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Stability results for backward heat equations with time-dependent coefficient in the Banach space $L_p(\mathbb{R})$

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ABSTRACT

In this paper, we investigate the problem of backward heat equations with time-dependent coefficient in the Banach space $L_p(\mathbb{R})$, $(1 < p < \infty)$. For this problem, we first prove the stability estimates of Hölder type. After that the Tikhonov-type regularization is applied to solve the problem. A priori and a posteriori parameter choice rules are investigated, which yield error estimates of Hölder type. Numerical implementations are presented to show the validity of the proposed scheme.

1. Introduction

Let $p \in (1, \infty)$, ε, E be given constants such that $0 < \varepsilon \leqslant E < \infty$ and φ be a function in $L_p(\mathbb{R})$. We consider the Cauchy problem for the heat equation backward in time with inexact final data

$$\begin{cases} \dfrac{\partial u}{\partial t} = a(t)\dfrac{\partial^2 u}{\partial x^2}, \quad (x,t) \in (-\infty; +\infty) \times (0; T), \\ \|u(\cdot, T) - \varphi(\cdot)\|_p \leqslant \varepsilon, \end{cases} \tag{1.1}$$

where $\|\cdot\|_p$ is L_p norm in $\mathbb{R}$ and $a(t)$ is a continuous function on $[0, T]$ satisfying

$$0 < \underline{a} \leqslant a(t) \leqslant \overline{a}, \ \forall t \in [0, T].$$

The Cauchy problem (1.1) is an inverse problem and well known to be ill-posed, i.e., a small perturbation in the Cauchy data may cause a very large error in the solution. It is therefore difficult to develop numerical methods for it, since errors of measurements in the Cauchy data, discretization errors, and round-off errors make numerical solutions unstable. To overcome this difficulty, we must apply a regularization method in order to solve problem (1.1) in a stable way.

There have been several results concerning inverse problems of parabolic equations with time-independent coefficients in Banach spaces (e.g. see [1–3,6,9–11,16] and the references therein). However, results concerning the case of time-dependent coefficients as in (1.1) are less popular (see [5]).

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In [9], Dinh Nho Hao used a mollification method to regularize backward heat equations with time-independent coefficient in the Banach space $L_p(\mathbb{R})$ $(1 < p \leqslant \infty)$. This method was extended in the works [10] and [16]. In [11], Dinh Nho Hao and co-workers used the Tikhonov-type regularization method and the non-local boundary value problem method to regularize backward parabolic equations with time-independent coefficients in Banach spaces. However, numerical methods were not considered in that work. By using the semigroup theory of operators, many researchers have also obtained many results for the inverse problems of parabolic equations in Banach spaces such as the backward parabolic equation, identifying an unknown source term of parabolic equation in Banach spaces ([1–3,6,5]).

In addition to the topic of the inverse problem, other related problems for parabolic equations have also been investigated, e.g., the forward-backward parabolic equations have been studied and obtained many profound results ([7,13–15]), the influence of the fourth-order diffusion term on the well-posedness of solution of Cauchy problems for fourth order parabolic equations has been studied in [8], the influence of external force source on the well-posedness of solution of semilinear pseudo-parabolic equation with Neumann boundary condition has been investigated in [17].

In this paper, we first establish stability estimates of Hölder type for backward heat equations with a time-dependent coefficient (1.1). Then, we apply the Tikhonov-type regularization method to regularize this problem. We suggest a priori and a posteriori parameter choice rules and obtain error estimates of Hölder type. Finally, we present numerical results to confirm the theory.

2. Stability estimates

Theorem 1. *Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of problem (1.1) satisfying*

$$\|u_i(\cdot,0)\|_p \leqslant E,\ i=1,2,\ 0<\varepsilon<E. \tag{2.1}$$

Then, there exists a constant $C>0$ such that the following stability estimate holds

$$\|u_1(\cdot,t)-u_2(\cdot,t)\|_p \leqslant C\varepsilon^{\nu(t)}E^{1-\nu(t)},\ \forall t\in[0,T]. \tag{2.2}$$

Here,

$$a_1(t)=\int_0^t a(\tau)d\tau,\ \nu(t)=\frac{a_1(t)}{a_1(T)},$$

for all $t\in[0,T]$.

Proof. Since $a(t)$ is a continuous function on $[0,T]$ satisfying $0<\underline{a}\leqslant a(t)\leqslant \overline{a}$ for all $t\in[0,T]$, $a_1(t)$ is continuous and strictly increasing on $[0,T]$. This implies that $\nu(t)$ is also continuous and strictly increasing on $[0,T]$ and $\nu(0)=0$, $\nu(T)=1$. It is reasonable to set

$$v_i(x,\eta) := u_i\left(x,\nu^{-1}(\eta)\right),\ x\in\mathbb{R}, \eta\in[0,1], i=1,2. \tag{2.3}$$

We have

$$\begin{aligned} v_i(x,0) &= u_i\left(x,\nu^{-1}(0)\right)=u_i(x,0),\ x\in\mathbb{R}, i=1,2,\\ v_i(x,1) &= u_i\left(x,\nu^{-1}(1)\right)=u_i(x,T),\ x\in\mathbb{R}, i=1,2. \end{aligned}$$

Therefore, we obtain

$$\|v_i(\cdot,1)-\varphi(\cdot)\|_p \leqslant \varepsilon, i=1,2, \tag{2.4}$$

$$\|v_i(\cdot,0)\|_p \leqslant E,\ i=1,2. \tag{2.5}$$

Furthermore, from (2.3) we have $v_i(x,\nu(t))=u_i(x,t),\ \forall t\in[0,T], i=1,2$. This implies that

$$\begin{aligned} \frac{\partial u_i}{\partial t}(x,t) &= \frac{\partial}{\partial t}\left(v_i\left(x,\nu\left(t\right)\right)\right)=\frac{\partial}{\partial t}\left(v_i(x,\eta)\right)\ (\eta=\nu(t))\\ &= \frac{\partial}{\partial \eta}\left(v_i(x,\eta)\right)\frac{\partial \eta}{\partial t}=\frac{a(t)}{a_1(T)}\frac{\partial}{\partial\eta}\left(v_i(x,\eta)\right), \end{aligned} \tag{2.6}$$

$$\frac{\partial^2 u_i}{\partial x^2}(x,t)=\frac{\partial^2 v_i}{\partial x^2}\left(x,\nu\left(t\right)\right)=\frac{\partial^2 v_i}{\partial x^2}(x,\eta). \tag{2.7}$$

Since $\frac{\partial u_i}{\partial t}(x,t)=a(t)\frac{\partial^2 u_i}{\partial x^2}(x,t)$, from (2.6) and (2.7), we obtain

$$\frac{\partial v_i}{\partial \eta}(x,\eta) = a_1(T)\frac{\partial^2 v_i}{\partial x^2}(x,\eta). \tag{2.8}$$

Now, we set $w_i(x,\theta) = v_i\left(x, \frac{\theta}{a_1(T)}\right), x \in \mathbb{R}, \theta \in [0, a_1(T)], i = 1, 2$. We have $w_i(x,\theta) = v_i(x,\eta)$ with $\theta = a_1(T)\eta$. Therefore, we obtain

$$\|w_i(\cdot, a_1(T)) - \varphi(\cdot)\|_p = \|v_i(\cdot, 1) - \varphi(\cdot)\|_p \leqslant \varepsilon, i = 1, 2, \tag{2.9}$$

$$\|w_i(\cdot, 0)\|_p = \|v_i(\cdot, 0)\|_p \leqslant E,\ i = 1, 2, \tag{2.10}$$

$$\frac{\partial w_i}{\partial \theta}(x,\theta) = \frac{\partial^2 w_i}{\partial x^2}(x,\theta). \tag{2.11}$$

Using Theorem 3.2 in [10], we conclude that there exists a constant $C_1 > 0$ such that

$$\|w_1(\cdot,\theta) - w_2(\cdot,\theta)\|_p \leqslant C_1 \varepsilon^{\frac{\theta}{a_1(T)}} E^{1-\frac{\theta}{a_1(T)}},\ \forall \theta \in [0, a_1(T)]. \tag{2.12}$$

This implies that there exists a constant $C > 0$ such that

$$\|u_1(\cdot,t) - u_2(\cdot,t)\|_p \leqslant C\varepsilon^{\nu(t)} E^{1-\nu(t)},\ \forall t \in [0, T].$$

The theorem is proved. □

Remark 1.

1) If c is a positive number and $a(t) = c,\ \forall t \in [0, T]$, then $\nu(t) = \frac{t}{T},\ \forall t \in [0, T]$.

