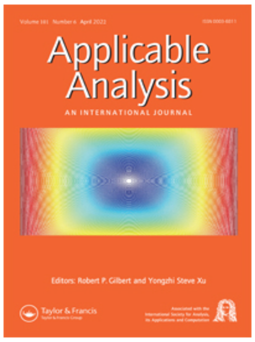

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Identifying an unknown source term of a parabolic equation in Banach spaces

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ABSTRACT
A new regularization method for an inverse source problem for a parabolic equation in a Banach space is proposed. Hölder-type error estimates for the regularized solutions are proved for both *a priori* and *a posteriori* regularization parameter choice rules. Some numerical examples are presented for illustrating the efficiency of the method.

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1. Introduction

Let T be a positive real number and X be a Banach space with norm $\|\cdot\|$. Let $A : D(A) \subset X \to X$ be a densely-defined linear operator in X such that $-A$ generates an analytic semigroup $\{S(t)\}_{t\geq 0}$ on X. By $D(A), \sigma(A), \rho(A)$ and $R(\lambda; A), \lambda \in \rho(A)$, we denote the domain, the spectrum set, the resolvent set, and the resolvent of operator A, respectively. We assume that the following condition holds:

$$\|S(T)\| < 1. \tag{1}$$

Let $u : [0, T] \to X$ be a function from $[0, T]$ to X and F be an element in X. We consider the inverse problem of determining the element F in the following problem

$$\begin{aligned} &u'(t) + Au(t) = F, \quad t \in (0, T),\\ &u(0) = 0,\\ &u(T) = g, \end{aligned} \tag{2}$$

with $g \in X$ being given. The first two equations in (2) form a Cauchy problem, which is considered as the forward problem in which u is to be determined with F being given. A solution $u(t)$ of the forward problem is assumed to be in $C^1([0, T]; X)$ and $u(t) \in D(A)$ for $t \in [0, T]$.

The last equation in (2) is considered as a measured datum in the inverse source problem. In practical situations, measurements usually contain error. Therefore, we assume that only a noisy datum

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g^δ is given, which is merely in X and satisfies

$$\|g - g^\delta\| \le \delta, \tag{3}$$

where $\delta > 0$ represents the noise level.

Throughout this paper, we assume that there exists a positive constant $E > \delta$ such that the following condition holds:

$$\|(I + A^p)F\| \le E, \tag{4}$$

where p is some positive real number such that $p \ge 1$ and A^p is a fractional power of A as defined in Definition 2.3. If A is a differential operator, as in the case when (2) is a parabolic equation, condition (4) means that the source term F needs to be regular enough as a function in X, see Example 2.6.

We note that the inverse source problem arises in several practical applications. Examples include the problem of determining heat sources in a heat transfer process and the problem of determining sources of pollution in air or water [1–5]. In [6], the authors considered an application of inverse source problems in solving a coefficient inverse problem.

There is an extensive literature on inverse source problems for parabolic equations in Hilbert spaces, see, e.g. [7–13] and the references therein. For an up-to-date list of inverse source problems for parabolic equations and corresponding references, we refer the reader to Hào et al. [8]. Some of the earliest works on inverse source problems for parabolic equations in Banach spaces were due to Iskenderov and Tagiev [14] and Rundell [15]. For other works on this topic, see [16–24]. The uniqueness of the inverse problem (2) has been proved under the assumption that $g \in D(A)$ and no number of the form $2\pi ik/T$, with integers $k \neq 0$, is an eigenvalue of A, see [17, 23]. The uniqueness has also been proved for a more general equation in which F is a function of time, see [15, 24]. For the case with noisy data g^δ, which is generally not in $D(A)$, problem (2) is ill-posed. Therefore, to solve it, regularization methods need to be applied. Although the theory on regularization of ill-posed linear equations in Banach spaces have been discussed by various authors, see, e.g. [25] and the references therein, results on the particular case of inverse source problems for parabolic equations are limited. In [19], the authors proposed a regularization method based on the conversion of the inverse source problem into a Fredholm equation of second kind and then approximate the original problem in an infinite dimensional space by a problem in a finite dimensional space. However, the convergence rate was not investigated and numerical implementation was not discussed.

In this work, we propose a regularization method, which approximates (2) by the following problem

$$\begin{aligned} &v'(t) + Av(t) = F_\alpha, \quad t \in (0, T),\\ &v(0) = 0,\\ &v(T) = (I + \alpha A^b)^{-1} g^\delta, \end{aligned} \tag{5}$$

where $\alpha > 0$ is a regularization parameter, b is a positive integer, and I is the identity operator in X. We will see that $(I + \alpha A^b)^{-1}$ is a bounded operator on X (Lemma 4.5) and problem (5) is well-posed (Remark 4.1). We obtain error estimates between F_α and F of Hölder type for both *a priori* and *a posteriori* choice rules for the regularization parameter α.

In the Hilbert space setting, a similar regularization method with $b = 1$ was proposed in [7] in which the authors derived a Hölder-type error estimate for an *a priori* parameter choice rule. The proof of the result in [7] was based on the eigenfunction expansion method in Hilbert spaces. In this paper, we obtain Hölder-type error estimates for the regularized solutions using both *a priori* and *a posteriori* parameter choice rules (Theorems 3.1 and 3.2) with a higher rate of convergence than that in [7] when $b > 1$. We note that our results are proven in Banach spaces where the techniques used in [7] are no more applicable.

The remainder of this work is arranged as follows. In Section 2 we present some auxiliary results which are needed in our analysis. In Section 3, we state error estimates for the regularized solutions

– the main results of this paper. The proofs of these results are presented in Section 4. In Section 5, we present some numerical examples for showing the efficiency of our method. Some concluding remarks are provided in Section 6. In the Appendix, we prove a result concerning eigenvalues of fractional powers of a linear operator in a Banach space.

2. Auxiliary results

In this section, we state some auxiliary results that we need to use in Section 3 for proving error estimates. For more details, see the cited references.

Definition 2.1 ([26], p. 93): We call a (possibly unbounded) operator A a generator if A generate a uniformly bounded strongly continuous holomorphic semigroup $\{e^{-zA}\}_{\mathrm{Re}\, z\geqslant 0}$. By switching to equivalent norm $|\|x\||=\sup_{\mathrm{Re}\, z\geqslant 0}\|e^{-zA}\|$, if necessary, we may assume that $\|e^{-zA}\|\leq 1$, for all $\mathrm{Re}\, z\geqslant 0$. For $s\geqslant 0$, define

$$G(s,A):=\int_{\mathbb{R}}\frac{1-\cos(sr)}{r^2}e^{irA}\frac{dr}{\pi}. \tag{6}$$

Remark 2.1 (See Proposition 10 in [26]): The following inequality holds

$$\|G(s,A)\|\leq s, \quad \forall s\geqslant 0.$$

Definition 2.2 ([26]): Let $AC^1[0,1]$ be the set defined by $AC^1[0,1]:=\{h:[0,1]\to\mathbb{R}|h'\text{is absolutely continuous on }[0,1]\}$ and $AC^1[0,\infty):=\{h\circ g|h\in AC^1[0,1]\}$, where $g(t)=(1+t)^{-1},\ t\in[0,\infty)$. For $f\in AC^1[0,\infty)$, we define the functional calculus

$$f(A):=\left(\lim_{t\to\infty}f(t)\right)I+\int_0^\infty f''(s)G(s,A)\,ds. \tag{7}$$

In (7), f'' is the pointwise second derivative of f. We remark that since f' is absolutely continuous on $[0,\infty)$, f'' exists almost everywhere.

Definition 2.3 ([27], p. 69): Let A be a densely-defined closed linear operator which satisfies Definition 2.1 such that $\rho(A)\supset\Sigma^+:=\{\lambda:0<\omega<|arg\lambda|\leq\pi\}\cup V$, where ω is a given positive real number and V is a neighborhood of zero in the complex plane $\mathbb{C}$. For $b>0$, the power A^{-b} of A is defined by:

$$A^{-b}:=\frac{1}{2\pi i}\int_C z^{-b}(A-Iz)^{-1}\,dz, \tag{8}$$

where C is a path running in the resolvent set of A from ∞e^{-iv} to ∞e^{iv}, with $\omega<v<\pi$. We also define $A^b:=(A^{-b})^{-1}$ and $A^0=I$.

