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The quasi-reversibility method for an inverse source problem for time-space fractional parabolic equations

Nguyen Van Duc [a,*], Nguyen Van Thang [b], Nguyen Trung Thành [c]

[a] *Department of Mathematics, Vinh University, Vinh City, Viet Nam*
[b] *Quan Hanh Secondary School, Quan Hanh Town, Nghi Loc District, Nghe An Province, Viet Nam*
[c] *Department of Mathematics, Rowan University, Glassboro, NJ, USA*

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Abstract

In this paper, we apply the quasi-reversibility method to solve an inverse source problem for a time-space fractional parabolic equation. Hölder-type error estimates for the regularized solutions are proved for both a priori and a posteriori regularization parameter choice rules. The theoretical error estimates are confirmed with numerical tests for one and two dimensional equations.

Keywords: Inverse source problems for time-space fractional parabolic equations; Quasi-reversibility method; Error estimates; Numerical implementation

1. Introduction

Fractional differential equations (FDEs) are an important tool in modeling many problems in biology [12], physics [9,16,30], finance [21], etc. For example, FDEs are used to describe underground fluid flows [14], fractal comb structures [28], extreme events like earthquakes [2], and stochastic processes [11,24,35]. FDEs can also be used to describe anomalous diffusion processes in viscoelastic materials, heterogeneous media, and plasma physics [22,23,25,20,29]. Fractional-order equations enable modeling of dynamical processes with memory [25,39].

* Corresponding author.
E-mail addresses: ducnv@vinhuni.edu.vn (N.V. Duc), nguyenvanthangk17@gmail.com (N.V. Thang), nguyent@rowan.edu (N.T. Thành).

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In this paper we consider an inverse source problem for an FDE of parabolic type using data measured at the final time instant. The statement of the inverse source problem is as follows. Let $\gamma \in (0, 1)$ and $\beta > 0$ be two given positive numbers. Let $\mathbb{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ be a self-adjoint closed operator in $\mathbb{H}$ such that $-A$ generates a compact contraction semi-group $\{S(t)\}_{t\geq 0}$ on $\mathbb{H}$ and A admits an orthonormal eigenbasis $\{\phi_i\}_{i\geq 1}$ in $\mathbb{H}$. The associated eigenvalues $\{\lambda_i\}_{i\geq 1}$ of A are such that

$$0 < \lambda_1 < \lambda_2 < \dots, \quad \text{and} \quad \lim_{i\to+\infty} \lambda_i = +\infty.$$

For $p > 0$, we define (see [8,27])

$$D(A^p) := \left\{ \psi \in \mathbb{H} : \sum_{n=1}^{\infty} \lambda_n^{2p} \langle \psi, \phi_n \rangle^2 < \infty \right\}$$

and the associated norm

$$\|\psi\|_p := \left(\sum_{n=1}^{\infty} \lambda_n^{2p} \langle \psi, \phi_n \rangle^2 \right)^{\frac{1}{2}}, \ \psi \in D(A^p).$$

Since $\lambda_n \to +\infty$, it follows that $\lambda_n^{2p} \to +\infty$. Hence, $D(A^p)$ is a proper subspace of $\mathbb{H}$ for $p > 0$. We consider the following inverse source problem of determining a function $f \in D(A^p)$ for the time-space fractional parabolic equation

$$\begin{cases} \dfrac{\partial^\gamma u}{\partial t^\gamma} + A^\beta u = f h(t), \quad 0 < t < T, \\ u(0) = 0, \\ u(T) = g, \end{cases} \tag{1.1}$$

with $g \in \mathbb{H}$ being given and $h(t) : [0, T] \to \mathbb{H}$ being a continuous time-dependent function. Here, the Caputo derivative $\dfrac{\partial^\gamma}{\partial t^\gamma}$ is defined as

$$\frac{\partial^\gamma u}{\partial t^\gamma} := \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} \frac{\partial u(\cdot, s)}{\partial s} ds, \quad \gamma \in (0, 1), \tag{1.2}$$

with $\Gamma(\cdot)$ being Euler's Gamma function (see [15,25]).

