DCL property);*
(f) *there exists $x_0 \in X$ such that $g(x_0) \preceq f(x_0)$ (resp., $g(x_0) \succeq f(x_0)$);*
(g) *there exists $\varphi \in \Omega$ such that*

$$d(fx, fy) \leq \varphi(d(gx, gy)), \quad \forall x, y \in X \text{ with } g(x) \prec\succ g(y).$$

Then f and g have a coincidence point. Moreover, if the following condition also holds:

(h) *for each pair $x, y \in X$, there exists $z \in X$ such that $g(x) \prec\succ g(z)$ and $g(y) \prec\succ g(z)$,*

then f and g have a unique point of coincidence, which remains also a unique common fixed point.

Lemma 17.2 *Let $(X, d, \preceq)$ be an ordered metric space and Y an $\overline{\mathrm{O}}$-complete (resp. $\underline{\mathrm{O}}$-complete) subspace of X. Let f and g be two self-mappings on X. Suppose that the following conditions hold:*

(a) *$f(X) \subseteq Y \subseteq g(X)$;*
(b) *f is g-increasing;*
(c) *either f is $(g, \overline{\mathrm{O}})$-continuous (resp. $(g, \underline{\mathrm{O}})$-continuous) or f and g are continuous or $(Y, d, \preceq)$ has the g-ICU property (resp., the g-DCL property);*
(d) *there exists $x_0 \in X$ such that $g(x_0) \preceq f(x_0)$ (resp., $g(x_0) \succeq f(x_0)$);*
(e) *there exists $\varphi \in \Omega$ such that*

$$d(fx, fy) \leq \varphi(d(gx, gy)), \quad \forall x, y \in X \text{ with } g(x) \prec\succ g(y).$$

Then f and g have a coincidence point. Moreover, if the following condition also holds:

(f) *for each pair $x, y \in X$, $\exists\, z \in X$ such that $g(x) \prec\succ g(z)$ and $g(y) \prec\succ g(z)$,*

then f and g have a unique point of coincidence.

We skip the proofs of above lemmas as they are proved in Alam et al. [24–26].

Lemma 17.3 ([7]) *Let X be a nonempty set, $Y \subseteq X$, $F : X^n \to X$, $g : X \to X$ two mappings and let $* \in \mathfrak{B}_{\mathfrak{n}}$.*

(1) *If $F(X^n) \subseteq g(X) \cap Y$, then $F_*(X^n) \subseteq (FX^n)^n \subseteq G(X^n) \cap Y^n$.*

(2) *If $F(X^n) \subseteq Y \subseteq g(X)$, then $F_*(X^n) \subseteq (FX^n)^n \subseteq Y^n \subseteq G(X^n)$.*

(3) *An element $(x_1, x_2, \ldots, x_n) \in X^n$ is $*$-coincidence point of F and g if and only if $(x_1, x_2, \ldots, x_n)$ is a coincidence point of F_* and G.*

(4) *An element $(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n) \in X^n$ is point of $*$-coincidence of F and g if and only if $(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ is a point of coincidence of F_* and G.*

(5) *An element $(x_1, x_2, \ldots, x_n) \in X^n$ is common $*$-fixed point of F and g if and only if $(x_1, x_2, \ldots, x_n)$ is a common fixed point of F_* and G.*

Lemma 17.4 *Let $(X, \preceq)$ be an ordered set, $g : X \to X$ a mapping and $* \in \mathfrak{B}_{\mathfrak{n}}$. If $G(\mathrm{U}) \sqsubseteq_n G(\mathrm{V})$ for some U,V$\in X^n$, then, for each $i \in I_n$, $G(\mathrm{U}_i^*) \sqsubseteq_n G(\mathrm{V}_i^*)$.*

Proof Let $\mathrm{U} = (x_1, x_2, \ldots, x_n)$ and $\mathrm{V} = (y_1, y_2, \ldots, y_n)$ be such that $G(\mathrm{U}) \sqsubseteq_n G(\mathrm{V})$, then we have

$$\begin{aligned}
&(gx_1, gx_2, \ldots, gx_n) \sqsubseteq_n (gy_1, gy_2, \ldots, gy_n)\\
&\Longrightarrow g(x_i) \preceq g(y_i) \text{ for each } i \in I_n\\
&\Longrightarrow g(x_{i_k}) \preceq g(y_{i_k}) \text{ for each } i \in I_n \text{and } k \in I_n\\
&\Longrightarrow (gx_{i_1}, gx_{i_2}, \ldots, gx_{i_n}) \sqsubseteq_n (gy_{i_1}, gy_{i_2}, \ldots, gy_{i_n}) \ \text{ for each } i \in I_n,
\end{aligned}$$

i.e.,

$$G(\mathrm{U}_i^*) \sqsubseteq_n G(\mathrm{V}_i^*) \ \text{ for each } i \in I_n.$$

This completes the proof.

Lemma 17.5 *Let $(X, \preceq)$ be an ordered set, $F : X^n \to X$, $g : X \to X$ two mappings and let $* \in \mathfrak{B}_{\mathfrak{n}}$. If F has the g-monotone property, then F_* is G-increasing in ordered set $(X^n, \sqsubseteq_n)$.*

Proof Take $\mathrm{U} = (x_1, x_2, \ldots, x_n)$, $\mathrm{V} = (y_1, y_2, \ldots, y_n) \in X^n$ with $G(\mathrm{U}) \sqsubseteq_n G(\mathrm{V})$. Using Lemma 17.4, we obtain

$$G(\mathrm{U}_i^*) \sqsubseteq_n G(\mathrm{V}_i^*) \ \text{ for each } i \in I_n,$$

which implies, for all $i \in I_n$, that

$$g(x_{i_k}) \preceq g(y_{i_k}) \ \text{ for each } k \in I_n. \tag{17.1}$$

On using (17.1) and the g-monotone property of F, we obtain that, for all $i \in I_n$,

$$F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \preceq F(y_{i_1}, y_{i_2}, \ldots, y_{i_n}),$$

i.e.,

$$F(\mathrm{U}_i^*) \preceq F(\mathrm{V}_i^*). \tag{17.2}$$

Using (17.2), we get

$$\begin{aligned} F_*(\mathrm{U}) &= (F\mathrm{U}_1^*, F\mathrm{U}_2^*, \ldots, F\mathrm{U}_n^*) \\ &\sqsubseteq_n (F\mathrm{V}_1^*, F\mathrm{V}_2^*, \ldots, F\mathrm{V}_n^*) \\ &= F_*(\mathrm{V}). \end{aligned}$$

Hence F_* is G-increasing. This completes the proof.

Lemma 17.6 ([7]) *Let (X, d) be a metric space, $g : X \to X$ a mapping and $* \in \mathfrak{B}_\mathfrak{n}$. Then, for any $U = (x_1, x_2, \ldots, x_n)$, $V = (y_1, y_2, \ldots, y_n) \in X^n$ and $i \in I_n$, we have*

(1) $\frac{1}{n}\sum\limits_{k=1}^{n} d(gx_{i_k}, gy_{i_k}) = \frac{1}{n}\sum\limits_{j=1}^{n} d(gx_j, gy_j) = \Delta_n(G\mathrm{U}, G\mathrm{V})$ *provided $*$ is permuted,*

(2) $\max\limits_{k\in I_n} d(gx_{i_k}, gy_{i_k}) = \max\limits_{j\in I_n} d(gx_j, gy_j) = \nabla_n(G\mathrm{U}, G\mathrm{V})$ *provided $*$ is permuted,*

(3) $\max\limits_{k\in I_n} d(gx_{i_k}, gy_{i_k}) \leq \max\limits_{j\in I_n} d(gx_j, gy_j) = \nabla_n(G\mathrm{U}, G\mathrm{V})$.

Proposition 17.3 ([7]) *Let (X, d) be a metric space. Then, for any sequence $\mathrm{U}^{(m)} \subset X^n$ and $\mathrm{U} \in X^n$, where $\mathrm{U}^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})$ and $\mathrm{U} = (x_1, x_2, \ldots, x_n)$, we have*

(1) $\mathrm{U}^{(m)} \xrightarrow{\Delta_n} \mathrm{U} \Longleftrightarrow x_i^{(m)} \xrightarrow{d} x_i$ for each $i \in I_n$,

(2) $\mathrm{U}^{(m)} \xrightarrow{\nabla_n} \mathrm{U} \Longleftrightarrow x_i^{(m)} \xrightarrow{d} x_i$ for each $i \in I_n$.

Lemma 17.7 ([7]) *Let (X, d) be a metric space, $F : X^n \to X$, $g : X \to X$ be two mappings and let $* \in \mathfrak{B}_\mathfrak{n}$.*

(1) *If g is continuous, then G is continuous in both metric spaces (X^n, Δ_n) and (X^n, ∇_n),*

(2) *If F is continuous, then F_* is continuous in both metric spaces (X^n, Δ_n) and (X^n, ∇_n).*

Proposition 17.4 ([7]) *Let $(X, d, \preceq)$ be an ordered metric space and $\{\mathrm{U}^{(m)}\}$ be a sequence in X^n, where $\mathrm{U}^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})$.*

(1) *If $\{\mathrm{U}^{(m)}\}$ is increasing (resp., decreasing) in $(X^n, \sqsubseteq_n)$, then each of $\{x_1^{(m)}\}$, $\{x_2^{(m)}\}$,..., $\{x_n^{(m)}\}$ is increasing (resp., decreasing) in $(X, \preceq)$,*

(2) *If $\{\mathrm{U}^{(m)}\}$ is a Cauchy sequence in (X^n, Δ_n) (similarly, in (X^n, ∇_n)), then each of $\{x_1^{(m)}\}$,$\{x_2^{(m)}\}$, ...,$\{x_n^{(m)}\}$ is a Cauchy sequence in (X, d).*

Lemma 17.8 *Let $(X, d, \preceq)$ be an ordered metric space, $Y \subseteq X$, $F : X^n \to X$, $g : X \to X$ be two mappings and let $* \in \mathfrak{B}_\mathfrak{n}$.*

(1) *If* $(Y, d, \preceq)$ *is* $\overline{\mathrm{O}}$*-complete (resp.,* $\underline{\mathrm{O}}$*-complete), then* $(Y^n, \Delta_n, \sqsubseteq_n)$ *and* $(Y^n, \nabla_n, \sqsubseteq_n)$ *both are* $\overline{\mathrm{O}}$*-complete (resp.,* $\underline{\mathrm{O}}$*-complete),*

(2) *If* F *and* g *are* $(*, \overline{\mathrm{O}})$*-compatible pair (resp.,* $(*, \underline{\mathrm{O}})$*-compatible pair), then* F_* *and* G *are* $\overline{\mathrm{O}}$*-compatible pair (resp.,* $\underline{\mathrm{O}}$*-compatible pair) in both ordered metric spaces* $(X^n, \Delta_n, \sqsubseteq_n)$ *and* $(X^n, \nabla_n, \sqsubseteq_n)$*,*

(3) *If* g *is* $\overline{\mathrm{O}}$*-continuous (resp.,* $\underline{\mathrm{O}}$*-continuous), then* G *is* $\overline{\mathrm{O}}$*-continuous (resp.,* $\underline{\mathrm{O}}$*-continuous) in both ordered metric spaces* $(X^n, \Delta_n, \sqsubseteq_n)$ *and* $(X^n, \nabla_n, \sqsubseteq_n)$*,*

(4) *If* F *is* $\overline{\mathrm{O}}$*-continuous (resp.,* $\underline{\mathrm{O}}$*-continuous), then* F_* *is* $\overline{\mathrm{O}}$*-continuous (resp.,* $\underline{\mathrm{O}}$*-continuous) in both ordered metric spaces* $(X^n, \Delta_n, \sqsubseteq_n)$ *and* $(X^n, \nabla_n, \sqsubseteq_n)$*,*

(5) *If* F *is* $(g, \overline{\mathrm{O}})$*-continuous (resp.,* $(g, \underline{\mathrm{O}})$*-continuous), then* F_* *is* $(G, \overline{\mathrm{O}})$*-continuous (resp.,* $(G, \underline{\mathrm{O}})$*-continuous) in both ordered metric spaces* $(X^n, \Delta_n, \sqsubseteq_n)$ *and* $(X^n, \nabla_n, \sqsubseteq_n)$*,*

(6) *If* $(Y, d, \preceq)$ *has the* g*-ICU property (resp., the* g*-DCL property), then both* $(Y^n, \Delta_n, \sqsubseteq_n)$ *and* $(Y^n, \nabla_n, \sqsubseteq_n)$ *have the* G*-ICU property (resp., the* G*-DCL property),*

(7) *If* $(Y, d, \preceq)$ *has the ICU property (resp., the DCL property), then both* $(Y^n, \Delta_n, \sqsubseteq_n)$ *and* $(Y^n, \nabla_n, \sqsubseteq_n)$ *have the ICU property (resp., the DCL property).*

Proof We prove above conclusions only for $\overline{\mathrm{O}}$-analogues and only for the ordered metric space $(X^n, \Delta_n, \sqsubseteq_n)$. Their $\underline{\mathrm{O}}$-analogues can analogously be proved. In the similar manner, one can prove same arguments in the framework of ordered metric space $(X^n, \nabla_n, \sqsubseteq_n)$.

(1) Let $\{\mathrm{U}^{(m)}\}$ be an increasing Cauchy sequence in $(Y^n, \Delta_n, \sqsubseteq_n)$. Denote $\mathrm{U}^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})$, then by Proposition 17.4, each of $\{x_1^{(m)}\}$,$\{x_2^{(m)}\}$,…,$\{x_n^{(m)}\}$ is an increasing Cauchy sequence in $(Y, d, \preceq)$. By $\overline{\mathrm{O}}$-completeness of $(Y, d, \preceq)$, there exist $x_1, x_2, \ldots, x_n \in Y$ such that

$$x_i^{(m)} \xrightarrow{d} x_i \text{ for each } i \in I_n,$$

which using Proposition 17.3, implies that

$$\mathrm{U}^{(m)} \xrightarrow{\Delta_n} \mathrm{U},$$

where $\mathrm{U} = (x_1, x_2, \ldots, x_n)$. It follows that $(Y^n, \Delta_n, \sqsubseteq_n)$ is $\overline{\mathrm{O}}$-complete.

(2) Take a sequence $\{\mathrm{U}^{(m)}\} \subset X^n$ such that $\{G\mathrm{U}^{(m)}\}$ and $\{F_*\mathrm{U}^{(m)}\}$ are increasing (w.r.t. partial order $\sqsubseteq_n$) and

$$G(\mathrm{U}^{(m)}) \xrightarrow{\Delta_n} \mathrm{W} \text{ and } F_*(\mathrm{U}^{(m)}) \xrightarrow{\Delta_n} \mathrm{W},$$

for some W$\in X^n$. Write $\mathrm{U}^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})$ and $\mathrm{W} = (z_1, z_2, \ldots, z_n)$. Then, by using Propositions 17.3 and 17.4, we obtain

$$g(x_i^{(m)}) \uparrow z_i \text{ and } F(x_{i_1}^{(m)}, x_{i_2}^{(m)}, \ldots, x_{i_n}^{(m)}) \uparrow z_i \text{ for each } i \in I_n. \tag{17.3}$$

On using (17.3) and $(*, \overline{\mathrm{O}})$-compatibility of the pair (F, g), we have

$$\lim_{m\to\infty} d(gF(x_{i_1}^{(m)}, x_{i_2}^{(m)}, \ldots, x_{i_n}^{(m)}), F(gx_{i_1}^{(m)}, gx_{i_2}^{(m)}, \ldots, gx_{i_n}^{(m)})) = 0 \ \text{ for each } i \in I_n,$$

i.e.,

$$\lim_{m\to\infty} d(g(F\mathrm{U}_i^{(m)*}), F(G\mathrm{U}_i^{(m)*})) = 0 \ \text{ for each } i \in I_n. \tag{17.4}$$

Now, owing to (17.4), we have

$$\begin{aligned} \Delta_n(GF_*\mathrm{U}^{(m)}, F_*G\mathrm{U}^{(m)}) &= \frac{1}{n}\sum_{i=1}^{n} d(g(F\mathrm{U}_i^{(m)*}), F(G\mathrm{U}_i^{(m)*})) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

It follows that (F_*, G) is $\overline{\mathrm{O}}$-compatible pair in ordered metric space $(X^n, \Delta_n, \sqsubseteq_n)$.

The procedure of the proofs of parts (3) and (4) are similar to Lemma 17.5 and the part (5) and hence is left for readers as an exercise.

(5) Take a sequence $\{\mathrm{U}^{(m)}\} \subset X^n$ and a $\mathrm{U} \in X^n$ such that $\{G\mathrm{U}^{(m)}\}$ is increasing (w.r.t. partial order $\sqsubseteq_n$) and

$$G(\mathrm{U}^{(m)}) \xrightarrow{\Delta_n} G(\mathrm{U}).$$

Write $\mathrm{U}^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})$ and $\mathrm{U} = (x_1, x_2, \ldots, x_n)$. Then, by using Propositions 17.3 and 17.4, we obtain

$$g(x_i^{(m)}) \uparrow g(x_i) \ \text{ for each } i \in I_n.$$

It follows for each $i \in I_n$ that

$$g(x_{i_1}^{(m)}) \uparrow g(x_{i_1}), g(x_{i_2}^{(m)}) \uparrow g(x_{i_2}), \ldots, g(x_{i_n}^{(m)}) \uparrow g(x_{i_n}). \tag{17.5}$$

Using (17.5) and the $(g, \overline{\mathrm{O}})$-continuity of F, we get

$$F(x_{i_1}^{(m)}, x_{i_2}^{(m)}, \ldots, x_{i_n}^{(m)}) \xrightarrow{d} F(x_{i_1}, x_{i_2}, \ldots, x_{i_n})$$

so that

$$F(\mathrm{U}_i^{(m)*}) \xrightarrow{d} F(\mathrm{U}) \ \text{ for each i} \in \mathrm{I_n},$$

which, by using Proposition 17.3, gives rise

$$F_*(\mathrm{U}^{(m)}) \xrightarrow{\Delta_n} F_*(\mathrm{U}).$$

Hence F_* is $(G, \overline{\mathrm{O}})$-continuous in ordered metric space $(X^n, \Delta_n, \sqsubseteq_n)$.

(6) Suppose that $(Y, d, \preceq)$ has the *g-ICU* property. Take a sequence $\{\mathrm{U}^{(m)}\} \subset Y^n$ and a $\mathrm{U} \in Y^n$ such that $\{\mathrm{U}^{(m)}\}$ is increasing (w.r.t. partial order $\sqsubseteq_n$) and

$$\mathrm{U}^{(m)} \xrightarrow{\Delta_n} \mathrm{U}.$$

Write $\mathrm{U}^{(m)} = (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})$ and $\mathrm{U} = (x_1, x_2, \ldots, x_n)$. Then, by Propositions 17.3 and 17.4, we obtain

$$x_i^{(m)} \uparrow x_i \text{ for each } i \in I_n,$$

which on using the *g-ICU* property of $(Y, d, \preceq)$, gives rise

$$g(x_i^{(m)}) \preceq g(x_i) \text{ for each } i \in I_n,$$

or, equivalently,

$$\mathrm{U}^{(m)} \sqsubseteq_{\iota_n} \mathrm{U}.$$

It follows that $(Y^n, \Delta_n, \sqsubseteq_n)$ has the *G-ICU* property.

Analogously, it can be proved that if $(Y, d, \preceq)$ has the *g-DCL* property, then $(Y^n, \Delta_n, \sqsubseteq_n)$ has the *G-DCL* property.

(7) This result is directly follows from (6) by setting $g = I$, the identity mapping. This completes the proof.

17.4 Multi-tupled Coincidence Theorems for Compatible Mappings

In this section, we prove the results regarding the existence and uniqueness of $*$-coincidence points in ordered metric spaces for compatible pair of mappings.

Theorem 17.1 *Let $(X, d, \preceq)$ be an ordered metric space, Y be an $\overline{\mathrm{O}}$-complete subspace of X and let $* \in \mathfrak{B}_{\mathfrak{n}}$. Let $F : X^n \to X$ and $g : X \to X$ be two mappings. Suppose that the following conditions hold:*

(a) *$F(X^n) \subseteq g(X) \cap Y$;*
(b) *F has g-monotone property;*
(c) *F and g are $(*, \overline{\mathrm{O}})$-compatible;*
(d) *g is $\overline{\mathrm{O}}$-continuous;*
(e) *either F is $\overline{\mathrm{O}}$-continuous or $(Y, d, \preceq)$ has the g-ICU property;*
(f) *there exist $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X$ such that*

$$g(x_i^{(0)}) \preceq F(x_{i_1}^{(0)}, x_{i_2}^{(0)}, \ldots, x_{i_n}^{(0)}) \text{ for each } i \in I_n;$$

(g) *there exists* $\varphi \in \Omega$ *such that*

$$\frac{1}{n}\sum_{i=1}^{n} d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) \leq \varphi\Big(\frac{1}{n}\sum_{i=1}^{n} d(gx_i, gy_i)\Big)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $g(x_i) \preceq g(y_i)$ *or* $g(x_i) \succeq g(y_i)$ *for each* $i \in I_n$*, or, alternately,*

(g$'$) *there exists* $\varphi \in \Omega$ *such that*

$$\max_{i \in I_n} d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) \leq \varphi\Big(\max_{i \in I_n} d(gx_i, gy_i)\Big)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $g(x_i) \preceq g(y_i)$ *or* $g(x_i) \succeq g(y_i)$ *for each* $i \in I_n$*. Then* F *and* g *have an* $*$*-coincidence point.*

Proof We can induce two metrics Δ_n and ∇_n, the partial order $\sqsubseteq_n$ and two self-mappings F_* and G on X^n defined as in Sect. 17.3. By item (1) of Lemma 17.8, both ordered metric subspaces $(Y^n, \Delta_n, \sqsubseteq_n)$ and $(Y^n, \nabla_n, \sqsubseteq_n)$ are $\overline{\text{O}}$-complete. Further,

(a) implies that $F_*(X^n) \subseteq G(X^n) \cap Y^n$ by (1) of Lemma 17.3;

(b) implies that F_* is G-increasing in ordered set $(X^n, \sqsubseteq_n)$ by Lemma 17.5;

(c) implies that F_* and G are $\overline{\text{O}}$-compatible in both $(X^n, \Delta_n, \sqsubseteq_n)$ and $(X^n, \nabla_n, \sqsubseteq_n)$ by (2) of Lemma 17.8;

(d) implies that G is $\overline{\text{O}}$-continuous in both $(X^n, \Delta_n, \sqsubseteq_n)$ and $(X^n, \nabla_n, \sqsubseteq_n)$ by (3) of Lemma 17.8;

(e) implies that either F_* is $\overline{\text{O}}$-continuous in both $(X^n, \Delta_n, \sqsubseteq_n)$ and $(X^n, \nabla_n, \sqsubseteq_n)$ or both $(Y^n, \Delta_n, \sqsubseteq_n)$ and $(Y^n, \nabla_n, \sqsubseteq_n)$ have the *G-MCB* property by (4) and (6) of Lemma 17.8;

(f) is equivalent to $G(\mathrm{U}^{(0)}) \sqsubseteq_n F_*(\mathrm{U}^{(0)})$ where $\mathrm{U}^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \in X^n$;

(g) means that $\Delta_n(F_*\mathrm{U}, F_*\mathrm{V}) \leq \varphi(\Delta_n(G\mathrm{U}, G\mathrm{V}))$ for all $\mathrm{U} = (x_1, x_2, \ldots, x_n)$, $\mathrm{V} = (y_1, y_2, \ldots, y_n) \in X^n$ with $\mathrm{U} \sqsubseteq_n \mathrm{V}$ or $\mathrm{U} \sqsupseteq_n \mathrm{V}$;

(g') means that $\nabla_n(F_*\mathrm{U}, F_*\mathrm{V}) \leq \varphi(\nabla_n(G\mathrm{U}, G\mathrm{V}))$ for all $\mathrm{U} = (x_1, x_2, \ldots, x_n)$, $\mathrm{V} = (y_1, y_2, \ldots, y_n) \in X^n$ with $\mathrm{U} \sqsubseteq_n \mathrm{V}$ or $\mathrm{U} \sqsupseteq_n \mathrm{V}$.

Therefore, the conditions (a)–(g) of Lemma 17.1 are satisfied in the context of ordered metric space $(X^n, \Delta_n, \sqsubseteq_n)$ or $(X^n, \nabla_n, \sqsubseteq_n)$ and two self-mappings F_* and G on X^n. Thus, by Lemma 17.1, F_* and G have a coincidence point, which is a $*$-coincidence point of F and g by (3) of Lemma 17.3. This completes the proof.

Now, we present a dual result corresponding to Theorem 17.1.

Theorem 17.2 *Theorem 17.1 remains true if certain involved terms, namely,* $\overline{\text{O}}$*-complete,* $(*, \overline{\text{O}})$*-compatible,* $\overline{\text{O}}$*-continuous and the g-ICU property are, respectively, replaced by* $\underline{\text{O}}$*-complete,* $(*, \underline{\text{O}})$*-compatible,* $\underline{\text{O}}$*-continuous and the g-DCL property provided the assumption* (f) *is replaced by the following (besides retaining the rest of the hypotheses):*

(f$'$) *there exist* $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X$ *such that*

$$g(x_i^{(0)}) \succeq F(x_{i_1}^{(0)}, x_{i_2}^{(0)}, \ldots, x_{i_n}^{(0)}) \text{ for each } i \in I_n.$$

Proof The procedure of the proof of this result is analogously followed, point by point, by the lines of the proof of Theorem 17.1.

Now, combining Theorems 17.1 and 17.2, we obtain the following result:

Theorem 17.3 *Theorem 17.1 remains true if certain involved terms, namely,* $\overline{\mathrm{O}}$-*complete,* $(*, \overline{\mathrm{O}})$-*compatible,* $\overline{\mathrm{O}}$-*continuous and the g-ICU property are, respectively, replaced by O-complete,* $(*, \mathrm{O})$-*compatible, O-continuous and the g-MCB property provided the assumption* (f) *is replaced by the following (besides retaining the rest of the hypotheses):*

(f″) *there exist* $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X$ *such that*

$$g(x_i^{(0)}) \preceq F(x_{i_1}^{(0)}, x_{i_2}^{(0)}, \ldots, x_{i_n}^{(0)}) \text{ for each } i \in I_n$$

or

$$g(x_i^{(0)}) \succeq F(x_{i_1}^{(0)}, x_{i_2}^{(0)}, \ldots, x_{i_n}^{(0)}) \text{ for each } i \in I_n.$$

Notice that Theorems 17.1, 17.2, and 17.3 provide their consequences, in which the $\overline{\mathrm{O}}$, $\underline{\mathrm{O}}$ and O analogous of metrical notions can be replaced by their usual senses.

Now, we present some consequences of Theorems 17.1, 17.2, and 17.3.

Corollary 17.1 *Theorem 17.1 (similarly, Theorems 17.2 and 17.3) remains true if we replace the condition* (g) *by the following condition:*

(g") *there exists* $\varphi \in \Omega$ *such that*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \le \varphi\Big(\frac{1}{n}\sum_{i=1}^{n} d(gx_i, gy_i)\Big)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $g(x_i) \preceq g(y_i)$ *or* $g(x_i) \succeq g(y_i)$ *for each* $i \in I_n$ *provided that* $*$ *is permuted.*

Proof Set $U = (x_1, x_2, \ldots, x_n)$ and $V = (y_1, y_2, \ldots, y_n)$, Then we have $G(\mathrm{U}) \sqsubseteq_n G(\mathrm{V})$ or $G(\mathrm{U}) \sqsupseteq_n G(\mathrm{V})$. As $G(\mathrm{U})$ and $G(\mathrm{V})$ are comparable, for each $i \in I_n$, $G(\mathrm{U}_i^*)$ and $G(\mathrm{V}_i^*)$ are comparable w.r.t. the partial order $\sqsubseteq_n$. Applying the contractivity condition (g’) on these points and using Lemma 17.6, for each $i \in I_n$, we obtain

$$\begin{aligned} d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) &\le \varphi\Big(\frac{1}{n}\sum_{k=1}^{n} d(gx_{i_k}, gy_{i_k})\Big) \\ &= \varphi\Big(\frac{1}{n}\sum_{j=1}^{n} d(gx_j, gy_j)\Big) \end{aligned}$$

as $*$ is permuted, so that

$$d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) \leq \varphi\Big(\frac{1}{n}\sum_{j=1}^{n} d(gx_j, gy_j)\Big) \text{ for each } i \in I_n.$$

Taking summation over $i \in I_n$ on both the sides of above inequality, we obtain

$$\sum_{i=1}^{n} d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) \leq n\varphi\Big(\frac{1}{n}\sum_{j=1}^{n} d(gx_j, gy_j)\Big),$$

so that

$$\frac{1}{n}\sum_{i=1}^{n} d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) \leq \varphi\Big(\frac{1}{n}\sum_{j=1}^{n} d(gx_j, gy_j)\Big)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ with $g(x_i) \preceq g(y_i)$ or $g(x_i) \succeq g(y_i)$ for each $i \in I_n$. Therefore, the contractivity condition (g) of Theorem 17.1 (similarly, Theorems 17.2 and 17.3) holds and hence Theorem 17.1 (similarly, Theorems 17.2 and 17.3) is applicable. This completes the proof.