Lemma 2.4 ([28], p. 105–106 and p. 158–160): *Assume that A is an operator as in Definition 2.3 and $\{S(t)\}_{t\geq 0}$ is the semigroup generated by $-A$. Then,*

(i) *For given numbers α,β, $0<\alpha<\beta<1$, there exists a constant $M_1=M_1(\alpha,\beta)\geqslant 1$ such that*

$$\|A^\alpha x\|\leq M_1\|A^\beta x\|, \quad x\in D(A^\beta). \tag{9}$$

(ii) *For any $\beta\geqslant 0$, there exits a constant $M_\beta\geqslant 1$ such that*

$$\|A^\beta S(t)\|=\|A^\beta e^{-tA}\|\leq M_\beta t^{-\beta}, \quad t>0. \tag{10}$$

(iii) *For* $\alpha < \beta < \gamma$, *there exists a constant* $C = C(\alpha, \beta, \gamma) > 0$ *such that*

$$\|A^{\beta}v\| \leq C(\alpha, \beta, \gamma)\|A^{\gamma}v\|^{((\beta-\alpha)/(\gamma-\alpha))}\|A^{\alpha}v\|^{((\gamma-\beta)/(\gamma-\alpha))}. \tag{11}$$

Lemma 2.5 (See [29], p. 42): *Assume that A is an operator as in Definition 2.3. The following properties hold*:

(a) $A^{-b} \in B(X)$ *and is injective for* $b > 0$, *where* $B(X)$ *is the space of bounded linear operators in X.*
(b) A^b *is a closed operator and* $D(A^b) \subset D(A^d)$ *for* $b > d > 0$.
(c) $A^b x = A^{(b-n)}A^n x$ *for* $x \in D(A^n)$ *and* $n > b, n \in \mathbb{N}$.

Example 2.6: To illustrate hypothesis (4), we consider the case when X is the Hilbert space $L^2[0, \pi]$ with norm $\|\cdot\|_2$ and inner product $\langle,\rangle$, $A = -(\partial^2/\partial x^2)$ with Dirichlet boundary conditions at $x = 0$ and $x = \pi$, then with $p = 1$, (4) is equivalent to the condition

$$F(0) = F(\pi) = 0, \quad \|F - F''\|_2 \leq E.$$

It follows from part (c) of Lemma 2.5 that $A^2 = A(A) = (\partial^4/\partial x^4)$. Therefore, with $p = 2$, (4) becomes

$$F(0) = F(\pi) = 0, \quad \|F + F^{(4)}\|_2 \leq E.$$

3. Error estimates for regularized solutions

In this section, we derive error estimates for the regularized solutions for both *a priori* and *a posteriori* parameter choice rules. Our main results are stated in Theorems 3.1 and 3.2. Since their proofs are quite technical, we present them in Section 4 to enable the reader to understand the results without being distracted by technical details. At the end of this section, we discuss our results in comparison with those in the literature.

First, we represent the solutions of the inverse source problem (2) and the regularized problem (5) via the semigroup $S(t)$. Since $u(0) = 0$, the solution $u(t)$ of (2) is given by

$$u(t) = S(t)u(0) + \int_0^t S(s)F\,\mathrm{d}s = \int_0^t S(s)F\,\mathrm{d}s.$$

Hence,

$$Au(T) = A\int_0^T S(s)F\,\mathrm{d}s = (S(T) - I)F.$$

For the last equality, see [27, p. 5]. Since $u(T) = g$, we have

$$F = (S(T) - I)^{-1}Ag. \tag{12}$$

In (12), the operator $(S(T) - I)^{-1}$ is bounded. The ill-posedness of (12) is due to the unboundedness of the operator A. Indeed, if g is replaced by the noisy data g^{δ}, Ag^{δ} may not exist or if it exists, a small error in g^{δ} will be amplified by A. We refer the reader to [30] for an extensive discussion on regularization methods for general linear operator equations in Hilbert spaces, and to [25] for those in Banach spaces.

In our approach, the solution of the regularized problem (5) is given by

$$F_{\alpha} = (S(T) - I)^{-1}A(I + \alpha A^b)^{-1}g^{\delta}. \tag{13}$$

Error estimates for the regularized solution F_{α} in case of *a priori* parameter choice rules are given in the following theorem.

Theorem 3.1: *Suppose that b is a positive integer and condition* (4) *is satisfied for some* $p \geq 1$. *Then there exist positive constants* C^*, C^{**} *such that*

$$\|F - F_\alpha\| \leq \begin{cases} C^*(\alpha^{-1/b}\delta + \alpha E) & \text{if } p \geq b, \\ C^{**}\left(\alpha^{-1/b}\delta + \alpha^{p/b}E\right) & \text{if } 1 \leq p < b. \end{cases} \tag{14}$$

In particular,

(i) *for* $p \geq b$ *and* $\alpha = (\delta/E)^{b/(b+1)}$, *we obtain*

$$\|F - F_\alpha\| \leq 2C^*\delta^{b/(b+1)}E^{1/(b+1)}; \tag{15}$$

(ii) *for* $p < b$ *and* $\alpha = (\delta/E)^{b/(1+p)}$, *we obtain*

$$\|F - F_\alpha\| \leq 2C^{**}\delta^{p/(1+p)}E^{1/(1+p)}. \tag{16}$$

For *a posteriori* parameter choice rules, we define the function $\rho(\alpha)$ by

$$\rho(\alpha) := \|(I + \alpha A^b)^{-1}g^\delta - g^\delta\|. \tag{17}$$

We assume that $0 < \delta < \|g^\delta\|$. This assumption is practically reasonable since the measured datum is useless when its error is too high. As proved in Lemma 4.6, $\rho(\alpha)$ is a continuous function on $(0, \infty)$ and $\lim_{\alpha\to 0}\rho(\alpha) \to 0$ and $\lim_{\alpha\to\infty}\rho(\alpha) \to \|g^\delta\|$.

Let τ be a positive constant such that $\tau > 1$ and $\tau\delta < \|g^\delta\|$. Then, there exists a parameter $\alpha = \alpha(\delta, \tau)$ such that

$$\rho(\alpha) = \tau\delta. \tag{18}$$

Error estimates for the regularized solution F_α with α being chosen using the *a posteriori* rule (18) are stated in the following theorem.

Theorem 3.2: *Suppose that b is an integer such that* $b > 1$ *and condition* (4) *is satisfied for some* $p \geq 1$. *Let* F_α *be the solution of the regularized problem* (5) *with regularization parameter* α *given by* (18). *Then, there exist positive constants* $C^\dagger$ *and* $C^{\dagger\dagger}$ *such that*

$$\|F - F_\alpha\| \leq C^\dagger\delta^{(b-1)/b}E^{1/b}, \quad \text{if } p \geq b - 1, \tag{19}$$

$$\|F - F_\alpha\| \leq C^{\dagger\dagger}\delta^{p/(1+p)}E^{1/(1+p)}, \quad \text{if } 1 \leq p < b - 1. \tag{20}$$

Remark 3.1: Since $\rho(\alpha)$ may not be monotonic, the regularization parameter satisfying (18) may not be unique. However, the error estimates (15) and (16) are valid for any α satisfying (18). A theoretical estimate of the effect of α on the accuracy of the regularized solution is still open.

Remark 3.2: We note that although representing the regularized solution F_α using (13) is convenient in proving the error estimates, its numerical realization is challenging since $S(T)$ is not easy to calculate. To find the regularized solution F_α, we first convert (5) into the following problem for $w(t) := (I + \alpha A^b)v$:

$$\begin{aligned} &w'(t) + Aw(t) = (I + \alpha A^b)F_\alpha, \\ &w(0) = 0, \\ &w(T) = g^\delta. \end{aligned} \tag{21}$$

Then F_α is calculated from (21).

Remark 3.3: As an example, consider the following one-dimensional inverse source problem, where X is a Banach space of functions defined on $[0,\pi]$,

$$\begin{aligned} &u_t(x,t) - u_{xx}(x,t) = F(x), \quad (x,t) \in (0,\pi) \times (0,T), \\ &u(0,t) = u(\pi,t) = 0, \quad t \in [0,T], \\ &u(x,0) = 0, \quad x \in (0,\pi), \\ &u(x,T) = g^\delta(x), \quad x \in [0,\pi]. \end{aligned} \tag{22}$$

Assume that A admits a set of positive eigenvalues $\{\lambda_n\}_{n\geq 1}$ and the corresponding eigenfunctions $\{\phi_n\}_{n\geq 1}$ which form a Schauder basis of the Banach space X. In this case, using Lemma A.2 in the appendix, we can write condition (4) in the following form:

$$\left\| \sum_{n=1}^{\infty} (1+\lambda_n^p) f_n \phi_n \right\| \leq E, \tag{23}$$

where $\{f_n\}_{n\geq 1}$ are the coordinates of F with respect to basis $\{\phi_n\}_{n\geq 1}$. Hence, under condition (23) we obtain the error estimates in Theorems 3.1 and 3.2. In Section 5 we present some numerical examples for this problem.