Function g in the last equation of (1.1) is considered as the measured data. In practice, the measured data always contain noise. Denote by $\varepsilon > 0$ the noise level and by g^ε the noisy data, which satisfies

$$\|g - g^\varepsilon\| \leq \varepsilon. \tag{1.3}$$

Several methods have been proposed to solve inverse source problems for fractional parabolic equations in the literature. Some of such methods are the truncation method by Zhang and Wei

[39], the generalized Tikhonov regularization based on a boundary element method by Zhang and Wei [34], the β-mollification method by Ruan and Wang [26], and the Landweber iterative regularization method by Yang et al. [38]. Furthermore, Jiang et al. [13], Tatar and Ulusoy [31], and Jiang et al. [13] proved the uniqueness of the solution. After that, Liu et al. [19] and Ali et al. [1] proved the uniqueness and the stability of the solution. Some more works related to problem (1.1) include [13,19,33,26,34,38,39], where a special setting $\beta = 1$ is considered.

In this paper, we consider problem (1.1) in a more general setting. In particular, the fractional order β is considered as an arbitrary positive real number and the source term is considered to be a time-dependent function (rather than a time-independent one as in some of the aforementioned works). It is known that the inverse source problem (1.1) is ill-posed (see Remark 2). In this work, we apply the quasi-reversibility method for solving this problem. This method was proposed by Lattes and Lions [17], and has been used for solving various types of ill-posed problems, e.g., sideways heat equation [3], backward heat conduction problem [36], backward parabolic equations [10] and inverse source problems [4,5,37]. The principle of this method is to approximate the unknown source function f in problem (1.1) by the function f_α in the following problem:

$$\begin{cases} \dfrac{\partial^\gamma v}{\partial t^\gamma} + A^\beta v = (I + \alpha A^b) f_\alpha h(t), \ 0 < t < T, \\ v(0) = 0, \\ v(T) = g^\varepsilon, \end{cases} \tag{1.4}$$

where $\alpha > 0$ is a regularization parameter and $b \geq \beta$ is an arbitrary positive real number.

To compare with existing results in the literature, we refer to the papers [4,5]. In [4], a simpler version (when $b = 1$) of the quasi-reversibility method was applied to an inverse source problem for a heat equation using final-time data. In [5], a similar method was also applied to an inverse source problem for parabolic equations in Banach spaces. Although the method we propose in this paper is similar to that in [4], the latter does not work in the case $\beta > 1$, see more details in Remark 5. Moreover, [4] only considered a classical integer-order heat equation. In [5], we used a different technique, which is based on the semigroup theory, in proving the convergence of the quasi-reversibility method. Moreover, in [5] we did not prove that the convergence rates of the regularized solutions to the exact one are of optimal order.

The main theoretical contributions of this paper include Hölder-type error estimates for the solution of the regularized problem (1.4) using both a priori and a posteriori regularization parameter choice rules. In addition, we also prove a stability estimate of optimal order for the inverse problem (1.1). We also propose a noniterative algorithm to solve the regularized inverse problem. We would like to mention that in our earlier work [33] similar results were also obtained using a mollification regularization method with more restricted conditions on the time-dependent source function $h(t)$.

This paper is organized as follows. In Section 2, we recall some definitions and present some inequalities for later use. Section 3 is devoted to the stability estimate for the solution of problem (1.1) with optimal order. In Section 4, we present the main results of the paper on convergence rates of regularized solutions to the exact solution with both a priori and a posteriori parameter choice rules. The proofs of these results are presented in Section 5. In Section 6, we present numerical solutions for illustrating the efficiency of our approach. Conclusions are drawn in Section 7.