2) If c is a positive number and $a(t) = T + ct,\ \forall t \in [0, T]$, then

$$\frac{2}{c+2}.\frac{t}{T} \leqslant \nu(t) \leqslant \frac{t}{T}\ \forall t \in [0, T].$$

3) If c is a positive number and $a(t) = (1+c)T - ct,\ \forall t \in [0, T]$, then

$$\frac{t}{T} < \nu(t) \leqslant \frac{2+2c}{2+c}.\frac{t}{T},\ \forall t \in [0, T).$$

4) In the general case, let $p = \min_{t\in[0,T]} a(t)$ and $q = \max_{t\in[0,T]} a(t)$ then $\nu(t) \geqslant \frac{p}{q}.\frac{t}{T},\ \forall t \in [0, T]$. Indeed, we have $0 < \underline{a} \leqslant p \leqslant a(t) \leqslant q \leqslant \bar{a},\ \forall t \in [0, T]$. This implies that $pt \leqslant a_1(t) = \int_0^t a(\tau)d\tau \leqslant qt,\ \forall t \in [0, T]$ and $pT \leqslant a_1(T) = \int_0^T a(\tau)d\tau \leqslant qT$. Therefore, we obtain

$$\nu(t) = \frac{a_1(t)}{a_1(T)} \geqslant \frac{pt}{qT} = \frac{p}{q}.\frac{t}{T},\ \forall t \in [0, T].$$

Since $\nu(t) \in (0, 1],\ \forall t \in (0, T]$ and $\nu(0) = 0$, Theorem 1 gives stability estimates of Hölder type for all $t \in (0, T]$ but does not give any information about the continuous dependence of the solution of (1.1) at $t = 0$ on the final data $t = T$. With our best knowledge there are not any result on the convergence rate at $t = 0$ under condition (2.1) even in Hilbert spaces. To establish this, we suppose further that there exists a positive γ satisfying

$$\omega(u(\cdot,0), h)_p \leqslant \tilde{E}h^{\gamma}, \forall h > 0. \tag{2.13}$$

Here, $\omega(f,h)_p = \sup_{|z|\leqslant h} \|f(\cdot) - f(\cdot - z)\|_p$ is the modulus of continuity of the function $f \in L_p(\mathbb{R})$ in the metric of $L_p(\mathbb{R})$ ([12, p. 147]). We will see that with assumptions (2.1) and (2.13) a stability estimate of logarithmic type at $t = 0$ is guaranteed.

Theorem 2. *Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of problem (1.1) satisfying conditions (2.1) and (2.13). Then*

$$\|u_1(\cdot,0) - u_2(\cdot,0)\|_p \leqslant O\left(\left(\ln\frac{E}{\varepsilon}\right)^{-\gamma/2}\right) \text{ as } \varepsilon \longrightarrow 0^+.$$

Proof. Let $v_i, w_i, i = 1, 2$ as in proof of Theorem 1. We have

$$u_i(x,0) = v_i(x,0) = w_i(x,0),\ \forall x \in \mathbb{R}, i = 1, 2.$$

Therefore, for $i = 1, 2$, we get

$$\|w_i(\cdot, a_1(T)) - \varphi(\cdot)\|_p \leqslant \varepsilon,$$
$$\|w_i(\cdot, 0)\|_p \leqslant E,$$
$$\frac{\partial w_i}{\partial \theta}(x,\theta) = \frac{\partial^2 w_i}{\partial x^2}(x,\theta),$$
$$\omega(w_i(\cdot, 0), h)_p \leqslant \tilde{E}h^\gamma, \forall h > 0.$$

Using Theorem 3.3 in [10] and the triangle inequality of norm, we conclude that

$$\|w_1(\cdot, 0) - w_2(\cdot, 0)\|_p \leqslant O\left(\left(\ln\frac{E}{\varepsilon}\right)^{-\gamma/2}\right) \text{ as } \varepsilon \longrightarrow 0^+.$$

This implies that

$$\|u_1(\cdot, 0) - u_2(\cdot, 0)\|_p \leqslant O\left(\left(\ln\frac{E}{\varepsilon}\right)^{-\gamma/2}\right) \text{ as } \varepsilon \longrightarrow 0^+.$$

The theorem is proved. □

3. Tikhonov-type regularization method and error estimates

We minimize the Tikhonov-type functional

$$J_\alpha(g) = \|v(\cdot, T, g) - \varphi(\cdot)\|_p^p + \alpha\|g\|_p^p \tag{3.1}$$

over $L_p(\mathbb{R})$, where $v = v(x,t,g)$ is the solution of the well-posed initial problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = a(t)\dfrac{\partial^2 v}{\partial x^2}, & (x,t) \in (-\infty; +\infty) \times (0; T) \\ v(x, 0) = g \in L_p(\mathbb{R}) \end{cases} \tag{3.2}$$

and consider the minimizers of the functional J_α as a regularized solution to problem (1.1). We propose a priori and a posteriori parameter choice rules for obtaining error estimates of the same order as that in Theorem 1 and Theorem 2.

In this section, we always assume that problem (1.1) has a solution $u(x,t)$ satisfying

$$\|u(\cdot, 0)\|_p \leqslant E, \tag{3.3}$$

with $E > \varepsilon$ being a given positive number.

Remark 2. In [11], we have used the Tikhonov-type functional $\|v(\cdot, T, g) - \varphi(\cdot)\|_p^2 + \alpha\|g\|_p^2$ and obtained error estimates of Hölder type. In this paper, we use the Tikhonov-type functional (3.1) with noting that $p > 1$. With this change, the functional in the discretized problem of (3.1) is Fréchet differentiable at any point (see in Subsection 4.2) and thus we can use some efficient numerical algorithms to solve it.

Theorem 3. *Problem (3.2) is well-posed.*

Proof. See [18]. □

Theorem 4. *There exists a unique solution to problem (3.1)-(3.2).*

The proof of this theorem is similar to that of Theorem 3 of [11]. Therefore, we skip its proof here.

Theorem 5. *Let $\{\varphi_n\}$ be a sequence converging to φ^* and $\{g_n\}$ be the minimizer of problem (3.1)-(3.2) with φ replaced by φ_n. Then, the sequence $\{g_n\}$ converges to the minimizer g^* of problem (3.1)-(3.2) with φ replaced by φ^*.*

The proof of this theorem is similar to that of Theorem 4 of [11]. Therefore, we skip its proof here.

3.1. Error estimate under an a priori parameter choice rule

Theorem 6. *Let $u(x,t)$ be a solution of problem (1.1) satisfying the condition (3.3) and g be the solution of problem (3.1)-(3.2) with the a priori parameter choice*

$$\alpha = \left(\frac{\varepsilon}{E}\right)^p.$$

Then, there exists a constant $\widetilde{C} > 0$ such that

$$\|u(\cdot,t) - v(\cdot,t,g)\|_p \leqslant \widetilde{C}\varepsilon^{\nu(t)}E^{1-\nu(t)}, \ \forall t \in [0,T].$$

Proof. Let $z(x,t) = u(x,t) - v(x,t,g), x \in \mathbb{R}, t \in [0,T]$. Then

$$\frac{\partial z}{\partial t} = a(t)\frac{\partial^2 z}{\partial x^2}, \ \ (x,t) \in (-\infty;+\infty) \times (0;T).$$

Set

$$m(x,\eta) := z\left(x, \nu^{-1}(\eta)\right), \ x \in \mathbb{R}, \eta \in [0,1], i = 1,2. \tag{3.4}$$

Using the same argument as in the proof of Theorem 1, we have

$$\frac{\partial m}{\partial \eta}(x,\eta) = a_1(T)\frac{\partial^2 m}{\partial x^2}(x,\eta), (x,\eta) \in (-\infty;+\infty) \times (0;1). \tag{3.5}$$

Now, we set $n(x,\theta) = m\left(x, \frac{\theta}{a_1(T)}\right), x \in \mathbb{R}, \theta \in [0, a_1(T)]$. We have $n(x,\theta) = m(x,\eta)$ with $\theta = a_1(T)\eta$. Furthermore, we obtain

$$\frac{\partial n}{\partial \eta}(x,\theta) = \frac{\partial^2 n}{\partial x^2}(x,\theta), (x,\theta) \in (-\infty;+\infty) \times (0;a_1(T)). \tag{3.6}$$

Since $n(x,\theta)$ solves equation (3.6) and $n(x,0) = m(x,0) = z(x,0) = u(x,0) - v(x,0,g) = u(x,0) - g \in L_p(\mathbb{R})$, there exists a positive constant C_1 such that (see [10,16])

$$\|n(\cdot,\theta)\|_p \leqslant C_1 \|n(\cdot,a_1(T))\|_p^{\frac{\theta}{a_1(T)}} \|n(\cdot,0)\|_p^{1-\frac{\theta}{a_1(T)}}, \ \forall \theta \in [0,a_1(T)]. \tag{3.7}$$

Since $n(\cdot,0) = m(\cdot,0) = z(\cdot,0)$ and $n(\cdot,a_1(T)) = m(\cdot,1) = z(\cdot,T)$, from (3.7) we have

$$\|n(\cdot,\theta)\|_p \leqslant C_1 \|z(\cdot,T)\|_p^{\frac{\theta}{a_1(T)}} \|z(\cdot,0)\|_p^{1-\frac{\theta}{a_1(T)}}, \ \forall \theta \in [0,a_1(T)]. \tag{3.8}$$

By choosing $\theta = a_1(T)\nu(t), t \in [0,T]$, we have $n(\cdot,\theta) = m\left(\cdot, \frac{\theta}{a_1(T)}\right) = m(\cdot,\nu(t)) = z(\cdot,t)$. From (3.8), we obtain

$$\|z(\cdot,t)\|_p \leqslant C_1 \|z(\cdot,T)\|_p^{\nu(t)} \|z(\cdot,0)\|_p^{1-\nu(t)}, \ \forall \theta \in [0,T].$$