Corollary 17.2 *Theorem 17.1 (similarly, Theorems 17.2 and 17.3) remains true if we replace the condition* (g$'$) *by the following condition:*

(g$'''$) *there exists* $\varphi \in \Omega$ *such that*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \varphi\Big(\max_{i \in I_n} d(gx_i, gy_i)\Big)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $g(x_i) \preceq g(y_i)$ *or* $g(x_i) \succeq g(y_i)$ *for each* $i \in I_n$ *provided that either* $*$ *is permuted or* φ *is increasing on* $[0, \infty)$.

Proof Set $U = (x_1, x_2, \ldots, x_n)$, $V = (y_1, y_2, \ldots, y_n)$. Then, similar to previous corollary, for each $i \in I_n$, $G(\mathrm{U}_i^*)$ and $G(\mathrm{V}_i^*)$ are comparable w.r.t. the partial order $\sqsubseteq_n$. Applying the contractivity condition (g$'''$) on these points and using Lemma 17.6, for each $i \in I_n$, we obtain

$$\begin{aligned} &d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) \\ &\leq \varphi\Big(\max_{k \in I_n} d(gx_{i_k}, gy_{i_k})\Big) \\ &\begin{cases} = \varphi\Big(\max_{j \in I_n} d(gx_j, gy_j)\Big) & \text{if } * \text{ is permuted,} \\ \leq \varphi\Big(\max_{j \in I_n} d(gx_j, gy_j)\Big) & \text{if } \varphi \text{ is increasing,} \end{cases} \end{aligned}$$

so that

$$d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) \leq \varphi\Big(\max_{i \in I_n} d(gx_i, gy_i)\Big) \text{ for each } i \in I_n.$$

Taking maximum over $i \in I_n$ on both the sides of above inequality, we obtain

$$\max_{i \in I_n} d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) \leq \varphi\Big(\max_{j \in I_n} d(gx_j, gy_j)\Big)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ with $g(x_i) \preceq g(y_i)$ or $g(x_i) \succeq g(y_i)$ for each $i \in I_n$. Therefore, the contractivity condition (g′) of Theorem 17.1 (similarly, Theorems 17.2 and 17.3) holds and hence Theorem 17.1 (similarly, Theorems 17.2 and 17.3) is applicable. This completes the proof.

Now, we present multi-tupled coincidence theorems for linear and generalized linear contractions.

Corollary 17.3 *In addition to the hypotheses* (a)–(f) *of Theorem 17.1* (*similarly, Theorems 17.2 and 17.3*), *suppose that one of the following conditions holds:*

(h) *there exists* $\alpha \in [0, 1)$ *such that*

$$\frac{1}{n} \sum_{i=1}^{n} d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) \leq \frac{\alpha}{n} \sum_{i=1}^{n} d(gx_i, gy_i)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $g(x_i) \preceq g(y_i)$ *or* $g(x_i) \succeq g(y_i)$ *for each* $i \in I_n$;

(i) *there exists* $\alpha \in [0, 1)$ *such that*

$$\max_{i \in I_n} d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) \leq \alpha \max_{i \in I_n} d(gx_i, gy_i)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $g(x_i) \preceq g(y_i)$ *or* $g(x_i) \succeq g(y_i)$ *for each* $i \in I_n$.

Then F *and* g *have an* $*$*-coincidence point.*

Proof On setting $\varphi(t) = \alpha t$ with $\alpha \in [0, 1)$, in Theorem 17.1 (similarly, Theorems 17.2 and 17.3), we get our result.

Corollary 17.4 *In addition to the hypotheses* (a)–(f) *of Theorem 17.1* (*similarly, Theorems 17.2 and 17.3*), *suppose that one of the following conditions holds:*

(j) *there exists* $\alpha \in [0, 1)$ *such that*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \alpha \max_{i \in I_n} d(gx_i, gy_i)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $g(x_i) \preceq g(y_i)$ *or* $g(x_i) \succeq g(y_i)$ *for each* $i \in I_n$;

(k) *there exist* $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, 1)$ *with* $\sum_{i=1}^{n} \alpha_i < 1$ *such that*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \sum_{i=1}^{n} \alpha_i d(gx_i, gy_i)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $g(x_i) \preceq g(y_i)$ *or* $g(x_i) \succeq g(y_i)$ *for each* $i \in I_n$*;*

(l) *there exists* $\alpha \in [0, 1)$ *such that*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \frac{\alpha}{n} \sum_{i=1}^{n} d(gx_i, gy_i)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $g(x_i) \preceq g(y_i)$ *or* $g(x_i) \succeq g(y_i)$ *for each* $i \in I_n$.

Then F and g have an $*$*-coincidence point.*

Proof Setting $\varphi(t) = \alpha t$ with $\alpha \in [0, 1)$, in Corollary 17.2, we get the result corresponding to the contractivity condition (j). Notice that here φ is increasing on $[0, \infty)$.

To prove the result corresponding to (k), let $\beta = \sum_{i=1}^{n} \alpha_i < 1$, then we have

$$\begin{aligned} d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) &\leq \sum_{i=1}^{n} \alpha_i d(gx_i, gy_i) \\ &\leq \Big(\sum_{i=1}^{n} \alpha_i \Big) \max_{j \in I_n} d(gx_j, gy_j) \\ &= \beta \max_{j \in I_n} d(gx_j, gy_j), \end{aligned}$$

so that result follows from the result corresponding to (j).

Finally, setting $\alpha_i = \frac{\alpha}{n}$ for all $i \in I_n$, where $\alpha \in [0, 1)$ in (k), we get the result corresponding to (l). Notice that here $\sum_{i=1}^{n} \alpha_i = \alpha < 1$. This completes the proof.

Now, we present uniqueness result corresponding to Theorem 17.1 (resp., Theorems 17.2 and 17.3), which runs as follows:

Theorem 17.4 *In addition to the hypotheses of Theorem 17.1 (resp., Theorems 17.2 and 17.3), suppose that, for every pair* $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n) \in X^n$, *there exists* $(z_1, z_2, \ldots, z_n) \in X^n$ *such that* $(gz_1, gz_2, \ldots, gz_n)$ *is comparable to* $(gx_1, gx_2, \ldots, gx_n)$ *and* $(gy_1, gy_2, \ldots, gy_n)$ *w.r.t. the partial order* $\sqsubseteq_n$*, then F and g have a unique point of* $*$*-coincidence, which remains also a unique common* $*$*-fixed point.*

Proof Set $U = (x_1, x_2, \ldots, x_n)$, $V = (y_1, y_2, \ldots, y_n)$ and $W = (z_1, z_2, \ldots, z_n)$. Then, by one of our assumptions $G(W)$ is comparable to $G(U)$ and $G(V)$. Therefore, all the conditions of Lemma 17.1 are satisfied. Hence, by Lemma 17.1, F_* and

G have a unique common fixed point, a unique point of coincidence as well as a unique common fixed point, which is indeed a unique point of $*$-coincidence as well as a unique common $*$-fixed point of F and g by (4) and (5) of Lemma 17.3. This completes the proof.

Theorem 17.5 *In addition to the hypotheses of Theorem 17.4, suppose that g is one-one, then F and g have a unique $*$-coincidence point.*

Proof Let $U = (x_1, x_2, \ldots, x_n)$ and $V = (y_1, y_2, \ldots, y_n)$ be two $*$-coincidence point of F and g then, using Theorem 17.4, we obtain

$$(gx_1, gx_2, \ldots, gx_n) = (gy_1, gy_2, \ldots, gy_n)$$

or, equivalently,

$$g(x_i) = g(y_i) \quad \text{for each } i \in I_n.$$

As g is one-one, we have

$$x_i = y_i \quad \text{for each } i \in I_n.$$

It follows that U=V, i.e., F and g have a unique $*$-coincidence point. This completes the proof.

17.5 Multi-tupled Coincidence Theorems Without Compatibility of Mappings

In this section, we prove the results regarding the existence and uniqueness of $*$-coincidence points in an ordered metric space X for a pair of mappings $F : X^n \to X$ and $g : X \to X$, which are not necessarily compatible.

Theorem 17.6 *Let $(X, d, \preceq)$ be an ordered metric space, Y an $\overline{\mathrm{O}}$-complete subspace of X and $* \in \mathfrak{B}_{\mathrm{n}}$. Let $F : X^n \to X$ and $g : X \to X$ be two mappings. Suppose that the following conditions hold:*

(a) $F(X^n) \subseteq Y \subseteq g(X)$*;*

(b) *F has g-monotone property;*

(c) *either F is $(g, \overline{\mathrm{O}})$-continuous or F and g are continuous or $(Y, d, \preceq)$ has the g-ICU property;*

(d) *there exist $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X$ such that*

$$g(x_i^{(0)}) \preceq F(x_{i_1}^{(0)}, x_{i_2}^{(0)}, \ldots, x_{i_n}^{(0)}) \quad \text{for each } i \in I_n;$$

(e) *there exists $\varphi \in \Omega$ such that*

$$\frac{1}{n}\sum_{i=1}^{n} d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) \leq \varphi\Big(\frac{1}{n}\sum_{i=1}^{n} d(gx_i, gy_i)\Big)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $g(x_i) \preceq g(y_i)$ *or* $g(x_i) \succeq g(y_i)$ *for each* $i \in I_n$

or, alternately,

(e′) *there exists* $\varphi \in \Omega$ *such that*

$$\max_{i \in I_n} d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) \leq \varphi\Big(\max_{i \in I_n} d(gx_i, gy_i)\Big)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $g(x_i) \preceq g(y_i)$ *or* $g(x_i) \succeq g(y_i)$ *for each* $i \in I_n$.

Then F *and* g *have an* $*$*-coincidence point.*

Proof We can induce two metrics Δ_n and ∇_n, the partial order $\sqsubseteq_n$ and two self-mappings F_* and G on X^n defined as in Sect. 17.3. By (1) of Lemma 17.8, both ordered metric subspaces $(Y^n, \Delta_n, \sqsubseteq_n)$ and $(Y^n, \nabla_n, \sqsubseteq_n)$ are $\overline{\text{O}}$-complete. Further,

(a) implies that $F_*(X^n) \subseteq Y^n \subseteq G(X^n)$ by (2) of Lemma 17.3;

(b) implies that F_* is G-increasing in an ordered set $(X^n, \sqsubseteq_n)$ by Lemma 17.5;

(c) implies that either F_* is $(G, \overline{\text{O}})$-continuous in both $(X^n, \Delta_n, \sqsubseteq_n)$ and $(X^n, \nabla_n, \sqsubseteq_n)$ or F_* and G are continuous in both (X^n, Δ_n) and (X^n, ∇_n) or both $(Y^n, \Delta_n, \sqsubseteq_n)$ and $(Y^n, \nabla_n, \sqsubseteq_n)$ have the *G-ICU* property by Lemma 17.7 and (5) and (7) of Lemma 17.8;

(d) is equivalent to $G(\text{U}^{(0)}) \sqsubseteq_n F_*(\text{U}^{(0)})$ where $\text{U}^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) \in X^n$;

(e) means that $\Delta_n(F_*\text{U}, F_*\text{V}) \leq \varphi(\Delta_n(G\text{U}, G\text{V}))$ for all $U = (x_1, x_2, \ldots, x_n)$, $V = (y_1, y_2, \ldots, y_n) \in X^n$ with $G(\text{U}) \sqsubseteq_n G(\text{V})$ or $G(\text{U}) \sqsupseteq_n G(\text{V})$;

(e′) means that $\nabla_n(F_*\text{U}, F_*\text{V}) \leq \varphi(\nabla_n(G\text{U}, G\text{V}))$ for all $U = (x_1, x_2, \ldots, x_n)$, $V = (y_1, y_2, \ldots, y_n) \in X^n$ with $G(\text{U}) \sqsubseteq_n G(\text{V})$ or $G(\text{U}) \sqsupseteq_n G(\text{V})$.

Therefore, the conditions (a)–(e) of Lemma 17.2 are satisfied in the context of ordered metric space $(X^n, \Delta_n, \sqsubseteq_n)$ or $(X^n, \nabla_n, \sqsubseteq_n)$ and two self-mappings F_* and G on X^n. Thus, by Lemma 17.2, F_* and G have a coincidence point, which is a $*$-coincidence point of F and g by (2) of Lemma 17.3.

Now, we present a dual result corresponding to Theorem 17.6.

Theorem 17.7 *Theorem 17.6 remains true if certain involved terms, namely:* $\overline{\text{O}}$-*complete,* $(g, \overline{\text{O}})$*-continuous and the g-ICU property are respectively replaced by* $\underline{\text{O}}$*-complete,* $(g, \underline{\text{O}})$*-continuous and the g-DCL property provided the assumption* (d) *is replaced by the following (besides retaining the rest of the hypotheses):*

(d′) *there exist* $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X$ *such that*

$$g(x_i^{(0)}) \succeq F(x_{i_1}^{(0)}, x_{i_2}^{(0)}, \ldots, x_{i_n}^{(0)}) \text{ for each } i \in I_n.$$

Proof The procedure of the proof of this result is analogously followed, point by point, by the lines of the proof of Theorem 17.6.

Now, combining Theorems 17.6 and 17.7, we obtain the following result:

Theorem 17.8 *Theorem 17.6 remains true if certain involved terms, namely,* $\overline{\mathrm{O}}$-*complete,* $(g, \overline{\mathrm{O}})$-*continuous and the g-ICU property are, respectively, replaced by* O-*complete,* (g, O)-*continuous and the g-MCB property provided the assumption* (d) *is replaced by the following* (*besides retaining the rest of the hypotheses*)*:*

(d″) *there exist* $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X$ *such that*

$$g(x_i^{(0)}) \preceq F(x_{i_1}^{(0)}, x_{i_2}^{(0)}, \ldots, x_{i_n}^{(0)}) \text{ for each } i \in I_n$$

or

$$g(x_i^{(0)}) \succeq F(x_{i_1}^{(0)}, x_{i_2}^{(0)}, \ldots, x_{i_n}^{(0)}) \text{ for each } i \in I_n.$$

Notice that Theorems 17.6, 17.7, and 17.8 provide their consequences, in which the $\overline{\mathrm{O}}$, $\underline{\mathrm{O}}$ and O analogues of metrical notions can be replaced by their usual senses.

Similar to Corollaries 17.1–17.4, the following consequences of Theorems 17.5, 17.6, and 17.7 hold:

Corollary 17.5 *Theorem 17.6* (*similarly, Theorem 17.7 or Theorem 17.8*) *remains true if we replace the condition* (e) *by the following condition:*

($\overline{\mathrm{e}}$) *there exists* $\varphi \in \Omega$ *such that*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \varphi\Big(\frac{1}{n}\sum_{i=1}^{n} d(gx_i, gy_i)\Big)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $g(x_i) \preceq g(y_i)$ *or* $g(x_i) \succeq g(y_i)$ *for each* $i \in I_n$ *provided that* $*$ *is permuted.*

Corollary 17.6 *Theorem 17.6* (*similarly, Theorem 17.7 or Theorem 17.8*) *remains true if we replace the condition* ($\overline{\mathrm{e}}$) *by the following condition:*

($\tilde{\mathrm{e}}$) *there exists* $\varphi \in \Omega$ *such that*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \varphi\Big(\max_{i \in I_n} d(gx_i, gy_i)\Big)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $g(x_i) \preceq g(y_i)$ *or* $g(x_i) \succeq g(y_i)$ *for each* $i \in I_n$ *provided that either* $*$ *is permuted or* φ *is increasing on* $[0, \infty)$.

Corollary 17.7 *In addition to the hypotheses* (a)–(d) *of Theorem 17.6* (*similarly, Theorem 17.7 or Theorem 17.8*)*, suppose that one of the following conditions holds:*

(f) *there exists* $\alpha \in [0, 1)$ *such that*

$$\frac{1}{n}\sum_{i=1}^{n} d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) \leq \frac{\alpha}{n}\sum_{i=1}^{n} d(gx_i, gy_i)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $g(x_i) \preceq g(y_i)$ *or* $g(x_i) \succeq g(y_i)$ *for each* $i \in I_n$*;*

(g) *there exists* $\alpha \in [0, 1)$ *such that*

$$\max_{i \in I_n} d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) \leq \alpha \max_{i \in I_n} d(gx_i, gy_i)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $g(x_i) \preceq g(y_i)$ *or* $g(x_i) \succeq g(y_i)$ *for each* $i \in I_n$.

Then F *and* g *have an* $*$*-coincidence point.*

Corollary 17.8 *In addition to the hypotheses* (a)–(d) *of Theorem 17.6* (*similarly, Theorem 17.7 or Theorem 17.8*), *suppose that one of the following conditions hold:*

(h) *there exists* $\alpha \in [0, 1)$ *such that*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \alpha \max_{i \in I_n} d(gx_i, gy_i)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $g(x_i) \preceq g(y_i)$ *or* $g(x_i) \succeq g(y_i)$ *for each* $i \in I_n$*;*

(i) *there exist* $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, 1)$ *with* $\sum\limits_{i=1}^{n} \alpha_i < 1$ *such that*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \sum_{i=1}^{n} \alpha_i d(gx_i, gy_i)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $g(x_i) \preceq g(y_i)$ *or* $g(x_i) \succeq g(y_i)$ *for each* $i \in I_n$*;*

(j) *there exists* $\alpha \in [0, 1)$ *such that*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \frac{\alpha}{n} \sum_{i=1}^{n} d(gx_i, gy_i)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $g(x_i) \preceq g(y_i)$ *or* $g(x_i) \succeq g(y_i)$ *for each* $i \in I_n$.

Then F *and* g *have an* $*$*-coincidence point.*

Now, we present uniqueness results corresponding to Theorems 17.6, 17.7 and 17.8, which run as follows:

Theorem 17.9 *In addition to the hypotheses of Theorem 17.6 (similarly, Theorem 17.7 or Theorem 17.8), suppose that, for every pair* $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n) \in X^n$, *there exists* $(z_1, z_2, \ldots, z_n) \in X^n$ *such that* $(gz_1, gz_2, \ldots, gz_n)$ *is comparable to* $(gx_1, gx_2, \ldots, gx_n)$ *and* $(gy_1, gy_2, \ldots, gy_n)$ *w.r.t. the partial order* $\sqsubseteq_n$, *then F and g have a unique point of ∗-coincidence.*

Proof Set $U = (x_1, x_2, \ldots, x_n)$, $V = (y_1, y_2, \ldots, y_n)$ and $W = (z_1, z_2, \ldots, z_n)$, then, by one of our assumptions $G(W)$ is comparable to $G(U)$ and $G(V)$. Therefore, all the conditions of Lemma 17.2 are satisfied. Hence, by Lemma 17.2, F_* and G have a unique point of coincidence, which is indeed a unique point of $*$-coincidence of F and g by (4) of Lemma 17.3. This completes the proof.

Theorem 17.10 *In addition to the hypotheses of Theorem 17.9, suppose that g is one-one, then F and g have a unique ∗-coincidence point.*

Proof Let $U = (x_1, x_2, \ldots, x_n)$ and $V = (y_1, y_2, \ldots, y_n)$ be two $*$-coincidence points of F and g then, using Theorem 17.9, we obtain

$$(gx_1, gx_2, \ldots, gx_n) = (gy_1, gy_2, \ldots, gy_n)$$

or, equivalently,

$$g(x_i) = g(y_i) \text{ for each } i \in I_n.$$

As g is one-one, we have

$$x_i = y_i \text{ for each } i \in I_n.$$

It follows that $U = V$, i.e., F and g have a unique $*$-coincidence point. This completes the proof.

Theorem 17.11 *In addition to the hypotheses of Theorem 17.9, suppose that F and g are* $(*, w)$*-compatible, then F and g have a unique common ∗-fixed point.*

Proof Let $(x_1, x_2, \ldots, x_n)$ be a $*$-coincidence point of F and g. Write $F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) = g(x_i) = \overline{x}_i$ for each $i \in I_n$. Then, by Proposition 17.2, $(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ being a point of $*$-coincidence of F and g is also a $*$-coincidence point of F and g. It follows from Theorem 17.9 that

$$(gx_1, gx_2, \ldots, gx_n) = (g\overline{x}_1, g\overline{x}_2, \ldots, g\overline{x}_n),$$

i.e., $\overline{x_i} = g(\overline{x_i})$ for each $i \in I_n$, which, for each $i \in I_n$, yields that

$$F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) = g(\overline{x_i}) = \overline{x_i}.$$

Hence $(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ is a common $*$-fixed point of F and g.

To prove the uniqueness, assume that $(x_1^*, x_2^*, \ldots, x_n^*)$ is another common $*$-fixed point of F and g. Then, again from Theorem 17.9, we have

$$(gx_1^*, gx_2^*, \ldots, gx_n^*) = (g\overline{x}_1, g\overline{x}_2, \ldots, g\overline{x}_n),$$

i.e.,

$$(x_1^*, x_2^*, \ldots, x_n^*) = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n).$$

This completes the proof.

17.6 Multi-tupled Fixed Point Theorems

On particularizing $g = I$, the identity mapping on X, in the foregoing results contained in Sects. 17.4 and 17.5, we obtain the corresponding $*$-fixed point results, which run as follows.

Theorem 17.12 *Let $(X, d, \preceq)$ be an ordered metric space, $F : X^n \to X$ be a mapping and let $* \in \mathfrak{B}_{\mathfrak{n}}$. Let Y be an $\overline{\mathrm{O}}$-complete subspace of X such that $F(X^n) \subseteq Y$. Suppose that the following conditions hold:*

(a) *F has the monotone property;*
(b) *either F is $\overline{\mathrm{O}}$-continuous or $(Y, d, \preceq)$ has the ICU property;*
(c) *there exist $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X$ such that*

$$x_i^{(0)} \preceq F(x_{i_1}^{(0)}, x_{i_2}^{(0)}, \ldots, x_{i_n}^{(0)}) \text{ for each } i \in I_n;$$

(d) *there exists $\varphi \in \Omega$ such that*

$$\frac{1}{n}\sum_{i=1}^{n} d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) = \varphi\Big(\frac{1}{n}\sum_{i=1}^{n} d(x_i, y_i)\Big)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ with $x_i \preceq y_i$ or $x_i \succeq y_i$ for each $i \in I_n$ or, alternately,

(d′) *there exists $\varphi \in \Omega$ such that*

$$\max_{i\in I_n} d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) = \varphi\Big(\max_{i\in I_n} d(x_i, y_i)\Big)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ with $x_i \preceq y_i$ or $x_i \succeq y_i$ for each $i \in I_n$. Then F has an $*$-fixed point.

Theorem 17.13 *Theorem 17.12 remains true if certain involved terms, namely:* $\overline{\mathrm{O}}$*-complete,* $\overline{\mathrm{O}}$*-continuous and the ICU property are, respectively, replaced by* $\underline{\mathrm{O}}$*-complete,* $\underline{\mathrm{O}}$*-continuous and the DCL property provided the assumption* (c) *is replaced by the following* (*besides retaining the rest of the hypotheses*)*:*

(c′) *there exist* $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X$ *such that*

$$x_i^{(0)} \succeq F(x_{i_1}^{(0)}, x_{i_2}^{(0)}, \ldots, x_{i_n}^{(0)}) \text{ for each } i \in I_n.$$

Theorem 17.14 *Theorem 17.12 remains true if certain involved terms, namely,* $\overline{\mathrm{O}}$*-complete,* $\overline{\mathrm{O}}$*-continuous and the ICU property are, respectively, replaced by O-complete, O-continuous and the MCB property provided the assumption* (c) *is replaced by the following* (*besides retaining the rest of the hypotheses*)*:*

(c″) *there exist* $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)} \in X$ *such that*

$$x_i^{(0)} \preceq F(x_{i_1}^{(0)}, x_{i_2}^{(0)}, \ldots, x_{i_n}^{(0)}) \text{ for each } i \in I_n$$

or

$$x_i^{(0)} \succeq F(x_{i_1}^{(0)}, x_{i_2}^{(0)}, \ldots, x_{i_n}^{(0)}) \text{ for each } i \in I_n.$$

Corollary 17.9 *Theorem 17.12* (*similarly, Theorem 17.13 or Theorem 17.14*) *remains true if we replace the condition* (d) *by the following condition:*

($\overline{\mathrm{d}}$) *there exists* $\varphi \in \Omega$ *such that*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \varphi\Big(\frac{1}{n}\sum_{i=1}^{n} d(x_i, y_i)\Big)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $x_i \preceq y_i$ *for each* $i \in I_n$ *or* $x_i \succeq y_i$ *for each* $i \in I_n$ *provided that* $*$ *is permuted.*

Corollary 17.10 *Theorem 17.12* (*similarly, Theorem 17.13 or Theorem 17.14*) *remains true if we replace the condition* ($\overline{\mathrm{d}}$) *by the following condition:*

($\tilde{\mathrm{d}}$) *there exists* $\varphi \in \Omega$ *such that*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \varphi\Big(\max_{i \in I_n} d(x_i, y_i)\Big)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $x_i \preceq y_i$ *for each* $i \in I_n$ *or* $x_i \succeq y_i$ *for each* $i \in I_n$ *provided that either* $*$ *is permuted or* φ *is increasing on* $[0, \infty)$.

Corollary 17.11 *Theorem 17.12* (*similarly, Theorem 17.13 or Theorem 17.14*) *remains true if we replace the condition* (d)(resp(d′)) *by the following condition:*

(e) *there exists* $\alpha \in [0, 1)$ *such that*

$$\frac{1}{n}\sum_{i=1}^{n} d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) \leq \frac{\alpha}{n}\sum_{i=1}^{n} d(x_i, y_i)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $x_i \preceq y_i$ *or* $x_i \succeq y_i$ *for each* $i \in I_n$*; or, alternately*

(e′) *there exists* $\alpha \in [0, 1)$ *such that*

$$\max_{i \in I_n} d(F(x_{i_1}, x_{i_2}, \ldots, x_{i_n}), F(y_{i_1}, y_{i_2}, \ldots, y_{i_n})) \leq \alpha \max_{i \in I_n} d(x_i, y_i)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $x_i \preceq y_i$ *or* $x_i \succeq y_i$ *for each* $i \in I_n$.

Corollary 17.12 *Theorem 17.12 (similarly, Theorem 17.13 or Theorem 17.14) remains true if we replace the conditions* (d) *and* (d′) *by one of the following conditions:*

(f) *there exists* $\alpha \in [0, 1)$ *such that*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \alpha \max_{i \in I_n} d(x_i, y_i)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $x_i \preceq y_i$ *or* $x_i \succeq y_i$ *for each* $i \in I_n$*;*

(g) *there exist* $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, 1)$ *with* $\sum\limits_{i=1}^{n} \alpha_i < 1$ *such that*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \sum_{i=1}^{n} \alpha_i d(x_i, y_i)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $x_i \preceq y_i$ *for each* $i \in I_n$ *or* $x_i \succeq y_i$ *for each* $i \in I_n$*;*

(h) *there exists* $\alpha \in [0, 1)$ *such that*

$$d(F(x_1, x_2, \ldots, x_n), F(y_1, y_2, \ldots, y_n)) \leq \frac{\alpha}{n}\sum_{i=1}^{n} d(x_i, y_i)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in X$ *with* $x_i \preceq y_i$ *or* $x_i \succeq y_i$ *for each* $i \in I_n$.

Theorem 17.15 *In addition to the hypotheses of Theorem 17.12 (similarly, Theorem 17.13 or Theorem 17.14), suppose that, for every pair* $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in X^n$*, there exists* $(z_1, z_2, \ldots, z_n) \in X^n$ *such that* $(z_1, z_2, \ldots, z_n)$ *is comparable to* $(x_1, x_2, \ldots, x_n)$ *and* $(y_1, y_2, \ldots, y_n)$ *w.r.t. the partial order* $\sqsubseteq_n$*. Then* F *has a unique* $*$*-fixed point.*

17.7 Conclusion

We have seen that $*$-fixed point theorems proved in Alam et al. [7] unify all multi-tupled fixed point theorems involving mixed monotone property. Analogously, all $*$-fixed point theorems proved in this chapter unify all multi-tupled fixed point theorems involving monotone property, which substantiate the utility of our results. For the sake of demonstration, in the following lines, we consider some special cases of our newly proved results by choosing suitable involved terms: n and $*$.