2. Auxiliary results

Denote by $E_{\gamma,\varpi}(z)$ the Mittag-Leffler function [15,25]:

$$E_{\gamma,\varpi}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\gamma + \varpi)}, z \in \mathbb{C}, \gamma > 0, \ \varpi > 0. \tag{2.1}$$

Lemma 1. ([27]) *Let $\gamma \in (0,1)$, $\lambda > 0$ and $t > 0$. We have*

$$a) \quad \frac{d}{dt} E_{\gamma,1}(-\lambda t^\gamma) = -\lambda t^{\gamma-1} E_{\gamma,\gamma}(-\lambda t^\gamma), \tag{2.2}$$

$$b) \quad \frac{d^\gamma}{dt^\gamma} E_{\gamma,1}(-\lambda t^\gamma) = -\lambda E_{\gamma,1}(-\lambda t^\gamma), \tag{2.3}$$

$$c) \quad \frac{d}{dt}\left(t E_{\gamma,2}(-\lambda t^\gamma)\right) = E_{\gamma,1}(-\lambda t^\gamma). \tag{2.4}$$

Remark 1. ([18]) $E_{\gamma,1}(-s)$ is a decreasing function on $(0,\infty)$.

Lemma 2. ([25]) *Let $0 < \gamma < 2$ and $\varpi > 0$ be arbitrary. We suppose that μ is such that $\pi\gamma/2 < \mu < \min\{\pi, \pi\gamma\}$. Then there exists a constant $C_1 = C_1(\gamma, \varpi, \mu) > 0$ such that*

$$E_{\gamma,\varpi}(z) \le \frac{C_1}{1+|z|}, \ \mu \le |arg z| \le \pi.$$

Lemma 3. ([25]) *Assume that $0 < \gamma < 1$. Then there exist constants $C_2, C_3 > 0$ such that*

$$\frac{C_2}{1-x} \le E_{\gamma,1}(x) \le \frac{C_3}{1-x}, \text{ for all } x \leqslant 0.$$

Lemma 4. (Young's inequality) *If a, b are nonnegative numbers and m, n are positive numbers such that $\frac{1}{m} + \frac{1}{n} = 1$, then $ab \le \frac{a^m}{m} + \frac{b^n}{n}$.*

Lemma 5. ([18]) *For $0 < \gamma < 1$, $\eta > 0$, we have $0 \le E_{\gamma,\gamma}(\eta) \le \frac{1}{\Gamma(\gamma)}$. Moreover, $E_{\gamma,\gamma}(-\eta)$ is a monotonic decreasing function with $\eta > 0$.*

Definition 1. Let $b \ge \beta$. For every $v \in \mathbb{H}$, we define

$$B_\alpha v := \sum_{n=1}^{\infty} \left(\frac{1}{1+\alpha\lambda_n^b}\right) \langle v, \phi_n \rangle \phi_n. \tag{2.5}$$

Lemma 6. *The inverse source problem (1.1) has a unique solution given by*

$$f = \sum_{n=1}^{\infty} \frac{\langle g, \phi_n \rangle \phi_n}{\int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds}.$$

Proof. The function $u(t)$ in (1.1), which is usually referred to as the solution of the forward problem, can be represented by (see [31], p. 2235)

$$u(t) = \sum_{n=1}^{\infty} \left\{ \int_0^t \langle f, \phi_n \rangle (t-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (t-s)^\gamma) h(s) ds \right\} \phi_n. \tag{2.6}$$

Taking $t = T$ and taking the inner product of (2.6) with ϕ_n, we obtain

$$\langle g, \phi_n \rangle = \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds \, \langle f, \phi_n \rangle . \tag{2.7}$$

Therefore,

$$f = \sum_{n=1}^{\infty} \frac{\langle g, \phi_n \rangle \phi_n}{\int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds}. \tag{2.8}$$