In the case when X is the Hilbert space $L^2[0,\pi]$, Dou et al. [7] also proposed the same regularized problem as (21) with $b=1$. That means, the right-hand side of the first equation of (21) is $F_\alpha(x) - \alpha F''_\alpha(x)$. Note that $A = -(\partial^2/\partial x^2)$ with Dirichlet boundary conditions. In this case, we have $\lambda_n = n$ and $\phi_n(x) = \sin(nx)$, $n = 1,2,\ldots$. Then, condition (23) is equivalent to the following condition:

$$\|F\|_{H^p} \leq E, \tag{24}$$

where $\|\cdot\|_p$ is the norm in the Sobolev space $H^p[0,\pi]$. Under the same condition, the authors in [7] obtained the following error estimate for the *a priori* parameter choice rule $\alpha = (\delta/E)^{2/(p+2)}$:

$$\|F - F_\alpha\| \leq \widetilde{C}\delta^{p/(p+2)}E^{2/(p+2)} + \max\{\alpha^{p/2}, \alpha\}E, \tag{25}$$

where $\tilde{C}$ is a positive constant. Note that $\max\{\alpha^{p/2},\alpha\}E \geq \delta^{1/2}E^{1/2}$, for all $p>0$. This error estimate is of the same order as our result in Theorem 3.1 for $b=1$ (see (15)). For $b\geq 2$, our result is better than [7]. Indeed, when $p \geq b \geq 2$ the error estimate (15) is $\delta^{b/b+1}E^{1/b+1}$, which is smaller than or equal to $\delta^{2/3}E^{1/3}$.

A posteriori parameter choice rules were not considered in [7].

4. Proofs of Theorems 3.1 and 3.2

The following results are needed for proving Theorem 3.1.

Lemma 4.1: *The following inequality holds*

$$\|(S(T)-I)^{-1}\| \leq \frac{1}{1-\|S(T)\|}.$$

Proof: Note that $S(T) - I$ is a bounded linear operator and

$$1 = \|I\| = \|I - S(T) + S(T)\| \leq \|I - S(T)\| + \|S(T)\|.$$

This implies that $\|I - S(T)\| \geqslant 1 - \|S(T)\| > 0$. Therefore, there exists $(S(T)-I)^{-1}$ which is also a bounded linear operator and

$$\|(S(T)-I)^{-1}\| \leq \frac{1}{1-\|S(T)\|}.$$

The lemma is proved. ∎

Lemma 4.2: *The following inequality holds*

$$\left\|A(I+\alpha A^b)^{-1}\right\| \le \begin{cases} 2\alpha^{-1} & \text{if } b=1 \\ (b-1)^{(b-1)/b}(b+1)^{1/b}\alpha^{-1/b} & \text{if } b>1. \end{cases} \tag{26}$$

Proof: Let $A_\alpha = A(I+\alpha A^b)^{-1}$ and $f(s) = s/(1+\alpha s^b)$. We have

$$f'(s) = \frac{1+(1-b)\alpha s^b}{(1+\alpha s^b)^2}, \quad f''(s) = \frac{b\alpha s^{b-1}\left((b-1)\alpha s^b - b - 1\right)}{\left(1+\alpha s^b\right)^3}. \tag{27}$$

If $b=1$, then

$$A_\alpha = f(A) = \frac{1}{\alpha}I + \int_0^\infty \frac{-2\alpha}{(1+\alpha s)^3} G(s,A)\,\mathrm{d}s.$$

Therefore, we have

$$\|A_\alpha\| \le \frac{1}{\alpha} + \int_0^\infty \frac{2\alpha s}{(1+\alpha s)^3}\,\mathrm{d}s = \frac{2}{\alpha}. \tag{28}$$

If $b>1$, then

$$A_\alpha = \int_0^\infty \frac{b\alpha s^{b-1}\left((b-1)\alpha s^b - b - 1\right)}{(1+\alpha s^b)^3} G(s,A)\,\mathrm{d}s.$$

This implies that

$$\begin{aligned} \|A_\alpha\| &\le \int_0^\infty \frac{b\alpha s^{b-1}\left|(b-1)\alpha s^b - b - 1\right|}{(1+\alpha s^b)^3}\,\mathrm{d}s \\ &= -\int_0^a \frac{b\alpha s^{b-1}\left((b-1)\alpha s^b - b - 1\right)}{(1+\alpha s^b)^3}\,\mathrm{d}s + \int_a^\infty \frac{b\alpha s^{b-1}\left((b-1)\alpha s^b - b - 1\right)}{(1+\alpha s^b)^3}\,\mathrm{d}s, \end{aligned} \tag{29}$$

where $a = ((b+1)/((b-1)\alpha))^{1/b}$. The first integral on the right-hand side of (29) is evaluated as

$$\begin{aligned} -\int_0^a \frac{b\alpha s^{b-1}\left((b-1)\alpha s^b - b - 1\right)}{(1+\alpha s^b)^3}\,\mathrm{d}s &= -s\frac{1+(1-b)\alpha s^b}{(1+\alpha s^b)^2}\Big|_0^a + \int_0^a \frac{1+(1-b)\alpha s^b}{(1+\alpha s^b)^2}\,\mathrm{d}s \\ &= -\left(\frac{b+1}{(b-1)\alpha}\right)^{1/b} \frac{-b}{\left(1+\frac{b+1}{b-1}\right)^2} + \frac{s}{1+\alpha s^b}\Big|_0^a \\ &= \frac{b-1}{2}\left(\frac{b+1}{(b-1)\alpha}\right)^{1/b}. \end{aligned} \tag{30}$$

By a similar argument, we have

$$\int_a^\infty \frac{b\alpha s^{b-1}\left((b-1)\alpha s^b - b - 1\right)}{(1+\alpha s^b)^3}\,\mathrm{d}s = \frac{b-1}{2}\left(\frac{b+1}{(b-1)\alpha}\right)^{1/b}. \tag{31}$$

From (29)–(31) we obtain

$$\|A_\alpha\| \le (b-1)\left(\frac{b+1}{(b-1)\alpha}\right)^{1/b} = (b-1)^{(b-1)/b}(b+1)^{1/b}\alpha^{-1/b}.$$

The lemma is proved. ∎

Lemma 4.3: *Suppose that $F_{1\alpha}, F_{2\alpha}$ are source terms of* (5) *corresponding to the final values g_1^δ, g_2^δ. There exists a positive constant C_1 such that*

$$\|F_{1\alpha} - F_{2\alpha}\| \le C_1\alpha^{-1/b}\|g_1^\delta - g_2^\delta\|.$$

Proof: From Lemmas 4.1 and 4.2, there exists a positive constant C_1 such that

$$\begin{aligned}\|F_{1\alpha} - F_{2\alpha}\| &= \|(I - S(T))^{-1}A(I+\alpha A^b)^{-1}(g_1^\delta - g_2^\delta)\| \\ &\le \|A(I+\alpha A^b)^{-1}\|\|(I - S(T))^{-1}\|\|g_1^\delta - g_2^\delta\| \\ &\le C_1\alpha^{-1/b}\|g_1^\delta - g_2^\delta\|.\end{aligned}$$

The lemma is proved. ∎

Remark 4.1: Problem (5) is well-posed. Indeed, from (13) it follows that problem (5) has a solution F_α. Moreover, Lemma 4.3 implies that this solution is unique and depends continuously on the data g^δ.