Therefore, we conclude that there exists a positive constant C_1 such that

$$\|u(\cdot,t) - v(\cdot,t,g)\|_p \leqslant C_1 \|u(\cdot,T) - v(\cdot,T,g)\|_p^{\nu(t)} \|u(\cdot,0) - g\|_p^{1-\nu(t)}, \ \forall t \in [0,T]. \tag{3.9}$$

On the other hand, we have

$$\begin{aligned}
\|v(\cdot,T,g) - \varphi\|_p^p &\leqslant \|v(\cdot,T,g) - \varphi\|_p^p + \left(\frac{\varepsilon}{E}\right)^p \|g\|_p^p \\
&\leqslant \|u(\cdot,T) - \varphi\|_p^p + \left(\frac{\varepsilon}{E}\right)^p \|u(\cdot,0)\|_p^p \\
&\leqslant \varepsilon^p + \left(\frac{\varepsilon}{E}\right)^p E^p = 2\varepsilon^p.
\end{aligned}$$

Therefore, we have $\|v(\cdot,T,g) - \varphi\|_p \leqslant 2^{\frac{1}{p}}\varepsilon$. This implies that

$$\|u(\cdot,T) - v(\cdot,T,g)\|_p \leqslant \|u(\cdot,T) - \varphi\|_p + \|v(\cdot,T,g) - \varphi\|_p \leqslant \varepsilon + 2^{\frac{1}{p}}\varepsilon \leqslant (1 + 2^{\frac{1}{p}})\varepsilon. \tag{3.10}$$

Furthermore, we have

$$\left(\frac{\varepsilon}{E}\right)^p \|g\|_p^p \leqslant \|v(\cdot,T,g)-\varphi\|_p^p + \left(\frac{\varepsilon}{E}\right)^2 \|g\|_p^p$$
$$\leqslant \|u(\cdot,T)-\varphi\|_p^p + \left(\frac{\varepsilon}{E}\right)^p \|u(0)\|_p^p$$
$$\leqslant 2\varepsilon^p.$$

Therefore, $\|g\|_p \leqslant 2^{\frac{1}{p}} E$. This implies that

$$\|u(\cdot,0)-g\|_p \leqslant \|u(0)\|_p + \|g\|_p \leqslant (1+2^{\frac{1}{p}})E. \tag{3.11}$$

From (3.9)–(3.11), we claim that there exists a constant $\widetilde{C} > 0$ such that

$$\|u(\cdot,t)-v(\cdot,t,g)\|_p \leqslant \widetilde{C}\varepsilon^{\nu(t)}E^{1-\nu(t)}, \ \forall t \in [0,T].$$

The theorem is proved. □

3.2. Error estimate under an a posteriori parameter choice rule

Lemma 1. *Suppose that $0 < \varepsilon < \|\varphi\|_p$. Let $\rho(\alpha) = \|v(\cdot,T,g_\alpha^*)-\varphi\|_p, \ \alpha > 0$ where g_α^* is the solution of problem (3.1)-(3.2). Then*

a) $\rho(\alpha)$ is a continuous function on $(0,+\infty)$;

b) $\lim_{\alpha\to 0^+} \rho(\alpha) \leqslant \varepsilon$;

c) $\lim_{\alpha\to+\infty} \rho(\alpha) = \|\varphi\|_p$.

Proof. a) We first prove that:

$$\text{If } 0 < \alpha_1 \le \alpha_2 \text{ then } \begin{cases} J_{\alpha_1}(g_{\alpha_1}^*) \le J_{\alpha_2}(g_{\alpha_2}^*) \\ \|g_{\alpha_1}^*\|_p \ge \|g_{\alpha_2}^*\|_p, \end{cases} \tag{3.12}$$

where $g_{\alpha_i}^*$ is the minimum of the functional $J_{\alpha_i}, i=1,2$ (see (3.1) for the definition of the function J_α).

Indeed, if $0 < \alpha_1 \le \alpha_2$, then

$$J_{\alpha_1}(g_{\alpha_1}^*) \le J_{\alpha_1}(g_{\alpha_2}^*)$$
$$= J_{\alpha_2}(g_{\alpha_2}^*) + (\alpha_1-\alpha_2)\|g_{\alpha_2}^*\|_p^p$$
$$\le J_{\alpha_2}(g_{\alpha_2}^*).$$

Furthermore, we have

$$J_{\alpha_1}(g_{\alpha_1}^*) \le J_{\alpha_1}(g_{\alpha_2}^*)$$
$$= J_{\alpha_2}(g_{\alpha_2}^*) + (\alpha_1-\alpha_2)\|g_{\alpha_2}^*\|_p^p$$
$$\le J_{\alpha_2}(g_{\alpha_1}^*) + (\alpha_1-\alpha_2)\|g_{\alpha_2}^*\|_p^p,$$

which implies

$$\alpha_1\|g_{\alpha_1}^*\|_p^p \le \alpha_2\|g_{\alpha_1}^*\|_p^p + (\alpha_1-\alpha_2)\|g_{\alpha_2}^*\|_p^p \quad \Leftrightarrow \quad (\alpha_1-\alpha_2)\left(\|g_{\alpha_2}^*\|_p^p - \|g_{\alpha_1}^*\|_p^p\right) \ge 0. \tag{3.13}$$

If $0 < \alpha_1 = \alpha_2$ then $g_{\alpha_1}^* = g_{\alpha_2}^*$ (since the minimizer of the functional J_α is unique). This implies that $\|g_{\alpha_1}^*\|_p = \|g_{\alpha_2}^*\|_p$. If $0 < \alpha_1 < \alpha_2$ then from (3.13), we conclude that $\|g_{\alpha_1}^*\|_p \le \|g_{\alpha_2}^*\|_p$.

Let $\alpha_0 > 0$ be an arbitrary real number. Next, we will prove that

$$\lim_{\alpha\to\alpha_0^+} \rho(\alpha) = \rho(\alpha_0). \tag{3.14}$$

Take any sequence $\{\alpha_n\}_{n=1}^\infty$ satisfying $\alpha_n > \alpha_0, \ \forall n \in \mathbb{N}^*$ and $\lim_{n\to\infty} \alpha_n = \alpha_0$. Let $h(x) = 0, \ \forall x \in \mathbb{R}$. Then $h \in L_p(\mathbb{R})$ and $v(\cdot,T,h) \equiv 0$. We have

$$\|v(\cdot,T,g_{\alpha_n}^*)-\varphi(\cdot)\|_p^p + \alpha_0\|g_{\alpha_n}^*\|_p^p \le \|v(\cdot,T,g_{\alpha_n}^*)-\varphi(\cdot)\|_p^p + \alpha_n\|g_{\alpha_n}^*\|_p^p$$
$$= J_{\alpha_n}(g_{\alpha_n}^*) = \min_{g\in L_p(\mathbb{R})} J_{\alpha_n}(g)$$
$$\le J_{\alpha_n}(h)$$
$$= \|v(\cdot,T,h)-\varphi\|_p^p + \alpha_n\|h\|_p^p$$
$$= \|\varphi\|_p^p.$$

Hence, $\|g^*_{\alpha_n}\|_p$, $\|v(.,T,g^*_{\alpha_n})-\varphi(\cdot)\|_p$ are uniformly bounded by $\|\varphi\|_p/{\alpha_0}^{1/p}$, $\|\varphi\|_p$, respectively, and there exists a subsequence α_{n_k} such that $g^*_{\alpha_{n_k}}$ converges weakly to some $g^*\in L_p(\mathbb{R})$. Since $L_p(\mathbb{R})$ is a reflexive Banach space and $g\mapsto v(.,T,g)$ is a continuous linear operator, it implies that $v(.,T,g^*_{\alpha_{n_k}})-\varphi$ also converges weakly to $v(.,T,g^*)-\varphi$. By the weak lower semicontinuity of the norm, we have

$$J_{\alpha_0}(g^*)\le \liminf_{k\to\infty} J_{\alpha_{n_k}}(g^*_{\alpha_{n_k}})\le \limsup_{k\to\infty} J_{\alpha_{n_k}}(g^*_{\alpha_{n_k}})$$
$$\le \limsup_{k\to\infty} J_{\alpha_{n_k}}(g^*_{\alpha_0}) = J_{\alpha_0}(g^*_{\alpha_0}).$$

Since the minimizer of $J_\alpha(g)$ is unique, we must have $g^*_{\alpha_{n_k}}=g^*$ and thus it also follows that

$$\lim_{k\to\infty} J_{\alpha_{n_k}}(g^*_{\alpha_{n_k}}) = J_{\alpha_0}(g^*_{\alpha_0}). \tag{3.15}$$

Since $\alpha_{n_k}>\alpha_0$ for all k we get by (3.12)

$$\|g^*_{\alpha_0}\|_p^p \le \liminf_{k\to\infty}\|g^*_{\alpha_{n_k}}\|_p^p \le \limsup_{k\to\infty}\|g^*_{\alpha_{n_k}}\|_p^p \le \|g^*_{\alpha_0}\|_p^p.$$

This implies that

$$\lim_{k\to\infty}\|g^*_{\alpha_{n_k}}\|_p^p = \|g^*_{\alpha_0}\|_p^p. \tag{3.16}$$