The proof is complete. □

Assumption H. A function $h : [0, T] \to \mathbb{R}$ is assumed to satisfy Assumption H if h is continuous on $[0, T]$ and there exists a constant $T_0 \in [0, T)$ such that $|h(t)| \geq \eta > 0,\ t \in [T_0, T]$ for some positive constant η. Furthermore, one of the following two conditions is satisfied:

H1 $h(t)$ does not change sign on $[0, T]$.
H2 If $h(t)$ changes sign on $[0, T]$ then $h(t)$ is differentiable and there exists a constant θ such that $|h_t(t)| \leq \theta,\ t \in [0, T]$. Moreover, $|h(t)| \leq \frac{\eta(T-T_0)}{T_0},\ t \in I$, where $I = \{t : h(t)h(T) \leq 0\}$.

Remark 2. Assumption H1 for function $h(t)$ was also used in [33,38]. However, under Assumption H the function $h(t)$ can take more general forms. Indeed, Assumption H2 allows $h(t)$ to change sign. In this case, Assumption H2 ensures that the denominator of the fraction on the right-hand side of formula (2.8) is non-zero, namely

$$\int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds \neq 0.$$

This condition is needed in our analysis.

We note that any continuous function h which does not change sign and $h(T) > 0$ will satisfy Assumption H1. As an example of a function $h(t)$ which satisfies Assumption H2, but not H1, consider $h(t) = t - \dfrac{T}{6},\ t \in [0, T]$. Clearly, this function does not satisfy H1 because it changes sign on $[0, T]$. To show that it satisfies H2, we note that $h(t) \leq 0,\ t \in [0, T/6]$. Therefore, $I = \{t : h(t)h(T) \leq 0\} = \{t : h(t) \leq 0\} = [0, T/6]$. Furthermore, $h(t) \geq T/3,\ t \in [T/2, T]$. Therefore, the function $h(t)$ satisfies H2 and Assumption H for $T_0 = T/2, \eta = T/3, \theta = 1$.

3. Stability estimate

Theorem 1. *Suppose that $h(t)$ satisfies Assumption H and f is a solution of problem (1.1). If $\|u(T)\| = \|g\| \le \varepsilon$ and $\|f\|_p \le E$ for some positive constants E and p, then there exists a constant $\overline{C}_1 > 0$ such that*

$$\|f\| \le \overline{C}_1 \varepsilon^{\frac{p}{p+\beta}} E^{\frac{\beta}{p+\beta}}. \tag{3.1}$$

Remark 3. The result in Theorem 1 is better than a stability result in [39]. The authors of [39] only considered problem (1.1) with $\beta = 1$ and a time-independent source function. Our result in Theorem 1 is valid where β is any positive real number and the source function may depend on time.

To prove Theorem 1, we need the following auxiliary result.

Lemma 7. *If $h(t)$ satisfies Assumption H then there exists a constant $C_4 > 0$ such that*

$$\left| \lambda_n^\beta \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds \right| \ge C_4, \ n = 1, 2, \ldots$$

Proof. Case 1. $h(t)$ satisfies Assumption H1. Since $h(t)$ is a function that does not change sign on $[0, T]$, we have

$$\begin{aligned}
&\left| \lambda_n^\beta \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds \right| \\
&= \lambda_n^\beta \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) |h(s)| ds \\
&\ge \eta \int_{T_0}^T \lambda_n^\beta (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) ds \\
&= \eta [1 - E_{\gamma,1}(-\lambda_n^\beta (T-T_0)^\gamma)] \\
&\ge \eta [1 - E_{\gamma,1}(-\lambda_1^\beta (T-T_0)^\gamma)].
\end{aligned}$$