The following family of control functions is introduced by Lakshmikantham and Ćirić [9]:

$$\Phi = \Big\{\varphi : [0,\infty) \to [0,\infty) : \varphi(t) < t, \quad \lim_{r\to t^+} \varphi(r) < t \ \text{ for each t} > 0\Big\}.$$

It is clear that the class Ω enlarges the class Φ, i.e., $\Phi \subset \Omega$.

Corollary 17.13 ([16]) *Let $(X, d, \preceq)$ be an ordered complete metric space and $F : X^2 \to X$ be a mapping. Suppose that the following conditions hold:*

(a) *F has the argumentwise monotone property;*
(b) *either F is continuous or $(X, d, \preceq)$ has the MCB property;*
(c) *there exist $x^{(0)}, y^{(0)} \in X$ such that $x^{(0)} \preceq F(x^{(0)}, y^{(0)})$ and $y^{(0)} \preceq F(y^{(0)}, x^{(0)})$;*
(d) *there exists $\alpha \in [0, 1)$ such that*

$$d(F(x, y), F(u, v)) \leq \frac{\alpha}{2}[d(x, u) + d(y, v)]$$

for all $x, y, u, v \in X$ with $x \preceq u$ and $y \preceq v$.
Then F has a coupled fixed point.

Here it can be pointed out that merely the *ICU* property can serve our purpose instead of the *MCB* property.

Corollary 17.14 ([17]) *Let $(X, d, \preceq)$ be an ordered metric space and $F : X^2 \to X$ and $g : X \to X$ two mappings. Assume that there exists $\varphi \in \Phi$ such that*

$$\max\{d(F(x, y), F(u, v)), d(F(y, x), F(v, u))\} \leq \varphi(\max\{d(gx, gu), d(gy, gv)\})$$

for all $x, y, u, v \in X$ with $g(x) \preceq g(u)$ and $g(y) \preceq g(v)$ or $g(x) \succeq g(u)$ and $g(y) \succeq g(v)$. If the following conditions hold:

(a) *$F(X^2) \subseteq g(X)$;*
(b) *F has the argumentwise g-monotone property;*
(c) *there exist $x^{(0)}, y^{(0)} \in X$ such that*

$$g(x^{(0)}) \preceq F(x^{(0)}, y^{(0)}), \quad g(y^{(0)}) \preceq F(y^{(0)}, x^{(0)})$$

or

$$g(x^{(0)}) \succeq F(x^{(0)}, y^{(0)}), \quad g(y^{(0)}) \succeq F(y^{(0)}, x^{(0)});$$

(d) *F and g are continuous and compatible and (X, d) is complete or*
(e) *$(X, d, \preceq)$ has the MCB property and one of $F(X^2)$ or $g(X)$ is complete.*

Then F and g have a coupled coincidence point.

Corollary 17.15 ([13]) *Let $(X, d, \preceq)$ be an ordered complete metric space and $F : X^3 \to X$ be a mapping. Suppose that the following conditions hold:*

(a) *F has the argumentwise monotone property;*
(b) *either F is continuous or $(X, d, \preceq)$ has the ICU property;*
(c) *there exist $x^{(0)}, y^{(0)}, z^{(0)} \in X$ such that*

$$x^{(0)} \preceq F(x^{(0)}, y^{(0)}, z^{(0)}), y^{(0)} \preceq F(y^{(0)}, x^{(0)}, z^{(0)})$$

and

$$z^{(0)} \preceq F(z^{(0)}, y^{(0)}, x^{(0)});$$

(d) *there exist $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$ such that*

$$d(F(x, y, z), F(u, v, w)) \leq \alpha d(x, u) + \beta d(y, v) + \gamma d(z, w)$$

for all $x, y, z, u, v, w \in X$ with $x \preceq u$, $y \preceq v$ and $z \preceq w$.

Then F has a tripled fixed point (in the sense of Borcut [13]), i.e., there exist $x, y, z \in X$ such that $F(x, y, z) = x$, $F(y, x, z) = y$ and $F(z, y, x) = z$.

Corollary 17.16 ([14]) *Let $(X, d, \preceq)$ be an ordered complete metric space and $F : X^3 \to X$ and $g : X \to X$ two mappings. Suppose that the following conditions hold:*

(a) *$F(X^3) \subseteq g(X)$;*
(b) *F has the argumentwise g-monotone property;*
(c) *F and g are commuting;*
(d) *g is continuous;*
(e) *either F is continuous or $(X, d, \preceq)$ has the g-ICU property;*
(f) *there exist $x^{(0)}, y^{(0)}, z^{(0)} \in X$ such that*

$$g(x^{(0)}) \preceq F(x^{(0)}, y^{(0)}, z^{(0)}), g(y^{(0)}) \preceq F(y^{(0)}, x^{(0)}, z^{(0)})$$

and

$$g(z^{(0)}) \preceq F(z^{(0)}, y^{(0)}, x^{(0)});$$

(g) *there exists $\varphi \in \Phi$ such that*

$$d(F(x, y, z), F(u, v, w)) \leq \varphi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\})$$

for all $x, y, z, u, v, w \in X$ *with* $g(x) \preceq g(u)$, $g(y) \preceq g(v)$ *and* $g(z) \preceq g(w)$.

Then F *and* g *have a tripled coincidence point* (*in the sense of Borcut* [14]), *i.e., there exist* $x, y, z \in X$ *such that* $F(x, y, z) = g(x)$, $F(y, x, z) = g(y)$ *and* $F(z, y, x) = g(z)$.

Corollary 17.17 ([15]) *Let* $(X, d, \preceq)$ *be an ordered complete metric space and* $F : X^4 \to X$ *and* $g : X \to X$ *two mappings. Suppose that the following conditions hold:*

(a) $F(X^4) \subseteq g(X)$;
(b) F *has the* g*-monotone property;*
(c) F *and* g *are commuting;*
(d) g *is continuous;*
(e) *either* F *is continuous or* $(X, d, \preceq)$ *has the g-ICU property;*
(f) *there exist* $x^{(0)}, y^{(0)}, z^{(0)}, w^{(0)} \in X$ *such that*

$$g(x^{(0)}) \preceq F(x^{(0)}, y^{(0)}, z^{(0)}, w^{(0)}),\ g(y^{(0)}) \preceq F(x^{(0)}, w^{(0)}, z^{(0)}, y^{(0)}),$$

$$g(z^{(0)}) \preceq F(z^{(0)}, y^{(0)}, x^{(0)}, w^{(0)}),\ g(w^{(0)}) \preceq F(z^{(0)}, w^{(0)}, x^{(0)}, y^{(0)});$$

(g) *there exists* $\varphi \in \Phi$ *such that*

$$d(F(x, y, z, w), F(u, v, r, t)) \leq \varphi\left(\frac{d(gx, gu) + d(gy, gv) + d(gz, gr) + d(gw, gt)}{4}\right)$$

for all $x, y, z, w, u, v, r, t \in X$ *with* $g(x) \preceq g(u)$, $g(y) \preceq g(v)$, $g(z) \preceq g(r)$ *and* $g(w) \preceq g(t)$.

Then F *and* g *have a quartet coincidence point* (*in the sense of Karapinar* [11]), *i.e., there exist* $x, y, z, w \in X$ *such that* $F(x, y, z, w) = g(x)$, $F(x, w, z, y) = g(y)$, $F(z, y, x, w) = g(z)$ *and* $F(z, w, x, y) = g(w)$.

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Chapter 18
Convergence Analysis of Solution Sets for Minty Vector Quasivariational Inequality Problems in Banach Spaces

Nguyen Van Hung, Dinh Huy Hoang, Vo Minh Tam, and Yeol Je Cho

Abstract In this paper, we consider convergence analysis of the solution sets for vector quasi-variational inequality problems of the Minty type. Based on the nonlinear scalarization function, we obtain a key assumption (H_h) by virtue of a sequence of gap functions. Then we establish the necessary and sufficient conditions for the Painlevé–Kuratowski lower convergence and Painlevé–Kuratowski convergence.

Keywords Minty vector quasivariational inequality · Gap function · Painlevé–Kuratowski convergence · Continuous convergence · Convergence analysis

18.1 Introduction

Vector variational inequality was first introduced and studied by Giannessi [19] in finite-dimensional spaces. Since then, vector variational inequality problems in finite and infinite dimensional spaces were studied by many authors. Recently, there has

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been an increasing interest in the study for the existence conditions and stability of solution sets, as the closedness, the lower semi-continuity and upper semi-continuity, some kinds of the continuity, the connectedness for different problem models as vector variational inequality problems, see [1, 11, 13, 21, 23, 25, 32, 40, 45], equilibrium problems, see [3, 5, 12, 14, 15, 24, 34, 36] and references therein.

It is well known that the notion of gap function for finite-dimensional variational inequalities was introduced by Auslender [8] in 1976. Using the gap function, a variational inequality can be reformulated as an optimization problem. Fukushima [18] and Yamashita et al. [42] developed various kind of regularized gap functions for variational inequalities. Based on regularized gap functions, they also established error bounds for variational inequalities under some suitable assumptions. Since then, the study of gap functions and error bounds for equilibrium problems, variational inequalities and hemivariational inequalities has become an interesting topic, see e.g. [2, 6, 9, 26–31] and the references therein.

In 1994, Luc et al. [38] established the Painlevé–Kuratowski convergence and Attouch-Wets convergence of the efficient and weak efficient solution sets for optimization problems. After that, many authors considered the convergence of the solution sets for various kinds of the optimization problems, variational inequality problems and equilibrium problems, see [16, 17, 20, 29, 35, 38, 39, 44]. In [20], Huang studied the Painlevé–Kuratowski convergence and Mosco convergence of the approximate sets to the efficient sets for optimization problems.

Recently, Li et al. [37] established Painlevé–Kuratowski convergence of the approximate solution sets for generalized Ky Fan inequality problems by continuous convergence of the bifunction sequence and Painlevé–Kuratowski convergence of the set sequence. Very recently, Hung et al. [22] extended and studied generalized Ky Fan inequality problems to generalized vector quasiequilibrium problems of the Minty type and Stampacchia type. After that, Hung et al. [22] discussed the Painlevé–Kuratowski upper convergence, lower convergence and convergence of the approximate solution sets for these problems by using a sequence of mappings Γ_C-converging.

On the other hand, in 2008, Fang et al. [17] used the nonlinear scalarization function method to study the Painlevé–Kuratowski convergence of the solution sets of the perturbed set-valued weak vector variational inequality problems of the Stampacchia type. The authors used the key hypothesis (H_g) to establish sufficient conditions for the Painlevé–Kuratowski lower convergence of the solution sets for these problems. Based on the approach of Fang et al. [17].

In 2017, Anh et al. [4] established necessary and sufficient conditions for the Painlevé–Kuratowski upper convergence, the Painlevé–Kuratowski lower convergence and the Painlevé–Kuratowski convergence of solution sets to generalized set-valued quasiequilibrium problems of the Stampacchia type by virtue of a sequence of gap functions based on the nonlinear scalarization function in metric spaces. However, to the best of our knowledge, up to now, there are not any works on establishing the necessary and sufficient conditions for Painlevé–Kuratowski lower convergence and Painlevé–Kuratowski convergence for the generalized vector quasivariational inequality problems of the Minty type by using nonlinear scalarization function method.

Motivated by the research works mentioned above, in this paper, we introduce generalized vector quasi-variational inequality problems of the Minty type (for short, (MQVIP) and (MQVIP)$_n$) in real Banach spaces. Based on the nonlinear scalarization function, we obtain a key assumption (H_h) by virtue of a sequence of gap functions. Then we establish the necessary and sufficient conditions for Painlevé–Kuratowski lower convergence and Painlevé–Kuratowski convergence of the solution sets of these problems. Our results are new and an improvement the existing ones in the literature. Some examples are given for the illustration of our results.

The structure of our paper is as follows: In Sect. 18.2, we introduce the problems (MQVIP) and (MQVIP)$_n$, recall some definitions and important properties. We also establish gap functions for the problems (MQVIP) and (MQVIP)$_n$ and consider their continuity. In Sect. 18.3, we prove that the hypothesis (H_h) is a sufficient and necessary condition for the Painlevé–Kuratowski lower convergence and Painlevé–Kuratowski convergence of the solution sets of these problems.

18.2 Preliminaries

Let $\mathscr{X}$ be a real Banach space. A nonempty subset $\mathscr{C}$ of $\mathscr{X}$ is called a *convex cone* if $\mathscr{C} + \mathscr{C} \subset \mathscr{C}$ and $\lambda\mathscr{C} \subset \mathscr{C}$ for all $\lambda > 0$. A cone $\mathscr{C}$ is said to be *pointed* if $\mathscr{C} \cap (-\mathscr{C}) = \{0\}$ and *solid* if it has nonempty interior, i.e., int$\mathscr{C} \neq \varnothing$.

Throughout this paper, we assume that X and Y be two real Banach spaces. Let the norm in X be denoted by $\|\cdot\|$, $A \subset X$ be a nonempty subset and let $x \in X$. Then *distance between* of the point x and the set A is defined by

$$dist(x, A) = \inf_{a \in A}\{\|x - a\|\}.$$

Let $L(X, Y)$ be the space of all linear continuous operators from X to Y. Let $K : X \rightrightarrows X$, $T : X \rightrightarrows L(X, Y)$ be set-valued mappings and $C : X \rightrightarrows Y$ be a set-valued mapping such that, for all $x \in X$, $C(x)$ is a pointed, closed convex and solid cone in Y with apex at 0. Denoted by $\langle z, x\rangle$ the value of a linear operator $z \in L(X, Y)$ at $x \in X$.

Now, we consider the following *generalized vector quasi-variational inequality problem of the Minty type* (for short, (MQVIP)):

(MQVIP) Find $\overline{x} \in K(\overline{x})$ such that

$$\langle z, y - \overline{x}\rangle \in Y \setminus -\mathrm{int}C(\overline{x}), \quad \forall y \in K(\overline{x}),\ z \in T(y).$$

For the sequences of set-valued mappings $K_n : X \rightrightarrows X$, $T_n : X \rightrightarrows L(X, Y)$, we consider the following sequence of vector quasi-variational inequality problems of the Minty type (for short, (MQVIP)$_n$):

(MQVIP)$_n$ Find $\overline{x}_n \in K_n(\overline{x}_n)$ such that

$$\langle z, y - \overline{x}_n \rangle \in Y \setminus -\text{int}C(\overline{x}_n), \quad \forall y \in K_n(\overline{x}_n),\ z \in T_n(y).$$

We denote the solution sets of the problems (MQVIP) and $(\text{MQVIP})_n$ by $S(T, K)$ and $S(T_n, K_n)$, respectively. Since the existence of solutions for vector quasi-variational inequality problems of the Minty type has been studied intensively (see, for example, [40]), we always assume that $S(T, K)$ and $S(T_n, K_n)$ are not equal empty sets.

In the following, we recall concepts related to the convergences of set and mapping sequences studied in Rockafellar et al. [41] and Durea [15].

For each $\varepsilon > 0$ and a subset $A \subset X$, let the open ε-neighbourhood of A be defined as $\mathscr{U}(A, \varepsilon) = \{x \in X \mid \exists y \in A\ :\ \|y - x\| < \varepsilon\}$. The notation $\mathscr{B}(x, r)$ denotes the open ball with center x and radius $r > 0$.

Let X be a normed space. A sequence of sets $\{D_n\}$, $D_n \subset X$, is said to be *upper convergent* (resp., *lower convergent*) in the sense of Painlevé–Kuratowski to D if $\limsup\limits_{n\to\infty} D_n \subset D$ (resp., $D \subset \liminf\limits_{n\to\infty} D_n$). $\{D_n\}$ is said to be *convergent* in the sense of Painlevé–Kuratowski to D if $\limsup\limits_{n\to\infty} D_n \subset D \subset \liminf\limits_{n\to\infty} D_n$ with

$$\limsup_{n\to\infty} D_n := \left\{x \in X : x = \lim_{k\to\infty} x_{n_k},\ \ x_{n_k} \in D_{n_k}, \{x_{n_k}\}\text{a subsequence of } \{x_n\}\right\},$$
$$\liminf_{n\to\infty} D_n := \left\{x \in X : x = \lim_{n\to\infty} x_n,\ \ x_n \in D_n \text{ for sufficiently large } n\right\}.$$

A set-valued mapping $G : X \rightrightarrows Y$ is said to be *outer semi-continuous* (resp. *inner semi-continuous*) at x_0 if $\limsup\limits_{x\to x_0} G(x) \subset G(x_0)$ (resp., $\liminf\limits_{x\to x_0} G(x) \supset G(x_0)$) with

$$\begin{aligned}
\limsup_{x\to x_0} G(x) &= \bigcup_{x_n\to x_0} \limsup_{n\to\infty} G(x_n)\\
&= \{y \in Y : \exists x_n \to x_0, \exists y_n \in G(x_n) : y_n \to y,\ \forall n \geq 1\},\\
\liminf_{x\to x_0} G(x) &= \bigcap_{x_n\to x_0} \liminf_{n\to\infty} G(x_n)\\
&= \{y \in Y : \forall x_n \to x_0, \exists y_n \in G(x_n) : y_n \to y,\ \forall n \geq 1\}.
\end{aligned}$$

Let $G_n : X \rightrightarrows Y$ be a sequence of set-valued mappings and $G : X \rightrightarrows Y$ be a set-valued mapping. $\{G_n\}$ is said to be *outer convergent continuously* (resp., *inner convergent continuously*) to G at x_0 if $\limsup\limits_{n\to\infty} G_n(x_n) \subset G(x_0)$ (resp., $G(x_0) \subset \liminf\limits_{n\to\infty} G_n(x_n)$) when $x_n \to x_0$. $\{G_n\}$ is said to be *convergent continuously* to G at x_0 if $\limsup\limits_{n\to\infty} G_n(x_n) \subset G(x_0) \subset \liminf\limits_{n\to\infty} G_n(x_n)$ when $x_n \to x_0$. If $\{G_n\}$ is convergent continuously to G at every $x_0 \in X$, then $\{G_n\}$ is said to be *convergent continuously* to G in X.

Definition 18.1 (*see* [7]) Let $G : X \rightrightarrows Y$ be a set-valued mapping and $x_0 \in X$ be a given point. Then we have the following:

(1) G is said to be *lower semi-continuous* in the sense of Berge (shortly, B-l.s.c.) at $x_0 \in X$ if, for any open set V with $G(x_0) \cap V \neq \varnothing$, there exists $\delta > 0$ such that, for all $x \in B(x_0, \delta)$, $G(x) \cap V \neq \varnothing$.

(2) G is said to be *upper semi-continuous* in the sense of Berge (shortly, B-u.s.c.) at $x_0 \in X$ if, for any open set V with $G(x_0) \subset V$, there exists $\delta > 0$ such that, for all $x \in B(x_0, \delta)$, $G(x) \subset V$.

(3) G is said to be *continuous* in the sense of Berge at $x_0 \in X$ if it is both lower semi-continuous and upper semi-continuous at x_0. G is said to be *continuous* in X if it is both lower semi-continuous and upper semi-continuous at each $x_0 \in X$.

(4) G is said to be *closed* at $x_0 \in X$ if, for each of the sequences $\{x_n\}$ in X converging to x_0 and $\{y_n\}$ in Y converging to y_0 such that $y_n \in G(x_n)$, we have $y_0 \in G(x_0)$. G is said to be *closed* on X if it is closed at each $x_0 \in X$.

Lemma 18.1 (see [7]) *Let $G : X \rightrightarrows Y$ be a set-valued mapping and $x_0 \in X$ be a given point. Then we have the following:*

(1) *G is lower semi-continuous at x_0 if and only if, for any sequence $x_n \to x_0$ and $y_0 \in G(x_0)$, there exists a sequence $\{y_n\} \subset G(x_n)$ such that $y_n \to y_0$.*

(2) *If G has compact values, then G is upper semi-continuous at x_0 if and only if, for any sequence $\{x_n\} \subset X$ which converges to x_0 and $\{y_n\} \subset G(x_n)$, there are $y \in G(x)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y$.*

Lemma 18.2 (see [10, 14]) *Let $e : X \to Y$ be a vector-valued mapping and, suppose that, for any $x \in X$, $e(x) \in C(x)$. Then the nonlinear scalarization function $\xi_e : X \times Y \to \mathbb{R}$ defined by*

$$\xi_e(x, y) := \inf\{r \in \mathbb{R} : y \in re(x) - C(x)\}, \quad \forall (x, y) \in X \times Y,$$

has the following properties:

(1) $\xi_e(x, y) < r \iff y \in re(x) - \mathrm{int}C(x)$.
(2) $\xi_e(x, y) \geq r \iff y \notin re(x) - \mathrm{int}C(x)$.

Lemma 18.3 (see [10, 14]) *Let X, Z be two locally convex Hausdorff topological vector spaces and $C : X \to 2^Z$ be a set-valued mapping such that, for any $x \in X$, $C(x)$ is a proper, closed and convex cone in Z with $\mathrm{int}C(x) \neq \varnothing$. Furthermore, let $e : X \to Z$ be the continuous selection of the set-valued mapping $\mathrm{int}C(\cdot)$. Define a set-valued mapping $V : X \rightrightarrows Z$ by $V(x) = Z \setminus \mathrm{int}C(x)$ for all $x \in X$. Then the nonlinear scalarization function $\xi_e : X \times Z \to \mathbb{R}$ defined by*

$$\xi_e(x, z) := \inf\{r \in \mathbb{R} : z \in re(x) - C(x)\}, \quad \forall (x, z) \in X \times Z,$$

has the following properties:

(1) *If V is upper semi-continuous in X, then ξ_e is upper semi-continuous in $X \times Z$.*
(2) *If C is upper semi-continuous in X, then ξ_e is lower semi-continuous in $X \times Z$.*

(3) *If V and C are both upper semi-continuous in X, then ξ_e is continuous in $X \times Z$.*

Now, we suppose that $K(x)$, $T(x)$, $K_n(x)$ and $T_n(x)$ are compact sets for all $x \in X$. We define functions $h : X \to \mathbb{R}$ and $h_n : X \to \mathbb{R}$ as follows:

$$h(x) = \max_{y \in K(x)} \max_{z \in T(y)} \{-\xi_e(x, \langle z, y - x \rangle)\}, \quad \forall x \in K(x),$$

and

$$h_n(x_n) = \max_{y \in K_n(x_n)} \max_{z \in T_n(y)} \{-\xi_e(x_n, \langle z, y - x_n \rangle)\}, \quad \forall x_n \in K_n(x_n).$$

Since $K(x)$, $T(x)$, $K_n(x)$, $T_n(x)$ are compact sets for all $x \in X$ and ξ_e is continuous, h and h_n are well-defined.

Proposition 18.1 *We have the following:*

(a) *If $x_0 \in K(x_0)$, $h(x_0) = 0$ if and only if $x_0 \in S(T, K)$.*
(b) *$h(x) > 0$ for all $x \in K(x) \setminus S(T, K)$.*
(c) *$h(x) \geq 0$ for all $x \in K(x)$.*

Proof (a) For any $x_0 \in K(x_0)$, by the definition of h, $h(x_0) = 0$ if and only if

$$\max_{y \in K(x_0)} \max_{z \in T(y)} \{-\xi_e(x, \langle z, y - x \rangle)\} = 0,$$

which shows that

$$-\xi_e(x_0, \langle z_0, y - x_0 \rangle) \leq 0, \quad \forall y \in K(x_0),\ z \in T(y),$$

or

$$\xi_e(x_0, \langle z_0, y - x_0 \rangle) \geq 0, \quad \forall y \in K(x_0),\ z \in T(y).$$

By Lemma 18.2 (2), this implies that

$$\langle z_0, y - x_0 \rangle \notin -\mathrm{int}C(x_0), \quad \forall y \in K(x_0),\ z \in T(y),$$

or

$$\langle z_0, y - x_0 \rangle \in Y \setminus -\mathrm{int}C(x_0), \quad \forall y \in K(x_0),\ z \in T(y),$$

i.e., $x_0 \in S(T, K)$.

(b) For any given $x \in K(x)$, but $x \notin S(T, K)$. Then there exist $y_0 \in K(x)$ and $z_0 \in T(y_0)$ such that

$$\langle z_0, y_0 - x \rangle \in -\mathrm{int}C(x).$$

So, it follows from Lemma 18.2 (2) that

$$\xi_e(x, \langle z_0, y_0 - x\rangle) < 0,$$

that is,

$$-\xi_e(x, \langle z_0, y_0 - x\rangle) > 0,$$

Hence we have

$$h(x) = \max_{y \in K(x)} \max_{z \in T(y)} \{-\xi_e(x, \langle z, y - x\rangle)\} > 0.$$

(c), From (a) and (b), we directly get $h(x) \geq 0$ for all $x \in K(x)$. This completes the proof. □

Remark 18.1 If the function h satisfies the properties (a)–(c) of Proposition 18.1, then h is called the *gap function* for the problem (MQVIP).

Similarly, we have the gap functions h_n for the problem $(\text{MQVIP})_n$ in the following:

Proposition 18.2 *We have the following:*

(a) *If* $x_n^0 \in K_n(x_n^0)$, *then* $h_n(x_n^0) = 0$ *if and only if* $x_n^0 \in S(T_n, K_n)$
(b) $h_n(x_n) > 0$ *for all* $x_n \in K_n(x_n) \setminus S(T_n, K_n)$.
(c) $h_n(x_n) \geq 0$ *for all* $x_n \in K_n(x_n)$.

Now, we consider the continuity of h and h_n as follows:

Proposition 18.3 *Consider the problem* $(\text{MQVIP})_n$. *If the following conditions hold:*

(a) K_n *is continuous with compact values in* X*;*
(b) T_n *is continuous with compact values in* X*;*
(c) V *and* C *are upper semi-continuous in* X *and* $e(\cdot) \in \text{int}C(\cdot)$ *is continuous in* X.

Then h_n *is continuous in* X.

Proof First, we prove that h_n is lower semi-continuous in X. Indeed, let $r \in \mathbb{R}$ and suppose that $\{x_n^k\} \subset X$ satisfies $h_n(x_n^k) \leq r$ and $x_n^k \to x_n^0$ as $k \to \infty$. Moreover, it follows from $h_n(x_n^k) \leq r$ that

$$h_n(x_n^k) = \max_{y \in K_n(x_n^k)} \max_{z \in T_n(y)} \{-\xi_e(x_n^k, \langle z, y - x_n^k\rangle)\} \leq r, \quad \forall x_n^k \in K_n(x_n^k),$$

and so

$$-\xi_e(x_n^k, \langle z, y - x_n^k\rangle) \leq r, \quad \forall y \in K_n(x_n^k),\ z \in T_n(y). \tag{18.1}$$

Since K_n is upper semi-continuous with compact values in X, we have $x_n^0 \in K_n(x_n^0)$. Since K_n is lower semi-continuous in X, for any $y_n^0 \in K_n(x_n^0)$, there exists $y_n^k \in K_n(x_n^k)$ such that $y_n^k \to y_n^0$ as $k \to \infty$. Since T_n is lower semi-continuous in X, for any $z_n^0 \in T_n(y_n^0)$, there exists $z_n^k \in T_n(y_n^k)$ such that $z_n^k \to z_n^0$ as $k \to \infty$. From $y_n^k \in K_n(x_n^k)$ and $z_n^k \in T_n(x_n^k)$, it follows from (18.1) that

$$-\xi_e(x_n^k, \langle z_n^k, y_n^k - x_n^k \rangle) \le r. \tag{18.2}$$

From the continuity of ξ_e, taking the limit in (18.2), we have

$$-\xi_e(x_n^0, \langle z_n^0, y_n^0 - x_n^0 \rangle) \le r. \tag{18.3}$$

Since $y_n^0 \in K_n(x_n^0)$ and $z_n^0 \in T_n(y_n^0)$ are arbitrary, it follows from (18.3) that

$$h_n(x_n^0) = \max_{y \in K_n(x_n^0)} \max_{z \in T_n(y)} \{-\xi_e(x_n^0, \langle z, y - x_n^0 \rangle)\} \le r.$$

This proves that, for each $r \in \mathbb{R}$, the level set $\{x_n^k \in X : h_n(x_n^k) \le r\}$ is closed. Hence h_n is lower semi-continuous in X.