From (3.15) and (3.16), we obtain

$$\begin{aligned}\lim_{k\to\infty}\|v(\cdot,T,g^*_{\alpha_{n_k}})-\varphi(\cdot)\|_p^p &= \lim_{k\to\infty}\left(\|v(\cdot,T,g^*_{\alpha_{n_k}})-\varphi(\cdot)\|_p^p+\alpha_{n_k}\|g^*_{\alpha_{n_k}}\|_p^p-\alpha_{n_k}\|g^*_{\alpha_{n_k}}\|_p^p\right)\\
&= \lim_{k\to\infty}\left(J_{\alpha_{n_k}}(g^*_{\alpha_{n_k}})-\alpha_{n_k}\|g^*_{\alpha_{n_k}}\|_p^p\right)\\
&= \lim_{k\to\infty} J_{\alpha_{n_k}}(g^*_{\alpha_{n_k}}) - \lim_{k\to\infty}\alpha_{n_k}\|g^*_{\alpha_{n_k}}\|_p^p\\
&= J_{\alpha_0}(g^*_{\alpha_0})-\alpha_0\|g^*_{\alpha_0}\|_p^p\\
&= \|v(\cdot,T,g^*_{\alpha_0})-\varphi(\cdot)\|_p^p.\end{aligned} \tag{3.17}$$

It follows from (3.17) that $\lim_{k\to\infty}\rho(\alpha_{n_k})=\rho(\alpha_0)$.

Since $\lim_{n\to\infty}\alpha_n=\alpha_0$, it follows that $\lim_{k\to\infty}\alpha_{n_k}=\alpha_0$. Let $\delta>0$ be an arbitrarily small real number. From (3.15), (3.16) and $\lim_{k\to\infty}\alpha_{n_k}=\alpha_0$, we conclude that there exists a positive integer k_0 such that

$$0<\alpha_{n_{k_0}}-\alpha_0<\frac{\delta}{3}.\frac{\alpha_0}{\|\varphi\|_p^p}, \tag{3.18}$$

$$0\le \|g^*_{\alpha_0}\|_p^p-\|g^*_{\alpha_{n_{k_0}}}\|_p^p<\frac{\delta}{3\alpha_0}, \tag{3.19}$$

$$0\le J_{\alpha_{n_{k_0}}}(g^*_{\alpha_{n_{k_0}}})-J_{\alpha_0}(g^*_{\alpha_0})<\frac{\delta}{3}. \tag{3.20}$$

Since $\alpha_n>\alpha_0,\ \forall n\in\mathbb{N}^*$ and $\lim_{n\to\infty}\alpha_n=\alpha_0$, there exists a positive integer n_0 such that $\alpha_n\in(\alpha_0,\alpha_{n_{k_0}}),\ \forall n\ge n_0$. From $\alpha_0<\alpha_n<\alpha_{n_{k_0}},\ \forall n\ge n_0$ and (3.12), we obtain for all $n\ge n_0$

$$\begin{cases} J_{\alpha_0}(g^*_{\alpha_0})\le J_{\alpha_n}(g^*_{\alpha_n})\le J_{\alpha_{n_{k_0}}}(g^*_{\alpha_{n_{k_0}}})\\ \|g^*_{\alpha_0}\|_p\ge \|g^*_{\alpha_n}\|_p\ge \|g^*_{\alpha_{n_{k_0}}}\|.\end{cases} \tag{3.21}$$

From (3.18), (3.19), (3.20) and (3.21), we get for all $n\ge n_0$

$$0<\alpha_n-\alpha_0<\frac{\delta}{3}.\frac{\alpha_0}{\|\varphi\|_p^p}, \tag{3.22}$$

$$0\le \|g^*_{\alpha_0}\|_p^p-\|g^*_{\alpha_n}\|_p^p<\frac{\delta}{3\alpha_0}, \tag{3.23}$$

$$0\le J_{\alpha_n}(g^*_{\alpha_n})-J_{\alpha_0}(g^*_{\alpha_0})<\frac{\delta}{3}. \tag{3.24}$$

We have for all $n \geq n_0$

$$\begin{aligned}
&\left| \|v(\cdot, T, g^*_{\alpha_n}) - \varphi(\cdot)\|_p^p - \|v(\cdot, T, g^*_{\alpha_0}) - \varphi(\cdot)\|_p^p \right| \\
&= \left| J_{\alpha_n}(g^*_{\alpha_n}) - J_{\alpha_0}(g^*_{\alpha_0}) + \alpha_0(\|g^*_{\alpha_0}\|_p^p - \|g^*_{\alpha_n}\|_p^p) + (\alpha_0 - \alpha_n)\|g^*_{\alpha_n}\|_p^p \right| \\
&\leq \left| J_{\alpha_n}(g^*_{\alpha_n}) - J_{\alpha_0}(g^*_{\alpha_0}) \right| + \alpha_0 \left| \|g^*_{\alpha_0}\|_p^p - \|g^*_{\alpha_n}\|_p^p \right| + |\alpha_0 - \alpha_n| \|g^*_{\alpha_n}\|_p^p \\
&\leq \frac{\delta}{3} + \alpha_0 . \frac{\delta}{3\alpha_0} + \frac{\delta}{3} . \frac{\alpha_0}{\|\varphi\|_p^p} . \|g^*_{\alpha_n}\|_p^p \\
&\leq \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta.
\end{aligned} \tag{3.25}$$

This implies that $\lim\limits_{n\to\infty} \|v(\cdot, T, g^*_{\alpha_n}) - \varphi(\cdot)\|_p^p = \|v(\cdot, T, g^*_{\alpha_0}) - \varphi(\cdot)\|_p^p$ or $\lim\limits_{n\to\infty} \rho(\alpha_n) = \rho(\alpha_0)$. (3.14) has been proven.

The proof of

$$\lim_{\alpha\to\alpha_0^-} \rho(\alpha) = \rho(\alpha_0) \tag{3.26}$$

is carried out in the same way as above, noting that for any sequence $\{\alpha_n\}_{n=1}^{\infty}$ satisfying $0 < \alpha_n < \alpha_0, \ \forall n \in \mathbb{N}^*$ and $\lim\limits_{n\to\infty} \alpha_n = \alpha_0$, there is a positive real number γ such that $0 < \gamma < \alpha_n < \alpha_0$ for all $n \in \mathbb{N}^*$.

From (3.14) and (3.26), we obtain $\lim\limits_{\alpha\to\alpha_0} \rho(\alpha) = \rho(\alpha_0)$. Therefore, $\rho(\alpha)$ is a continuous function at α_0. Since α_0 is an arbitrary positive real number, we conclude that $\rho(\alpha)$ is a continuous function on $(0, +\infty)$.

The proof of part a) is complete.

b) Let $u(x,t)$ be a solution of problem (1.1) satisfying the condition (3.3). We have

$$\begin{aligned}
\rho^p(\alpha) &= \|v(\cdot, T, g^*_\alpha) - \varphi\|_p^p \\
&\leqslant \|v(\cdot, T, g^*_\alpha) - \varphi\|_p^p + \alpha\|g^*_\alpha\|_p^p \\
&= J_\alpha(g^*_\alpha) = \min_{g\in L_p(\mathbb{R})} J_\alpha(g) \\
&\leqslant J_\alpha(u(\cdot,0)) \\
&= \|v(\cdot, T, u(\cdot,0)) - \varphi\|_p^p + \alpha\|u(\cdot,0)\|_p^p \\
&= \|u(\cdot,T) - \varphi\|_p^p + \alpha\|u(\cdot,0)\|_p^p \\
&\leqslant \varepsilon^p + \alpha E^p.
\end{aligned}$$

This implies that

$$0 \leqslant \rho(\alpha) \leqslant \left(\varepsilon^p + \alpha E^p\right)^{\frac{1}{p}}. \tag{3.27}$$

Since $\lim\limits_{\alpha\to 0^+} \alpha E^p = 0$, from (3.27) we obtain $\lim\limits_{\alpha\to 0^+} \rho(\alpha) \leqslant \varepsilon$.

c) Let $h(x) = 0, \ \forall x \in \mathbb{R}$. Then $h \in L_p(\mathbb{R})$ and $v(\cdot, T, h) \equiv 0$. Therefore, we obtain

$$\begin{aligned}
\rho^p(\alpha) &= \|v(\cdot, T, g^*_\alpha) - \varphi\|_p^p \\
&\leqslant \|v(\cdot, T, g^*_\alpha) - \varphi\|_p^p + \alpha\|g^*_\alpha\|_p^p \\
&= J_\alpha(g^*_\alpha) = \min_{g\in L_p(\mathbb{R})} J_\alpha(g) \\
&\leqslant J_\alpha(h) \\
&= \|v(\cdot, T, h) - \varphi\|_p^p + \alpha\|h\|_p^p \\
&= \|0 - \varphi\|_p^p + \alpha\|0\|_p^p \\
&= \|\varphi\|_p^p.
\end{aligned}$$

This implies that

$$\rho(\alpha) \leqslant \|\varphi\|_p. \tag{3.28}$$

Further, we have

$$\rho(\alpha) = \|v(\cdot, T, g^*_\alpha) - \varphi\|_p \geqslant \|\varphi\|_p - \|v(\cdot, T, g^*_\alpha)\|_p. \tag{3.29}$$