Case 2. $h(t)$ satisfies Assumption H2. Let $D = \{t : h(t)h(T) \ge 0\}$,

$$T_1 = \max\{t : t \in [0, T], h(t) = 0\},$$

and

$$C_5 = \left(\frac{2C_1 \big(|h(0)| + \theta T \big)}{\eta T^\gamma} \right)^{\frac{1}{\beta}},$$

where C_1 is given in Lemma 2. We have $0 < T_1 < T_0$, $[T_1, T] \subseteq D$, and $I \subseteq [0, T_1]$. With $\lambda_n \leq C_5$, by Lemma 5 we obtain

$$
\begin{aligned}
&\Big|\lambda_n^\beta \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds\Big| \\
&= \lambda_n^\beta \Big| \int_D (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds + \int_I (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds\Big| \\
&\geq \lambda_1^\beta \int_D (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) |h(s)| ds \\
&\quad - \lambda_1^\beta \int_I (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) |h(s)| ds \\
&\geq \lambda_1^\beta \int_{T_1}^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) |h(s)| ds \\
&\quad - \frac{\eta(T-T_0)}{T_0} \lambda_1^\beta \int_0^{T_1} (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) ds \\
&\geq \lambda_1^\beta (T-T_1)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-T_1)^\gamma) \left(\int_{T_1}^T |h(s)| ds - \frac{\eta(T-T_0)}{T_0} \int_0^{T_1} ds \right) \\
&\geq \lambda_1^\beta (T-T_1)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-T_1)^\gamma) \left(\int_{T_1}^{T_0} |h(s)| ds + \int_{T_0}^T \eta ds - \frac{\eta T_1 (T-T_0)}{T_0} \right) \\
&\geq \lambda_1^\beta (T-T_1)^{\gamma-1} E_{\gamma,\gamma}(-C_5^\beta (T-T_1)^\gamma) \left(\int_{T_1}^{T_0} |h(s)| ds + \frac{\eta(T_0-T_1)(T-T_0)}{T_0} \right). \qquad (3.2)
\end{aligned}
$$

With $\lambda_n \geq C_5$, by integration by parts we get

$$
\begin{aligned}
&\Big|\lambda_n^\beta \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds\Big| \\
&= \Big| h(T) - h(0) E_{\gamma,1}(-\lambda_n^\beta T^\gamma) - \int_0^T E_{\gamma,1}(-\lambda_n^\beta (T-s)^\gamma) h_s(s) ds \Big|
\end{aligned}
$$

$$\geq |h(T)| - |h(0)|E_{\gamma,1}(-\lambda_n^\beta T^\gamma) - \int_0^T E_{\gamma,1}(-\lambda_n^\beta (T-s)^\gamma)|h_s(s)|ds$$

$$\geq |h(T)| - |h(0)|E_{\gamma,1}(-\lambda_n^\beta T^\gamma) - \theta \int_0^T E_{\gamma,1}(-\lambda_n^\beta (T-s)^\gamma)ds$$

$$= |h(T)| - |h(0)|E_{\gamma,1}(-\lambda_n^\beta T^\gamma) - \theta T E_{\gamma,2}(-\lambda_n^\beta T^\gamma).$$

By Lemma 2 and $\lambda_n \geq C_5$ we have

$$\begin{aligned}
&\left|\lambda_n^\beta \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma)h(s)ds\right| \\
&\geq |h(T) - \frac{C_1|h(0)|}{1+\lambda_n^\beta T^\gamma} - \frac{\theta C_1 T}{1+\lambda_n^\beta T^\gamma} \\
&\geq \eta - \frac{C_1\big(|h(0)| + \theta T\big)}{\lambda_n^\beta T^\gamma} \\
&\geq \frac{\eta}{2}.
\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), there exists a constant $C_4 > 0$ such that

$$\left|\lambda_n^\beta \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma)h(s)ds\right| \geq C_4, \ n = 1, 2, ... \tag{3.4}$$