Lemma 4.4: *For $b \in \mathbb{N}, b > 0$ and $p \ge 1$, then there exists a positive constant C_2 such that*

$$\|A^b(I + A^p)^{-1}\| \le C_2, \quad p \ge b.$$

Proof: Let $f_1(A) = A^b(I+A^p)^{-1}$, we have $f_1(s) = s^b/(1+s^p)$ and

$$\begin{aligned}f_1'(s) &= \frac{bs^{b-1} + (b-p)s^{b+p-1}}{(1+s^p)^2}, \\ f_1''(s) &= \frac{b(b-1)s^{b-2} + (b-p)(b+p-1)s^{b+p-2}}{(1+s^p)^2} - \frac{2ps^{p-1}(bs^{b-1} + (b-p)s^{b+p-1})}{(1+s^p)^3}.\end{aligned}$$

When $b = 1$, it follows from Lemma 4.2, with $\alpha = 1$, that there exists a positive constant C_3 such that

$$\|A^b(I+A^p)^{-1}\| \le C_3.$$

Now, we consider $b > 1$. If $p = b$, then

$$f_1(A) = \int_0^\infty \left(\frac{p(p-1)s^{p-2}}{(1+s^p)^2} - \frac{2p^2s^{2p-2}}{(1+s^p)^3}\right) G(s, A)\,\mathrm{d}s.$$

Therefore,

$$\begin{aligned}\|f_1(A)\| &\le 1 + \int_0^\infty \left|\frac{p(p-1)s^{p-2}}{(1+s^p)^2} - \frac{2p^2s^{2p-2}}{(1+s^p)^3}\right| s\,\mathrm{d}s \\ &\le 1 + \int_0^\infty \left(\frac{p(p-1)s^{p-1}}{(1+s^p)^2} + \frac{2p^2s^{2p-1}}{(1+s^p)^3}\right)\mathrm{d}s \\ &\le 1 + \int_0^\infty \frac{3p^2s^{p-1}}{(1+s^p)^2}\,\mathrm{d}s \\ &= 1 + 3p.\end{aligned}$$

If $p > b$, then $f_1(A) = \int_0^\infty f_1''(s)G(s,A)\,\mathrm{d}s$. Therefore,

$$\|f_1(A)\| \le \int_0^\infty \left|\frac{b(b-1)s^{b-2} + (b-p)(b+p-1)s^{b+p-2}}{(1+s^p)^2} - \frac{2ps^{p-1}(bs^{b-1} + (b-p)s^{b+p-1})}{(1+s^p)^3}\right| s\,\mathrm{d}s$$

$$\leq \int_0^\infty \left(\frac{b(b-1)s^{b-1} + (p-b)(b+p-1)s^{b+p-1}}{(1+s^p)^2} + \frac{2ps^p(bs^{b-1} + (p-b)s^{b+p-1})}{(1+s^p)^3} \right) \mathrm{d}s$$
$$\leq \int_0^\infty \left(\frac{pbs^{b-1} + (p-b)(b+p-1)s^{b+p-1}}{(1+s^p)^2} + \frac{2p(bs^{b-1} + (p-b)s^{b+p-1})}{(1+s^p)^2} \right) \mathrm{d}s$$
$$= \int_0^\infty \frac{3pbs^{b-1} + (p-b)(b+3p-1)s^{b+p-1}}{(1+s^p)^2} \mathrm{d}s$$
$$\leq \int_0^\infty \frac{3pbs^{p-1}}{(1+s^p)^2} \mathrm{d}s + \int_0^\infty \frac{(p-b)(b+3p-1)s^{b-1}}{1+s^p} \mathrm{d}s$$
$$= 3b + \int_0^1 \frac{(p-b)(b+3p-1)s^{b-1}}{1+s^p} + \int_1^\infty \frac{(p-b)(b+3p-1)s^{b-1}}{1+s^p} \mathrm{d}s$$
$$\leq 3b + (p-b)(b+3p-1) + \int_1^\infty (p-b)(b+3p-1)s^{b-p-1} \mathrm{d}s$$
$$= 3b + (p-b)(b+3p-1) + b + 3p - 1$$
$$\leq 3b + p(b+3p) + b + 3p.$$

The lemma is proved. ■

Lemma 4.5: *For $b \in \mathbb{N},\ b > 0,\ p > 0$, there exist positive constants C_4, C_5 such that*

(a) $\|(I+\alpha A^b)^{-1}\| \leq C_4$,
(b) $\|A^p(I+\alpha A^b)^{-1}\| \leq C_5\alpha^{-p/b},\ p \leq b$.

Proof: Let $f_2(A) = (I+\alpha A^b)^{-1}$. We have $f_2(s) = 1/(1+\alpha s^b)$ and

$$f_2''(s) = \frac{b(b-1)\alpha s^{b-2} - b(b+1)\alpha^2 s^{2b-2}}{(1+\alpha s^b)^3}.$$

Therefore

$$f_2(A) = \int_0^\infty f_2''(s) G(s,A)\, \mathrm{d}s.$$

We have

$$\|f_2(A)\| \leq \int_0^\infty \left| f_2''(s) \right| s\, \mathrm{d}s$$
$$\leq \int_0^\infty \frac{b(b-1)\alpha s^{b-1} + b(b+1)\alpha^2 s^{2b-1}}{(1+\alpha s^b)^3} \mathrm{d}s$$
$$\leq \int_0^\infty \frac{b(b-1)\alpha s^{b-1}}{(1+\alpha s^b)^3} \mathrm{d}s + \int_0^\infty \frac{b(b+1)\alpha s^{b-1}}{(1+\alpha s^b)^2} \mathrm{d}s$$
$$= \frac{b-1}{2} + b + 1.$$

b) Using a similar argument as in the proof of Lemmas 4.2 and 4.4, there exists a positive constant C_5 such that

$$\|A^p(I+\alpha A^b)^{-1}\| \leq C_5\alpha^{-p/b}, \quad p \leq b.$$

The lemma is proved. ■

Lemma 4.6: *Set*

$$\rho(\alpha) = \|(I+\alpha A^b)^{-1}g^\delta - g^\delta\|.$$

For $b \in \mathbb{N}, b > 0$, *If* $0 < \delta < \|g^\delta\|$, *then*

(i) $\rho(\alpha)$ *is a continuous function,*
(ii) $\lim_{\alpha\to\infty} \rho(\alpha) = \|g^\delta\|$,
(iii) $\lim_{\alpha\to 0} \rho(\alpha) = 0$.

Proof: (i) We have

$$\rho(\alpha) = \|(I+\alpha A^b)^{-1}g^\delta - g^\delta\| = \|\alpha A^b(I+\alpha A^b)^{-1}g^\delta\|.$$

With $\alpha, \alpha_0 > 0$, we obtain

$$\begin{aligned}|\rho(\alpha)-\rho(\alpha_0)| &= |\|\alpha A^b(I+\alpha A^b)^{-1}g^\delta\| - \|\alpha_0 A^b(I+\alpha_0 A^b)^{-1}g^\delta\||\\ &\le \|\alpha A^b(I+\alpha A^b)^{-1}g^\delta - \alpha_0 A^b(I+\alpha_0 A^b)^{-1}g^\delta\|\\ &= |\alpha-\alpha_0|\|A^b((I+\alpha_0 A^b)^{-1}(I+\alpha A^b)^{-1}g^\delta\|\end{aligned}$$

Case 1. If $b = 1$, it follows from Lemmas 4.2 and 4.4 that there exists a positive constant C_6 such that

$$\begin{aligned}|\rho(\alpha)-\rho(\alpha_0)| &\le |\alpha-\alpha_0|\|A((I+\alpha_0 A)^{-1}\|\|(I+\alpha A^b)^{-1}\|\|g^\delta\|\\ &\le C_6\alpha_0^{-1}|\alpha-\alpha_0|.\end{aligned}$$

Case 2. If $b > 1$, it follows from Lemmas 4.2 and 4.4, then there exists a positive constant C_7 such that

$$\begin{aligned}|\rho(\alpha)-\rho(\alpha_0)| &= |\alpha-\alpha_0|\|A(I+\alpha_0 A^b)^{-1}A^{b-1}(I+\alpha A^b)^{-1}g^\delta\|\\ &\le |\alpha-\alpha_0|\|A(I+\alpha_0 A)^{-1}\|\|A^{b-1}(I+\alpha A^b)^{-1}\|\|g^\delta\|\\ &\le C_7\alpha_0^{-1/b}\alpha^{(1-b)/b}|\alpha-\alpha_0|.\end{aligned}$$

Therefore $\rho(\alpha)$ is a continuous function.