Next, we show that h_n is upper semi-continuous in X, i.e., $-h_n$ is lower semi-continuous in X. Indeed, let $r \in \mathbb{R}$ and suppose that $\{x_n^k\} \subset X$ satisfies $-h_n(x_n^k) \le r$ and $x_n^k \to x_n^0$ as $k \to \infty$. Moreover, it follows from $-h_n(x_n^k) \le r$ that

$$-\max_{y \in K_n(x_n^k)} \max_{z \in T_n(y)} \{-\xi_e(x_n^k, \langle z, y - x_n^k \rangle)\} \le r, \quad \forall x_n^k \in K_n(x_n^k),$$

that is,

$$\min_{y \in K_n(x_n^k)} \min_{z \in T_n(y)} \xi_e(x_n^k, \langle z, y - x_n^k \rangle) \le r, \quad \forall x_n^k \in K_n(x_n^k). \tag{18.4}$$

Since K_n is upper semi-continuous with compact values in X, we have $x_n^0 \in K_n(x_n^0)$. Since K_n and T_n have compact values in X, from (18.4), there exist $\overline{y}_n^k \in K_n(x_n^k)$ and $\overline{z}_n^k \in T_n(\overline{y}_n^k)$ such that

$$\xi_e(x_n^k, \langle \overline{z}_n^k, \overline{y}_n^k - x_n^k \rangle) = \min_{y \in K_n(x_n^k)} \min_{z \in T_n(y)} \xi_e(x_n^k, \langle z, y - x_n^k \rangle) \le r. \tag{18.5}$$

Since K_n is upper semi-continuous with compact values in X, there exists $y_n^0 \in K_n(x_n^0)$ such that $\overline{y}_n^k \to y_n^0$ (taking a subsequence if necessary) as $k \to \infty$. Since T_n is upper semi-continuous with compact values in X, there exists $z_n^0 \in T_n(y_n^0)$ such that $\overline{z}_n^k \to z_n^0$ (taking a subsequence if necessary) as $k \to \infty$. From the continuity of ξ_e, taking $k \to \infty$ in (18.5), we have

$$\xi_e(x_n^0, \langle z_n^0, y_n^0 - x_n^0 \rangle) \le r. \tag{18.6}$$

Thus, for any $y \in K_n(x_n^0)$ and $z \in T_n(y)$, it follows from (18.6) that

$$\min_{y \in K_n(x_n^0)} \min_{z \in T_n(y)} \xi_e(x_n^0, \langle z, y - x_n^0 \rangle) \le r,$$

that is,

$$-h_n(x_n^0) = -\max_{y \in K_n(x_n^0))} \max_{z \in T_n(y)} \{-\xi_e(x_n^0, \langle z, y - x_n^0 \rangle)\} \le r,$$

This proves that, for $r \in \mathbb{R}$, the level set $\{x_n^k \in X : -h_n(x_n^k) \le r\}$ is closed. Hence $-h_n$ is lower semi-continuous in X. This completes the proof. $\square$

For the problem (MQVIP), we also obtain a similar conclusion as well as Proposition 18.3.

Proposition 18.4 *Consider the problem* (MQVIP). *If the following conditions hold:*

(a) *K is continuous with compact values in X;*
(b) *T is continuous with compact values in X;*
(c) *V and C are upper semi-continuous in X and $e(\cdot) \in \mathrm{int}C(\cdot)$ is continuous in X.*

Then h is continuous in X.

Remark 18.2 Noting that Fang et al. [17] and Anh et al. [4] discussed the gap functions for set-valued weak vector variational inequality problems of the Stampacchia type and generalized set-valued quasiequilibrium problems of the Stampacchia type, respectively, while we consider the gap functions for generalized vector quasivariational inequality problems of the Minty type. Therefore, our propositions in this section are new and different from the results from Fang et al. [17] and Anh et al. [4].

18.3 Main Results

Motivated by the hypothesis (H_1) of [33, 43], the assumption (H_g) in [11, 17, 36] and the assumption (H_h) in [2, 4], by virtue of the gap functions h and h_n, we introduce the following key hypothesis and employ it to study the Painlevé–Kuratowski lower convergence and Painlevé–Kuratowski convergence of the solution sets for the problems (MQVIP) and (MQVIP)$_n$:

(H_h): For any $\varepsilon > 0$, there exists $\alpha > 0$ and an $\overline{n}$ such that $h_n(x_n) \ge \alpha$ for all $n > \overline{n}$ and $x_n \in K_n(x_n) \setminus \mathscr{U}(S(T_n, K_n), \varepsilon)$.

To illustrate assumption (H_h), we give the following example:

Example 18.1 Let $X = Y = \mathbb{R}$ and $C(x) = \mathbb{R}_+$ for all $x \in X$. Define the set-valued mappings $K, K_n : X \rightrightarrows X$ and $T, T_n : X \rightrightarrows L(X, Y)$ as follows;

$$K(x) = [0, 1 + x^2], \quad K_n(x_n) = \left[0, 1 + \frac{1}{2n} + x_n^2\right],$$

$$T(y) = \left[\frac{1}{2}, 1 + 2^y\right], \quad T_n(y) = \left[\frac{1}{2}, 1 + \frac{1}{n} + 2^y\right].$$

Consider the problems (MQVIP) and (MQVIP)$_n$. It follows from the direct computation that

$$S(T, K) = S(T_n, K_n) = \{0\}.$$

Now, we show that $h_n(x_n)$ is a gap function of the problem (MQVIP)$_n$. Indeed, we taking $e(\cdot) = 1 \in \operatorname{int}\mathbb{R}_+$, we have

$$\begin{aligned} h_n(x_n) &= \max_{y \in K_n(x_n)} \max_{z \in T_n(y)} \{-\xi_e(x_n, \langle z, y - x_n\rangle)\} \\ &= \max_{y \in \left[0, 1 + \frac{1}{2n} + x_n^2\right]} \max_{z \in \left[\frac{1}{2}, 1 + \frac{1}{n} + 2^y\right]} \{z(x_n - y)\} \\ &= \Big(2 + \frac{1}{n}\Big) x_n. \end{aligned}$$

Clearly, $h_n(x_n^0) = 0$ if and only if $x_n^0 = 0 \in S(T_n, K_n)$. Moreover, for all $x_n \in K_n(x_n))$, $h_n(x_n) \geq 0$ and, for all $x_n \in K_n(x_n) \setminus S(T_n, K_n) = (0, +\infty)$, $h_n(x_n) > 0$. Thus $h_n(x_n)$ is a gap function of the problem (MQVIP)$_n$.

For any $\varepsilon > 0$, we take $\alpha = 2\varepsilon > 0$ and $\overline{n} = 1$. Then, for all $n > \overline{n}$ and $x_n \in K_n(x_n) \backslash \mathscr{U}(S(T_n, K_n), \varepsilon) = [\varepsilon, +\infty)$, it follows that $h_n(x_n) = \big(2 + \frac{1}{n}\big) x_n \geq \alpha$ and the assumption H_h holds.

Lemma 18.4 *Suppose that*

(a) $\{K_n\}$ *converges continuously to* K *with compact values in* X*;*
(b) $\{T_n\}$ *converges continuously to* T *with compact values in* X*;*
(c) V, C *are upper semi-continuous in* X *and* $e(\cdot) \in \operatorname{int}C(\cdot)$ *is continuous in* X.

Then, for any $\delta > 0$, $x_0 \in K(x_0)$ *and the sequence* $\{x_n\}$ *with* $x_n \in K_n(x_n)$ *and* $x_n \to x_0$*, there exists* $n_0 > 0$ *such that* $h_n(x_n) - \delta \leq h(x_0) \leq h_n(x_n) + \delta$*, for all* $n \geq n_0$.

Proof For any $x_0 \in K(x_0)$ and sequence $\{x_n\}$ with $x_n \in K_n(x_n)$ and $x_n \to x_0$, since K_n and T_n have compact values in X, there exist $y_n \in K_n(x_n)$ and $z_n \in T_n(y_n)$ such that

$$\max_{y \in K_n(x_n)} \max_{z \in T_n(y)} \{-\xi_e(x_n, \langle z_n, y - x_n\rangle)\} = -\xi_e(x_n, \langle z_n, y_n - x_n\rangle). \tag{18.7}$$

From the compactness of $K_n(x_n)$, we may assume, without loss of generality, that $y_n \to y_0$ (can take a subsequence if necessary). From $\limsup\limits_{n \to \infty} K_n(x_n) \subset K(x_0)$, we have $y_0 \in K(x_0)$. Similarly, by the compactness of $T_n(y_n)$, we may assume, without loss of generality, that $z_n \to z_0$ (can take a subsequence if necessary). From $\limsup\limits_{n \to \infty} T_n(y_n) \subset T(x_0)$, we get $z_0 \in T(y_0)$. By the continuity of $\langle \cdot, \cdot \rangle$ and $(x_n, z_n, y_n) \to (x_0, z_0, y_0)$, we have

$$\langle z_n, y_n - x_n\rangle \to \langle z_0, y_0 - x_0\rangle.$$

Since ξ_e is continuous, we take the limit in (18.7) as follows:

$$\begin{aligned}\lim_{n\to\infty}\{-\xi_e(x_n, \langle z_n, y_n - x_n\rangle)\} &= -\xi_e(x_0, \langle z_0, y_0 - x_0\rangle)\\ &\leq \max_{y\in K(x_0)}\max_{z\in T(y)}\{-\xi_e(x_0, \langle z, y - x_0\rangle)\}\\ &= h(x_0).\end{aligned}$$

So, for all $\delta > 0$, there exists $n_0 > 0$ such that

$$-\xi_e(x_n, \langle z_n, y_n - x_n\rangle) - \delta \leq h(x_0), \quad \forall n \geq n_0.$$

From (18.7), we have

$$\begin{aligned}h_n(x_n) - \delta &= \max_{y\in K_n(x_n)}\max_{z\in T_n(y)}\{-\xi_e(x_n, \langle z_n, y - x_n\rangle)\} - \delta\\ &= -\xi_e(x_n, \langle z_n, y_n - x_n\rangle) - \delta\\ &\leq h(x_0).\end{aligned}$$

On the other hand, since K and T have compact values in X, there exist $y_0 \in K(x_0)$ and $z_0 \in T(y_0)$ such that

$$h(x_0) = \max_{y\in K(x)}\max_{z\in T(y)}\{-\xi_e(x_0, \langle z_0, y_0 - x_0\rangle)\} = -\xi_e(x_0, \langle z_0, y_0 - x_0\rangle). \tag{18.8}$$

From $K(x_0) \subset \liminf_{n\to\infty} K_n(x_n)$, we can assume that there exist $y_n \in K_n(x_n)$ such that $y_n \to y_0$. Similarly, since $T(y_0) \subset \liminf_{n\to\infty} T_n(y_n)$, there exist $z_n \in T_n(y_n)$ such that $z_n \to z_0$. By the continuity of $\langle\cdot,\cdot\rangle$ and $(x_n, z_n, y_n) \to (x_0, z_0, y_0)$, we have

$$\langle z_n, y_n - x_n\rangle \to \langle z_0, y_0 - x_0\rangle.$$

It follows from the continuity of ξ_e and (18.8) that

$$\lim_{n\to\infty}\{-\xi_e(x_n, \langle z_n, y_n - x_n\rangle)\} = -\xi_e(x_0, \langle z_0, y_0 - x_0\rangle) = h(x_0).$$

From $h_n(x_n) \geq \{-\xi_e(x_n, \langle z_n, y_n - x_n\rangle)\}$, for any $\delta > 0$, there is $n_0 \in \mathbb{N}$ such that

$$h_n(x_n) + \delta \geq \lim_{n\to\infty}\{-\xi_e(x_n, \langle z_n, y_n - x_n\rangle)\} = -\xi_e(x_0, \langle z_0, y_0 - x_0\rangle) = h(x_0)$$

for all $n \geq n_0$. This completes the proof. □

Example 18.2 Let $X = Y = \mathbb{R}$ and $C(x) = \mathbb{R}_+$ for all $x \in X$. Define the set-valued mappings $K, K_n : X \rightrightarrows X$ and $T, T_n : X \rightrightarrows L(X, Y)$ as follows:

$$K(x) = [0, 1 + x^2], \quad K_n(x_n) = \left[0, 1 + \frac{1}{2n} + x_n^2\right],$$

$$T(y) = \left[\frac{1}{2}, 1 + 2^y\right], \quad T_n(y) = \left[\frac{1}{2}, 1 + \frac{1}{n} + 2^y\right].$$

The conditions (a)–(c) in Lemma 18.4 are satisfied. From Example 18.1, we have

$$h_n(x_n) = \left(2 + \frac{1}{n}\right) x_n$$

is a gap function of the problem $(\text{MQVIP})_n$. Similarly, we have the following function, that is,

$$h(x) = 2x$$

is a gap function of the problem (MQVIP). For any $x_0 \in K(x_0)$ and $x_n \in K_n(x_n)$ such that $x_n \to x_0$, we have

$$\left(2 + \frac{1}{n}\right)x_n \to 2x_0.$$

Thus we have $h_n(x_n) \to h(x_0)$ and, for any $\delta > 0$,

$$h_n(x_n) - \delta \le h(x_0) \le h_n(x_n) + \delta.$$

Next, we prove the compactness of the solution sets for the problems (MQVIP) and $(\text{MQVIP})_n$.

Proposition 18.5 *Suppose that*

(a) *K is inner semi-continuous with compact values in X;*
(b) *T is inner semi-continuous in X;*
(c) *K and $V(\cdot) = Y \setminus -\text{int}C(\cdot)$ are closed in X.*

Then $S(T, K)$ is a compact set.

Proof First, we prove that $S(T, K)$ is a closed set. Take any $x_n \in S(T, K)$ with $x_n \to x_0$. Since K is closed in X, we have $x_0 \in K(x_0)$.

Now, we show that $x_0 \in S(T, K)$. Suppose that $x_0 \notin S(T, K)$. Then there exist $y_0 \in K(x_0)$ and $z_0 \in T(y_0)$ such that

$$\langle z_0, y_0 - x_0 \rangle \in -\text{int}C(x_0). \tag{18.9}$$

By the inner semi-continuity of K and T in X, there exist $y_n \in K(x_n)$ and $z_n \in T(y_n)$ such that $y_n \to y_0$ and $z_n \to z_0$. Since $x_n \in S(T, K)$, we have

$$\langle z_n, y_n - x_n \rangle \in Y \setminus -\text{int}C(x_n). \tag{18.10}$$

From (18.10), the continuity of $\langle \cdot, \cdot \rangle$ and the closedness of $V(\cdot) = Y \setminus -\mathrm{int}C(\cdot)$, it follows that

$$\langle z_0, y_0 - x_0 \rangle \in Y \setminus -\mathrm{int}C(x_0),$$

which contradicts (18.9). Hence it follows that $x_0 \in S(T, K)$ and $S(T, K)$ is a closed set. Further, since $S(T, K) \subset K(x)$ and $K(x)$ is compact for all $x \in X$, it follows that $S(T, K)$ is a compact. This completes the proof. □

Using the proof lines for Proposition 18.5, we have the following result:

Proposition 18.6 *Suppose that*

(a) K_n *is inner semi-continuous with compact values in* X*;*
(b) T_n *is inner semi-continuous in* X*;*
(c) K_n *and* $V(\cdot) = Y \setminus -\mathrm{int}C(\cdot)$ *are closed in* X.

Then $S(T_n, K_n)$ *is a compact set.*

Lemma 18.5 *Suppose that all the conditions in Propositions 18.5 and 18.6 are satisfied. Then* $S(T, K) \subset \liminf_{n\to\infty} S(T_n, K_n)$ *if and only if, for all* $\varepsilon > 0$*, there exists* $N > 0$ *such that* $S(T, K) \subset \mathscr{U}(S(T_n, K_n), \varepsilon)$ *for all* $n \geq N$.

Proof We assume that $S(T, K) \subset \liminf_{n\to\infty} S(T_n, K_n)$ and there exists $\varepsilon_0 > 0$ such that, for all $N > 0$, there exists $N_n \geq N$ satisfying

$$S(T, K) \not\subset \mathscr{U}(S(T_{N_n}, K_{N_n}), \varepsilon_0).$$

Then there exists a sequence $\{x_n\}$ with $x_n \in S(T, K)$, but $x_n \notin \mathscr{U}(S(T_{N_n}, K_{N_n}), \varepsilon_0)$. From Proposition 18.5, we know that $S(T, K)$ is a compact set. Without loss of generality, we assume that $x_n \to x$ and $x \in S(T, K)$. Thus, for any sequence $\{t_n\}$ satisfying $t_n \to t$ with $t_n \in S(T_n, K_n)$, we have $\|t_{N_n} - x_n\| \geq \varepsilon_0 > 0$. Taking $n \to \infty$, we get $\|t - x\| \geq \varepsilon_0 > 0$. Therefore, there does not exist $t_n \in S(T_n, K_n)$ satisfying $t_n \to x$. This is a contradiction to $S(T, K) \subset \liminf_{n\to\infty} S(T_n, K_n)$.

Conversely, suppose that, for any $\varepsilon > 0$, there exists $N > 0$ such that

$$S(T, K) \subset \mathscr{U}(S(T_n, K_n), \varepsilon), \quad \forall n \geq N.$$

From Proposition 18.6, we derive that $S(T_n, K_n)$ is compact. Thus, for any $x \in S(T, K)$, there exists $x_n \in S(T_n, K_n)$ such that

$$\|x_n - x\| = d(x, S(T_n, K_n)) \leq \varepsilon, \quad \forall n \geq N$$

and hence $x_n \to x$ and $S(T, K) \subset \liminf_{n\to\infty} S(T_n, K_n)$. Therefore, the result of this lemma follows. This completes the proof. □

Lemma 18.6 *Suppose that all the conditions in Proposition 18.3 are satisfied. Then* (H_h) *holds if and only if, for any* $\varepsilon > 0$,

$$\liminf_{x_n \in K_n(x_n)\setminus \mathscr{U}(S(T_n,K_n),\varepsilon)} h_n(x_n) > 0,$$

where

$$\liminf_{x_n \in K_n(x_n)\setminus \mathscr{U}(S(T_n,K_n),\varepsilon)} h_n(x_n) = \liminf_{n\to\infty}\Big(\inf_{x_n \in K_n(x_n)\setminus \mathscr{U}(S(T_n,K_n),\varepsilon)} h_n(x_n)\Big).$$

Proof If (H_h) holds, then, for any $\varepsilon > 0$, there exist $\alpha > 0$ and $\overline{n}$ such that $h_n(x_n) \geq \alpha$ for all $n > \overline{n}$ and $x_n \in K_n(x_n) \setminus \mathscr{U}(S(T_n, K_n), \varepsilon)$. This implies that

$$\liminf_{x_n \in K_n(x_n)\setminus \mathscr{U}(S(T_n,K_n),\varepsilon)} h_n(x_n) \geq \alpha > 0.$$

Conversely, suppose that, for any $\varepsilon > 0$, there exists $\overline{n} \in \mathbb{N}$ such that

$$\tau := \liminf_{x_n \in K_n(x_n)\setminus \mathscr{U}(S(T_n,K_n),\varepsilon)} h_n(x_n) > 0, \quad \forall n > \overline{n},$$

where $\alpha := \frac{1}{2}\tau$. Hence, for any $x_n \in K_n(x_n) \setminus \mathscr{U}(S(T_n, K_n), \varepsilon)$, we have $h_n(x_n) \geq \alpha > 0$, which shows that (H_h) holds. This completes the proof. □

Theorem 18.1 *Suppose that all the assumptions in Propositions 18.5 and 18.6 are satisfied and the following additional conditions:*

(a) $\{K_n\}$ *converges continuously to* K *with compact values in* X*;*
(b) $\{T_n\}$ *converges continuously to* T *with compact values in* X*;*
(c) V *and* C *are upper semi-continuous in* X *and* $e(\cdot) \in \mathrm{int}C(\cdot)$ *is continuous in* X.

Then $S(T, K) \subset \liminf_{n\to\infty} S(T_n, K_n)$ *if and only if* (H_h) *holds.*

Proof First, we prove the sufficient condition. Suppose to the contrary that (H_h) holds, but $S(T, K) \not\subset \liminf_{n\to\infty} S(T_n, K_n)$. Then, by Lemma 18.5, that there exists $\varepsilon_0 > 0$ such that, for any $\overline{m} > 0$, $m_n \geq \overline{m}$ satisfying

$$S(T, K) \not\subset \mathscr{U}(S(T_{m_n}, K_{m_n}), \varepsilon_0),$$

that is, there exists a sequence $\{x_{m_n}\}$ such that

$$x_{m_n} \in S(T, K) \setminus \mathscr{U}(S(T_{m_n}, K_{m_n}, \varepsilon_0). \tag{18.11}$$

From the compactness of $S(T, K)$, we can assume that $x_{m_n} \to x \in S(T, K)$. Then there exists $\overline{m}_1 > 0$ such that

$$\|x_{m_n} - x\| \leq \varepsilon_0/4, \quad \forall n > \overline{m}_1.$$

It is clear that, for all $n > 0$, $\mathscr{B}(x, \varepsilon_0/m_n) \bigcap K(x) \neq \varnothing$. By the assumption (a), there exists a sequence $a_{m_n} \in K_{m_n}(x_{m_n})$ satisfying $a_{m_n} \to x$. Then there exists $\overline{m}_2 > 0$ such that

$$a_{m_n} \in \mathscr{B}(x, \varepsilon_0/m_n) \bigcap K_{m_n}(x_{m_n}), \quad \forall n > \overline{m}_2.$$

Now, we claim that $a_{m_n} \notin \mathscr{U}(S(T_{m_n}, K_{m_n}), \varepsilon_0/4)$. Otherwise, there exists $t_{m_n} \in S(T_{m_n}, K_{m_n})$ such that $\|a_{m_n} - t_{m_n}\| < \varepsilon_0/4$. Consequently, for $\overline{m}_0 = \max\{\overline{m}_1, \overline{m}_2\}$, we get

$$\begin{aligned}\|x_{m_n} - t_{m_n}\| &\leq \|x_{m_n} - x\| + \|x - a_{m_n}\| + \|a_{m_n} - t_{m_n}\| \\ &\leq \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{m_n} + \frac{\varepsilon_0}{4} < \varepsilon_0, \quad \forall n > \overline{m}_0.\end{aligned}$$

This implies that $x_{m_n} \in \mathscr{U}(S(T_{m_n}, K_{m_n}), \varepsilon_0)$, which contradicts (18.11). Thus we have

$$a_{m_n} \notin \mathscr{U}\Big(S(T_{m_n}, K_{m_n}), \frac{\varepsilon_0}{4}\Big).$$

By the assumption (H_h), there exists $\beta > 0$ such that $h_n(a_n) \geq \beta$. By Lemma 18.4, with n large enough, for any $\delta > 0$, we have

$$h_{m_n}(a_{m_n}) - \delta \leq h(x).$$

We can take δ such that $\beta - \delta > 0$. Thus we have

$$h(x) \geq h_{m_n}(a_{m_n}) - \delta \geq \beta - \delta > 0$$

and so

$$h(x) = \max_{y \in K(x)} \max_{z \in T(y)} \{-\xi_e(x, \langle z, y - x\rangle)\} > 0,$$

which contradicts $x \in S(T, K)$ (by Proposition 18.1(a)). Therefore, we have

$$S(T, K) \subset \liminf_{n\to\infty} S(T_n, K_n).$$

Conversely, we prove the necessary condition. Suppose to the contrary that

$$S(T, K) \subset \liminf_{n\to\infty} S(T_n, K_n),$$

but (H_h) does not hold. By Lemma 18.6, there exists $\varepsilon_0 > 0$ such that, for any $\overline{n} > 0$ and $n \geq \overline{n}$,

$$\lim \inf_{x_n \in K_n(x_n) \setminus \mathscr{U}(S(T_n, K_n), \varepsilon_0)} h_n(x_n) = 0.$$

Using the compactness of $K_n(x_n) \setminus \mathscr{U}(S(T_n, K_n), \varepsilon_0)$ and the continuity of h_n from Proposition 18.3, we can assume that there exists $x_n' \in K_n(x_n') \setminus \mathscr{U}(S(T_n, K_n), \varepsilon_0)$

such that $\lim_{n\to\infty} h_n(x_n') = 0$. Since $\{K_n\}$ converges continuously to K with compact values in X, we can assume that $x_n' \to x_0 \in K(x_0)$. From Lemma 18.4, we have $h(x_0) = 0$. Indeed, if $h(x_0) = \sigma > 0$, we take $\delta := \frac{\sigma}{4}$. Since $\{h_n(x_n')\}$ converges to 0, there exists $n_1 \in \mathbb{N}$ such that $h_n(x_n') \leq \frac{\sigma}{4}$ for all $n \geq n_1$. By Lemma 18.4, we can assume that there exists $n_0 \in \mathbb{N}$ satisfying $h_n(x_n') + \delta \geq h(x_0)$ for all $n \geq n_0$. Putting $n_2 = \max\{n_0, n_1\}$, we have

$$\frac{\sigma}{4} + \frac{\sigma}{4} = \frac{\sigma}{2} \geq h_n(x_n') + \delta \geq h(x_0) = \sigma > 0, \quad \forall n \geq n_2,$$

which is impossible. So, it follows from Proposition 18.1 that $x_0 \in S(T, K)$. Since $S(T, K) \subset \liminf_{n\to\infty} S(T_n, K_n)$, there exists $w_n \in S(T_n, K_n)$ such that $\{w_n\}$ converges to x_0. Since $x_n' \in K_n(x_n') \setminus (S(T_n, K_n) + U)$, $w_n - x_n' \notin U$ for all n, which is impossible since $\{w_n\}$ and $\{x_n'\}$ converge to the same point x_0. Thus (H_h) holds. This completes the proof. $\square$

Now, we give the following examples to illustrate Theorem 18.1:

Example 18.3 Let $X = Y = \mathbb{R}$ and $C(x) = \mathbb{R}_+$ for all $x \in X$. Define the set-valued mappings $K, K_n : X \rightrightarrows X$ and $T, T_n : X \rightrightarrows L(X, Y)$ as follows:

$$K(x) = [0, 1 + x^2], \quad K_n(x_n) = \left[0, 1 + \frac{1}{2n} + x_n^2\right],$$

$$T(y) = \left[\frac{1}{2}, 1 + 2^y\right], \quad T_n(y) = \left[\frac{1}{2}, 1 + \frac{1}{n} + 2^y\right].$$

The assumption (H_h) hold by Example 18.1 and so are all the conditions of Theorem 18.1. From Example 18.1, we have $\liminf\limits_{n\to\infty} S(T_n, K_n) = \{0\} = S(T, K)$. Thus the solution sets of the problem (MQVIP)$_n$ is lower convergent in the sense of Painlevé–Kuratowski to the solution set of the problem (MQVIP).

Example 18.4 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $C(x) = \mathbb{R}^2_+$. Define the set-valued mappings $K, K_n : X \rightrightarrows X$ and $T, T_n : X \rightrightarrows L(X, Y)$ as follows:

$$K(x) = K_n(x_n) = \left[-\frac{1}{2}, \frac{1}{2}\right],$$

$$T(y) = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}, \quad T_n(y) = \begin{pmatrix} [0, \frac{y^2}{n}] \\ \frac{1}{n} \end{pmatrix}.$$

Consider the problems (MQVIP) and (MQVIP)$_n$. It follows from the direct computation that $S(T, K) = [-\frac{1}{2}, \frac{1}{2}]$ and $S(T_n, K_n) = \{-\frac{1}{2}\}$. Hence $S(T_n, K_n)$ is not lower convergent to $S(T, K)$ in the sense of Painlevé–Kuratowski.

Now, we show that the condition (H_h) does not hold. Taking $e = (1, 1) \in \mathrm{int}\mathbb{R}^2_+$, we have

$$
\begin{aligned}
h_n(x_n) &= \max_{y\in K_n(x_n)} \max_{z\in T_n(y)} \{-\xi_e(x_n, \langle z, y-x_n\rangle)\} \\
&= \max_{y\in K_n(x_n)} \max_{z\in T_n(y)} \max_{1\le i\le 2} [\langle z, y-x_n\rangle)]_i \\
&= \max_{y\in[-\frac{1}{2},\frac{1}{2}]} \max_{z\in[0,\frac{y^2}{n}]} \left\{ z(x_n-y), \frac{1}{n}(x_n-y) \right\} \\
&= \max_{y\in[-\frac{1}{2},\frac{1}{2}]} \left\{ \frac{1}{n}(x_n-y) \right\} \\
&= \frac{1}{n}\Big(x_n+\frac{1}{2}\Big).
\end{aligned}
$$

It follows that h_n is a gap function of the problem (MQVIP)$_n$. For any ε with $0<\varepsilon<\frac{1}{2}$ and $\alpha>0$, taking $\overline{n}\in\mathbb{N}$ such that, for all $n>\overline{n}$, $0<\frac{1}{2n}<\alpha$ and $x_n=0\in K_n(x_n)\setminus\mathscr{U}(S(T_n,K_n),\varepsilon)=[-\frac{1}{2}+\varepsilon,\frac{1}{2}]$, we have $h_n(x_n)=\frac{1}{2n}<\alpha$. Hence (H_h) does not hold.

Next, we discuss the sufficient and necessary conditions for the Painlevé–Kuratowski convergence of the solution sets for the problems (MQVIP) and (MQVIP)$_n$.