From (3.28) and (3.29) we obtain

$$\|\varphi\|_p - \|v(\cdot,T,g^*_\alpha)\|_p \leqslant \rho(\alpha) \leqslant \|\varphi\|_p. \tag{3.30}$$

We have

$$\begin{aligned}\alpha\|g^*_\alpha\|_p^p &\leqslant \|v(\cdot,T,g^*_\alpha)-\varphi\|_p^p + \alpha\|g^*_\alpha\|_p^p\\ &= J_\alpha(g^*_\alpha) = \min_{g\in L_p(\mathbb{R})} J_\alpha(g)\\ &\leqslant J_\alpha(h)\\ &= \|v(\cdot,T,h)-\varphi\|_p^p + \alpha\|h\|_p^p\\ &= \|0-\varphi\|_p^p + \alpha\|0\|_p^p\\ &= \|\varphi\|_p^p.\end{aligned}$$

This implies that

$$0 \leqslant \|g^*_\alpha\|_p \leqslant \left(\frac{1}{\alpha}\right)^{\frac{1}{p}} \|\varphi\|_p. \tag{3.31}$$

From (3.31) and

$$\lim_{\alpha\to+\infty} \left(\frac{1}{\alpha}\right)^{\frac{1}{p}} \|\varphi\|_p = 0,$$

we obtain

$$\lim_{\alpha\to+\infty} \|g^*_\alpha\|_p = 0. \tag{3.32}$$

Let $A(t) = \int_0^t a(s)ds$, we can check that the solution of the problem

$$\begin{cases}\dfrac{\partial v}{\partial t} = a(t)\dfrac{\partial^2 v}{\partial x^2}, & (x,t)\in(-\infty;+\infty)\times(0;T)\\ v(x,0) = g^*_\alpha \in L_p(\mathbb{R})\end{cases}$$

is

$$v(x,t,g^*_\alpha) = \frac{1}{\sqrt{4\pi A(t)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4A(t)}} g^*_\alpha(y)dy,\ t\in(0,T].$$

Therefore, we have

$$v(x,T,g^*_\alpha) = \frac{1}{\sqrt{4\pi A(T)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4A(T)}} g^*_\alpha(y)dy.$$

Let $K(x) = \frac{1}{\sqrt{4\pi A(T)}} e^{-\frac{x^2}{4A(T)}}$. Then

$$v(x,T,g^*_\alpha) = \int_{-\infty}^{\infty} K(x-y)g^*_\alpha(y)dy.$$

Therefore, $v(\cdot,T,g^*_\alpha) = K * g^*_\alpha$, where $K * g^*_\alpha$ denotes the convolution of K with g^*_α. Since $K \in L_1(\mathbb{R})$ and $g^*_\alpha \in L_p(\mathbb{R})$, we have

$$\begin{aligned}\|v(\cdot,T,g^*_\alpha)\|_p &= \|K * g^*_\alpha\|_p\\ &\leqslant \|K\|_1\|g^*_\alpha\|_p.\end{aligned} \tag{3.33}$$

Furthermore,

$$\|K\|_1 = \frac{1}{\sqrt{4\pi A(T)}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4A(T)}} dx. \tag{3.34}$$

Let $y = \frac{x}{\sqrt{4A(T)}}$. We have

$$\begin{aligned}\|K\|_1 &= \frac{1}{\sqrt{4\pi A(T)}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4A(T)}} dx \\ &= \frac{1}{\sqrt{4\pi A(T)}} \int_{-\infty}^{\infty} e^{-y^2} \sqrt{4A(T)} dy \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy \\ &= 1. \end{aligned} \tag{3.35}$$

From (3.33) and (3.35) we obtain

$$0 \leqslant \|v(\cdot, T, g^*_\alpha)\|_p \leqslant \|g^*_\alpha\|_p. \tag{3.36}$$

From (3.32) and (3.36) we conclude that

$$\lim_{\alpha\to+\infty} \|v(\cdot, T, g^*_\alpha)\|_p = 0. \tag{3.37}$$

From (3.30) and (3.37) we obtain $\lim_{\alpha\to+\infty} \rho(\alpha) = \|\varphi\|_p$.

The lemma is proved. $\square$

Theorem 7. *Let $u(x,t)$ be a solution of problem (1.1) satisfying the condition (3.3). Suppose that $0 < \varepsilon < \|\varphi\|_p$ and $\tau > 1$ is chosen such that $0 < \tau\varepsilon < \|\varphi\|_p$. Then, there exists a number $\alpha_\varepsilon > 0$ such that*

$$\|v(\cdot, T, g^*_{\alpha_\varepsilon}) - \varphi\|_p = \tau\varepsilon, \tag{3.38}$$

where $g^*_{\alpha_\varepsilon}$ is the solution of problem (3.1)-(3.2) with $\alpha = \alpha_\varepsilon$. Furthermore, there exists a constant $C^* > 0$ such that

$$\|u(\cdot, t) - v(\cdot, t, g^*_{\alpha_\varepsilon})\|_p \leqslant C^* \varepsilon^{\nu(t)} E^{1-\nu(t)}, \ \forall t \in [0, T].$$

Proof. From Lemma 1 and $0 < \varepsilon < \tau\varepsilon < \|\varphi\|_p$ we conclude that there exists a number $\alpha_\varepsilon > 0$ satisfying (3.38). We have

$$\begin{aligned}\|u(\cdot, T) - v(\cdot, T, g^*_{\alpha_\varepsilon})\|_p &\leqslant \|u(\cdot, T) - \varphi\|_p + \|v(\cdot, T, g^*_{\alpha_\varepsilon}) - \varphi\|_p \\ &\leqslant (\tau + 1)\varepsilon. \end{aligned} \tag{3.39}$$

It is obvious that

$$\begin{aligned}\tau^p \varepsilon^p &= \|v(T, g^*_{\alpha_\varepsilon}) - \varphi\|_p^p \\ &\leqslant \|v_{\alpha_\varepsilon}(T, g^*_{\alpha_\varepsilon}) - \varphi\|^p + \alpha_\varepsilon \|g^*_{\alpha_\varepsilon}\|^p \\ &\leqslant \|u(T) - \varphi\|_p^p + \alpha_\varepsilon \|u(0)\|_p^p \\ &\leqslant \varepsilon^p + \alpha_\varepsilon E^p. \end{aligned}$$

This implies that $(\tau^p - 1)\varepsilon^p \leqslant \alpha_\varepsilon E^p$ or $\varepsilon^p \leqslant \frac{\alpha_\varepsilon E^p}{\tau^p - 1}$. On the other hand, we have

$$\begin{aligned}\alpha_\varepsilon \|g^*_{\alpha_\varepsilon}\|_p^p &\leqslant \|v(T, g^*_{\alpha_\varepsilon}) - \varphi\|_p^p + \alpha_\varepsilon \|g^*_{\alpha_\varepsilon}\|_p^p \\ &\leqslant \|u(T) - \varphi\|_p^p + \alpha_\varepsilon \|u(0)\|_p^p \\ &\leqslant \varepsilon^p + \alpha_\varepsilon E^p \\ &\leqslant \frac{\alpha_\varepsilon E^p}{\tau^p - 1} + \alpha_\varepsilon E^p. \end{aligned}$$

This implies that $\|g^*_{\alpha_\varepsilon}\|_p \leqslant \left(\frac{\tau^p}{\tau^p-1}\right)^{\frac{1}{p}} E$. Therefore,

$$\begin{aligned}\|u(0)-g^*_{\alpha_\varepsilon}\|_p &\leqslant \|u(0)\|_p + \|g^*_{\alpha_\varepsilon}\|_p \\ &\leqslant \left(1+\left(\frac{\tau^p}{\tau^p-1}\right)^{\frac{1}{p}}\right)E. \end{aligned} \tag{3.40}$$

Using the same arguments as in the proof of Theorem 6, we conclude that there exists a positive constant C_1 such that

$$\|u(\cdot,t)-v(\cdot,t,g^*_{\alpha_\varepsilon})\|_p \leqslant C_1\|u(\cdot,T)-v(\cdot,T,g^*_{\alpha_\varepsilon})\|_p^{\nu(t)}\|u(\cdot,0)-g^*_{\alpha_\varepsilon}\|_p^{1-\nu(t)},\ \forall t\in[0,T]. \tag{3.41}$$

From (3.39), (3.40) and (3.41), it follows that there exists a constant $C^*>0$ such that

$$\|u(\cdot,t)-v(\cdot,t,g^*_{\alpha_\varepsilon})\|_p \leqslant C^*\varepsilon^{\nu(t)}E^{1-\nu(t)},\ \forall t\in[0,T].$$

The theorem is proved. □

4. Numerical solution

4.1. Solution of the forward problem

We notice that the solution to the problem

$$\begin{cases}\dfrac{\partial v}{\partial t} = a(t)\dfrac{\partial^2 v}{\partial x^2}, & (x,t)\in(-\infty;+\infty)\times(0;T),\\ v(x,0)=g\in L_p(\mathbb{R}) \end{cases} \tag{4.1}$$

is

$$v(x,t)=\frac{1}{\sqrt{4\pi A(t)}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4A(t)}}g(y)dy, \tag{4.2}$$

where $A(t)=\int_0^t a(s)ds$.