(ii) From Lemma 4.4, for $f_2(A) = (I+\alpha A^b)^{-1}$ we have

$$f_2(A) = \int_0^\infty \frac{b(b-1)\alpha s^{b-2} - b(b+1)\alpha^2 s^{2b-2}}{(1+\alpha s^b)^3} G(s,A)\,\mathrm{d}s.$$

We obtain

$$\|f_2(A)\| \le \int_0^\infty \left|\frac{b(b-1)\alpha s^{b-2} - b(b+1)\alpha^2 s^{2b-2}}{(1+\alpha s^b)^3}\right| s\,\mathrm{d}s.$$

By Lemma 4.4, we have

$$\int_0^\infty \left|\frac{b(b-1)\alpha s^{b-2} - b(b+1)\alpha^2 s^{2b-2}}{(1+\alpha s^b)^3}\right| s\,\mathrm{d}s \le \frac{b-1}{2} + b + 1.$$

Therefore, there exists a number $\eta_\epsilon > 0$ such that

$$\|g^\delta\|\int_0^{\eta_\epsilon} \left|\frac{b(b-1)\alpha s^{b-2} - b(b+1)\alpha^2 s^{2b-2}}{(1+\alpha s^b)^3}\right| s\,\mathrm{d}s \le \epsilon/2.$$

Hence

$$\begin{aligned}\|(I+\alpha A^b)^{-1}g^\delta\| &\le \epsilon/2 + \|g^\delta\| \int_{\eta_\epsilon}^\infty \left| \frac{b(b-1)\alpha s^{b-2} - b(b+1)\alpha^2 s^{2b-2}}{(1+\alpha s^b)^3} \right| s\, \mathrm{d}s \\ &\le \epsilon/2 + \|g^\delta\| \left(\int_{\eta_\epsilon}^\infty \frac{b(b-1)\alpha s^{b-1}}{(1+\alpha s^b)^3}\, \mathrm{d}s + \int_{\eta_\epsilon}^\infty \frac{b(b+1)\alpha^2 s^{2b-1}}{(1+\alpha s^b)^3}\, \mathrm{d}s \right) \\ &\le \epsilon/2 + \|g^\delta\| \left(\int_{\eta_\epsilon}^\infty \frac{b(b-1)\alpha s^{b-1}}{(1+\alpha s^b)^2}\, \mathrm{d}s + \int_{\eta_\epsilon}^\infty \frac{b(b+1)\alpha s^{b-1}}{(1+\alpha s^b)^2}\, \mathrm{d}s \right) \\ &= \epsilon/2 + 3b^2\|g^\delta\| \int_{\eta_\epsilon}^\infty \frac{\alpha s^{b-1}}{(1+\alpha s^b)^2}\, \mathrm{d}s \\ &\le \epsilon/2 + 3b^2\|g^\delta\| \int_{\eta_\epsilon}^\infty \frac{1}{\alpha s^{b+1}}\, \mathrm{d}s = \epsilon/2 + \frac{3b\|g^\delta\|}{\alpha \eta_\epsilon^b}.\end{aligned}$$

With $\alpha \ge ((6b\|g^\delta\|)/(\epsilon\eta_\epsilon^b))$, we obtain

$$\|(I+\alpha A^b)^{-1}g^\delta\| \le \epsilon.$$

This implies that

$$\lim_{\alpha\to\infty} \|(I+\alpha A^b)^{-1}g^\delta\| = 0.$$

On the other hand

$$\begin{aligned}\|g^\delta\| - \|(I+\alpha A^b)^{-1}g^\delta\| \le \rho(\alpha) &= \|(I+\alpha A^b)^{-1}g^\delta - g^\delta\| \\ &\le \|g^\delta\| + \|(I+\alpha A^b)^{-1}g^\delta\|.\end{aligned}$$

We obtain

$$\lim_{\alpha\to\infty} \rho(\alpha) = \|g^\delta\|.$$

iii) Note that

$$\overline{R(S(T))} = X. \tag{32}$$

Let $\epsilon > 0$. By (32), there exists a $\psi \in X$ such that

$$\|S(T)\psi - g^\delta\| \le \frac{\epsilon}{2C_5}. \tag{33}$$

where C_5 is the constant in part b) of Lemma 4.5. We have

$$\begin{aligned}\rho(\alpha) &= \|\alpha A^b(I+\alpha A^b)^{-1}g^\delta\| \\ &= \|\alpha A^b(I+\alpha A^b)^{-1}(S(T)\psi - g^\delta + S(T)\psi)\| \\ &\le \|\alpha A^b(I+\alpha A^b)^{-1}(S(T)\psi - g^\delta)\| + \|\alpha A^b(I+\alpha A^b)^{-1}(S(T)\psi)\| \\ &\le \alpha\|A^b(I+\alpha A^b)^{-1}\|\|S(T)\psi - g^\delta\| + \|\alpha(I+\alpha A^b)^{-1}\|\|A^bS(T)\|\|\psi\| \\ &\le C_5\|S(T)\psi - g^\delta\| + C_4\alpha\|A^bS(T)\|\|\psi\|.\end{aligned} \tag{34}$$

From (10), (33), (34) and Lemma 4.4, there exists a positive constant C_8 such that

$$\rho(\alpha) \le \epsilon/2 + C_8\alpha\|\psi\|. \tag{35}$$

For $0 < \alpha < (\epsilon/(2C_{11}\|\psi\| + 1))$, (35) implies that $\rho(\alpha) < \epsilon$. Therefore

$$\lim_{\alpha\to 0} \rho(\alpha) = 0.$$

The lemma is proved. ■

4.1. Proof of Theorem 3.1

We have

$$\begin{aligned}
\|F - F_\alpha\| &= \|(I - S(T))^{-1}A(I + \alpha A^b)^{-1}g^\delta - A(I - S(T))^{-1}g\| \\
&= \Big\|(I - S(T))^{-1}A(I + \alpha A^b)^{-1}(g^\delta - g) \\
&\quad + (I - S(T))^{-1}A(I + \alpha A^b)^{-1}g - A(I - S(T))^{-1}g\Big\| \\
&\le \|(I - S(T))^{-1}A(I + \alpha A^b)^{-1}(g^\delta - g)\| \\
&\quad + \|(I - S(T))^{-1}A(I + \alpha A^b)^{-1}g - A(I - S(T))^{-1}g\| \\
&\le \|A(I + \alpha A^b)^{-1}\|\|(I - S(T))^{-1}\|\|g^\delta - g\| \\
&\quad + \|\alpha A^{b+1}(I + \alpha A^b)^{-1}(I - S(T))^{-1}g\|.
\end{aligned}$$

From Lemmas 4.1 and 4.2, there exists a positive constant C_9 such that

$$\begin{aligned}
\|F - F_\alpha\| &\le C_9\alpha^{-1/b}\delta + \alpha\|A^b(I + \alpha A^b)^{-1}F\| \\
&= C_9\alpha^{-1/b}\delta + \alpha\|A^b(I + \alpha A^b)^{-1}(I + A^p)^{-1}(I + A^p)F\|.
\end{aligned}$$

If $p \ge b$, from Lemma 4.4, there exists a positive constant C^* such that

$$\begin{aligned}
\|F - F_\alpha\| &\le C_9\alpha^{-1/b}\delta + \alpha\|(I + \alpha A^b)^{-1}\|\|A^b(I + A^p)^{-1}\|\|(I + A^p)F\| \\
&\le C^*(\alpha^{-1/b}\delta + \alpha E).
\end{aligned}$$

Choose $\alpha = (\frac{\delta}{E})^{b/(b+1)}$, we obtain

$$\|F - F_\alpha\| \le 2C^*\delta^{b/(b+1)}E^{1/(b+1)}.$$

If $1 \le p < b$, from Lemmas 4.3 and 4.4, there exists a positive constant C^{**} such that

$$\begin{aligned}
\|F - F_\alpha\| &\le C_9\alpha^{-1/b}\delta + \alpha\|A^{b-p}(I + \alpha A^b)^{-1}A^p(I + A^p)^{-1}(I + A^p)F\| \\
&\le C_9\alpha^{-1/b}\delta + \alpha\|A^{b-p}(I + \alpha A^b)^{-1}\|\|A^p(I + A^p)^{-1}\|\|(I + A^p)F\| \\
&\le C^{**}\Big(\alpha^{-1/b}\delta + \alpha^{p/b}E\Big).
\end{aligned}$$

Choose $\alpha = (\frac{\delta}{E})^{b/(1+p)}$, we obtain

$$\|F - F_\alpha\| \le 2C^{**}\delta^{p/(1+p)}E^{1/(1+p)}.$$

The theorem is proved.