Theorem 18.2 *Suppose that all the conditions in Theorem 18.1 are satisfied. Then $\{S(T_n,K_n)\}$ converges to $S(T,K)$ in the sense of Painlevé–Kuratowski if and only if (H_h) holds.*

Proof From Theorem 18.1, we only need to prove that

$$\limsup_{n\to\infty} S(T_n,K_n)\subset S(T,K).$$

Indeed, we suppose to the contrary that $\limsup_{n\to\infty} S(T_n,K_n)\not\subset S(T,K)$, i.e., there exists $x_0\in\limsup\limits_{n\to\infty} S(T_n,K_n)$, but $x_0\notin S(T,K)$. Since $x_0\in\limsup\limits_{n\to\infty} S(T_n,K_n)$, there exists a sequence $\{x_{n_k}\}$, $x_{n_k}\in S(T_{n_k},K_{n_k})$, such that $x_{n_k}\to x_0$. Then, for all $y\in K_{n_k}(x_{n_k})$ and $z\in T_{n_k}(y)$, we have

$$\langle z, y-x_{n_k}\rangle\in Y\setminus -\mathrm{int}C(x_{n_k}). \tag{18.12}$$

From $\limsup\limits_{k\to\infty} K_{n_k}(x_{n_k})\subset K(x_0)$ and $x_{n_k}\in K_{n_k}(x_{n_k})$, we have $x_0\in K(x_0)$.

Now, we prove that $x_0\in S(T,K)$. If $x_0\notin S(T,K)$, then there exist $y_0\in K(x_0)$ and $z_0\in T(y_0)$ such that

$$\langle z_0, y_0-x_0\rangle\in -\mathrm{int}C(x_0). \tag{18.13}$$

Since $\{K_n\}$ is inner convergent continuously to K and $\{T_n\}$ is inner convergent continuously to T, for all $y_0\in K(x_0)$, $z_0\in T(y_0)$, there exist $y_{n_k}\in K_{n_k}(x_{n_k})$, $z_{n_k}\in T_{n_k}(y_{n_k})$ such that $y_{n_k}\to y_0$, $z_{n_k}\to z_0$ as $k\to\infty$. From $x_{n_k}\in S(T_{n_k},K_{n_k})$, we have

$$\langle z_{n_k}, y_{n_k} - x_{n_k} \rangle \in Y \setminus -\mathrm{int}C(x_{n_k}). \tag{18.14}$$

From (18.14) and the continuity of $\langle \cdot, \cdot \rangle$, since $V(\cdot) = Y \setminus -\mathrm{int}C(\cdot)$ is closed, we have

$$\langle z_0, y_0 - x_0 \rangle \in Y \setminus -\mathrm{int}C(x_0),$$

which is a contradiction to (18.13), and so $x_0 \in S(T, K)$. This completes the proof. □

18.4 Conclusion

In this paper, we first established gap functions for the problems (MQVIP) and (MQVIP)$_n$ and consider their continuity. Then we proved that the hypothesis (H_h) is a sufficient and necessary condition for the lower convergence of Painlevé–Kuratowski and the convergence of Painlevé–Kuratowski of solution sets of these problems. As mentioned in Introduction and Remark 18.2, up to now, there have not been any works on the sufficient and necessary conditions for the lower convergence of Painlevé–Kuratowski and the convergence of Painlevé–Kuratowski of solution sets for generalized vector quasivariational inequality problems of the Minty type by the gap function method. Hence our main results are new and different from the results in Fang et al. [17] and Anh et al. [4].

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Chapter 19
Common Solutions for a System of Functional Equations in Dynamic Programming Passing Through the $JCLR$-Property in S_b-Metric Spaces

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Abstract In this chapter, we introduce the new concept of the joint common limit in the range property (shortly, ($JCLR$)-property) in S_b-metric spaces and prove some common fixed point theorems by using the $JCLR$-property in S_b-metric spaces without the completeness of S_b-metric spaces. We also give some examples to illustrate our results. As applications of our results, we show the existence of common solutions for a system of functional equations in dynamic programming.

19.1 Introduction and Preliminaries

Throughout this chapter, we denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ the sets of positive integers, non-negative real numbers and real numbers, respectively.

In 1993, Czerwik [1] introduced the concept of b-metric spaces as a generalization of metric spaces and proved the Banach contraction principle in b-metric spaces, which is a generalization of the Banach contraction principle in metric spaces. For more details on the Banach contraction principle, refer to [2].

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Later, several researchers have studied many results in b-metric spaces (see in [3–5] and references therein).

Next, we recall some definitions from b-metric spaces as follows:

Definition 19.1 ([1]) Let X be a nonempty set and $b \geq 1$ be a fixed real number. Suppose that $d : X \times X \to \mathbb{R}_+$ is a mapping satisfying the following conditions for all $x, y, z \in X$:

(BM1) $d(x, y) = 0$ if and only if $x = y$;
(BM2) $d(x, y) = d(y, x)$;
(BM3) $d(x, z) \leq b[d(x, y) + d(y, z)]$.

Then the mapping d is called a *b-metric* and the pair (X, d) is called a *b-metric space* with coefficient b.

In 2015, Sedghi et al. [6] introduced the concept of S-metric spaces as follows:

Definition 19.2 ([6]) Let X be a nonempty set. Suppose that $S : X \times X \times X \to \mathbb{R}_+$ is a mapping satisfying the following conditions for all $x, y, z, a \in X$:

(SM1) $0 < S(x, y, z)$ with $x \neq y \neq z \neq x$;
(SM2) $S(x, y, z) = 0$ if and only if $x = y = z$;
(SM3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then the mapping S is called an *S-metric* on X and the pair (X, S) is called an *S-metric space*.

Recently, Sedghi et al. [7] introduced the concept of S_b-metric spaces as a generalization of S-metric spaces by replacing the right-hand side of (SM3) with the generalized condition as follows:

Definition 19.3 ([7]) Let X be a nonempty set and $b \geq 1$ be a real number. Suppose that $S_b : X \times X \times X \to \mathbb{R}_+$ is a mapping satisfying the following conditions for all $x, y, z, a \in X$:

(SbM1) $0 < S_b(x, y, z)$ with $x \neq y \neq z \neq x$;
(SbM2) $S_b(x, y, z) = 0$ if and only if $x = y = z$;
(SbM3) $S_b(x, y, z) \leq b[S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)]$.

Then the mapping S_b is called an *S_b-metric* on X and the pair (X, S_b) is called an *S_b-metric space* with the coefficient b.

It should be noted that the class of S_b-metric spaces is effectively larger than that of S-metric spaces. Indeed, each S-metric space is an S_b-metric space with $b = 1$. A known example of an S_b-metric space is as follows:

Example 19.1 ([7]) Let (X, S) be an S-metric space, $p > 1$ be a real number and $S_* : X \times X \times X \to \mathbb{R}_+$ be a mapping defined by

$$S_*(x, y, z) = [S(x, y, z)]^p$$

for all $x, y, z \in X$. Therefore, S_* is an S_b-metric with the coefficient $b = 2^{2(p-1)}$ and so (X, S_*) is an S_b-metric space.

Next, we give some definitions and lemma in S_b-metric spaces which are needed for our results.

Definition 19.4 ([7]) Let (X, S_b) be an S_b-metric space, $x \in X$ and $r > 0$. The *open ball* $B_S(x, r)$ and *closed ball* $B_S[x, r]$ with center x and radius r are defined as follows, respectively:

$$B_S(x, r) := \{y \in X : S_b(y, y, x) < r\}$$

and

$$B_S[x, r] := \{y \in X : S_b(y, y, x) \leq r\}.$$

Lemma 19.1 ([7]) *Let (X, S_b) be an S_b-metric space with the coefficient $b \geq 1$. Then the following assertions hold:*

(1) $S_b(x, x, y) \leq bS_b(y, y, x)$ *for all* $x, y \in X$*;*
(2) $S_b(y, y, x) \leq bS_b(x, x, y)$ *for all* $x, y \in X$*;*
(3) $S_b(x, x, z) \leq 2bS_b(x, x, y) + b^2S_b(y, y, z)$ *for all* $x, y, z \in X$*.*

Definition 19.5 ([7]) Let (X, S_b) be an S_b-metric space.

(1) A sequence $\{x_n\}$ in X is said to be an S_b-*Cauchy sequence* if, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$S_b(x_n, x_n, x_m) < \varepsilon$$

for all $m, n \geq N$.

(2) A sequence $\{x_n\}$ in X is said to be S_b-*convergent* to a point $x \in X$ if, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$S_b(x_n, x_n, x) < \varepsilon \quad \text{or} \quad S_b(x, x, x_n) < \varepsilon$$

for all $n \geq N$, which is denoted by

$$\lim_{n\to\infty} x_n = x.$$

(3) An S_b-metric space X is said to be *complete* if and only if every S_b-Cauchy sequence is S_b-convergent in X.

Remark 19.1 For a sequence $\{x_n\}$ in an S_b-metric space (X, S_b) and $x \in X$, we obtain the following assertions:

(1) $\{x_n\}$ is an S_b-Cauchy sequence if and only if $\lim_{m,n\to\infty} S_b(x_n, x_n, x_m) = 0$;
(2) $\lim_{n\to\infty} x_n = x \Longleftrightarrow \lim_{n\to\infty} S_b(x_n, x_n, x) = 0$.

Lemma 19.2 ([7]) *Let (X, S_b) be an S_b-metric space with the coefficient $b \geq 1$. Suppose that $\{x_n\}$ is an S_b-convergent to a point $x \in X$. Then we have*

$$\frac{1}{b^2} S_b(x, x, y) \leq \liminf_{n \to \infty} S_b(x_n, x_n, y) \leq \limsup_{n \to \infty} S_b(x_n, x_n, y) \leq b^2 S_b(x, x, y)$$

for all $y \in X$. In particular,

$$\lim_{n \to \infty} S_b(x_n, x_n, x) = 0.$$

Definition 19.6 Let X be a nonempty set and $f, g : X \to X$ be two given mappings. The pair (f, g) is said to be *weakly compatible* if $fz = gz$ for some $z \in X$, then

$$fgz = gfz.$$

The aim of this chapter is to present the idea of the joint common limit in the range property in S_b-metric spaces. With the help of this property, we prove some unique common fixed point theorems in S_b-metric spaces without completeness. We also present an example to illustrate our results. Finally, we show the existence of a common solution for a system of functional equations in dynamic programming.

19.2 Main Results

First, we introduce the idea of the joint common limit in the range property in S_b-metric spaces and then we prove some common fixed point theorem for the generalized nonlinear contractive-type mappings in S_b-metric spaces using the joint common limit in the range property.

Definition 19.7 Let (X, S_b) be an S_b-metric space with the coefficient $b \geq 1$ and $f, g, H, T : X \to X$ be four given mappings.

(1) The pairs (f, H) and (g, T) are said to satisfy the *joint common limit in the range of H and T property* (shortly, $(JCLR_{HT})$*-property*) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \to \infty} fx_n = \lim_{n \to \infty} Hx_n = \lim_{n \to \infty} gy_n = \lim_{n \to \infty} Ty_n = Hu = Tu \tag{19.1}$$

for some $u \in X$.

(2) The pair (f, H) is said to satisfy the *common limit in the range of H property* (shortly, (CLR_H)*-property*) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to \infty} fx_n = \lim_{n \to \infty} Hx_n = Hu \tag{19.2}$$

for some $u \in X$.

Now, we give the main result in this chapter.

Theorem 19.1 *Let (X, S_b) be an S_b-metric space with the coefficient $b \geq 1$ and $f, g, H, T : X \to X$ be four mappings. Suppose that the pairs (f, H) and (g, T) satisfy the $(JCLR_{HT})$-property and*

$$S_b(fx, fx, gy) \leq \frac{q}{b^4} \max\Big\{S_b(Hx, Hx, Ty), S_b(fx, fx, Hx), S_b(gy, gy, Ty), \frac{1}{2}(S_b(Hx, Hx, gy) + S_b(fx, fx, Ty))\Big\} \tag{19.3}$$

for all $x, y \in X$, where $0 < q < \frac{b^3}{2}$. Then f, g, H and T have a coincidence point in X. If the pairs (f, H) and (g, T) are weakly compatible, then f, g, H and T have a unique common fixed point in X.

Proof Since the pairs (f, H) and (g, T) satisfy the $(JCLR_{HT})$-property, there exist the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Hx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = Hu = Tu \tag{19.4}$$

for some $u \in X$.

Now, we will show that $gu = Tu$. By using (19.3) with $x = x_n$ and $y = u$, we have

$$\begin{aligned}
&S_b(fx_n, fx_n, gu)\\
&\leq \frac{q}{b^4} \max\Big\{S_b(Hx_n, Hx_n, Tu), S_b(fx_n, fx_n, Hx_n), S_b(gu, gu, Tu),\\
&\qquad \frac{1}{2}(S_b(Hx_n, Hx_n, gu) + S_b(fx_n, fx_n, Tu))\Big\}\\
&\leq \frac{q}{b^4} \max\Big\{S_b(Hx_n, Hx_n, Tu),\\
&\qquad b(S_b(fx_n, fx_n, Tu) + S_b(fx_n, fx_n, Tu) + S_b(Hx_n, Hx_n, Tu)),\\
&\qquad b(S_b(gu, gu, fx_n) + S_b(gu, gu, fx_n) + S_b(Tu, Tu, fx_n)),\\
&\qquad \frac{1}{2}[b(S_b(Hx_n, Hx_n, Tu) + S_b(Hx_n, Hx_n, Tu) + b(S_b(gu, gu, fx_n)\\
&\qquad + S_b(gu, gu, fx_n) + S_b(Tu, Tu, fx_n))) + S_b(fx_n, fx_n, Tu)]\Big\}
\end{aligned}$$

for all $n \in \mathbb{N}$. Taking the limit superior as $n \to \infty$ in the above inequality, we obtain

$$\limsup_{n\to\infty} S_b(fx_n, fx_n, gu) \leq \frac{2q}{b^3} \limsup_{n\to\infty} S_b(fx_n, fx_n, gu).$$

This implies that

$$\Big(1 - \frac{2q}{b^3}\Big)\limsup_{n\to\infty} S_b(fx_n, fx_n, gu) \leq 0$$

and so

$$\lim_{n\to\infty} S_b(fx_n, fx_n, gu) = 0.$$

This yields that

$$Tu = \lim_{n\to\infty} fx_n = gu. \tag{19.5}$$

Next, we will claim that $fu = Tu$. From (19.3) with $x = u$ and $y = y_n$, we have

$$\begin{aligned}
&S_b(fu, fu, gy_n)\\
&\le \frac{q}{b^4} \max\Big\{S_b(Hu, Hu, Ty_n), S_b(fu, fu, Hu), S_b(gy_n, gy_n, Ty_n),\\
&\qquad \frac{1}{2}(S_b(Hu, Hu, gy_n) + S_b(fu, fu, Ty_n))\Big\}\\
&\le \frac{q}{b^4} \max\Big\{S_b(Hu, Hu, Ty_n), S_b(2S_b(fu, fu, gy_n) + S_b(Hu, Hu, gy_n)),\\
&\qquad b(2S_b(gy_n, gy_n, Hu) + S_b(Ty_n, Ty_n, Hu)),\\
&\qquad \frac{1}{2}(S_b(Hu, Hu, gy_n) + b(2S_b(fu, fu, gy_n) + S_b(Ty_n, Ty_n, gy_n))\Big\}
\end{aligned}$$

for all $n \in \mathbb{N}$. Taking the limit superior as $n \to \infty$ in the above inequality, we have

$$\limsup_{n\to\infty} S_b(fu, fu, gy_n) \le \frac{2q}{b^3} \limsup_{n\to\infty} S_b(fu, fu, gy_n).$$

This yields that

$$\left(1 - \frac{2q}{b^3}\right) \limsup_{n\to\infty} S_b(fu, fu, gy_n) \le 0$$

and so

$$\lim_{n\to\infty} S_b(fu, fu, gy_n) = 0.$$

This implies that

$$Tu = \lim_{n\to\infty} gy_n = fu. \tag{19.6}$$

Thus, from (19.4), (19.5) and (19.6), it follows that u is a coincident point of f, g, H and T.

Next, we will show that f, g, H and T have a common fixed point provided that the pairs (f, H) and (g, T) are weakly compatible. Assume that $z = fu = gu = Tu = Hu$. Since the pair (f, H) is weakly compatible, we have

$$fHu = Hfu$$

and then

$$fz = fHu = Hfu = Hz.$$

Also, since the pair (g, T) is weakly compatible, we obtain

$$gTu = Tgu$$

and hence

$$gz = gTu = Tgu = Tz.$$

Now, we will show that $z = fz$. To prove this, using (19.3) with $x = z$ and $y = u$, we have

$$S_b(fz, fz, gu) \leq \frac{q}{b^4} \max\Big\{S_b(Hz, Hz, Tu), S_b(fz, fz, Hz), S_b(gu, gu, Tu), \frac{1}{2}(S_b(Hz, Hz, gu) + S_b(fz, fz, Tu))\Big\}$$

and so

$$S_b(fz, fz, z) \leq \frac{q}{b^4} S_b(fz, fz, z).$$

Since $0 < q < \frac{b^3}{2}$, we have

$$S_b(fz, fz, z) \leq \frac{1}{2b} S_b(fz, fz, z).$$

This yields that

$$S_b(fz, fz, z) = 0$$

and so

$$z = fz = Hz.$$

Using (19.3) with $x = u$ and $y = z$, we have

$$S_b(fu, fu, gz) \leq \frac{q}{b^4} \max\Big\{S_b(Hu, Hu, Tz), S_b(fu, fu, Hu), S_b(gz, gz, Tz), \frac{1}{2}(S_b(Hu, Hu, gz) + S_b(fu, fu, Tz))\Big\}$$

and so

$$S_b(z, z, gz) \leq \frac{q}{b^4} S_b(z, z, gz).$$

From $0 < q < \frac{b^3}{2}$, we have

$$S_b(z, z, gz) \leq \frac{1}{2b} S_b(z, z, gz)$$

and hence

$$S_b(z, z, gz) = 0.$$

This implies that

$$z = gz = Tz.$$

Therefore, we have

$$z = fz = gz = Tz = Hz$$

and hence f, g, H and T have a common fixed point $z \in X$.

For the uniqueness of the common fixed point z, let w be another common fixed point of f, g, H and T. On using (19.3) with $x = z$ and $y = w$, we have

$$S_b(fz, fz, gw) \leq \frac{q}{b^4} \max \Big\{ S_b(Hz, Hz, Tw), S_b(fz, fz, Hz), S_b(gw, gw, Tw), \frac{1}{2}(S_b(Hz, Hz, gw) + S_b(fz, fz, Tw)) \Big\}.$$

It follows that

$$S_b(z, z, w) \leq \frac{q}{b^4} S_b(z, z, w) \leq \frac{1}{2b} S_b(z, z, w).$$

This implies that

$$S_b(z, z, w) = 0,$$

that is,

$$z = w.$$

Hence f, g, H and T have a unique common fixed point in X. This completes the proof.

Next, we give an example to illustrate Theorem 19.1.

Example 19.2 Let $X = [0, 4)$ and $S_b : X \times X \times X \to \mathbb{R}_+$ be defined by

$$S_b(x, y, z) = (|2x - y - z| + |y - z|)^2$$

for all $x, y, z \in X$. Then (X, S_b) is an S_b-metric space with $b = 4$. Define the mappings $f, g, H, T : X \to X$ by

$$fx = \left(\frac{x}{8}\right)^{10}, \quad gx = \left(\frac{x}{16}\right)^{8}, \quad Hx = \left(\frac{x}{8}\right)^{5}, \quad Tx = \left(\frac{x}{16}\right)^{4}$$

for all $x \in X$.

To prove that the pairs (f, H) and (g, T) are weakly compatible, we suppose that $fz = Hz$ and $gz = Tz$. Then $z = 0$ and so

$$fH(0) = f(0) = H(0) = Hf(0), \quad gT(0) = g(0) = T(0) = Tg(0).$$

Therefore, (f, H) and (g, T) are weakly compatible.

Next, we will show that the pairs (f, H) and (g, T) satisfy the $(JCLR_{HT})$-property. Define two sequences $\{x_n\}$ and $\{y_n\}$ in X by $x_n = \frac{1}{n+1}$ and $y_n = \frac{1}{n}$ for each $n \in \mathbb{N}$. Then there exists $0 \in X$ such that

$$\lim_{n\to\infty} S_b(fx_n, fx_n, 0) = \lim_{n\to\infty} (2fx_n)^2 = \lim_{n\to\infty} 4\left(\frac{1}{8(n+1)}\right)^{20} = 0,$$

$$\lim_{n\to\infty} S_b(Hx_n, Hx_n, 0) = \lim_{n\to\infty} (2Hx_n)^2 = \lim_{n\to\infty} 4\left(\frac{1}{8(n+1)}\right)^{10} = 0,$$

$$\lim_{n\to\infty} S_b(gy_n, gy_n, 0) = \lim_{n\to\infty} (2gy_n)^2 = \lim_{n\to\infty} 4\left(\frac{1}{16n}\right)^{16} = 0,$$

$$\lim_{n\to\infty} S_b(Ty_n, Ty_n, 0) = \lim_{n\to\infty} (2Ty_n)^2 = \lim_{n\to\infty} 4\left(\frac{1}{16n}\right)^{8} = 0,$$

$$H(0) = 0 = T(0).$$

This implies that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Hx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = H(0) = T(0).$$

Therefore, the pairs (f, H) and (g, T) satisfy the $(JCLR_{HT})$-property.

Now, for all $x, y \in X$, we have

$$\begin{aligned} S_b(fx, fx, gy) &= (|2fx - fx - gy| + |fx - gy|)^2 \\ &= (2|fx - gy|)^2 \\ &= 4\left|\left(\frac{x}{8}\right)^{10} - \left(\frac{y}{16}\right)^{8}\right|^2 \\ &= 4\left|\left(\frac{x}{8}\right)^{5} + \left(\frac{y}{16}\right)^{4}\right|^2 \left|\left(\frac{x}{8}\right)^{5} - \left(\frac{y}{16}\right)^{4}\right|^2 \\ &\le \left|\left(\frac{1}{2}\right)^{5} + \left(\frac{1}{4}\right)^{4}\right|^2 S_b(Hx, Hx, Ty) \\ &= \frac{81}{4^8} S_b(Hx, Hx, Ty). \end{aligned}$$

This yields that

$$S_b(fx, fx, gy) \le \frac{q}{b^4} \max\Big\{S_b(Hx, Hx, Ty), S_b(fx, fx, Hx), S_b(gy, gy, Ty), \\ \frac{1}{2}(S_b(Hx, Hx, gy) + S_b(fx, fx, Ty))\Big\},$$

where $0 < q = \frac{81}{256} < \frac{b^3}{2}$. Therefore, f, g, H and T satisfy all the conditions of Theorem 19.1. Hence f, g, H and T have a unique common fixed point, i.e., a point $0 \in X$.

From Theorem 19.1, we immediately have the following result:

Corollary 19.1 *Let (X, S_b) be an S_b-metric space with the coefficient $b \geq 1$ and $f, H : X \to X$ be two mappings. Suppose that the pair (f, H) satisfies the (CLR_H)-property and*

$$S_b(fx, fx, fy) \leq \frac{q}{b^4} \max \Big\{ S_b(Hx, Hx, Hy), S_b(fx, fx, Hx), S_b(fy, fy, Hy), \frac{1}{2}(S_b(Hx, Hx, fy) + S_b(fx, fx, Hy)) \Big\} \tag{19.7}$$

for all $x, y \in X$, where $0 < q < \frac{b^3}{2}$. Then f and H have a coincidence point in X. If the pair (f, H) is weakly compatible, then f and H have a unique common fixed point in X.

19.3 Applications to the Dynamic Programming

Throughout this section, "*opt*" stands for sup or inf, U and V are Banach spaces, $W \subseteq U$ is a state space and $D \subseteq V$ is a decision space.

In 1978, Bellman and Lee [8] investigated the existence of solutions for the following functional equations:

$$q(x) = \sup_{y \in D} \{G(x, y, q(\tau(x, y)))\}$$

for all $x \in W$, where $q : W \to \mathbb{R}$ is an unknown function, $G : W \times D \times \mathbb{R} \to \mathbb{R}$ and $\tau : W \times D \to W$ are given mappings. This equation can be obtained from the basic form of functional equations in dynamic programming.

Later, Bhakta and Mitra [9] guaranteed the existence of solutions for the following functional equations which arise in a multistage decision process related to dynamic programming:

$$q(x) = \sup_{y \in D} \{g(x, y) + G(x, y, q(\tau(x, y)))\}$$

for all $x \in W$, where $q : W \to \mathbb{R}$ is an unknown function, $g : W \times D \to \mathbb{R}$, $G : W \times D \times \mathbb{R} \to \mathbb{R}$ and $\tau : W \times D \to W$ are given mappings. For more details on the existence of solutions for systems of functional equations in dynamic programming, refer to [10–12] and others.

The aim of this section is to apply the theoretical results in the previous section for claiming the existence of common solutions for the following system of functional equations:

$$\begin{cases} q_1(x) = \underset{y \in D}{opt}\{g(x, y) + G_1(x, y, q_1(\tau(x, y)))\}, \\ q_2(x) = \underset{y \in D}{opt}\{g(x, y) + G_2(x, y, q_2(\tau(x, y)))\} \end{cases} \tag{19.8}$$

for all $x \in W$, where $q_1, q_2 : W \to \mathbb{R}$ is an unknown function, $g : W \times D \to \mathbb{R}$, $G_1, G_2 : W \times D \times \mathbb{R} \to \mathbb{R}$ and $\tau : W \times D \to W$ are given mappings.

Theorem 19.2 *Consider the system (19.8) of functional equations, where $g : W \times D \to \mathbb{R}$ and $G_1, G_2 : W \times D \times \mathbb{R} \to \mathbb{R}$ are bounded functions. Let $p \geq 1$, $B(W)$ be the set of all bounded real-valued functions on W and $T_1, T_2 : B(W) \to B(W)$ be the mappings defined by*

$$\begin{aligned} (T_1 h_1)(x) &= \underset{y \in D}{opt}\{g(x, y) + G_1(x, y, h_1(\tau(x, y)))\}, \\ (T_2 h_2)(x) &= \underset{y \in D}{opt}\{g(x, y) + G_2(x, y, h_2(\tau(x, y)))\} \end{aligned} \tag{19.9}$$

for $h_1, h_2 \in B(W)$ and $x \in W$, respectively. Suppose that the following conditions hold:

(A1) *there exists $p \geq 1$ and $0 < q < 2^{6p-7}$ such that*

$$|G_1(x, y, h(x)) - G_1(x, y, k(x))|^p \leq \frac{q}{2^{9p-8}} \Theta_p(h, k)$$

for all $x \in W$, $y \in D$, $h, k \in B(W)$, where

$$\begin{aligned} \Theta_p(h, k) := \max \Big\{ & [2d(T_2 h, T_2 k)]^p, [2d(T_1 h, T_2 h)]^p, [2d(T_1 k, T_2 k)]^p, \\ & \frac{[2d(T_2 h, T_1 k)]^p + [2d(T_1 h, T_2 k)]^p}{2} \Big\} \end{aligned}$$

such that d is a Chebyshev metric on $B(W)$;

(A2) *there exists a sequence $\{h_n\}$ in $B(W)$ and a function $h^* \in B(W)$ such that*

$$d(T_1 h_n, T_1 h^*) \to 0, \quad d(T_2 h_n, T_1 h^*) \to 0 \quad n \to \infty$$

and $T_2 h_n \neq T_2 h^*$ for all $n \in \mathbb{N}$;

(A3) *$T_1 T_1 h = T_1 h$, whenever $T_1 h = T_2 h$ for some $h \in B(W)$.*

Then the system (19.8) of functional equations has a bounded common solution.

Proof First, we define a mapping $S_b : B(W) \times B(W) \times B(W) \to \mathbb{R}_+$ by

$$S_b(x, y, z) = \Big[\sup_{t \in w}|x(t) - y(t)| + \sup_{t \in w}|x(t) - z(t)| + \sup_{t \in w}|y(t) - z(t)|\Big]^p.$$

Therefore, $(B(W)), d)$ is an S_b-metric space with the coefficient $b = 2^{2(p-1)}$. From (A2), we know that the hybrid pair (T_1, T_2) satisfies the CLR_{T_1}-property.