The lemma is proved. □

Now we prove Theorem 1.
Using Hölder's inequality, we have

$$\begin{aligned}
\|f\|^2 = \sum_{n=1}^\infty \langle f, \phi_n\rangle^2 &= \sum_{n=1}^\infty \left(\lambda_n^{\frac{2p\beta}{p+\beta}} |\langle f, \phi_n\rangle|^{\frac{2\beta}{p+\beta}}\right)\left(\lambda_n^{\frac{-2p\beta}{p+\beta}} |\langle f, \phi_n\rangle|^{\frac{2p}{p+\beta}}\right) \\
&\leq \left(\sum_{n=1}^\infty \lambda_n^{2p} |\langle f, \phi_n\rangle|^2\right)^{\frac{\beta}{p+\beta}} \left(\sum_{n=1}^\infty \lambda_n^{-2\beta} |\langle f, \phi_n\rangle|^2\right)^{\frac{p}{p+\beta}} \\
&\leq E^{\frac{2\beta}{p+\beta}} \Big[\sum_{n=1}^\infty \frac{\langle g, \phi_n\rangle^2}{\Big(\lambda_n^\beta \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma)h(s)ds\Big)^2}\Big]^{\frac{p}{p+\beta}}.
\end{aligned}$$

From Lemma 7 it follows that there exists a constant $\overline{C}_1 > 0$ such that

$$\|f\|^2 \le \overline{C}_1^2 E^{\frac{2\beta}{p+\beta}} \|g\|^{\frac{2p}{p+\beta}}.$$

Hence,

$$\|f\| \le \overline{C}_1 \varepsilon^{\frac{p}{p+\beta}} E^{\frac{\beta}{p+\beta}}.$$

The theorem is proved.

Remark 4. The inverse source problem (1.1) is ill-posed. More precisely, the solution f of (1.1), if exists, may not depend continuously on the final-time data. Indeed, since $h(t)$ is a continuous function on $[0, T]$, there exists a constant $C_6 > 0$ such that $C_6 = \sup_{t\in[0,T]} |h(t)| < +\infty$. We have

$$\begin{aligned}
&\left| \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds \right| \\
&\le C_6 \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) ds \\
&= \frac{C_6}{\lambda_n^\beta}(1 - E_{\gamma,1}(-\lambda_n^\beta T^\gamma)) \\
&\le \frac{C_6}{\lambda_n^\beta}.
\end{aligned} \tag{3.5}$$

Therefore

$$\left(\int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds \right)^{-1} \ge \frac{\lambda_n^\beta}{C_6}. \tag{3.6}$$

From (3.6) and Lemma 7, we have $\left(\int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds \right)^{-1}$ behaves like λ_n^β as $n \to \infty$. Hence, a small error in g is amplified by the factor λ_n^β in the formula (2.8) of f. Since $\lambda_n^\beta \to \infty$, f does not depend continuously on the data. Consequently, the inverse source problem (1.1) is ill-posed.

Theorem 2. *In Theorem 1, the estimate*

$$\|f\| \le \overline{C}_1 \varepsilon^{\frac{p}{\beta+p}} E^{\frac{\beta}{\beta+p}}$$

is of optimal order.

Proof. From (3.4), we get

$$\begin{aligned}
&\frac{1}{\int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s)ds} \\
&= \frac{\lambda_n^\beta}{\lambda_n^\beta \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-(T-s)^\gamma \lambda_n^\beta) ds} \\
&\le \frac{\lambda_n^\beta}{C_4}.
\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$\frac{\lambda_n^\beta}{C_6} \le \frac{1}{\int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s)ds} \le \frac{\lambda_n^\beta}{C_4} \tag{3.8}$$

or

$$\frac{\lambda_n^{2p}}{C_6^{2p/\beta}} \le \left(\int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s)ds \right)^{-2p/\beta} \le \frac{\lambda_n^{2p}}{C_4^{2p/\beta}}.$$