4.2. Proof of Theorem 3.2

From Lemma 4.5, we have

$$\begin{aligned}\tau\delta &= \alpha\|A^b(I+\alpha A^b)^{-1}g^\delta\| \\ &= \alpha\|A^b(I+\alpha A^b)^{-1}(g^\delta - g) + A^b(I+\alpha A^b)^{-1}g\| \\ &\geq \|\alpha A^b(I+\alpha A^b)^{-1}g\| - \alpha\|A^b(I+\alpha A^b)^{-1}(g^\delta - g)\| \\ &\geq \|\alpha A^{b-1}(I+\alpha A^b)^{-1}(I-S(T))F\| - C_5\|g^\delta - g\| \\ &\geq C_{10}\|\alpha A^{b-1}(I+\alpha A^b)^{-1}F\| - C_5\delta, \end{aligned} \tag{36}$$

where $C_{10} = 1 - \|S(T)\|$. Therefore, there exists a positive constant C_{11} such that

$$\|\alpha A^{b-1}(I+\alpha A^b)^{-1}F\| \leq C_{11}\delta. \tag{37}$$

From (36), there exists a positive constant C_{12} such that

$$\|F - F_{\alpha_\delta}\| \leq C_{12}\alpha^{-1/b}\delta + \alpha\|A^b(I+\alpha A^b)^{-1}F\| \tag{38}$$

Let $w = \alpha A^{b-1}(I+\alpha A^b)^{-1}F$, applying (11) with $\alpha = 0, \beta = 1, \gamma = p+1$, we obtain

$$\begin{aligned}\|Aw\| &\leq \|A^{p+1}w\|\|w\|^{p/(p+1)} \\ &\leq \|\alpha A^{p+b}(I+\alpha A^b)^{-1}F\|^{1/(p+1)}\|\alpha A^{b-1}(I+\alpha A^b)^{-1}F\|^{p/(p+1)} \\ &= \|\alpha A^{p+b}(I+\alpha A^b)^{-1}(I+A^p)^{-1}(I+A^p)F\|^{1/(p+1)}\|\alpha A^{b-1}(I+\alpha A^b)^{-1}F\|^{p/(p+1)} \\ &\leq \left(\|\alpha A^{p+b}(I+\alpha A^b)^{-1}(I+A^p)^{-1}\|\|(I+A^p)F\|\right)^{1/(p+1)}\|\alpha A^{b-1}(I+\alpha A^b)^{-1}F\|^{p/(p+1)} \\ &\leq \left(\|\alpha A^{b}(I+\alpha A^b)^{-1}\|\|A^p(I+A^p)^{-1}\|E\right)^{1/(p+1)}\|\alpha A^{b-1}(I+\alpha A^b)^{-1}F\|^{p/(p+1)}. \end{aligned} \tag{39}$$

From (37) and (39) and Lemma 4.5 there exists a positive constant C_{13} such that

$$\|Aw\| = \alpha\|A^b(I+\alpha A^b)^{-1}F\| \leq C_{13}E^{1/(p+1)}\delta^{p/(p+1)}. \tag{40}$$

From (38) and (40), we get

$$\|F - F_{\alpha_\delta}\| \leq C_{12}\alpha^{-1/b}\delta + C_{13}E^{1/(p+1)}\delta^{p/(p+1)}. \tag{41}$$

On the other hand, there exists a positive constant C_{14} such that

$$\begin{aligned}\tau\delta &= \alpha\|A^b(I+\alpha A^b)^{-1}g^\delta\| \\ &= \alpha\|A^b(I+\alpha A^b)^{-1}(g^\delta - g) + A^b(I+\alpha A^b)^{-1}g\| \\ &\leq \alpha\|A^b(I+\alpha A^b)^{-1}(g^\delta - g)\| + \|\alpha A^b(I+\alpha A^b)^{-1}g\| \\ &\leq C_5\|g^\delta - g\| + \|\alpha A^{b-1}(I+\alpha A^b)^{-1}(I-S(T))F\| \\ &\leq C_5\delta + C_{14}\|\alpha A^{b-1}(I+\alpha A^b)^{-1}F\|. \end{aligned} \tag{42}$$

Case 1. If $p \geq b-1$, from (42), there exists a positive constant C_{15} such that

$$\tau\delta \leq C_5\delta + C_{18}\|\alpha A^{b-1}(I+\alpha A^b)^{-1}(I+A^p)^{-1}(I+A^p)F\|$$

$$\leq C_5\delta + C_{18}\|\alpha A^{b-1}(I+\alpha A^b)^{-1}(I+A^p)^{-1}\|E$$
$$\leq C_5\delta + C_{15}\alpha E. \tag{43}$$

From (41), (43) and $E \geq \delta$, we obtain

$$\|F - F_{\alpha_\delta}\| \leq C^{\dagger} E^{1/b}\delta^{(b-1)/b}. \tag{44}$$

Case 2. If $1 \leq p < b-1$, then from (42), there exists a positive constant C_{16} such that

$$\tau\delta \leq C_5\delta + C_{18}\|\alpha A^{b-1}(I+\alpha A^b)^{-1}(I+A^p)^{-1}(I+A^p)F\|$$
$$\leq C_5\delta + C_{18}\|\alpha A^{b-1}(I+\alpha A^b)^{-1}(I+A^p)^{-1}\|E$$
$$\leq C_5\delta + C_{16}\alpha^{(p+1)/b}E. \tag{45}$$

From (41) and (45), we obtain

$$\|F - F_{\alpha_\delta}\| \leq C^{\dagger\dagger} E^{1/(p+1)}\delta^{p/(p+1)}. \tag{46}$$

The theorem is proved.

5. Numerical examples

To demonstrate how the proposed regularization method works, we consider here the one-dimensional problem (22) again. In this case, we solve problem (21) using the eigenfunction expansion method. Denote by $\{\lambda_n, \phi_n\}_{n\geq 1}$ the set of eigenvalues and eigenfunctions of A. Since $\{\phi_n\}_{n\geq 1}$ forms a Schauder basis of X, the following representations of w and F_α are unique:

$$w(t) = \sum_{n=1}^{\infty} b_n(t)\phi_n, \quad F_\alpha = \sum_{n=1}^{\infty} f_{\alpha n}\phi_n.$$

The first equation of (21) can be rewritten as

$$\sum_{n=1}^{\infty}[b_n'(t) + \lambda_n b_n(t)]\phi_n = \sum_{n=1}^{\infty}(1+\alpha\lambda_n^b)f_{\alpha n}\phi_n.$$

Again, due to the uniqueness of the representation with respect to the Schauder basis, it follows that $b_n'(t) + \lambda_n b_n(t) = (1+\alpha\lambda_n^b)f_{\alpha n},\ n = 1,2,\ldots,$ which results in

$$b_n(t) = c_n \mathrm{e}^{-\lambda_n t} + \frac{(1+\alpha\lambda_n^b)f_{\alpha n}}{\lambda_n}, \quad n = 1,2,\ldots,$$

where $c_n,\ n = 1,2,\ldots,$ are coefficients to be determined. Thus,

$$w(t) = \sum_{n=1}^{\infty}\left[c_n \mathrm{e}^{-\lambda_n t} + \frac{(1+\alpha\lambda_n^b)f_{\alpha n}}{\lambda_n}\right]\phi_n.$$

The zero initial condition implies that $c_n = -(((1+\alpha\lambda_n^b)f_{\alpha n})/\lambda_n)$. Therefore, at $t = T$, we have

$$g^\delta = \sum_{n=1}^{\infty}\left(-\mathrm{e}^{-\lambda_n T} + 1\right)\frac{(1+\alpha\lambda_n^b)f_{\alpha n}}{\lambda_n}\phi_n. \tag{47}$$

Denote by $\{g_n^\delta\}_{n\geq 1}$ the coordinates of g^δ with respect to basis $\{\phi_n\}_{n\geq 1}$. Then from (47) we have

$$f_{\alpha n} = \frac{\lambda_n g_n^\delta}{(1-\mathrm{e}^{-\lambda_n T})(1+\alpha\lambda_n^b)}.$$

Finally, F_α is approximated by the truncated series:

$$F_\alpha \approx F_{\alpha N} := \sum_{n=1}^{N} \frac{\lambda_n g_n^\delta}{(1-\mathrm{e}^{-\lambda_n T})(1+\alpha\lambda_n^b)}\phi_n,$$

where N is a positive integer to be chosen numerically.

As examples, we show below the reconstruction of $F(x)$ in (22) when $X = L^\infty[0,\pi]$ for the following functions.

(1) $F(x) = F_1(x) := \sin(x) + \sin(4x)$. In this case, we have $g(x) = (1-\mathrm{e}^{-T})\sin(x) + (1-\mathrm{e}^{-4^2 t})\sin(4x)$.