Next, we will show that the condition (19.7) holds. Consider the following two cases:

Case I: Suppose that $opt = \sup$. Let $x \in W$, $h_1, h_2 \in B(W)$ and $\lambda > 0$. Then there exist $y_1, y_2 \in D$ such that

$$T_1h_1(x) \leq g(x, y_1) + G_1(x, y_1, h_1(\tau(x, y_1))), \tag{19.10}$$

$$T_1h_2(x) \leq g(x, y_2) + G_1(x, y_2, h_2(\tau(x, y_2))), \tag{19.11}$$

$$T_1h_1(x) \geq g(x, y_2) + G_1(x, y_2, h_1(\tau(x, y_2))), \tag{19.12}$$

$$T_1h_2(x) \geq g(x, y_1) + G_1(x, y_1, h_2(\tau(x, y_1))). \tag{19.13}$$

From (19.10) and (19.13), it follows that

$$\begin{aligned} T_1h_1(x) - T_1h_2(x) &< G_1(x, y_1, h_1(\tau(x, y_1))) - G_1(x, y_1, h_2(\tau(x, y_1))) \\ &\leq |G_1(x, y_1, h_1(\tau(x, y_1))) - G_1(x, y_1, h_2(\tau(x, y_1)))|. \end{aligned} \tag{19.14}$$

Similarly, by using (19.11) and (19.12), we have

$$T_1h_2(x) - T_1h_1(x) < |G_1(x, y_1, h_1(\tau(x, y_1))) - G_1(x, y_1, h_2(\tau(x, y_1)))|. \tag{19.15}$$

Combining (19.14) and (19.15), we obtain

$$|T_1h_1(x) - T_1h_2(x)| < |G_1(x, y_1, h_1(\tau(x, y_1))) - G_1(x, y_1, h_2(\tau(x, y_1)))|$$

and so

$$\begin{aligned} |T_1h_1(x) - T_1h_2(x)|^p &\leq |G_1(x, y_1, h_1(\tau(x, y_1))) - G_1(x, y_1, h_2(\tau(x, y_1)))|^p \\ &\leq \frac{q}{2^p b^4}\Theta_p(h_1, h_2). \end{aligned}$$

Taking the supremum on $x \in W$, we have

$$\begin{aligned} [d(T_1h_1, T_1h_2)]^p &= \Big[\sup_{x \in w}|T_1h_1(x) - T_1h_2(x)|\Big]^p \\ &= \sup_{x \in w}\big[|T_1h_1(x) - T_1h_2(x)|^p\big] \\ &\leq \frac{q}{2^p b^4}\Theta_p(h_1, h_2). \end{aligned}$$

Therefore, we have

$$S_b(T_1h_1, T_1h_1, T_1h_2) = [2d(T_1h_1, T_1h_2)]^p \leq \frac{q}{b^4}\Theta_p(h_1, h_2).$$

Thus, all the hypotheses of Corollary 19.1 hold for the hybrid pair (T_1, T_2). Moreover, by (A3), the hybrid pair (T_1, T_2) is weakly compatible. Therefore, T_1 and T_2 have a common fixed point. Hence, the system (19.8) of functional equations has a bounded solution.

Case II: Assume that $opt = \inf$. Similar to Case I, we have that the system (19.8) of functional equations has a bounded common solution.

19.4 Conclusions

The new concept of the joint common limit in the range property (shortly, $(JCLR_{HT})$-property) in S_b-metric spaces is introduced in this chapter, and it can be used to apply for proving common fixed point results in S_b-metric spaces without the completeness of S_b-metric spaces. This brings to one of the advantages of our results. Here, we give another one of the advantages of our results. In many common fixed theorems in various spaces like metric spaces, b-metric spaces, S-metric spaces, S_b-metric spaces, partial metric spaces, semimetric spaces, multiplicative metric spaces, complex-valued metric spaces, vector metric spaces, cone metric spaces, modular metric spaces, convex metric spaces and fuzzy metric spaces, the closedness of the underlying space (or subspaces) together with conditions on the continuity in respect of any one of the involved mappings is required for proving such common fixed point results. However, our results never require these conditions, and so the reader can apply our results to many problems in the weak form from the past.

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Chapter 20
A General Approach on Picard Operators

Nicolae Adrian Secelean and Dariusz Wardowski

Abstract In the chapter there are presented the recent investigations concerning the existence and the uniqueness of fixed points for the mappings in the setting of spaces which are not metric with different functions of measuring the distance and in consequence with the various convergence concepts. In this way we obtain the systematized knowledge of fixed point tools which are, in some situations, more convenient to apply than the known theorems with an underlying usual metric space. The appropriate illustrative examples are also presented.

Keywords Fixed point · Picard operator · Generalized metric space · ρ-metric space · Convergence

20.1 Introduction and Preliminaries

The notations used throughout the chapter are the following:

$\mathbb{N}$—the set of all positive integers
$\mathbb{R}$—the set of all real numbers
$\mathbb{R}_+$—the set of all nonnegative real numbers
$f : A \to B$—a mapping defined on a domain A with values in B
$f \circ g$—the composition of functions
T^n—the n-th iterate of T

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$x \in A$—x is an element of a set A
$\varnothing$—the empty set
$A \times B$—the Cartesian product of the sets A and B
$A \cup B$—the union of the sets A and B
$\bigcup_{A \in \mathscr{A}} A$—the union of all the sets from $\mathscr{A}$
$A \cap B$—the intersection of the sets A and B
$A \setminus B$—the set difference of A and B
$\complement X$—the complement of X
$A \subset B$—A is a subset of B
$\sup A$—the supremum of a set A
$\inf A$—the infimum of a set A
$\min A$—the minimum of a set A
$\max A$—the maximum of a set A
$\Longleftrightarrow$—if and only if
$\Longrightarrow$—implies
$[a, \infty)$—the closed interval $\{x \in \mathbb{R} \colon a \leq x\}$
(a, ∞)—the open interval $\{x \in \mathbb{R} \colon a < x\}$
$(-\infty, a]$—the closed interval $\{x \in \mathbb{R} \colon x \leq a\}$
$(-\infty, a)$—the open interval $\{x \in \mathbb{R} \colon x < a\}$
(a, b)—the open interval $\{x \in \mathbb{R} \colon a < x < b\}$
$[a, b)$—the half-open interval $\{x \in \mathbb{R} \colon a \leq x < b\}$
$(a, b]$—the half-open interval $\{x \in \mathbb{R} \colon a < x \leq b\}$
$B_r(x_0)$—the open ball centred at x_0 with radius r
$(x_n) \subset X$—a sequence (x_n) with elements from X
$\lim_{n\to\infty} x_n = x$, $\lim_n x_n = x$ or $x_n \to x$—the sequence (x_n) converges to x
$x_n \searrow \lambda$, $x_n \to \lambda^+$—a sequence (x_n) is decreasing and convergent to λ

For the convenience of the reader, we recall some standard definitions which are used throughout the chapter.

A mapping $f \colon X \to Y$ is said to be *injective* if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all $x_1, x_2 \in X$. An element $\xi \in X$ is a *fixed point* of f if $f(\xi) = \xi$. If X is a nonempty set on which a convergence property is defined, we say that a mapping $T : X \to X$ is a *Picard Operator* (P.O. for short) if it has a unique fixed point $\xi \in X$ and $\xi = \lim_n T^n x$ for all $x \in X$. For a mapping $T \colon X \to X$ and $x_0 \in X$ the *orbit* of T starting at the point x_0, denoted by $O(T, x_0)$, is the set

$$O(T, x_0) = \{x_0, Tx_0, T^2x_0, \cdots\}.$$

Let X be a set and τ the family of subsets of X satisfying the conditions: (i) $\varnothing, X \in \tau$, (ii) for every collection $\mathscr{A}$ of members of τ, $\bigcup_{A \in \mathscr{A}} A \in \tau$, (iii) for every $A, B \in \tau$, $A \cap B \in \tau$. Then τ is called a *topology* on X and the pair (X, τ) is called a *topological space*. Elements of τ are called *open sets*. A subset $V \subset X$ is called a *neighbourhood of* $p \in X$ if there exists $U \in \tau$ such that $p \in U \subset V$. A collection $\mathscr{B}$ of open sets is called a *base* for the topology of X if each open set in X is the union of some of the elements of $\mathscr{B}$. A collection $\mathscr{S}$ of open sets is called a *subbase* for the topology of X if the collection of all finite intersections of sets in $\mathscr{S}$ is a base for the topology of X. Let $Y \subset X$. A *subspace topology on* Y induced from the topological space X is the collection of all intersections of Y with open sets of τ. A topological space (X, τ) is said to be a *Hausdorff space* if, for every two different elements $p, q \in X$, there exist disjoint neighbourhoods of p and q.

Let (X, τ), (Y, ρ) be topological spaces, let $f: X \to Y$ be a mapping and let $x_0 \in X$. The mapping f is *continuous* at x_0 if, for every $V \in \rho$ such that $f(x_0) \in V$, there exists $U \in \tau$ satisfying $x_0 \in U$ and $f(U) \subset V$.

Let X be a set. A function $d: X \times X \to [0, \infty)$ is called a *metric* or *standard metric* on X if, for all $x, y, z \in X$, the following conditions are satisfied: (i) $d(x, y) = 0$ if and only if $x = y$, (ii) $d(x, y) = d(y, x)$ (symmetry condition), (iii) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality). If d is a metric on X, then a pair (X, d) is called a *metric space*. A metric space (X, d) is said to be *bounded* if there exists $M > 0$ satisfying $d(x, y) \leq M$ for all $x, y \in X$.

A sequence (x_n) of elements from X is a *Cauchy sequence* if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n > N$. A metric space (X, d) is said to be *complete* if every Cauchy sequence of elements from X is convergent in X. For $x \in X$ and $\varepsilon > 0$, a set $B_\varepsilon(x) = \{u \in X;\ d(x, u) < \varepsilon\}$ is called the *open ball centred at* x *with radius* ε. The topology with a base as a collection of all open balls $B_\varepsilon(x)$ for all $x \in X$, $\varepsilon > 0$ is called a *topology generated by* d. A topological space (X, τ) is called *metrizable* if there exists a metric d on X which generates the topology τ.

20.2 Some Generalized Metric Spaces

20.2.1 ν-Generalized Metric Space

In [6], Branciari, by extending the triangle inequality (iii) in the definition of a metric space and involving four or more points, proposed more general case. We present this definition as follows.

Definition 20.1 (Branciari [6]) Let X be a nonempty set, let $d: X \times X \to [0, \infty)$ be a function and let $\nu \in \mathbb{N}$. If the following conditions are satisfied:

(gm1) $d(x, y) = 0 \Leftrightarrow x = y$ for all $x, y \in X$,
(gm2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(gm3) $d(x, y) \leq d(x, u_1) + d(u_1, u_2) + \cdots + d(u_\nu, y)$ for all $x, u_1, \cdots, u_\nu, y \in X$,

then (X, d) is called a *ν-generalized metric space*.

From the above definition we can make the following simple observation.

Remark 20.1 Every metric space (X, d) is a ν-generalized metric space for every $\nu \in \mathbb{N}$. The case for $\nu = 1$ is trivial since Definition 20.1 reduces to the definition of metric space.

Suppose that (X, d) is a $(\nu - 1)$-generalized metric space for fixed $\nu > 1$ and take any $x, u_1, \dots, u_\nu, y \in X$. We have:

$$\begin{aligned} d(x, y) &\le d(x, u_\nu) + d(u_\nu, y) \\ &\le d(x, u_1) + d(u_1, u_\nu) + d(u_\nu, y) \\ &\le d(x, u_1) + \sum_{i=1}^{\nu-1} d(u_i, u_{i+1}) + d(u_\nu, y). \end{aligned}$$

By induction, we obtain that (X, d) is a ν-generalized metric space.

In the following example there is presented a 2-generalized metric space which is not a metric space.

Example 20.1 (*Suzuki* [30]) Let $X = \{(0, 0)\} \cup ((0, 1] \times [0, 1])$ and let $d \colon X \times X \to [0, \infty)$ be defined as follows:

$$\begin{aligned} &d(x, x) = 0 \ \text{ if } x \in X, \\ &d((0, 0), (x, 0)) = d((x, 0), (0, 0)) = x \ \text{ if } x \in (0, 1], \\ &d((x, 0), (y, z)) = d((y, z), (x, 0)) = |x - y| + z \ \text{ if } x, y, z \in (0, 1], \\ &d(x, y) = 3 \ \text{ otherwise.} \end{aligned}$$

The function d is not a metric since

$$d((0, 0), (1, 0)) + d((1, 0), (1, 1)) = 2 < 3 = d((0, 0), (1, 1)).$$

The function d in Example 20.1 is a 2-generalized metric space which can be concluded from the following result due to Suzuki [30].

Proposition 20.1 (Suzuki [30]) *Let (X, δ) be a bounded metric space and let M be a real number satisfying*

$$\sup\{\delta(x, y) \colon x, y \in X\} \le M.$$

Let $A, B \subset X$, $A, B \neq \varnothing$ be such that $X = A \cup B$ and $A \cap B = \varnothing$. Define a function $d \colon X \times X \to [0, \infty)$ as follows:

$$\begin{aligned} &d(x, x) = 0 \ \text{ if } x \in X, \\ &d(x, y) = d(y, x) = \delta(x, y) \ \text{ if } x \in A, y \in B, \\ &d(x, y) = M \ \text{ otherwise.} \end{aligned}$$

Then (X, d) is a 2-generalized metric space.

Analogously like for metric spaces, we can define the following notions: Let (X, d) be a ν-generalized metric space.

Definition 20.2 A sequence $(x_n) \subset X$ is said to be *convergent* to $x \in X$ and it is denoted by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} d(x_n, x) = 0$.

Definition 20.3 A sequence $(x_n) \subset X$ is said to be a *Cauchy sequence* if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n > m > N$.

Definition 20.4 X is said to be *complete* if every Cauchy sequence in X is convergent to some point in X.

Definition 20.5 Let τ be a topology in X. We say that τ is *compatible* with d if, for every sequence $(x_n) \subset X$ and $x \in X$, $\lim_{n\to\infty} x_n = x$ (in (X, d)) if and only if (x_n) is convergent to x in τ.

The definition of ν-generalized metric space and its related notions listed above are very similar to the usual metric case, however it is interesting to note that the properties of spaces introduced by Branciari are significantly different.

Remark 20.2 The function d in a ν-generalized metric space (X, d) need not be continuous.

Example 20.2 (*Sarma et al.* [19]) Let $A = \{0, 2\}$, $B = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $X = A \cup B$. Define $d : X \times X \to [0, \infty)$ as follows:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \text{ and } \{x, y\} \subset A \text{ or } \{x, y\} \subset B, \\ y & \text{if } x \in A, y \in B, \\ x & \text{if } x \in B, y \in A. \end{cases}$$

Then (X, d) is a 2-generalized metric space (called *rectangular metric space*, *RMS* for short) and $\lim_n d(\frac{1}{n}, \frac{1}{2}) \neq d(0, \frac{1}{2})$.

Remark 20.3 The sequence in a ν-generalized metric space need not have a unique limit.

Example 20.3 If we consider a ν-generalized metric space from Example 20.2 and take $x_n = 1/n$, then we get $d(1/n, 0) = d(1/n, 2) = 1/n \to 0$ as $n \to \infty$, in consequence, for any positive r, $B_r(0) \cap B_r(2) \neq \emptyset$, where $B_r(0)$, $B_r(2)$ are the open balls defined analogously like in metric spaces.

Remark 20.4 A sequence convergent in a ν-generalized metric space need not be a Cauchy sequence.

Example 20.4 Consider the sequence (x_n) from Example 20.3. Then we have $d(1/n, 1/(n + 1)) = 1$ for all $n \in \mathbb{N}$, which means that (x_n) is not a Cauchy sequence.

Remark 20.5 In general, a ν-generalized metric space (X, d) does not necessarily have a topology compatible with d. See the following example given by Suzuki.

Example 20.5 (*Suzuki* [30]) Let

$$X = \{(0, 0)\} \cup ((0, 1] \times [0, 1]).$$

Define a function d from $X \times X$ into $[0, \infty)$ by

$$\begin{aligned}
&d(x, x) = 0,\\
&d((0, 0), (s, 0)) = d((s, 0), (0, 0)) = s \text{ if } s \in (0, 1],\\
&d((s, 0), (p, q)) = d((p, q), (s, 0)) = |s - p| + q \text{ if } s, p, q \in (0, 1],\\
&d(x, y) = 3 \text{ otherwise.}
\end{aligned}$$

Then the following hold:

(1) (X, d) is not a metric space,
(2) (X, d) is a 2-generalized metric space,
(3) X does not have a topology which is compatible with d.

In [29], Suzuki et al. proved that ν-generalized metric spaces have a compatible topology only for $\nu = 1$ and $\nu = 3$. The case $\nu = 1$ is trivial. In the setting of 3-generalized metric spaces, the authors proved the following results:

Theorem 20.1 (Suzuki et al. [29]) *Let (X, d) be a 3-generalized metric space. Define a function $\delta\colon X \times X \to [0, \infty)$ by*

$$\delta(x, y) = \inf \left\{ \sum_{j=0}^{n} d(u_j, u_{j+1})\colon n \in \mathbb{N} \cup \{0\},\ u_0 = x, u_1, \cdots, u_n \in X,\ u_{n+1} = y \right\}.$$

Then (X, δ) is a metric space and for every $x \in X$ and $(x_n) \subset X$, $\lim_n d(x, x_n) = 0$ if and only if $\lim_n \rho(x, x_n) = 0$.

Theorem 20.2 (Suzuki et al. [29]) *Let (X, d) be a 3-generalized metric space. Let $A, B \subset X$ be defined as follows: $x \in A$ if and only if there exists a sequence $(x_n) \subset X \setminus \{x\}$ converging to x, and, respectively, $x \in B$ if and only if there exists a sequence $(x_n) \subset A \setminus \{x\}$ converging to x. For $x \in X$ define $\delta_x > 0$ by*

$$\delta_x = \begin{cases} \inf\{d(x, y)\colon y \in X \setminus \{x\}\} & \text{if } x \in X \setminus A,\\ \inf\{d(x, y)\colon y \in A \setminus \{x\}\} & \text{if } x \in A \setminus B,\\ \infty & \text{if } x \in B. \end{cases}$$

Define a subset N_x of X by

$$N_x = \{B_r(x)\colon 0 < r < \delta_x\}.$$

Then the topology induced by a subbase $\{N_x : x \in X\}$ *is compatible with* d.

Moreover, in [29], the authors gave a construction of a ν-generalized metric space (X, d), $\nu > 3$, which does not have a topology compatible with d.

Example 20.6 (*Suzuki et al.* [29]) Let

$$X = \{(0, 0)\} \cup ((0, 2] \times [0, 2]).$$

Define a function d from $X \times X$ into $[0, \infty)$ by

$$\begin{aligned}
&d(x, x) = 0,\\
&d((0, 0), (s, 0)) = d((s, 0), (0, 0)) = s \ \text{ if } s \in (0, 2],\\
&d((s, 0), (s, t)) = d((s, t), (s, 0)) = t \ \text{ if } s, t \in (0, 2],\\
&d(x, y) = 6 \ \text{ otherwise.}
\end{aligned}$$

Then the following hold:

(1) (X, d) is not a ν-generalized metric space for each $\nu = 1, 2, 3$.
(2) (X, d) is a ν-generalized metric space for each $\nu \geq 4$.
(3) X does not have a topology which is compatible with d.

We end this section by the following fixed point results:

Theorem 20.3 (Suzuki [30]) *Let* (X, d) *be a complete* ν*-generalized metric space and let* T *be a CJM contraction on* X *(Ćirić–Jachymski–Matkowski contraction), that is, the following hold:*

(a) *for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $d(x, y) < \varepsilon + \delta$ *implies* $d(Tx, Ty) \leq \varepsilon$ *for every* $x, y \in X$*,*
(b) $x \neq y$ *implies* $d(Tx, Ty) < d(x, y)$ *for every* $x, y \in X$.

Then T *has a unique fixed point* $\xi \in X$*. Moreover,* $\lim_{n\to\infty} d(T^n x, \xi) = 0$ *for every* $x \in X$.

The consequence of the above result is a standard generalization of the Banach contraction principle in a ν-generalized metric space.

Theorem 20.4 *Let* (X, d) *be a complete* ν*-generalized metric space and suppose that there exists* $r \in [0, 1)$ *such that* $d(Tx, Ty) \leq rd(x, y)$ *for every* $x, y \in X$*. Then* T *has a unique fixed point* $\xi \in X$ *and* $\lim_n d(T^n x, \xi) = 0$ *for every* $x \in X$.

Let Θ be the family of functions $\theta : (0, \infty) \to (1, \infty)$ that satisfy the following conditions:

(Θ_1) θ is nondecreasing,
(Θ_2) for each sequence $(t_n) \subset (0, \infty)$, $\lim_n \theta(t_n) = 1$ if and only if $t_n \to 0^+$,
(Θ_3) there exist $r \in (0, 1)$ and $l > 0$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

Theorem 20.5 (Jleli et al. [12]) *Let (X, d) be a complete RMS and $T : X \to X$ be a given map. Suppose that there exist $\theta \in \Theta$ and $\lambda \in (0, 1)$ such that*

$$x, y \in X, \ \ d(Tx, Ty) \neq 0 \implies \theta\big(d(Tx, Ty)\big) \leq \big[\theta\big(d(x, y)\big)\big]^{\lambda}.$$

Then T has a unique fixed point.

20.2.2 $\mathfrak{D}$-Generalized Metric Spaces

In the following we describe another generalization of the standard notion of metric given by Jleli and Samet in [11], which extends some other generalized metric structures such as: b-metric spaces introduced by Bakhtin [3] and dislocated metric defined by Hitzler and Seda in [8]. In these spaces several fixed point results are improved.

Definition 20.6 Let X be a nonempty set. We consider a function $d : X \times X \to \mathbb{R}_+$ and the following conditions:

(c_1) for every $x, y \in X$, $d(x, y) = 0 \implies x = y$,
(c_2) for every $x \in X$, $d(x, x) = 0$,
(c_3) for every $x, y \in X$, $d(x, y) = d(y, x)$,
(c_4) for every $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$,
(c_5) there exists $s \geq 1$ such that, for every $x, y, z \in X$, $d(x, z) \leq s\big(d(x, y) + d(y, z)\big)$.

(1) We say that d is a *b-metric* if it satisfies (c_1), (c_2), (c_3) and (c_5).
(2) We say that d is a *dislocated metric* if it satisfies (c_1), (c_3) and (c_4).
(3) If d satisfies (c_1), (c_3) and (c_5) then it is called a *dislocated b-metric*.
(4) Accordingly, (X, d) will be *b-metric, dislocated* and *dislocated b-metric space*, respectively.

Remark 20.6 Every dislocated metric is a dislocated b-metric with $s = 1$ and each b-metric is a dislocated b-metric. Also, the standard metric is a particular case of each of the metrics defined above.

Example 20.7 (*Bakhtin* [3]) For $0 < p < 1$, the space

$$l_p = \left\{(x_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\right\}$$

endowed with $d : l_p \times l_p \to \mathbb{R}_+$ given by

$$d(x, y) = \Big(\sum_{n=1}^{\infty} |x_n - y_n|^p\Big)^{\frac{1}{p}},$$

where $x = (x_n),\ y = (y_n) \in l_p$ is a b-metric space with $s = 2^{\frac{1}{p}}$.

Example 20.8 (*Bakhtin* [3]) The space L_p $(0 < p < 1)$ of all real functions $x(t)$, $t \in [0, 1]$, such that

$$\int_0^1 |x(t)|^p dt < \infty,$$

becomes a b-metric space if we take

$$d(x, y) = \Big(\int_0^1 |x(t) - y(t)|^p \Big)^{\frac{1}{p}} dt \quad \text{for all } x, y \in L_p,$$

the constant being $s = 2^{\frac{1}{p}}$.

The following example shows that a b-metric on a set X need not be a metric on X.

Example 20.9 (*Singh et al.* [28]) Let $X = \{x_1, x_2, x_3, x_4\}$ and $d(x_1, x_2) = p \geq 2$, $d(x_1, x_3) = d(x_1, x_4) = d(x_2, x_3) = d(x_2, x_4) = d(x_3, x_4) = 1,\ d(x_i, x_j) = d(x_j, x_i)$ for each $i, j = 1, 2, 3, 4$ and $d(x_i, x_i) = 0$ for each $i = 1, 2, 3, 4$. Then we have

$$d(x_i, x_k) \leq \frac{p}{2}\big(d(x_i, x_j) + d(x_j, x_k)\big), \quad \text{for all } i, j, k = 1, 2, 3, 4,$$

and, if $p > 2$, the triangle inequality (c_4) does not hold.

Example 20.10 If $X = \mathbb{R}_+$, then $d(x, y) = x + y$ is a dislocated metric which is not a standard metric.

Definition 20.7 A function $\mathfrak{D} : X \times X \to [0, \infty]$ is said to be a *generalized metric* if it satisfies the conditions (c_1) and (c_3) (see Definition 20.6) and

(c_6) there exists $C > 0$ such that

$$x, y \in X,\ (x_n) \subset X,\ \lim_{n\to\infty} \mathfrak{D}(x_n, x) = 0 \implies \mathfrak{D}(x, y) \leq C \limsup_{n\to\infty} \mathfrak{D}(x_n, y).$$

For no confusion, we will call this function $\mathfrak{D}$-*metric* and the pair $(X, \mathfrak{D})$ $\mathfrak{D}$-*metric space*.

Obviously, if there are not a sequence $(x_n) \subset X$ and a point $x \in X$ such that $\lim_{n\to\infty} \mathfrak{D}(x_n, x) = 0$, then $\mathfrak{D}$ is a generalized metric if and only if (c_1) and (c_3) are satisfied.

The next proposition highlights that this new concept of $\mathfrak{D}$-metric covers a large class of existing metrics in the literature.

Proposition 20.2 (Jleli et al. [11]) *Every b-metric, respectively dislocated metric, dislocated b-metric, standard metric is a* $\mathfrak{D}$*-metric.*

Proof According to Remark 20.6 it is enough to prove that every dislocated b-metric d on X satisfies (c_6). Let $(x_n) \subset X$ and $x \in X$ be such that $\mathfrak{D}(x_n, x) \to 0$. Then, for every $y \in X$, by (c_5), one has

$$d(x, y) \leq sd(x, x_n) + sd(x_n, y) \quad \text{for all } n = 1, 2, \ldots.$$

Hence we have

$$d(x, y) \leq s \limsup_{n\to\infty} d(x_n, y)$$

and so (c_6) is satisfied for $C = s$.

Definition 20.8 Let $(X, \mathfrak{D})$ be a $\mathfrak{D}$-metric space and $(x_n) \subset X$. We say that (x_n) *converges* to some $x \in X$ if $\mathfrak{D}(x_n, x) \to 0$. Also, (x_n) is a *Cauchy sequence* if $\mathfrak{D}(x_{n+p}, x_n) \to 0$, $n, p \to \infty$. $(X, \mathfrak{D})$ is *complete* if every Cauchy sequence is convergent.

Notice that the concepts of convergence and completeness with respect to these metrics are similar to those in standard metric spaces.

From (c_6) and (c_1) it is easy to deduce that in a $\mathfrak{D}$-metric space every convergent sequence has a unique limit.

Senapati et al. [25] proved that, in a $\mathfrak{D}$-metric space, a sequence may be convergent without being Cauchy unlike in metric spaces, b-metric spaces, dislocated metric spaces and dislocated b-metric spaces, where every convergent sequence must be Cauchy. This shows that $\mathfrak{D}$-metric is a proper generalization of b-metric, dislocated metric and dislocated b-metric.

Example 20.11 (*Senapati et al.* [25]) Let $X = \mathbb{R}_+ \cup \{\infty\}$ and $\mathfrak{D} : X \times X \to [0, \infty]$ be defined as follows:

$$\mathfrak{D}(x, y) = \begin{cases} x + y & \text{if at least one of } x \text{ or } y \text{ is } 0, \\ x + y + 1 & \text{otherwise.} \end{cases}$$

Then $(X, \mathfrak{D})$ is a $\mathfrak{D}$-metric and the sequence $x_n = \frac{1}{n}$ for each $n \geq 1$ converges to 0. However, $\lim_{m,n\to\infty} \mathfrak{D}(x_n, x_{m+n}) = 1$ and so (x_n) is not a Cauchy sequence.

Theorem 20.6 (Jleli et al. [11]) *Let $(X, \mathfrak{D})$ be a complete $\mathfrak{D}$-metric space and $T : X \to X$ be a mapping for which there is $k \in (0, 1)$ such that*

$$\mathfrak{D}(Tx, Ty) \leq k\mathfrak{D}(x, y) \quad \textit{for every } x, y \in X.$$

If there is $x_0 \in X$ such that $\sup_{i,j\in\mathbb{N}} \mathfrak{D}(T^i x_0, T^j x_0) < \infty$, then $(T^n x_0)$ converges to a fixed point ω of T. Moreover, if ω' is another fixed point of T such that $\mathfrak{D}(\omega, \omega') < \infty$, then $\omega = \omega'$.

Senapati et al. [25] generalized the notion of F-contraction (see Sect. 20.3.1) to a $\mathfrak{D}$-metric space and proved a fixed point theorem.