4.2. Discretized problem and quasi-Newton algorithm

We approximate the Tikhonov-type regularization problem on a finite domain $\Omega=[-M,M]\times[0,T]$. We divide the intervals, $[-M,M]$ and $[0,T]$, into equally sub-intervals by the grid points $x_i, i=0,\dots,N$ and $t_j, j=0,\dots,m$, respectively. Then, for each function $g\in L^p(\mathbb{R})$ we denote $g_i=g(x_i), v_i=v(x_i,T), i=0,\dots,N$ ($v(x,t)$ is given by (4.2)). We use boldfaced letters to denote vectors, for examples $\mathbf{g}=[g_0\ g_1\ \cdots\ g_N]^T, \mathbf{v}(\mathbf{g})=[v_0\ v_1\ \cdots\ v_N]^T$. We also denote $\mathbf{g}^\alpha=[g_0^\alpha\ g_1^\alpha\ \cdots\ g_N^\alpha]^T$. Using the trapezoidal rule for approximating integrals, we have

$$\int_{-M}^{M} g(x)dx \approx \mathbf{w}^T\mathbf{g}, \mathbf{v}(\mathbf{g})=(K\circ W)\,\mathbf{g}:=H\mathbf{g}, \tag{4.3}$$

where $\mathbf{w}=\frac{M}{N}[1\ 2\ \cdots\ 2\ 1]^T$, W is the square matrix whose rows are equal to w^T (repeated $N+1$ times) and $K=(k_{ij})$ with $k_{ij}=\frac{1}{\sqrt{4\pi A(T)}}e^{-\frac{(x_i-x_j)^2}{4A(T)}}$, $i,j=0,1,\dots,N$. Here, $K\circ W$ is the Hadamard product (also known as the elementwise, entry-wise or Schur product) that takes two matrices of the same dimensions and produces another matrix of the same dimension as the operands, where each element i,j is the product of elements i,j of the original two matrices.

The Tikhonov-type regularization functional is approximated by

$$\begin{aligned} J_\alpha^N(\mathbf{g}) &:= \mathbf{w}^T\,|\mathbf{v}(\mathbf{g})-\boldsymbol{\varphi}|^p+\alpha\mathbf{w}^T|\mathbf{g}|^p \\ &= \mathbf{w}^T\,|H\mathbf{g}-\boldsymbol{\varphi}|^p+\alpha\mathbf{w}^T|\mathbf{g}|^p. \end{aligned} \tag{4.4}$$

For $p\in(1,\infty)$ the function $J_\alpha^N(\mathbf{g})$ is strongly convex, and it is differentiable with respect to g. By directly computing, the gradient of $J_\alpha^N(\mathbf{g})$ is given by

$$\nabla J_\alpha^N(\mathbf{g}) = H^T\left(\mathbf{w}\circ\operatorname{sign}(H\mathbf{g}-\boldsymbol{\varphi})\circ|H\mathbf{g}-\boldsymbol{\varphi}|^{p-1}\right)+\alpha\left(\mathbf{w}\circ\operatorname{sign}(\mathbf{g})\circ|\mathbf{g}|^{p-1}\right). \tag{4.5}$$

The discretized version of problem (3.1) is as follows:

$$\min_{\mathbf{g}\in\mathbb{R}^N} J_\alpha^N(\mathbf{g}). \tag{4.6}$$

There are many algorithms to find the minimizer of the function (4.4) such as the steepest gradient method, conjugate gradient methods, and quasi-Newton methods [4]. In this paper, we use the quasi-Newton method with BFGS update in [4]. This algorithm is described below.

Algorithm 4.1 Quasi-Newton method with BFGS's update.

Input: Initial guess g_0, $H_0 = I$ and $tol = 10^{-9}$
1: **while** $\|\nabla J_\alpha^N(\mathbf{g_n})\| > tol$ **do**
2: $d_n = -H_n \nabla J_\alpha^N(\mathbf{g_n})$
3: $g_{n+1} = g_n + d_n$
4: $y_n = \nabla J_\alpha^N(\mathbf{g}_{n+1}) - \nabla J_\alpha^N(\mathbf{g}_n)$
5: $H_{n+1} = \left(I - \frac{d_n y_n^T}{y_n^T d_n}\right) H_n \left(I - \frac{y_n d_n^T}{y_n^T d_n}\right) + \frac{d_n d_n^T}{y_n^T d_n}$
6: $n = n+1$
7: **end while**
Output: $g = g_n$

4.3. Numerical examples

In this section, we present some numerical examples to illustrate our theoretical results. Here, we only make a comparison between theoretical results in Theorem 6 and numerical results. First, we consider the following example.

Example 1. We assume that the coefficients of the forward problem (4.1) are given by $a(t) = 1 + t$, $g(x) = e^{-x^2}$, and the discretized domain is $\Omega = [-M, M] \times [0, T]$ with $M = 20$ and $T = 1$. The interval $[-M, M]$ is divided equally by 601 points. We choose p-norm with $p = 1.8$. The noiseless data u^T is computed approximately by (4.3), and a noisy data is generated by

$$u_{noise} = u^T + \varepsilon \frac{R}{\|R\|_p},$$

where $R = randn(size(u^T))$, where $randn()$ is a Mathlab function that generates a random vector.

Numerical solutions with noiseless data: With $\varepsilon = 0$ and $\alpha_1 = 10^{-9}$ the exact solution u and the recovered solution u^α are illustrated in Fig. 1. We see that the recovered solution is a good approximation to the exact solution at $t > 0$. However, at $t = 0$ the quality of reconstruction is reduced. By increasing the number of grid points, the approximate error is reduced, but the maximum value of recovered solution at $t = 0$ still has a violation and could not reach the maximum value of the exact solution.

We look closer at the solution at several values of t. Fig. 2 illustrates the numerical exact solution and the recovered solution at $t = 0, 0.24, 0.75$ and $t = 1$. We see that for $t > 0$ the recovered solution and the exact solution are almost identical, but at $t = 0$ the maximum value of the recovered solution is smaller than one and there is a violation around

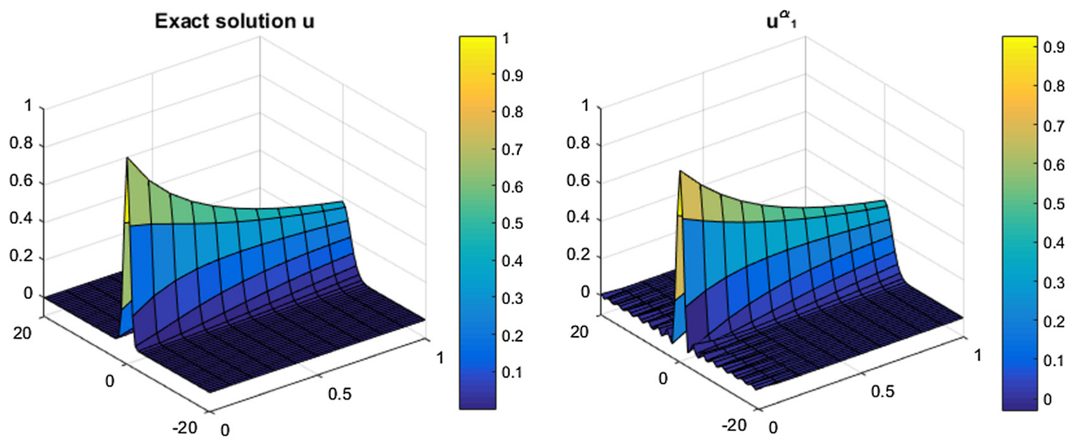

Fig. 1. Numerical exact solution u and the recovered solution u^α for noiseless data.

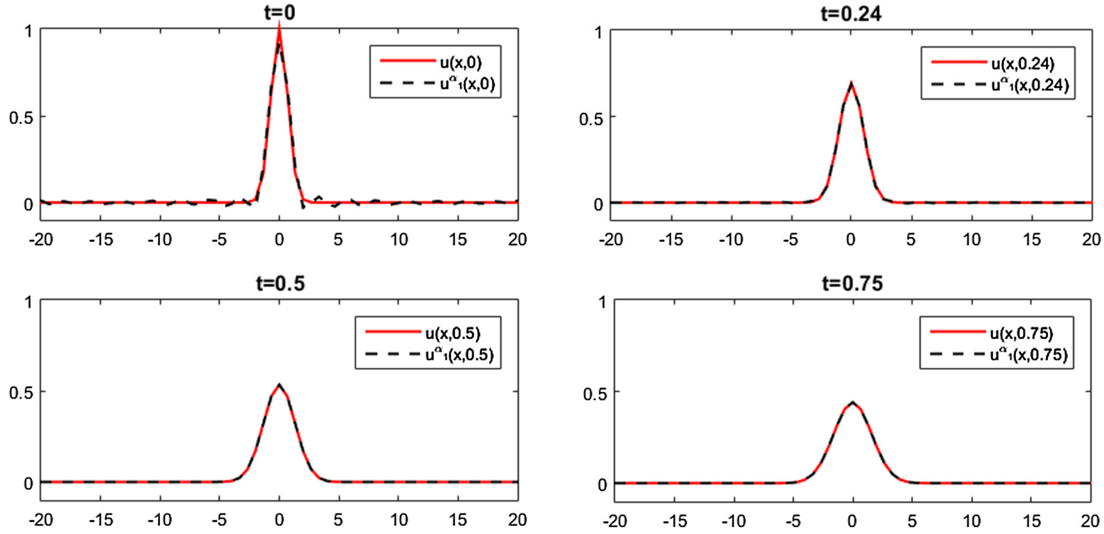

Fig. 2. Numerical exact solution u and the recovered solution u^{α_1} for noiseless data. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

the exact solution. This observation is suitable with the result in Theorem 6. Note that Theorem 6 does not consider the convergence rate at $t = 0$.