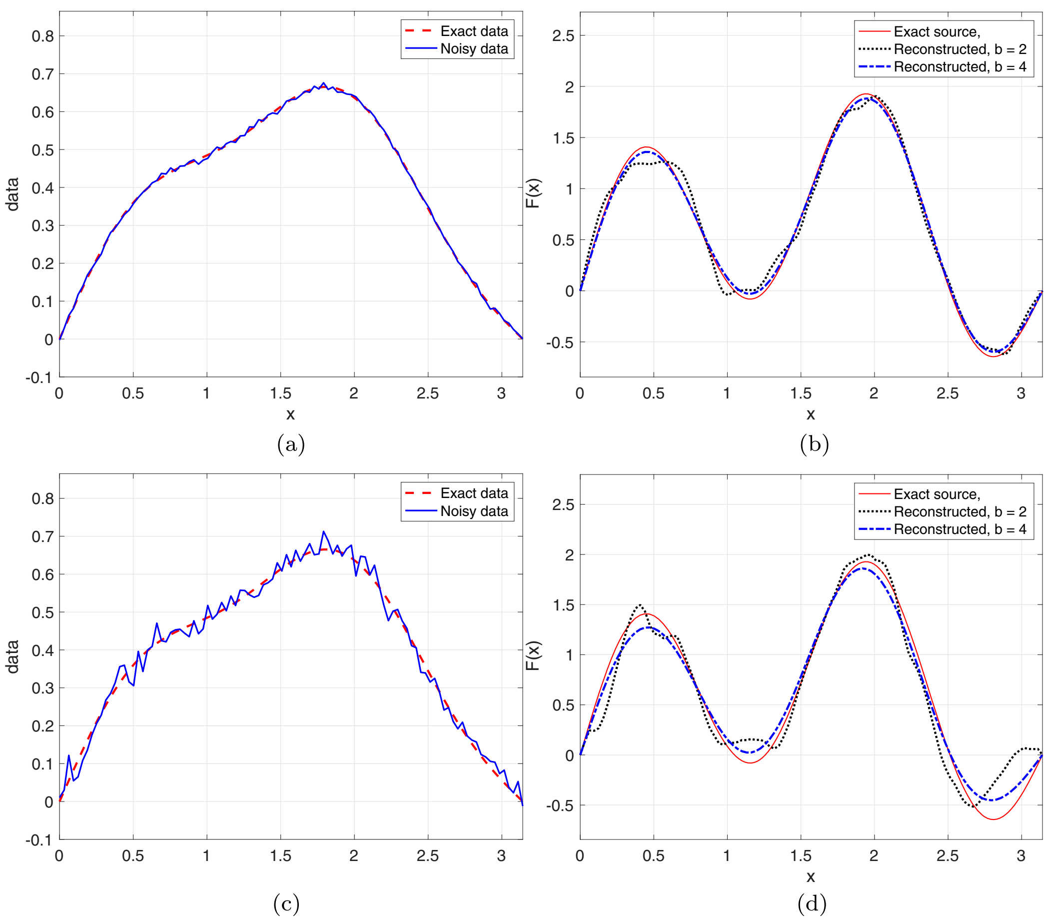

Figure 1. Reconstruction of the source function $F(x) = F_1(x) := \sin(x) + \sin(4x)$, $x \in [0,\pi]$, using data at $T = 1$. (a) Data at 2% noise. (b) Reconstruction at 2% noise. (c) Data at 10% noise. (d) Reconstruction at 10% noise.

(2)

$$F(x) = F_2(x) := \begin{cases} x, & \pi/3 \le x \le \pi/2, \\ \pi/2 - x, & \pi/2 \le x \le 2\pi/3, \\ 0 & \text{otherwise}. \end{cases} \tag{48}$$

(3) $F(x) = F_3(x) := \dfrac{1}{0.1\sqrt{2\pi}} \mathrm{e}^{-((x/\pi - 0.5)^2)/0.1^2}, \ x \in [0, \pi].$

Functions F_2 and F_3 in cases (2) and (3) were also used in [7]. In these cases, the data at the final time $t = T$ was calculated using the eigenfunction expansion method using 200 Fourier terms, while in the inverse problem we used $N = 100$ Fourier terms. The data was assumed to be measured at $T = 1$. Parameter p was chosen to be $p = 2$. To simulate measurement error, we added a pseudorandom noise of normal distribution, with zero mean and the standard deviation equal to a chosen noise

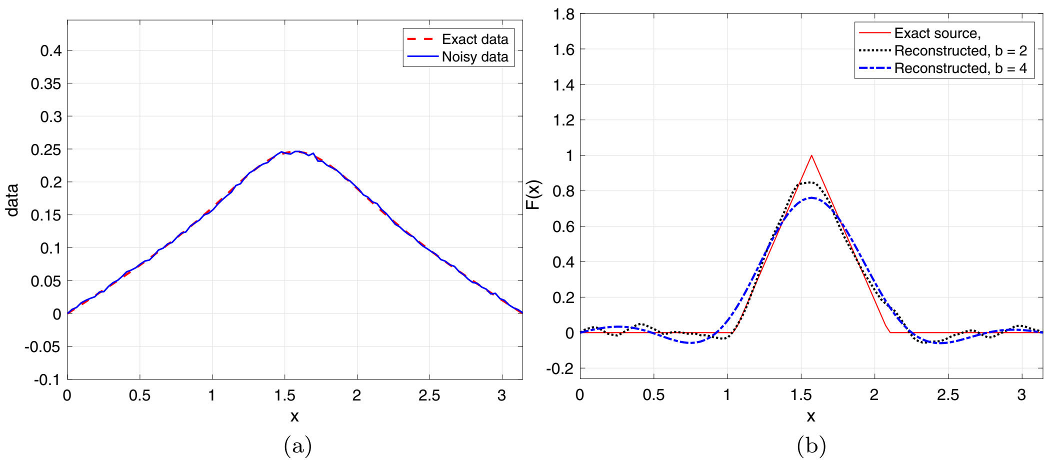

Figure 2. Reconstruction of the piecewise linear source function $F(x) = F_2(x)$ given by (48) using data at $T = 1$. (a) Data at 2% noise. (b) Reconstruction at 2% noise.

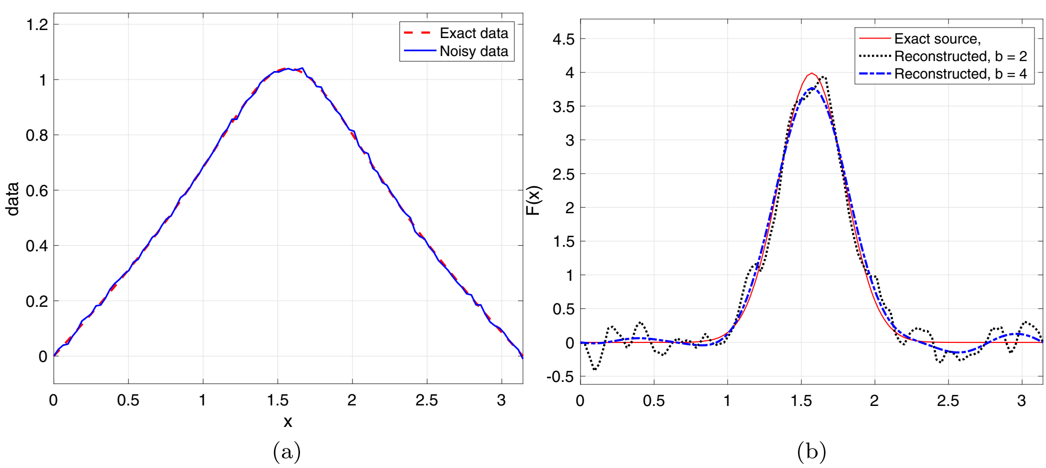

Figure 3. Reconstruction of the Gaussian source function $F(x) = F_3(x) := (1/(0.1\sqrt{2\pi}))\mathrm{e}^{-(((x/\pi - 0.5)^2)/0.1^2)}$, $x \in [0, \pi]$, using data at $T = 1$. (a) Data at 2% noise. (b) Reconstruction at 2% noise.

level, to the "exact" data, i.e.

$$g^{\delta} = g + \delta \cdot \text{randn},$$

where *randn* is the function which generate pseudorandom numbers of normal distribution. The noise level δ represents the L^{∞} norm of of $g^{\delta} - g$.

We have observed in our numerical tests that for functions F_1 and F_3, the accuracy was improved when b was increased, as depicted by the theoretical results. For F_2, however, the accuracy near the corners could not be improved by increasing b.

We depict in Figures 1–3 the reconstructions of the functions F_1, F_2, F_3, respectively. In these tests, the bound E in (23) was numerically chosen as 10^4. We can observe from these figures that the reconstruction were quite accurate at locations where the function F was smooth, even at rather high noise levels. This shows the good performance of the proposed regularization algorithm in the case of smooth source functions.

6. Concluding remarks

We proposed a regularization method for an inverse source problem in the Banach space setting and proved the Hölder-type error estimates for the regularized solution using both *a priori* and *a posteriori* parameter choice rules. Numerical examples showed good reconstruction results for a simple case. Numerical realization in more general cases is under consideration and will be reported in a future work.

The proposed regularization method should be able to be extended to the case of nonzero initial condition of the form $u(0) = u_0$, where $u_0 \in D(A)$. We are also investigating the case when the source term depends on t.