Definition 20.9 A self-mapping T on a $\mathfrak{D}$-metric space is said to be an *F-contraction* if there exists $\tau > 0$ such that, for all $x, y \in X$,

$$\mathfrak{D}(x, y) \in (0, \infty),\ \mathfrak{D}(Tx, Ty) \in (0, \infty) \implies \tau + F\big(\mathfrak{D}(Tx, Ty)\big) \leq F\big(\mathfrak{D}(x, y)\big),$$

where $F : (0, \infty) \to \mathbb{R}$ satisfies (F1) and (F2) (see Sect. 20.3.1).

Theorem 20.7 (Senapati et al. [25]) *Let $(X, \mathfrak{D})$ be a $\mathfrak{D}$-complete metric space and $T : X \to X$ be an F-contraction. If there is $x_0 \in X$ such that $\sup_{i,j\in\mathbb{N}} \mathfrak{D}(T^i x_0, T^j x_0) < \infty$, then $(T^n x_0)$ converges to a fixed point ω of T. Moreover, if ω' is another fixed point of T such that $\mathfrak{D}(\omega, \omega') < \infty$, then $\omega = \omega'$.*

In [2, 14], one can find more detailed and recent informations about fixed points in generalized spaces.

20.2.3 Asymmetric Metric Space

The quasi-metric (asymmetric) space notion was apparently introduced by Wilson [37]. This is defined as metric space (X, d) but without the symmetry requirement for d. Quasi-metric spaces have numerous recent applications both in pure and applied mathematics.

In quasi-metric spaces, some concepts, such as convergence, continuity, compactness and completeness, are different from those in metric case.

Definition 20.10 Let X be a nonempty set. A nonnegative real-valued function $d : X \times X \to \mathbb{R}$ is called *quasi-metric* or *asymmetric metric* if it satisfies the following axioms:

(Q1) for every $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,
(Q2) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Accordingly, the pair (X, d) is called a *quasi-metric (asymmetric metric) space*.

It is easy to see that, if d is a quasi-metric, then $\alpha d(x, y) + \beta d(y, x)$ is also a quasi-metric and $\max\big\{d(x, y), d(y, x)\big\}$, $\alpha\big(d(x, y) + d(y, x)\big)$ are metrics, where $\alpha, \beta > 0$.

Definition 20.11 The *forward topology* $\mathfrak{T}_{\mathrm{f}}$ induced by d is the topology generated by the *forward open balls* $\mathrm{B}_{\mathrm{f}}(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Likewise, the *backward topology* $\mathfrak{T}_{\mathrm{b}}$ induced by d is the topology generated by the *backward open balls* $\mathrm{B}_{\mathrm{b}}(y, \varepsilon) = \{x \in X : d(x, y) < \varepsilon\}$ for all $y \in X$ and $\varepsilon > 0$.

In the following, we present some usual examples of quasi-metric spaces.

Example 20.12 Let $\alpha > 0$ and $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Then $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ defined by

$$d(x, y) = \begin{cases} f(y) - f(x) & \text{if } y \geq x, \\ \alpha\big(f(x) - f(y)\big) & \text{if } y < x \end{cases}$$

is a quasi-metric on $\mathbb{R}$. If f is continuous then both $\mathfrak{T}_\mathrm{f}$ and $\mathfrak{T}_\mathrm{b}$ are the usual topology on $\mathbb{R}$.

Example 20.13 The function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ defined by

$$d(x, y) = \begin{cases} e^y - e^x & \text{if } y \geq x, \\ e^{-y} - e^{-x} & \text{if } y < x \end{cases}$$

is a quasi-metric. Both $\mathfrak{T}_\mathrm{f}$ and $\mathfrak{T}_\mathrm{b}$ are the usual topology on $\mathbb{R}$.

Example 20.14 Let (X, d) be a quasi-metric space and $f : X \to X$ be a mapping. Then the function $\delta : X \times X \to \mathbb{R}_+$ defined by

$$\delta(x, y) = d\big(f(x), f(y)\big) \quad \text{for all } x, y \in X,$$

is a quasi-metric if and only if f is injective.

Example 20.15 The function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ defined by

$$d(x, y) = \begin{cases} y - x & \text{if } y \geq x, \\ 1 & \text{if } y < x \end{cases}$$

is a quasi-metric named *Sorgenfrey quasi-metric*. Here $\mathfrak{T}_\mathrm{f}$ is the lower limit topology on $\mathbb{R}$ and it is well known that $\mathfrak{T}_\mathrm{f}$ is not metrizable (see, e.g., [7]). At the same time, $\mathfrak{T}_\mathrm{b}$ is the upper limit topology.

Definition 20.12 A sequence (x_n) *forward converges* (*f-converges* for short) to $x_0 \in X$, respectively, *backward converges* (*b-converges* for short) to $x_0 \in X$ if it converges with respect to the topology $\mathfrak{T}_\mathrm{f}$, $\mathfrak{T}_\mathrm{b}$ respectively. Accordingly, (x_n) f-converges, b-converges to x_0 if and only if

$$d(x_0, x_n) \to 0, \quad d(x_n, x_0) \to 0 \text{ respectively.}$$

We emphasize that the topologies $\mathfrak{T}_\mathrm{f}$ and $\mathfrak{T}_\mathrm{b}$ are not generally Hausdorff (see, e.g., [13, Ex. 5.7]). However, the bitopological space $(X, \mathfrak{T}_\mathrm{f}, \mathfrak{T}_\mathrm{b})$ is a *pairwise Hausdorff space*, that is, for each two distinct points $x, y \in X$, there are a $\mathfrak{T}_\mathrm{f}$-neighbourhood U of x and a $\mathfrak{T}_\mathrm{b}$-neighbourhood V of y such that $U \cap V = \varnothing$ (see [13, Prop. 4.2]).

It is easy to prove that, if, in a quasi-metric space (X, d), f-convergence implies b-convergence, then $(X, \mathfrak{T}_\mathrm{f})$ is a Hausdorff space.

Definition 20.13 We say that a sequence $(x_n) \subset X$ is *forward Cauchy* (resp. *backward Cauchy*) if, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for every $m \geq n \geq N$, one has

$$d(x_n, x_m) < \varepsilon, \quad (\text{resp. } d(x_m, x_n) < \varepsilon).$$

Note that the quasi-metric space (X, d) is *forward complete* (resp. *backward complete*) if every forward (resp. backward) Cauchy sequence is f-convergent (resp. b-convergent).

In the following, we present the Banach contraction principle in the settings of quasi-metric spaces (for more details, see [15]).

Definition 20.14 Let (X, d) be a quasi-metric space. A mapping $T : X \to X$ is called *forward* (resp. *backward*) *contraction* if there exists $0 < \alpha < 1$ such that

$$d(Tx, Ty) \le \alpha d(x, y) \ \big(\text{resp. } d(Tx, Ty) \le \alpha d(y, x)\big), \quad \text{for all } x, y \in X.$$

Theorem 20.8 (Khorshidvandpour et al. [15]) *Let (X, d) be a forward complete quasi-metric space and $T : X \to X$ be a forward contraction. If the forward convergence implies the backward convergence, then T has a unique fixed point ξ. Moreover, for each $x \in X$, the sequence $(T^n x)$ forward and backward converges to ξ.*

20.3 ρ-Metric Spaces

Metric fixed point theory plays a crucial role in many branches of mathematics and in many other sciences. Also, many applications are covered with the help of fixed point tools. The far-famed Banach contraction principle gave rise to the great development in many directions, including the research on very general conditions on the mappings and on the spaces where they are defined on. Some examples of the recent contributions in this direction one can find e.g. in [38], where there are considered the contractive mappings of various type in uniform spaces equipped with the so called *generalized pseudodistances*. Another example of such research is [5], where there are taken into consideration the contractions of Suzuki type [31]. A profound research on the equivalence of the existing contractive conditions can be found e.g. in [9, 10].

In the present section, we will describe the concept of a mapping, called ρ-metric, defined on the Cartesian product $X \times X$ which is more general than many known distance functions. We will also present the proves of some fixed point theorems concerning ρ-metric, which cover many known results in the literature. At the end of the section, there will be discussed the application to the certain non-metrizable topological space. The notions and the results presented in this section are based on the article [24].

20.3.1 Contractions in an Implicit Form

We recall the notion of F-contraction which was introduced by Wardowski [34]. He considered the function $F : (0, \infty) \to \mathbb{R}$ satisfying the following three conditions:

(F1) F is increasing,
(F2) $F(t_n) \to -\infty$ if and only if $t_n \searrow 0$,
(F3) $\lim_{t\to 0} t^{\lambda} F(t) = 0$ for some $\lambda \in (0, 1)$.

A mapping $T: X \to X$ defined on a metric space (X, d) is said to be an *F-contraction* if there exist $\tau > 0$ and a function F fulfilling (F1)–(F3) such that

$$F\big(d(Tx, Ty)\big) + \tau \leq F(d(x, y)) \tag{20.1}$$

for all $x, y \in X$ with $Tx \neq Ty$.

Theorem 20.9 (Wardowski [34]) *If (X, d) is a complete metric space and $T : X \to X$ an F-contraction, then T is a P.O.*

When we consider in (20.1) some concrete forms of F, then we obtain different known types of contractions, including Banach contraction. In the literature there are many papers, where the contraction condition is in an implicit form described by the function F. The examples of such articles are e.g. [17, 21, 22, 26, 35]. In [32], Turinici showed that some class of F-contractions are contractions of Matkowski type [16]. In the recent article [23], Secelean and Wardowski extended the family of F-contractions by introducing so called *ψF-contractions* which include even the Picard operators without nonexpansiveness condition. Also, there appeared many works, where different versions of nonlinear F-contractions have been applied to some nonlinear phenomena. The recent examples of such contributions are [1, 20, 27, 33, 36].

20.3.2 Definition of ρ-Metric Space and Its Properties

Let X be a nonempty set and consider the diagonal set $\Delta = \big\{(x, x);\ x \in X\big\}$. Denote by $\mathfrak{F}$ the class of all functions $F : (0, \infty) \to \mathbb{R}$ that satisfy $(F2)$.

Definition 20.15 Let $\rho : X \times X \setminus \Delta \to \mathbb{R}$ be a function.

(1) We say that a sequence $(x_n) \subset X$ *left ρ-converges* (resp. *right ρ-converges*) to some $x \in X$ if for each $M > 0$, there exists $N_M \in \mathbb{N}$ such that, for every $n \geq N_M$, one has

$$x_n \neq x \implies \rho(x_n, x) < -M \text{ (resp. } x_n \neq x \implies \rho(x, x_n) < -M). \tag{20.2}$$

A sequence (x_n) is called *ρ-convergent* if it is left and right ρ-convergent. The defined types of convergence we denote $x_n \xrightarrow{l} x$ (resp. $x_n \xrightarrow{r} x$, $x_n \longrightarrow x$).

(2) If each left or right ρ-convergent sequence has a unique limit, then ρ is called a *ρ-metric* and the pair (X, ρ) is said to be a *ρ-space*.

(3) A self mapping T on a ρ-space X is called *left Picard operator* (resp. *right Picard operator*, *Picard operator*), l-P.O. (resp. r-P.O., P.O.) for short, provided that it has a unique fixed point $\xi \in X$ and, for every $x \in X$, $T^n x \xrightarrow{l} \xi$ (resp. $T^n x \xrightarrow{r} \xi$, $T^n x \longrightarrow \xi$).

Note that, if ρ is symmetric, then the left and right ρ-convergence coincide.

Using the notion of ρ-convergence we can naturally impose a topological structure in a ρ-space.

Definition 20.16 If (X, ρ) is a ρ-space and τ is a topology on X such that a sequence (x_n) converges to $x \in X$ in the topology τ if and only if it left (right) ρ-converges to x, then we say that (X, τ, ρ) is a *topological ρ-space*.

Remark 20.7 (1) If (X, ρ) is a ρ-space and $(x_n) \subset X$ is such that, there are $x \in X$ and $N \in \mathbb{N}$ such that $x_n = x$ for all $n \geq N$, then it left and right ρ-converges to x. Also, if (x_n) is such that $x_n \neq x$ for all n greater that some $N \in \mathbb{N}$, then

$$x_n \xrightarrow{l} x \iff \rho(x_n, x) \to -\infty \text{ (resp. } x_n \xrightarrow{r} x \iff \rho(x, x_n) \to -\infty).$$

(2) In the above settings, if the set $A = \{n_k : \ x_{n_k} \neq x, \ k = 1, 2, \cdots\}$ is infinite and $\rho(x_{n_k}, x) \xrightarrow[k]{} -\infty$ (resp. $\rho(x, x_{n_k}) \xrightarrow[k]{} -\infty$), then, by Definition 20.15, $x_n \xrightarrow{l} x$ (resp. $x_n \xrightarrow{r} x$).

Example 20.16 (1) If (X, d) is a metric space, then one can observe that $\rho(x, y) = -1/d(x, y)$ is a symmetric ρ-metric. More generally, if $F \in \mathfrak{F}$, then $\rho(x, y) = F\big(d(x, y)\big)$ is a symmetric ρ-metric and X is a topological ρ-space.

(2) If (X, d) is an RMS in which all convergent sequences have a unique limit and $F \in \mathfrak{F}$, then $\rho(x, y) = F\big(d(x, y)\big)$ is a symmetric ρ-metric which may not be topological.

(3) Also, if $(X, \mathfrak{D})$ is a $\mathfrak{D}$-metric space and $F \in \mathfrak{F}$, then, taking $\rho(x, y) = F\big(\mathfrak{D}(x, y)\big)$ for all $x, y \in X$ with $x \neq y \in X$, we obtain a symmetric ρ-space.

(4) If (X, d) is a quasi-metric space such that the topologies $\mathfrak{T}_\mathrm{f}$ and $\mathfrak{T}_\mathrm{b}$ are Hausdorff and $F \in \mathfrak{F}$, then $\rho(x, y) = F\big(d(x, y)\big)$ for all $x, y \in X$ with $x, y \in X$, $x \neq y$, is a ρ-metric. In this case, the left ρ-convergence coincides with the b-convergence and, analogously, the right ρ-convergence coincides with the f-convergence.

(5) Generally, for every variety of generalized metrics $d : X \times X \to \mathbb{R}_+$ on a set X such that a sequence $(x_n) \subset X$ converges to a unique $x \in X$ if and only if $d(x_n, x) \to 0$, then the function $\rho(x, y) = F\big(d(x, y)\big)$ for all $x, y \in X$, $x \neq y$, and $F \in \mathfrak{F}$ is a ρ-metric on X.

When there is considered a topological space (X, τ) which is non-metrizable, ρ-metric can be applied to measure the "distance" between the elements in the topological ρ-space (X, τ, ρ). Such a situation is illustrated by the following example:

Example 20.17 Consider the Sorgenfrey line, that is, the topology τ_l on the set $\mathbb{R}$ generated by the basis of all half-open intervals

$$\mathscr{B} = \{[a, b) : a, b \in \mathbb{R},\ a < b\}.$$

The topological space $(\mathbb{R}, \tau_l)$ is Hausdorff and it is not metrizable (see [7]). In this topology, a given sequence converges whenever it converges in the standard topology and at most a finite number of elements are less than the limit.

Consider now the mapping $\rho\colon \mathbb{R} \times \mathbb{R} \setminus \Delta \to \mathbb{R}$ given by

$$\rho(x, y) = \frac{1}{y - x}, \quad \text{for all } x, y \in \mathbb{R},\ x \neq y.$$

Taking any sequence (x_n) convergent to x, $x_n \neq x$, with respect to the topology τ_l, we have $x_n \searrow x$ in the standard topology and, due to the fact that $x_n > x$ for all $n \in \mathbb{N}$, we get $\rho(x_n, x) \to -\infty$. On the other hand, taking any $(x_n) \subset \mathbb{R}$, $x \in \mathbb{R}$, $x_n \neq x$, such that $\rho(x_n, x) \to -\infty$, we must have $x_n > x$ except for finitely many n and $\frac{1}{x - x_n} \to -\infty$ and so $x_n \searrow x$ in the standard topology and hence $x_n \underset{\tau_l}{\to} x$.

Summarizing, $(\mathbb{R}, \tau_l, \rho)$ is a topological ρ-space. The analogous conclusion can be also obtained for the mapping:

$$\rho(x, y) = \begin{cases} c & \text{if } y > x, \\ \frac{1}{y-x} & \text{if } y < x, \end{cases}$$

where $c \in \mathbb{R}$. Note that, in both cases, the mapping ρ is asymmetric.

Definition 20.17 Let (X, ρ) be a ρ-space.

(1) A sequence $(x_n) \subset X$ is said to be a *ρ-backward-Cauchy sequence* (resp. *ρ-forward-Cauchy sequence*) whenever, for every $M > 0$, there is $N \in \mathbb{N}$ such that, for all $n \geq N$ and $p \geq 1$, one has

$$x_{n+p} \neq x_n \implies \rho(x_{n+p}, x_n) < -M \ \ (\text{resp. } \rho(x_n, x_{n+p}) < -M).$$

(2) A sequence $(x_n) \subset X$ is said to be a *ρ-Cauchy sequence* if, for every $M > 0$, there is $N \in \mathbb{N}$ such that, for all $m, n \geq N$ one has

$$x_m \neq x_n \implies \rho(x_m, x_n) < -M.$$

(3) We say that ρ is *backward complete* (resp. *forward complete*, *complete*) if every ρ-backward-Cauchy sequence (resp. ρ-forward-Cauchy sequence, ρ-Cauchy sequence) left ρ-converges (resp. right ρ-converges, ρ-converges).

Remark 20.8 (1) If ρ is symmetric, then the ρ-backward-Cauchy and the ρ-forward-Cauchy properties coincide.

(2) A ρ-space is backward complete (resp. forward complete, complete) if and only if every sequence of distinct elements $(x_n) \subset X$ such that

$$\rho(x_{n+p}, x_n) \underset{n,p}{\longrightarrow} -\infty, \text{ (resp. } \rho(x_n, x_{n+p}) \underset{n,p}{\longrightarrow} -\infty, \; \rho(x_m, x_n) \underset{m\neq n}{\longrightarrow} -\infty)$$

is ρ-convergent.

(3) Let (x_n) be a sequence of different elements in a ρ-space. Then

$$\rho(x_n, x_{n+p}) \underset{n,p}{\longrightarrow} -\infty \text{ and } \rho(x_{n+p}, x_n) \underset{n,p}{\longrightarrow} -\infty \tag{20.3}$$

if and only if $\rho(x_m, x_n) \underset{m\neq n}{\longrightarrow} -\infty$. Therefore, (x_n) is a ρ-Cauchy sequence if and only if it is simultaneously a ρ-backward-Cauchy sequence and a ρ-forward-Cauchy sequence.

Proof (1) and (2) are obvious.

(3) If (x_n) is a ρ-Cauchy sequence, one has clearly (20.3). Conversely, the properties from (20.3) imply that, for all $M > 0$, there exists $N \in \mathbb{N}$ such that

$$\rho(x_{n+p}, x_n) < -M \text{ and } \rho(x_n, x_{n+p}) < -M, \quad \text{for all } n \geq N, \; p \geq 1.$$

Choose $m, n \geq N$ with $m \neq n$. If $m > n$, then $\rho(x_m, x_n) = \rho(x_{n+p}, x_n) < -M$, where $p = m - n$. Analogously, if $m < n$, then one obtains $\rho(x_m, x_n) < -M$. Therefore (x_n) is a ρ-Cauchy sequence.

Example 20.18 Each of the ρ-metrics defined in Example 20.17 is forward complete, while it is not backward complete. Indeed, if $(x_n) \subset \mathbb{R}$ is a sequence of distinct elements such that $\rho(x_n, x_{n+p}) \underset{n,p}{\longrightarrow} -\infty$, then there is $n_0 \in \mathbb{N}$ such that $(x_n)_{n\geq n_0}$ is decreasing and $|x_n - x_{n+p}| \underset{n,p}{\longrightarrow} 0$. Hence (x_n) is a Cauchy sequence with respect to the Euclidean metric and so it is convergent. Therefore (x_n) converges in the Sorgenfrey topology.

Next, if $\rho(x_{n+p}, x_n) \underset{n,p}{\longrightarrow} -\infty$, then $(x_n)_{n\geq n_0}$ is increasing for some $n_0 \in \mathbb{N}$ and so it does not converge in the Sorgenfrey line.

Remark 20.9 Let us consider a metric space (X, d), a function $F \in \mathfrak{F}$ and the ρ-metric $\rho = F \circ d$ (see Example 20.16). Then d is complete if and only if ρ is complete. The same assertion holds if we consider an RMS, a $\mathfrak{D}$-metric or a quasi-metric space instead of a metric space.

Proof Suppose that d is complete and $(x_n) \subset X$ is a ρ-Cauchy sequence. Assume that (x_n) is not a d-Cauchy sequence. Then there exist $\varepsilon > 0$ and the subsequences (x_{n_k}) and (x_{m_k}) of (x_n) such that

$$t_k = d(x_{m_k}, x_{n_k}) > \varepsilon \quad \text{for all } k \in \mathbb{N}.$$

By (F2), $\rho(x_{m_k}, x_{n_k}) = F(t_k) \not\to -\infty$, which contradicts the fact that (x_n) is a ρ-Cauchy sequence (see Remark 20.8). Therefore, (x_n) is a d-Cauchy sequence and hence convergent, i.e., $d(x_n, x) \to 0$ for some $x \in X$. Now, using again (F2), one can easily see that (x_n) is ρ-convergent.

The proof of the second assertion is very similar.

Remark 20.10 A ρ-convergent sequence need not be a ρ-Cauchy sequence as we can see in the following example:

Set $X = [0, \infty)$ and let $\rho : X \times X \setminus \Delta \to \mathbb{R}$ be given by

$$\rho(x, y) = \begin{cases} \ln(x + y) & \text{if } x = 0 \text{ or } y = 0, \\ \ln(x + y + 1) & \text{otherwise.} \end{cases}$$

Then ρ is a ρ-metric. If $x_n = \frac{1}{n}$ for each $n = 1, 2, \ldots$, then $\rho(x_n, 0) = \rho(0, x_n) = -\ln n \to -\infty$ and so (x_n) is ρ-convergent. However, $\rho(x_{n+p}, x_n) = \rho(x_n, x_{n+p}) = \ln\left(\frac{1}{n+p} + \frac{1}{n} + 1\right) \underset{n,p}{\longrightarrow} 0$ and hence (x_n) is neither ρ-backward-Cauchy nor ρ-forward-Cauchy.

A nonempty subset $B \subset X$ is said to be *ρ-bounded* whenever there exists $M > 0$ such that $\rho(u, v) \le M$ for all $u, v \in B$ with $u \neq v$.

For any $\mu \in [0, \infty]$, let us denote by Ψ_μ the family of all nondecreasing mappings $\psi : (-\infty, \mu) \to (-\infty, \mu)$ such that $\psi^n(t) \to -\infty$ for all $t \in (-\infty, \mu)$. In the following, wherever we have $\mu > \infty$, we mean $\mu = \infty$.

Definition 20.18 Let (X, ρ) be a ρ-metric space, $\mu > \sup\{\rho(x, y) : x, y \in X,\ x \neq y\}$ and $\psi \in \Psi_\mu$. A mapping $T : X \to X$ is called *$\rho\psi$-contraction*, respectively, *weak-$\rho\psi$-contraction, $\alpha\beta$-weak-$\rho\psi$-contraction* if, for every $x, y \in X$ with $Tx \neq Ty$, one has

$$\rho(Tx, Ty) \le \psi\big(\rho(x, y)\big), \tag{20.4}$$

respectively,

$$\rho(Tx, Ty) \le \psi\Big(\max\big\{\rho(x, y), \rho(y, x)\big\}\Big), \tag{20.5}$$

$$\rho(Tx, Ty) \le \psi\big(\alpha\rho(x, y) + \beta\rho(y, x)\big)$$

for some $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.

Remark 20.11 Every $\rho\psi$-contraction is an $\alpha\beta$-weak-$\rho\psi$-contraction and every $\alpha\beta$-weak-$\rho\psi$-contraction is a weak-$\rho\psi$-contraction.

20.3.3 Fixed Point Theorems in ρ-Metric Spaces

Theorem 20.10 *Let* (X, ρ) *be a complete* ρ*-space and* T *be a self-mapping on* X *with a* ρ*-bounded orbit* $O(T, x_0)$ *for some* $x_0 \in X$. *Assume that* T *is a weak-*$\rho\psi$*-contraction, where* $\psi \in \Psi_\mu$ *and* $\mu > \sup\{\rho(x, y) : x, y \in X,\ x \neq y\}$. *Then* T *is a P.O.*

Proof First, note that $\psi(t) < t$ for all $t \in (-\infty, \mu)$. Indeed, if there is $t_0 \in (-\infty, \mu)$ such that $\psi(t_0) \geq t_0$, then $\psi^2(t_0) \geq \psi(t_0) \geq t_0$ and, inductively, $\psi^n(t_0) \geq t_0$ for all $n \in \mathbb{N}$. This contradicts $\psi^n(t_0) \to -\infty$. Since

$$\rho(Tx, Ty) \leq \psi\big(\max\big\{\rho(x, y), \rho(y, x)\big\}\big) < \max\big\{\rho(x, y), \rho(y, x)\big\} \tag{20.6}$$

and also

$$\rho(Ty, Tx) \leq \psi\big(\max\big\{\rho(x, y), \rho(y, x)\big\}\big) < \max\big\{\rho(x, y), \rho(y, x)\big\} \tag{20.7}$$

for all $x, y \in X$ and $Tx \neq Ty$, it follows that T has at most one fixed point.

In order to establish the existence of fixed point of T and also its successive approximation, we have to investigate two cases.

Case I If there exist $n, p \geq 1$ such that $T^{n+p}x_0 = T^p x_0$, then $T^p x_0$ is a fixed point for T^n. Next, we have $T^{n+p+1}x_0 = T^{p+1}x_0$ and hence $T^{p+1}x_0$ is also a fixed point of T^n.

Now, we claim that T^n has only one fixed point. Indeed, on the contrary, there exist $\xi, \eta \in X$ with $\xi \neq \eta$ such that $T^n\xi = \xi$ and $T^n\eta = \eta$. One has

$$\begin{aligned}
\rho(\xi, \eta) = \rho(T^n\xi, T^n\eta) &\leq \psi\big(\max\{\rho(T^{n-1}\xi, T^{n-1}\eta), \rho(T^{n-1}\eta, T^{n-1}\xi)\}\big) \\
&= \max\big\{\psi\big(\rho(T^{n-1}\xi, T^{n-1}\eta)\big), \psi\big(\rho(T^{n-1}\eta, T^{n-1}\xi)\big)\big\} \\
&< \max\big\{\rho(T^{n-1}\xi, T^{n-1}\eta), \rho(T^{n-1}\eta, T^{n-1}\xi)\big\} \\
&< \cdots < \max\big\{\rho(\xi, \eta), \rho(\eta, \xi)\big\}
\end{aligned}$$

and, analogously,

$$\rho(\eta, \xi) < \max\big\{\rho(\xi, \eta), \rho(\eta, \xi)\big\},$$

which is a contradiction. Consequently, $T^{p+1}x_0 = T^p x_0$ and so $\xi = T^p x_0$ is the unique fixed point of T.

Case II Assume that, for every $n, p \geq 1$, one has $T^{n+p}x_0 \neq T^n x_0$. Then, using the monotonicity of ψ and (20.5), for every $n, p \geq 1$, we have

$$\begin{aligned}\rho(T^{n+p}x_0, T^n x_0) &\le \psi\Big(\max\big\{\rho(T^{n+p-1}x_0, T^{n-1}x_0), \rho(T^{n-1}x_0, T^{n+p-1}x_0)\big\}\Big)\\ &= \max\big\{\psi\big(\rho(T^{n+p-1}x_0, T^{n-1}x_0)\big), \psi\big(\rho(T^{n-1}x_0, T^{n+p-1}x_0)\big)\big\}\\ &\le \psi^2\Big(\max\big\{\rho(T^{n+p-2}x_0, T^{n-2}x_0), \rho(T^{n-2}x_0, T^{n+p-2}x_0)\big\}\Big)\\ &\le \cdots \le \psi^n\Big(\max\big\{\rho(T^p x_0, x_0), \rho(x_0, T^p x_0)\big\}\Big)\\ &\le \psi^n(M) \underset{n}{\longrightarrow} -\infty,\end{aligned}$$

where $M = \sup\{\rho(x, y) : x, y \in O(T, x_0),\ x \ne y\}$, which means that $(T^n x_0)$ is a ρ-backward-Cauchy sequence. Analogously, we obtain

$$\rho(T^n x_0, T^{n+p}x_0) \le \cdots \le \psi^n\Big(\max\big\{\rho(T^p x_0, x_0), \rho(x_0, T^p x_0)\big\}\Big)$$

and so $(T^n x_0)$ is a ρ-forward-Cauchy sequence. Therefore, $(T^n x_0)$ is ρ-Cauchy. By hypothesis, there exists $\xi \in X$ such that $T^n x_0 \underset{n}{\longrightarrow} \xi$. Set $A = \{n \in \mathbb{N} : T^{n+1}x_0 \ne T\xi\}$. If A is finite, then $T^{n+1}x_0 \to T\xi$. Assume that A is infinite. Then $A = (n_k)_{k\in\mathbb{N}}$ and $T^{n_k}x_0 \ne \xi$ for all $k \ge 1$. Hence $\rho(T^{n_k}x_0, \xi) \underset{k}{\longrightarrow} -\infty$. Using (20.6) and (20.7), we get

$$\max\big\{\rho(T^{n_k+1}x_0, T\xi), \rho(T\xi, T^{n_k+1}x_0)\big\} < \max\big\{\rho(T^{n_k}x_0, \xi), \rho(\xi, T^{n_k}x_0)\big\}$$

for all $k \in \mathbb{N}$, that is $T^{n_k+1}x_0 \underset{k}{\longrightarrow} T\xi$. Therefore, by Remark 20.7 (2), $T^{n+1}x_0 \underset{n}{\longrightarrow} T\xi$. Since all convergent sequences in X have a unique limit, one obtains $T\xi = \xi$ and so ξ is a fixed point of T.