Numerical solutions with noisy data: We now illustrate the results in Theorem 6 and Theorem 7. First, we set $\varepsilon = 10^{-2}$ and denote by α_1 and α_2 the regularization parameters which are chosen by the a priori and a posteriori parameter choice rules. We have $\alpha_1 = (\varepsilon/E)^p$ as in Theorem 6. For α_2 it is computed approximately as follows: from Theorem 7 α_2 is a solution of the equation

$$m(\alpha_2) := \|v(\cdot, T, g^*_{\alpha_2}) - \varphi\|_p - \tau\varepsilon = 0. \tag{4.7}$$

Since $\lim_{\alpha\to 0^+} m(\alpha) \le \varepsilon - \tau\varepsilon < 0$, $\lim_{\alpha\to+\infty} m(\alpha) = \|\varphi\|_p - \tau\varepsilon > 0$, equation (4.7) has at least one solution in $(0, +\infty)$. Thus, we can use the bisection method to find such a solution. Combining the Tikhonov-type regularization and bisection method, we obtain the following algorithm.

Algorithm 4.2 Tikhonov-type regularization with a posteriori parameter choice rule.

Input: Initials: $k = 0$, $\tau \in \left(1, \frac{\|\varphi\|_p}{\varepsilon}\right)$, $tol = 10^{-4}$ and $a = 0, b = 10^{-1}$.

```
α = b; g_α = argmin_g J_α^N(g); tg = m(α)
while tg < 0 do
    a = b; b = 2b
    α = b; g_α = argmin_g J_α^N(g); tg = m(α)
end while
α = (a + b)/2; g_α = argmin_g J_α^N(g); tg = m(α)
while abs(tg) > tol do
    if tg > 0 then
        b = α
    else
        a = α
    end if
    α = (a + b)/2; g_α = argmin_g J_α^N(g); tg = m(α)
end while
```

Output: $\alpha_2 = \alpha$; $\mathbf{g}_{\alpha_2} = \mathbf{g}_\alpha$.

With $\varepsilon = 10^{-2}$ the numerical exact solution u and recovered solutions, $u^{\alpha_1}, u^{\alpha_2}$, are illustrated in Fig. 3. The recovered solutions u^{α_1} and u^{α_2} at some values of t are illustrated in Fig. 4. At $t > 0$ they are good approximations to the exact solution, but at $t = 0$ the recovered solutions have larger error near the maximum of the exact solution.

The L_p errors between the regularized solutions and the numerical exact solution with respect to t are presented in Fig. 5. We observe that on one hand for each noise level the errors increase when t decreases to zero. On the other hand they are reduced when noise levels are getting smaller. This is suitable with the result in Theorems 6 and 7. It also shows that the constant $\widetilde{C} > 0$ in Theorem 6 should be close to one.

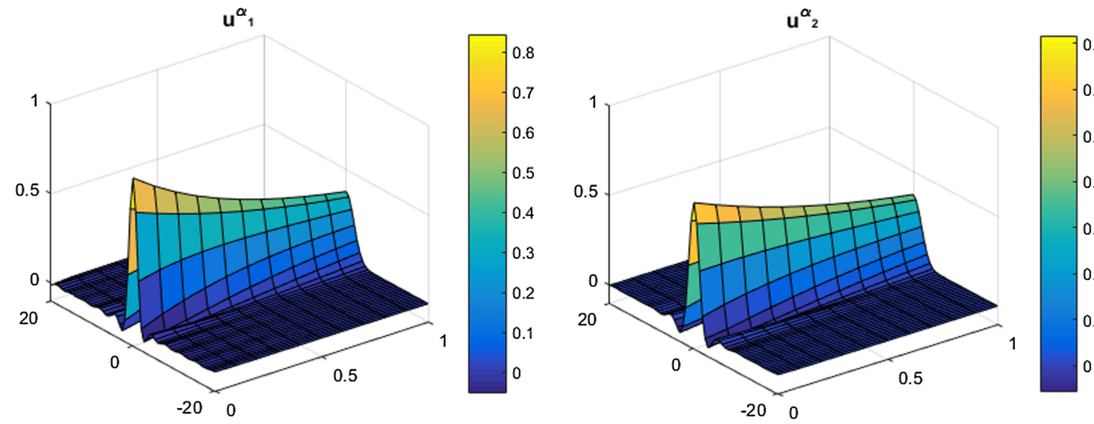

Fig. 3. Recovered solutions u^{α_1} and u^{α_2} in Example 1 with noise $\varepsilon = 10^{-2}$.

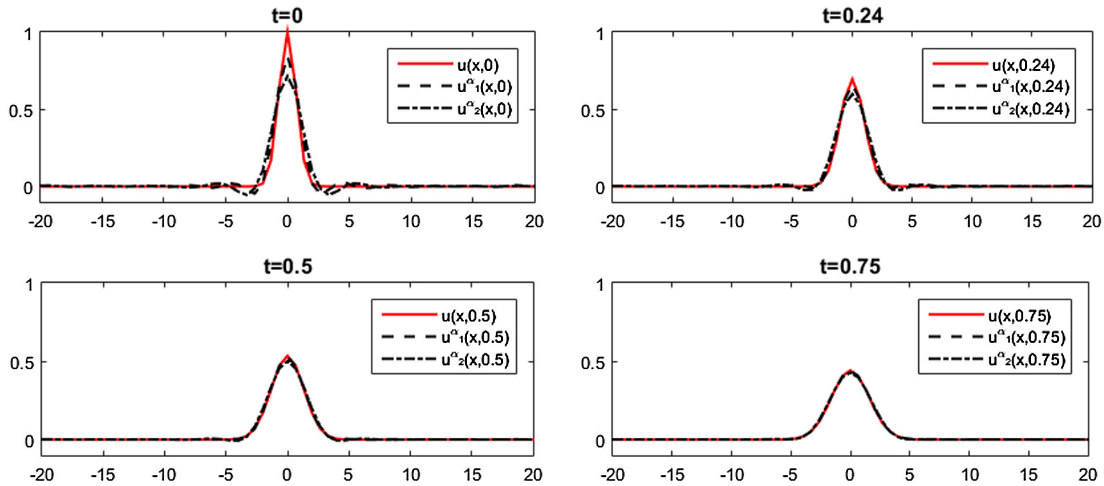

Fig. 4. Numerical exact solution u and the recovered solutions u^{α_1} and u^{α_2} in Example 1 with noise $\varepsilon = 10^{-2}$.

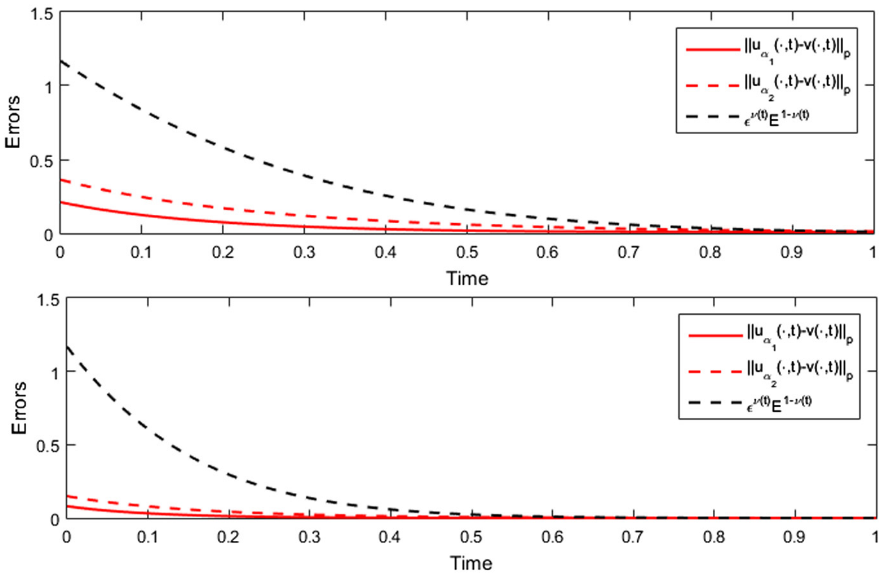

Fig. 5. The error $\|u^{\alpha_1}(\cdot,t) - v(\cdot,t)\|_p$ and $\|u^{\alpha_2}(\cdot,t) - v(\cdot,t)\|_p$ with respect to $\varepsilon = 10^{-2}$ (above) and $\varepsilon = 10^{-4}$ (below) in Example 1.