This implies that the condition

$$\|f\|_p^2 = \sum_{n=1}^\infty \lambda_n^{2p} \langle f, \phi_n \rangle^2 \le E^2$$

is equivalent to the condition

$$\sum_{n=1}^\infty \left(\int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s)ds \right)^{-2p/\beta} \langle f, \phi_n \rangle^2 \le \widetilde{E}^2 \tag{3.9}$$

with $\widetilde{E} = E C_4^{-p/\beta}$. On the other hand

$$\sum_{n=1}^\infty \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s)ds \, \langle f, \phi_n \rangle \, \phi_n = g.$$

Let us formulate this equation as an operator equation $Bf = g$. We have

$$Bf = \sum_{n=1}^\infty \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s)ds \, \langle f, \phi_n \rangle \, \phi_n.$$

Then B is a continuous linear and self-adjoint operator. Let B^* be the adjoint operator of B. Since $B^* = B$, we have $BB^* = \sum_{n=1}^{\infty} \left(\int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma h(s) ds \right)^2$. Therefore, (3.9) is equivalent to

$$\|(BB^*)^{-p/\beta} f\|^2 \le \widetilde{E}^2.$$

Define $\psi(\lambda) : (0, a] \to \mathbb{R}_+$ with $a \ge \|BB^*\|$ by $\psi(\lambda) = \lambda^{p/\beta}$ and $\rho(\lambda) = \lambda\psi^{-1}(\lambda)$. Then, $\psi(\lambda)$ is strongly monotonic increasing on $(0, a]$ and $\rho(\lambda) = \lambda^{(p+\beta)/p}$ is convex on $(0, a]$. Therefore, $\psi(\lambda)$ and $\rho(\lambda)$ satisfy Assumption 1.1 (p. 379) in [32]. Hence, by Theorem 2.1 in [32], the optimal order has the form

$$\widetilde{E}\sqrt{\rho^{-1}\left(\frac{\varepsilon^2}{\widetilde{E}^2}\right)} = \widetilde{E}^{\frac{\beta}{p+\beta}} \varepsilon^{\frac{p}{\beta+p}}.$$

The proof is complete. □

4. Error estimates for the regularized solution

In this section, we state error estimates for the regularization of problem (1.1) by problem (1.4). We propose a priori and a posteriori methods for choosing the regularization parameter α which yield error estimates of Hölder type. The theoretical results of this paper are stated in Theorem 3 and Theorem 4 below.

4.1. A priori parameter choice rule

Theorem 3. *Suppose that $h(t)$ satisfies Assumption H. For $b \ge \beta$, problem (1.4) is well-posed. Moreover, if the solution f of problem (1.1) satisfies*

$$\|f\|_p \le E, \quad p > 0, \ E > \varepsilon, \tag{4.1}$$

and f_α is solution of problem (1.4) then the following statements hold:
(i) If $0 < p < b$, then with $\alpha = \left(\frac{\varepsilon}{E}\right)^{\frac{b}{p+\beta}}$, there exists a constant $\overline{C}_2$ such that

$$\|f_\alpha - f\| \le \overline{C}_2 \varepsilon^{\frac{p}{p+\beta}} E^{\frac{\beta}{p+\beta}}.$$

(ii) If $p \ge b$, then with $\alpha = \left(\frac{\varepsilon}{E}\right)^{\frac{b}{b+\beta}}$, there exists a constant $\overline{C}_3$ such that

$$\|f_\alpha - f\| \le \overline{C}_3 \varepsilon^{\frac{b}{b+\beta}} E^{\frac{\beta}{b+\beta}}.$$

4.2. A posteriori parameter choice rule

Theorem 4. *Suppose that $h(t)$ satisfies Assumption H. Let $b > 0$ and $\sigma \in (0, 1)$. Suppose that $0 < \varepsilon^\sigma < \|g^\varepsilon\|$. Choose $\tau > 1$ such that $0 < \tau\varepsilon^\sigma \le \|g^\varepsilon\|$. If f satisfies (4.1) and ε is sufficiently small, then the following statements hold:*