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References

[1] El Kadi Abderrezzak K, Ata R, Zaoui F. One-dimensional numerical modelling of solute transport in streams: the role of longitudinal dispersion coefficient. J Hydrol. 2015;527:978–989.
[2] Li K, Zhang L, Li Y, et al. A three-dimensional water quality model to evaluate the environmental capacity of nitrogen and phosphorus in Jiaozhou Bay, China. Mar Pollut. 2015;91(1):306–316.
[3] Nguyen VT, Nguyen HT, Tran TB, et al. On an inverse problem in the parabolic equation arising from groundwater pollution problem. Boundary Value Problems. 2015;2015(1):95.
[4] Shou-dong W, Yong-ming S. Three high-order splitting schemes for 3d transport equation. Appl Math Mech. 2005;26(8):1007–1016.
[5] Zhang S, Xin X. Pollutant source identification model for water pollution incidents in small straight rivers based on genetic algorithm. Appl Water Sci. Jan. 2016;7(4):1955–1963.
[6] Nguyen PM, Nguyen LH. A numerical method for an inverse source problem for parabolic equations and its application to a coefficient inverse problem. J Inverse Ill-Posed Probl. 2020;28(3):323–339.
[7] Dou F-F, Fu C-L, Yang F. Identifying an unknown source term in a heat equation. Inverse Probl Sci Eng. 2009;17:901–913.
[8] Hào DN, Huong BV, Oanh NTN, et al. Determination of a term in the right-hand side of parabolic equations. J Comput Appl Math. 2017;309:28–43.
[9] Hào DN, Thanh PX, Lesnic D, et al. Determination of a source in the heat equation from integral observations. J Comput Appl Math. 2014;264:82–98.

[10] Hasanov A, Mukanova B. Relationship between representation formulas for unique regularized solutions of inverse source problems with final overdetermination and singular value decomposition of input-output operators. IMA J Appl Math. 2015;80(3):676–696.
[11] Isakov V. Inverse source problems. Providence (RI): American Mathematical Society; 1990.
[12] Trong DD, Dinh APN, Nam PT. Determine the spatial term of two-dimensional heat source. Appl Anal. 2009;88:457–474.
[13] Trong DD, Quan P, Dinh APN. Determination of a two-dimensional heat source: uniqueness, regularization and error estimate. Comput Math Appl. 2006;191:50–67.
[14] Iskenderov AD, Tagiev RG. The inverse problem on the determination of right sides of evolution equations in Banach space. Nauchn Tr Azerb Gos Univ. 1979;1:51–56.
[15] Rundell W. Determination of an unknown nonhomogeneous term in a linear partial differential equation from overspecified boundary data. Appl Anal. 1980;10(3):231–242.
[16] Eidelman Y. Boundary value problems for differential equations with parameters [PhD thesis]. Voronezh State University; 1984.
[17] Eidelman YS. An inverse problem for an evolution equation. Mat Zametki. 1991;49(5):135–141. 160.
[18] Guidetti D. Determining the source term in an abstract parabolic problem from a time integral of the solution. Mediterr J Math. 2012;9(4):611–633.
[19] Prilepko A, Piskarev S, Shaw S-Y. On approximation of inverse problem for abstract parabolic differential equations in Banach spaces. J Inverse Ill-Posed Probl. 2007;15(8):831–851.
[20] Prilepko AI, Kostin AB, Solov'ev VV. Inverse problems on finding the source and coefficients for elliptic and parabolic equations in Hölder and Sobolev spaces. Sib Zh Chist Prikl Mat. 2017;17(3):67–85.
[21] Prilepko AI, Tikhonov IV. Reconstruction of the inhomogeneous term in an abstract evolution equation. Izv Ross Akad Nauk Ser Mat. 1994;58(2):167–188.
[22] Prilepko AI, Tkachenko DS. The Fredholm property of the inverse source problem for parabolic systems. Differ Uravn. 2003;39(12):1693–1700. 1727.
[23] Tikhonov IV, Eidelman YS. Uniqueness of the solution of a two-point inverse problem for an abstract differential equation with an unknown parameter. Differ Uravn. 2000;36(8):1132–1133. 1151.
[24] Tikhonov IV, Eidelman YS. A uniqueness criterion in an inverse problem for an abstract differential equation with a nonstationary inhomogeneous term. Mat Zametki. 2005;77(2):273–290.
[25] Schuster T, Kaltenbacher B, Hofmann B, et al. Regularization methods in Banach spaces. Berlin: Walter de Gruyter GmbH & Co. KG; 2012. (Radon Series on Computational and Applied Mathematics; 10).
[26] de Laubenfels R. Functional calculus for generators of uniformly bounded holomorphic semigroups. Semigroup Forum. 1989;38(1):91–103.
[27] Pazy A. Semigroups of linear operators and applications to partial differential equations. New York: Springer; 1983. (Applied Mathematical Sciences; 44).
[28] Friedman A. Partial Differential Equations. Huntington (NY): Robert E. Krieger Publishing Co.; 1976.
[29] Huang Y, Zheng Q. Regularization for ill-posed Cauchy problems associated with generators of analytic semigroups. J Differ Equ. 2004;203(1):38–54.
[30] Engl HW, Hanke M, Neubauer A. Regularization of inverse problems. Dordrecht: Kluwer Academic Publishers Group; 1996.
[31] Diethelm K. The analysis of fractional differential equations. Heidelberg: Springer; 2004.

Appendix. Eigenvalues of fractional powers of a linear operator

In this appendix, we prove a result concerning eigenvalues of fractional powers of the operator A used in this paper. First, we need the following identity:

Lemma A.1: *Let γ be a real number in $(0,1)$. Then, the following identity holds:*

$$\int_0^\infty \frac{s^{\gamma-1}}{1+s}\,\mathrm{d}s = \frac{\pi}{\sin(\pi\gamma)}. \tag{A1}$$

Proof: We observed that this result has been used in some references. However, we could not find a proof. For the sake of completeness, we prove it here again.

Denoting $t = (1/(s+1))$, we have $s = (1/t) - 1$. Therefore, $\mathrm{d}s = -\dfrac{1}{t^2}dt$. We have

$$\int_0^\infty \frac{s^{\gamma-1}}{1+s}\,\mathrm{d}s = \int_0^1 t^{\gamma-1}(1-t)^{-\gamma}\,\mathrm{d}t = \int_0^1 t^{\gamma-1}(1-t)^{\beta-1}\,\mathrm{d}t,$$

where $\beta = 1 - \gamma$. By Theorems D.3 and D.6 in [31] (p. 228–229), we have

$$\int_0^1 t^{\gamma-1}(1-t)^{\beta-1}\,\mathrm{d}t = B(\gamma,\beta) = B(\gamma, 1-\gamma) = \frac{\pi}{\sin(\pi\gamma)},$$

where $B(\gamma,\beta)$ is the beta function. The lemma is proved. ■

Lemma A.2: *Assume that A is a linear operator in a Banach space X such that $-A$ is the infinitesimal generator of an analytic semigroup. Assume that A has a positive eigenvalue λ and ϕ is the corresponding eigenfunction, i.e. $A\phi = \lambda\phi$. Then, for an arbitrary positive real number γ, λ^γ is an eigenvalue of A^γ, where A^γ is the power of A defined in Definition* 2.3.

Proof: The assertion is obvious if γ is a positive integer. Now we consider the case $0 < \gamma < 1$. Since $-A$ generates an analytic semigroup, it follows from (6.16) of [27] (p. 72) that

$$A^\gamma\phi = \frac{\sin\pi\gamma}{\pi}\int_0^\infty t^{\gamma-1}A(tI+A)^{-1}\phi\,\mathrm{d}t. \tag{A2}$$

On the other hand, since $(tI + A)\phi = (t+\lambda)\phi$, we have

$$A(tI+A)^{-1}\phi = A\left(\frac{1}{t+\lambda}\phi\right) = \frac{\lambda}{t+\lambda}\phi. \tag{A3}$$

From (A2) and (A3), we obtain

$$A^\gamma\phi = \frac{\sin\pi\gamma}{\pi}\int_0^\infty \frac{t^{\gamma-1}}{t+\lambda}(\lambda\phi)\,\mathrm{d}t.$$

Using the change of variable $s := t/\lambda$ and (A1), we obtain

$$A^\gamma\phi = \left(\frac{\sin\pi\gamma}{\pi}\int_0^\infty \frac{s^{\gamma-1}}{1+s}\,\mathrm{d}s\right)\lambda^\gamma\phi = \lambda^\gamma\phi.$$

For $n < \gamma < n+1$, where n is an integer, we have

$$A^\gamma\phi = A^{\gamma-n}A^n\phi = \lambda^n A^{\gamma-n}\phi = \lambda^\gamma\phi.$$

The proof is complete. ■