In order to show the successive approximations of ξ, choose $x \in X$. If there is $n_0 \in \mathbb{N}$ such that $T^{n_0}x = \xi$, then the conclusion is obvious. Suppose that $T^n x \ne \xi$ for all $n \ge 1$. Then, as before,

$$\begin{aligned}\rho(T^n x, \xi) = \rho(T^n x, T^n \xi) &\le \psi\Big(\max\big\{\rho(T^{n-1}x, T^{n-1}\xi), \rho(T^{n-1}\xi, T^{n-1}x)\big\}\Big)\\ &\le \cdots \le \psi^n\Big(\max\big\{\rho(x, \xi), \rho(\xi, x)\big\}\Big) \underset{n}{\longrightarrow} -\infty\end{aligned}$$

and so $T^n x \xrightarrow{l} \xi$. Analogously, it follows that $T^n x \xrightarrow{r} \xi$ and hence $T^n x \longrightarrow \xi$. This completes the proof.

The following result may be proved in much the same way as the previous theorem:

Theorem 20.11 *Let (X, ρ) be a ρ-space and assume that ρ is backward complete (resp. forward complete) and that $\psi \in \Psi_\mu$, $\mu > \sup\{\rho(x, y) : x, y \in X,\ x \ne y\}$. If T is a $\rho\psi$-contraction with a ρ-bounded orbit $O(T, x_0)$ for some $x_0 \in X$, then it is a l-P.O. (resp. r-P.O.)*

Lemma 20.1 *If $\psi : (-\infty, \mu) \to (-\infty, \mu)$, $\mu \in [0, \infty]$, is an upper semi-continuous function (or continuous) with $\psi(t) < t$ for all $t < \mu$, then $\lim_n \psi^n(t) = -\infty$ for all $t < \mu$.*

Proof Fix $t \in (-\infty, \mu)$. Then $\psi^{n+1}(t) < \psi^n(t)$ for every $n \in \mathbb{N}$ and hence the sequence $\big(\psi^n(t)\big)$ is decreasing and so it has a limit $l \in [-\infty, \mu)$. If $l \in \mathbb{R}$, then $l \leq \limsup_{t \to l} \psi(t) \leq \psi(l)$, which is a contradiction. So $l = -\infty$.

Remark 20.12 Theorem 20.5 follows from Theorem 20.10 or Theorem 20.11 as a corollary, taking $\rho(x, y) = \frac{1}{1-\theta(d(x,y))}$ for all $x, y \in X$ with $x \neq y$ and $\psi : (-\infty, 0) \to (-\infty, 0)$, $\psi(t) = \frac{(-t)^\lambda}{(-t)^\lambda - (1-t)^\lambda}$.

Proof First note that, following the proof of Theorem 20.10, one can suppose for the function ρ only that every ρ-Cauchy sequence has a unique limit instead of the uniqueness of the limit of all ρ-convergent sequences. Next, since $F(t) = \frac{1}{1-\theta(t)} \in \mathfrak{F}$, it follows that ρ satisfies (20.2) in the Definition 20.15. We also note that, d being complete, ρ is complete.

A trivial verification shows that ψ is nondecreasing and that $\psi(t) < t$ for all $t \in (-\infty, 0)$. So, by Lemma 20.1, ψ satisfies the conditions from Theorem 20.10 or Theorem 20.11. Now, since $\theta\big(d(x, y)\big) = 1 - \frac{1}{\rho(x,y)}$, one has

$$\theta\big(d(Tx, Ty)\big) \leq \big[\theta\big(d(x, y)\big)\big]^\lambda \iff 1 - \frac{1}{\rho(Tx, Ty)} \leq \Big(1 - \frac{1}{\rho(x, y)}\Big)^\lambda$$

$$\iff \rho(Tx, Ty) \leq \frac{1}{1 - \big(1 - \frac{1}{\rho(x,y)}\big)^\lambda} = \psi\big(\rho(x, y)\big)$$

for all $x, y \in X$ with $Tx \neq Ty$. Finally, note that $\rho < 0$ and hence every orbit of T is ρ-bounded and hence the proof is complete.

Remark 20.13 Theorems 20.6 and [8, Th. 2.7] are simple consequences of Theorem 20.10 or Theorem 20.11 if we take $\rho(x, y) = -1/\mathfrak{D}(x, y)$ for all $x, y \in X$ with $x \neq y$ and $\psi(t) = \frac{1}{k}t$ for all $t \in (-\infty, 0)$.

Likewise, Theorem 20.8 can be deduced from Theorem 20.10 or Theorem 20.11 by taking $\rho(x, y) = -1/d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $\psi(t) = \frac{1}{\alpha}t$ for $t \in (-\infty, 0)$.

Remark 20.14 Theorem 20.7 is a particular case of Theorems 20.10 and 20.11 by taking $\rho(x, y) = (F \circ \mathfrak{D})(x, y)$ for all $x, y \in X$, $x \neq y$ and $\psi(t) = t - \tau$, $t < 0$.

Before we formulate further results, let us recall the concept of ϕ-contraction. A mapping $\phi\colon [0, \infty) \to [0, \infty)$ is said to be a *comparison function* if it is nondecreasing and $\phi^n(t) \to 0$ as $n \to \infty$ for every $t > 0$. A mapping $T : X \to X$ defined on a metric space (X, d) is called a *ϕ-contraction* if there exists a comparison function ϕ such that $d(Tx, Ty) \leq \phi(d(x, y))$ for every $x, y \in X$. For more details on ϕ-contractions the reader can refer to [4].

The following proposition states that the class of $\rho\psi$-contraction mappings on a complete metric space includes those of ϕ-contractions.

Proposition 20.3 *(1) If $\mu = 0$, then a mapping $\psi : (-\infty, \mu) \to (-\infty, \mu)$ is nondecreasing with $\psi^n(t) \to -\infty$ for all $t < \mu$ if and only if $\phi : (0, \infty) \to (0, \infty)$, $\phi(t) = -1/\psi(-t^{-1})$, is a comparison function.*

(2) Given a metric space (X, d), a mapping $T : X \to X$ is a ϕ-contraction if and only if it is a $\rho\psi$-contraction, where $\rho(x, y) = -1/d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $\psi(t) = -1/\phi(-t^{-1})$.

Proof (1) If ψ is nondecreasing, then clearly ϕ is nondecreasing too. Next, it is easy to see that $\phi^n(t) = -1/\psi^n(-t^{-1})$ for every $t > 0$ and $n \geq 1$. So, we have

$$\phi^n(t) \to 0 \iff \psi^n(-t^{-1}) \to -\infty.$$

Since $\psi(s) = -1/\phi(-s^{-1})$ for all $s < 0$, the converse implication is obvious.

(2) According to Example 20.16, ρ is a ρ-metric. For every $x, y \in X$ with $x \neq y$ and $Tx \neq Ty$, one has

$$\rho(Tx, Ty) \leq \psi\big(\rho(x, y)\big) \iff \frac{-1}{d(Tx, Ty)} \leq \psi\Big(\frac{-1}{d(x, y)}\Big) = \frac{-1}{\phi\big(d(x, y)\big)}$$
$$\iff d(Tx, Ty) \leq \phi\big(d(x, y)\big),$$

as required.

For a given ρ-space (X, ρ), we will need the existence of a function $\Gamma\colon (-\infty, \mu) \to (0, \infty)$, $\mu = \sup_{x,y \in X, x \neq y} \rho(x, y)$, such that

(Γ1) Γ is increasing,

(Γ2) $(\Gamma \circ \rho)(x, y) \leq (\Gamma \circ \rho)(x, z) + (\Gamma \circ \rho)(z, y)$ for all $x, y, z \in X$, $x \neq y \neq z \neq x$,

(Γ3) $t_n \to -\infty$ implies $\Gamma(t_n) \to 0$.

If X is a metrizable space, then a simple example of functions ρ and Γ can be found in what follows.

Example 20.19 Consider a metric space (X, d) and two functions $\Gamma : (-\infty, 0) \to (0, \infty)$ which satisfies (Γ1), (Γ3) and its inverse $\Gamma^{-1} : \Gamma\big((-\infty, 0)\big) \to (-\infty, 0)$. If $\rho : X \times X \setminus \Delta \to \mathbb{R}$ is given by $\rho(x, y) = \Gamma^{-1}\big(d(x, y)\big)$, then ρ is a symmetric ρ-metric and (Γ1)–(Γ3) hold. In particular, we can consider $\Gamma(t) = -1/t$ and $\rho(x, y) = -1/d(x, y)$ or $\Gamma(t) = e^t$ and $\rho(x, y) = \ln d(x, y)$. Moreover, in both cases, the mapping Γ is continuous.

Proposition 20.4 *Consider a ρ-space X, a function $\Gamma\colon (-\infty, \mu) \to (0, \infty)$ satisfying $(\Gamma 1) - (\Gamma 3)$ and $\Omega \subset (-\infty, \mu)$ such that $(-\infty, \mu) \setminus \Omega$ is dense in $(-\infty, \mu)$. For every sequence $(x_n) \subset X$ of different elements, if $\rho(x_n, x_{n+1}) \to -\infty$, $\rho(x_{n+1}, x_n) \to -\infty$ and (x_n) is not a ρ-Cauchy sequence, then there exist $M \in (-\infty, \mu) \setminus \Omega$ and the sequences (m_k), (n_k) of positive integers such that*

(a) $\Gamma\big(\rho(x_{m_k}, x_{n_k})\big) \searrow \Gamma(M)$ *as* $k \to \infty$,
(b) $\Gamma\big(\rho(x_{m_k+1}, x_{n_k+1})\big) \underset{k}{\to} \Gamma(M)$.

Proof Since (x_n) is not a ρ-Cauchy sequence and $(-\infty, \mu) \setminus \Omega$ is dense, there exists $M \in (-\infty, \mu) \setminus \Omega$ such that, for each $k \in \mathbb{N}$, one can find $m, n \in \mathbb{N}, k \le m < n$ such that $\rho(x_m, x_n) > M$. Denote

$$m_k = \min\big\{m \in \mathbb{N} : \exists n \in \mathbb{N},\ k \le m < n,\ \rho(x_m, x_n) > M\big\},$$

$$n_k = \min\big\{n \in \mathbb{N} : k \le m_k < n,\ \rho(x_{m_k}, x_n) > M\big\}.$$

Let $n_0 \in \mathbb{N}$ be such that $\rho(x_n, x_{n+1}) < M$ and $\rho(x_{n+1}, x_n) < M$ for all $n \ge n_0$. By the definitions of m_k and n_k for all $k \ge n_0$, one must have $n_k \ge m_k + 2$ and $\rho(x_{m_k}, x_{n_k-1}) \le M$. Therefore, using $(\Gamma 2)$ and $(\Gamma 3)$, for all $k \ge n_0$, we get

$$\Gamma(M) \le \Gamma\big(\rho(x_{m_k}, x_{n_k})\big) \le \Gamma\big(\rho(x_{m_k}, x_{n_k-1})\big) + \Gamma\big(\rho(x_{n_k-1}, x_{n_k})\big)$$

$$\le \Gamma(M) + \Gamma\big(\rho(x_{n_k-1}, x_{n_k})\big).$$

In consequence, since $\Gamma\big(\rho(x_{n_k-1}, x_{n_k})\big) > 0$, $\rho(x_{n_k-1}, x_{n_k}) \underset{k}{\to} -\infty$ and, due to $(\Gamma 3)$, we obtain

$$\Gamma\big(\rho(x_{m_k}, x_{n_k})\big) \searrow \Gamma(M),\ k \to \infty.$$

Also, observe that, for all $k \in \mathbb{N}$, using couple times $(\Gamma 2)$, we have the inequalities:

$$\Gamma\big(\rho(x_{m_k}, x_{n_k})\big) - \Gamma\big(\rho(x_{m_k}, x_{m_k+1})\big) - \Gamma\big(\rho(x_{n_k+1}, x_{n_k})\big) \le \Gamma\big(\rho(x_{m_k+1}, x_{n_k+1})\big)$$

$$\le \Gamma\big(\rho(x_{m_k+1}, x_{m_k})\big) + \Gamma\big(\rho(x_{m_k}, x_{n_k})\big) + \Gamma\big(\rho(x_{n_k}, x_{n_k+1})\big).$$

Letting $k \to \infty$ and applying $(\Gamma 3)$, finally we obtain

$$\Gamma(\rho(x_{m_k+1}, x_{n_k+1})) \underset{k}{\longrightarrow} \Gamma(M).$$

This completes the proof.

Theorem 20.12 *Let $T\colon X \to X$ be a weak-$\rho\psi$-contraction defined on a complete ρ-space, where $\psi\colon (-\infty, \mu) \to (-\infty, \mu)$, $\mu > \sup_{x,y \in X, x \ne y} \rho(x, y)$, is an upper semicontinuous function satisfying $\psi(t) < t$ for all $t < \mu$. Assume that there exists a map $\Gamma\colon (-\infty, \mu) \to (0, \infty)$ which satisfies* $(\Gamma 1)$-$(\Gamma 3)$. *Then T is a P.O.*

Proof We first show that T has at most one fixed point. Suppose, contrary to our claim, that there are $\xi, \eta \in X$, $\xi \ne \eta$, such that $T\xi = \xi$, $T\eta = \eta$. Then

$$\rho(\xi, \eta) = \rho(T\xi, T\eta) \le \psi\big(\max\{\rho(\xi, \eta), \rho(\eta, \xi)\}\big) < \max\{\rho(\xi, \eta), \rho(\eta, \xi)\}$$

and, analogously,

$$\rho(\eta, \xi) < \max\{\rho(\xi, \eta), \rho(\eta, \xi)\}.$$

Thus we obtain a contradiction.

In order to prove the existence of the fixed point, consider any $x_0 \in X$ and denote $x_n = T^n x_0$. If $x_{n_0} = x_{n_0-1}$ for some $n_0 \in \mathbb{N}$, then one can see that $T^{n_0-1}x_0$ is a fixed point of T. Assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$ and denote $\delta_n = \max\{\rho(x_n, x_{n+1}), \rho(x_{n+1}, x_n)\}$ for each $n \in \mathbb{N}$. We have

$$\begin{aligned}
\rho(x_n, x_{n+1}) = \rho(Tx_{n-1}, Tx_n) &\leq \psi\big(\max\{\rho(x_{n-1}, x_n), \rho(x_n, x_{n-1})\}\big) \\
&< \max\{\rho(x_{n-1}, x_n), \rho(x_n, x_{n-1})\}, \\
\rho(x_{n+1}, x_n) = \rho(Tx_n, Tx_{n-1}) &\leq \psi\big(\max\{\rho(x_{n-1}, x_n), \rho(x_n, x_{n-1})\}\big) \\
&< \max\{\rho(x_{n-1}, x_n), \rho(x_n, x_{n-1})\}.
\end{aligned}$$

Hence

$$\delta_n \leq \psi(\delta_{n-1}) < \delta_{n-1}, \quad \forall n \in \mathbb{N}.$$

Set $\lambda = \lim_n \delta_n$. If $-\infty < \lambda$, then, by the above, we get

$$\lambda = \lim_n \psi(\delta_{n-1}) \leq \limsup_{t \to \lambda} \psi(t) \leq \psi(\lambda),$$

which is a contradiction. Consequently $\delta_n \searrow -\infty$.

Now, suppose that (x_n) is not a ρ-Cauchy sequence. Let A be the set of all elements of $(-\infty, \mu)$ where Γ is continuous. From $(\Gamma 1)$ we know that A is dense in $(-\infty, \mu)$. Taking $\Omega = (-\infty, \mu) \setminus A$ in Proposition 20.4, it follows that there exist $M \in A$ and the sequences (m_k), (n_k) such that

$$\Gamma\big(\rho(x_{m_k}, x_{n_k})\big) \searrow \Gamma(M), \;\; \Gamma\big(\rho(x_{m_k+1}, x_{n_k+1})\big) \underset{k}{\to} \Gamma(M).$$

The continuity of Γ in M and its monotonicity imply $\rho(x_{m_k}, x_{n_k}) \searrow M$ and $\rho(x_{m_k+1}, x_{n_k+1}) \underset{k}{\to} M$. Thus we obtain

$$\rho(x_{m_k+1}, x_{n_k+1}) \leq \psi\big(\rho(x_{m_k}, x_{n_k})\big), \quad \forall k \in \mathbb{N}.$$

Letting $k \to \infty$ and using the upper semicontinuity of ψ, we get

$$M \leq \limsup_{k \to \infty} \psi\big(\rho(x_{m_k}, x_{n_k})\big) \leq \limsup_{t \to M} \psi(t) \leq \psi(M),$$

which is impossible. Therefore (x_n) is ρ-Cauchy and hence convergent. We can now proceed analogously to the proof of Theorem 20.10. This completes the proof.

As a particular case, we obtain:

Corollary 20.1 (Piri et al. [18]) *Let T be a self mapping on a complete metric space (X, d). Suppose that $F : (0, \infty) \to \mathbb{R}$ is a continuous function which satisfies* (F1) *and* (F2)*. If there exists $\tau > 0$ such that* (20.1) *holds, then T is a P.O.*

Proof The function $F : (0, \infty) \to (-\infty, M)$, where $M = \sup_{t>0} F(t)$, is invertible and $\Gamma := F^{-1}$ satisfies $(\Gamma 1)$–$(\Gamma 3)$. Next, taking $\rho := F \circ d$ and $\psi(t) = t - \tau$, the conclusion follows immediately from Theorem 20.12.

Remark 20.15 Corollary 20.1 generalizes Theorem 20.5.

20.3.4 Example

In the following, we provide an example of non-metrizable topological space in which Theorem 20.11 can be applied.

Take any $\lambda \in (0, 1)$ and set

$$X = \bigcup_{n=1}^{\infty} [\lambda^{2n-1}, \lambda^{2n-2}] \cup \{0\}$$

with the topology τ_l induced from the Sorgenfrey line. Let us consider a mapping $T : X \to X$ given by

$$Tx = \begin{cases} 0, & \text{if } x \in \bigcup_{n=1}^{\infty} [\lambda^{2n-1}, \lambda^{2n-2}) \cup \{0\}, \\ \lambda^{2n}, & \text{if } x = \lambda^{2n-2},\ n \in \mathbb{N} \end{cases}$$

and the function $\psi : (-\infty, \mu) \to (-\infty, \mu)$, $\mu > 0$, $\psi(t) = t + 2 \ln \lambda$. Denote $\Lambda = \{\lambda^{2n-2} : n \in \mathbb{N}\}$ and define $\rho : X \times X \setminus \Delta \to \mathbb{R}$ as follows:

$$\rho(x, y) = \begin{cases} \ln|x - y|, & \text{if } [x, y \in \Lambda \cup \{0\}] \text{ or } [x, y \notin \Lambda \text{ and } y < x], \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

(1) (X, τ_l) is a non-metrizable Hausdorff topological space,

(2) ρ is an asymmetric forward complete topological ρ-metric,

(3) T is continuous in (X, τ_l) while it is discontinuous with respect to the standard topology τ_d generated by the Euclidean metric d on $\mathbb{R}$,

(4) T is a $\rho\psi$-contraction and r-P.O.

Proof (1) The Sorgenfrey line is a Hausdorff space and so is its subspace (X, τ_l).

Now, we will prove the non-metrizability of X using a direct way. Suppose that (X, τ_l) is metrizable, that is, $\tau_l = \tau_\delta$, where τ_δ denotes the topology induced by some metric δ. Fix $x \in [\lambda, 1) \subset X$. Since $[x, 1) \in \tau_l = \tau_\delta$, there exists $n \in \mathbb{N}$ such that

$B(x, \frac{1}{n}) \subset [x, 1)$. Since $B(x, \frac{1}{n}) \in \tau_l$, there is also $m \in \mathbb{N}$ such that $[x, x + \frac{1}{m}) \subset B(x, \frac{1}{n})$. Summarizing, for each $a \in [\lambda, 1)$, one can find $s, t > 0$ such that $a \in V_{m,n}$, where

$$V_{m,n} = \left\{x \in \left[\lambda, 1\right): \left[x, x + \frac{1}{m}\right) \subset B\left(x, \frac{1}{n}\right) \subset [x, 1)\right\}.$$

In consequence, we get $[\lambda, 1) \subset \bigcup_{m,n \in \mathbb{N}} V_{m,n}$. The set $[\lambda, 1)$ is uncountable and so there must exist $m_0, n_0 \in \mathbb{N}$ such that V_{m_0,n_0} is uncountable. Let (x_k) be any sequence of elements in $[\lambda, 1)$ such that $\lambda < x_1 < x_2 < \cdots < 1$, $x_k \to 1$, $x_1 - \lambda < \frac{1}{m_0}$ and $x_{k+1} - x_k < \frac{1}{m_0}$ for all $k \in \mathbb{N}$. In one of the intervals $[\lambda, x_1]$ or $[x_k, x_{k+1}]$ for each $k \in \mathbb{N}$, there are uncountable many elements of V_{m_0,n_0}. Thus we can choose $u, v \in V_{m_0,n_0}$ such that $u > v$ and $u - v < \frac{1}{m_0}$. Hence we have

$$u \in \left[v, v + \frac{1}{m_0}\right) \subset B\left(v, \frac{1}{n_0}\right),$$

and, in consequence,

$$v \in B\left(u, \frac{1}{n_0}\right) \subset [u, 1).$$

From the above, we obtain $v \geq u$ which contradicts the choice of u, v. Hence (X, τ_l) is non-metrizable.

(2) In order to show that ρ is a ρ-metric, consider a sequence of different elements $(x_n) \subset X$ which converges to $x \in X$ with respect to τ_l. Then we can assume that (x_n) is decreasing and $|x_n - x| \to 0$, that is $x_n \to x$ with respect to the Euclidean metric. Clearly $x \notin \Lambda$.

Now, we claim that $\rho(x_n, x) \to -\infty$. Indeed, if there is $N \in \mathbb{N}$ such that $x_n \notin \Lambda$ for every $n \geq N$, then $\rho(x_n, x) = \ln |x_n - x|$. On the contrary, one can find a subsequence $(x_{n_k})_k \subset \Lambda$. In this case $x = 0$ and $\rho(x_n, 0) = \ln x_n$ for all $n \geq 1$. In both cases, $\rho(x_n, x) \to -\infty$.

Conversely, let suppose that $(x_n) \subset X$ has different elements and satisfies $\rho(x_n, x) \to -\infty$, where $x \in X$. It follows that $x_n > x \geq 0$ except for a finite set of $n \in \mathbb{N}$.

Now, we will prove that $\rho(x_n, x) = \ln |x_n - x|$ hence $|x_n - x| \to 0$, that is $x_n \to x$ with respect to the topology τ_l. If there exists $N \in \mathbb{N}$ such that $x_n, x \notin \Lambda$ for all $n \geq N$, then $\rho(x_n, x) = \ln |x_n - x|$. Assume that there is a subsequence $(x_{n_k}) \subset \Lambda$ of (x_n). Then $x_{n_k} \xrightarrow[k]{} 0$ and so $x = 0$ and $\rho(x_n, x) = \ln x_n$.

It remains to prove the forward completeness of ρ. For this purpose, let $(x_n) \subset X$ be a sequence of different numbers such that $\rho(x_n, x_{n+p}) \xrightarrow[n,p]{} -\infty$. Then one can find $N \in \mathbb{N}$ such that $\rho(x_n, x_{n+p}) = \ln |x_n - x_{n+p}|$ for all $n \geq N$, $p \geq 1$. Moreover, $(x_n)_{n \geq N}$ is decreasing because, if this is not so, one can find a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} < x_{n_{k+1}}$ for every $k \geq 1$. Hence $\rho(x_{n_k}, x_{n_{k+1}}) = 0$ for all $k \in \mathbb{N}$, which is a contradiction. Therefore, $|x_n - x_{n+p}| \xrightarrow[n,p]{} 0$ and hence (x_n) is a Cauchy sequence in $(\mathbb{R}, d)$ and so it converges to some $x \in \mathbb{R}$. Since $\complement X \in \tau_d$, we deduce that X is closed in $(\mathbb{R}, \tau_d)$ and so $x \in X$. Next, $(x_n)_{n \geq N}$ being decreasing, it follows that $x_n \to x$ with respect to τ_l. Consequently ρ is forward complete.

(3) Obviously, for every $n \in \mathbb{N}$, T is discontinuous in λ^{2n-2} with respect to the topology τ_d. However, it is continuous with respect to τ_l. Indeed, for every $n \in \mathbb{N}$, it is enough to consider the open neighbourhood U_n of λ^{2n-2} of the form

$$U_n = \{\lambda^{2n-2}\}.$$

Then, taking any $V_n \in \tau_l$ such that $\lambda^{2n} = T(\lambda^{2n-2}) \in V_n$, we have $T(U_n) \subset V_n$. The continuity of T with respect to τ_l on the set $X \setminus \Lambda$ is easy to be verified.

(4) In order to show that T is a $\rho\psi$-contraction, consider first $x = \lambda^{2m-2}$ and $y = \lambda^{2n-2}$ for each $m, n \in \mathbb{N}$ with $x \neq y$. We have

$$\begin{aligned}\rho(x, y) - \rho(Tx, Ty) &= \rho(\lambda^{2m-2}, \lambda^{2n-2}) - \rho(\lambda^{2m}, \lambda^{2n}) \\ &= \ln \frac{|\lambda^{2m-2} - \lambda^{2n-2}|}{|\lambda^{2m} - \lambda^{2n}|} = \ln \frac{\lambda^{2m-2}|1 - \lambda^{2n-2m}|}{\lambda^{2m}|1 - \lambda^{2n-2m}|} = -2\ln\lambda.\end{aligned}$$

For $x = 0$ and $y = \lambda^{2n-2}$, we get

$$\rho(x, y) - \rho(Tx, Ty) = \rho(0, \lambda^{2n-2}) - \rho(0, \lambda^{2n}) = \ln \frac{\lambda^{2n-2}}{\lambda^{2n}} = -2\ln\lambda.$$

Next, taking $x \notin \Lambda$, $x \neq 0$, and $y = \lambda^{2n-2}$ for some $n \geq 1$, we obtain

$$\rho(x, y) - \rho(Tx, Ty) = \rho(x, \lambda^{2n-2}) - \rho(0, \lambda^{2n}) = -\ln\lambda^{2n} \geq -2\ln\lambda.$$

In the other cases in which $x, y \notin \Lambda$, one has $Tx = Ty = 0$.

Finally, observe that every orbit of T is ρ-bounded. Consequently, all the assumptions of Theorem 20.11 are satisfied so T is a r-P.O.

Remark 20.16 In the space X endowed with the standard metric d, the operator T presented in the above example is neither nonexpansive nor expansive. This emphasizes that Theorem 20.11 offers a new method to establish that a self-mapping is a P.O.

Proof Take, for some $n \in \mathbb{N}$, $x_1 = 0$, $\max\left\{\lambda^{2n-1}, \lambda^{2n-2}(1-\lambda^2)\right\} < x_2 < \lambda^{2n-2}$ and $y = \lambda^{2n-2}$. The conclusion follows from the following inequalities:

$$d(Tx_1, Ty) < d(x_1, y) \iff \lambda^{2n} < \lambda^{2n-2}$$

and

$$d(Tx_2, Ty) > d(x_2, y) \iff \lambda^{2n} > \lambda^{2n-2} - x_2 \iff x_2 > \lambda^{2n-2}(1-\lambda^2).$$

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