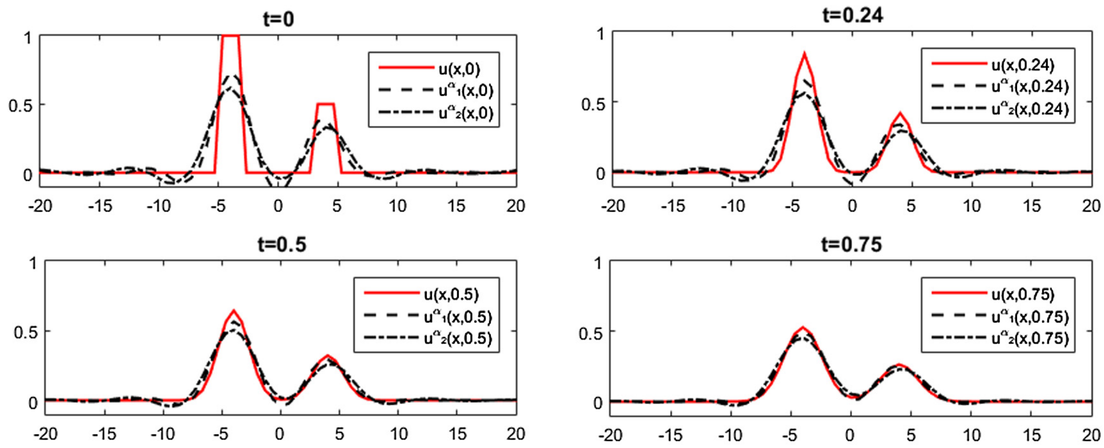

Fig. 6. Numerical exact solution u and the recovered solutions u^{α_1} and u^{α_2} in Example 2 with respect to $\varepsilon = 10^{-2}$.

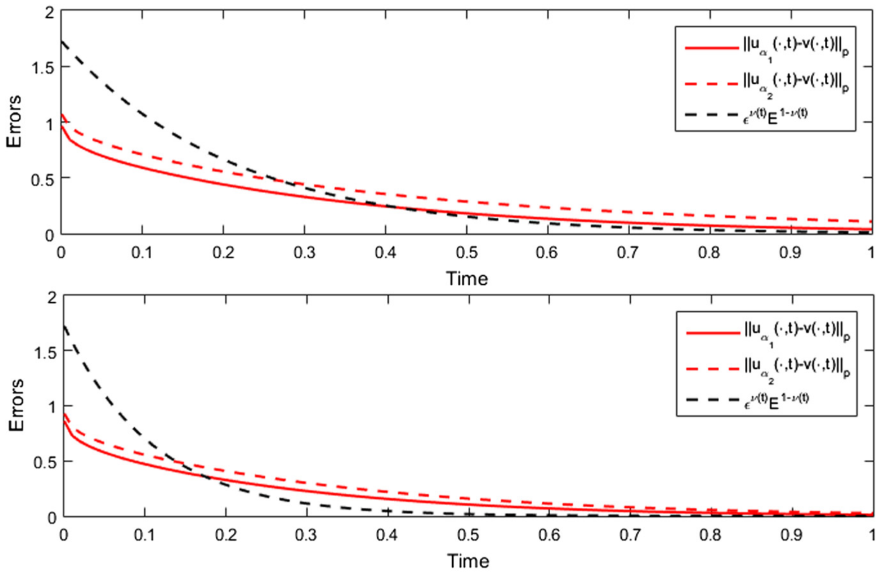

Fig. 7. The error $\|u^{\alpha_1}(\cdot,t) - v(\cdot,t)\|_p$ and $\|u^{\alpha_2}(\cdot,t) - v(\cdot,t)\|_p$ with respect to $\varepsilon = 10^{-2}$ (above) and $\varepsilon = 10^{-4}$ (below) in Example 2.

Example 2. We assume that the coefficients of the forward problem (4.1) are given by

$$a(t) = 4T - t^2, \; g(x) = \begin{cases} 1, & \text{if } x \in [-5,-3], \\ 0.5, & \text{if } x \in [3,5], \\ 0, & \text{otherwise.} \end{cases}$$

In this example, the initial condition g is not smooth. We make the same setting as in Example 1. To generate noise data, we set $\varepsilon = 10^{-2}$. We denote by α_1 and α_2 the regularization parameters which are chosen by the a priori parameter choice rule and a posteriori parameter choice rule, respectively. Fig. 6 illustrates the solutions at $t = 0, 0.24, 0.5$ and 0.75. Unlike Example 1, it is harder to obtain good approximations in this example (the case of nonsmooth initial condition). Here, the recovered solutions are good approximations to the exact solution for large values of t, but they are getting worse approximations for small values of t. Furthermore, in neighborhoods of discontinuous points, the qualities of approximation are bad. These situations are evident, and they are similar in numerical computations, i.e., if the solution is smoother, then the recovered solutions are better.

We now consider the L_p errors between the exact solution and the recovered solutions in L^p-norm, which are given in Fig. 7. The figure shows that the values $\varepsilon^{\nu(t)}E^{1-\nu(t)}$ are below the error values for t larger, which is different from Example 1. This implies that the constants in Theorems 6 and 7 must be larger than one.

Before closing this section, we want to show that the error values $\varepsilon^{\nu(t)}E^{1-\nu(t)}$ in this example are larger than those in Example 1. This is pointed out in Remark 1.

5. Conclusion

In this paper, we investigated the Cauchy problem for the heat equation backward in time with time-dependent coefficient $a(t)$ from the final data. We proved some results on stability estimates, which are given in Theorems 1 and 2. To recover the initial condition, we used the Tikhonov-type regularization that leads to a smooth and convex optimization problem. The wellposedness of the regularized problem proved. Then, we proposed two methods for choosing the regularized parameter, a priori parameter choice rule and a posteriori parameter choice rule. With these rules, the convergent rates of the recovered solutions to the exact solution were obtained in Theorems 6 and 7 for $t > 0$, respectively. Note that the convergence rate at $t = 0$, i.e., the convergence rate of the recovered initial condition to the exact one is still open. Finally, two numerical examples were shown to illustrate the performance of our approach and the theoretical results.

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References

[1] Karen A. Ames, Rhonda J. Hughes, Structural Stability for Ill-Posed Problems in Banach Space, Semigroup Forum, vol. 70, Springer, 2005, pp. 127–145.
[2] De-Han Chen, Bernd Hofmann, Jun Zou, Regularization and convergence for ill-posed backward evolution equations in Banach spaces, J. Differ. Equ. 265 (8) (2018) 3533–3566.
[3] Nguyen Van Duc, Nguyen Van Thang, Luong Duy Nhat Minh, Nguyen Trung Thành, Identifying an unknown source term of a parabolic equation in Banach spaces, Appl. Anal. (2020), https://doi.org/10.1080/00036811.2020.1800650.
[4] Roger Fletcher, Practical Methods of Optimization, John Wiley & Sons, 2013.
[5] Matthew A. Fury, Nonautonomous ill-posed evolution problems with strongly elliptic differential operators, Electron. J. Differ. Equ. 2013 (92) (2013) 1.
[6] Matthew A. Fury, Logarithmic well-posed approximation of the backward heat equation in Banach space, J. Math. Anal. Appl. 475 (2) (2019) 1367–1384.
[7] Brian H. Gilding, Alberto Tesei, The Riemann problem for a forward–backward parabolic equation, Phys. D, Nonlinear Phenom. 239 (6) (2010) 291–311.
[8] Hans-Christoph Grunau, Nobuhito Miyake, Shinya Okabe, Positivity of solutions to the Cauchy problem for linear and semilinear biharmonic heat equations, Adv. Nonlinear Anal. 10 (1) (2021) 353–370.
[9] Dinh Nho Hào, A mollification method for ill-posed problems, Numer. Math. 68 (1994) 469–506.
[10] Dinh Nho Hào, Nguyen Van Duc, Stability results for the heat equation backward in time, J. Math. Anal. Appl. 353 (2) (2009) 627–641.
[11] Dinh Nho Hào, Nguyen Van Duc, Regularization of backward parabolic equations in Banach spaces, J. Inverse Ill-Posed Probl. 20 (5–6) (2012) 745–763.
[12] Sergei Mihailovic Nikol'skii, Approximation of Functions of Several Variables and Imbedding Theorems, vol. 205, Springer Science & Business Media, 2012.
[13] Flavia Smarrazzo, Alberto Tesei, Degenerate regularization of forward–backward parabolic equations: the regularized problem, Arch. Ration. Mech. Anal. 204 (1) (2012) 85–139.
[14] Flavia Smarrazzo, Alberto Tesei, Degenerate regularization of forward–backward parabolic equations: the vanishing viscosity limit, Math. Ann. 355 (2) (2013) 551–584.
[15] Bui Le Trong Thanh, Flavia Smarrazzo, Alberto Tesei, Sobolev regularization of a class of forward–backward parabolic equations, J. Differ. Equ. 257 (5) (2014) 1403–1456.
[16] Nguyen Van Duc, An a posteriori mollification method for the heat equation backward in time, J. Inverse Ill-Posed Probl. 25 (4) (2017) 403–422.
[17] Xingchang Wang, Runzhang Xu, Global existence and finite time blowup for a nonlocal semilinear pseudo-parabolic equation, Adv. Nonlinear Anal. 10 (1) (2021) 261–288.
[18] Atsushi Yagi, Abstract Parabolic Evolution Equations and Their Applications, Springer Science & Business Media, 2009.