(i) If $b > \beta$ then there exists a unique number $\alpha_\varepsilon > 0$ such that

$$\|B_{\alpha_\varepsilon} g^\varepsilon - g^\varepsilon\| = \tau\varepsilon. \tag{4.2}$$

Furthermore, if f_{α_ε} satisfies problem (1.4) then there exist constants $\overline{C}_4, \overline{C}_5$ such that

$$\|f - f_{\alpha_\varepsilon}\| \le \begin{cases} \overline{C}_4 \varepsilon^{\frac{p}{p+\beta}} E^{\frac{\beta}{p+\beta}} & \text{if} \quad 0 < p < b - \beta, \\ \overline{C}_5 \varepsilon^{\frac{b}{b+\beta}} E^{\frac{\beta}{b+\beta}} & \text{if} \quad p \ge b - \beta > 0. \end{cases}$$

(ii) If $b = \beta$, then there exists a unique number $\alpha_\varepsilon > 0$ such that

$$\|B_{\alpha_\varepsilon} g^\varepsilon - g^\varepsilon\| = \tau\varepsilon^\sigma. \tag{4.3}$$

Furthermore, if f_{α_ε} satisfies problem (1.4) then there exists a constant $\overline{C}_6$ such that

$$\|f - f_{\alpha_\varepsilon}\| \le \overline{C}_6 \left(\varepsilon^{\frac{\sigma p}{p+\beta}} E^{\frac{\beta}{p+\beta}} + \varepsilon^{1-\sigma} E \right).$$

Remark 5. Our results in Theorems 3 and 4 are better than those of Dou, Fu and Yang in [4]. Indeed, in the latter, the authors applied the quasi-reversibility regularization method for identifying a space-dependent unknown source function for the classical heat equation but they only consider $\beta = 1$ and $h(t) = 1,\ t \in [0, T]$, whereas we consider β to be an arbitrary positive real number and $h(t)$ only satisfies Assumption H. The regularized function f_α in [4] is given by

$$f_\alpha = \sum_{n=1}^{\infty} \frac{\lambda_n^\beta \langle g^\varepsilon, \phi_n \rangle \phi_n}{(1 + \alpha\lambda_n)\left(1 - E_{\gamma,1}(-\lambda_n^\beta T^\gamma)\right)} \tag{4.4}$$

with $\beta = 1$ and $\gamma = 1$. Note that, for $\beta > 1$ we have $\dfrac{\lambda_n^\beta}{(1 + \alpha\lambda_n)\left(1 - E_{\gamma,1}(-\lambda_n^\beta T^\gamma)\right)} \ge \widehat{C}\alpha^{-1}\lambda_n^{\beta-1} \to +\infty$ as $n \to +\infty$. Therefore, the function f_α determined by formula (4.4) is not stable. This shows that the method in [4] cannot be applied to the case $\beta > 1$.

Concerning the convergence rates, [4] proposed an a priori parameter choice rule and obtained a convergence rate of the form

$$\varepsilon^{\frac{p}{p+2}} E^{\frac{1}{p+2}} \max\{1, \varepsilon^{\frac{2-p}{p+2}}\}. \tag{4.5}$$

The order of this error estimate does not exceed 1/2 for all $p > 0$. On the other hand, by choosing $b \ge \max\{\beta, p\}$ as in Theorem 3 and $b \ge \beta + p$ as in Theorem 4, we achieve in this paper a

convergence rate of optimal order of the form $\varepsilon^{\frac{p}{p+\beta}} E^{\frac{\beta}{p+\beta}}$, which is generally better than that in (4.5).

5. Proofs of the main results

5.1. Proof of Theorem 3

First, we present some auxiliary results.

Lemma 8. *For $b \geq \beta$, problem (1.4) is well-posed. Furthermore, $f_\alpha \in D(A^{b-\beta})$, $v(t) \in D(A^b)$, $t \in [0, T)$ and there exists a constant C_7 such that*

$$\|f_\alpha\| \leq C_7 \alpha^{-\beta/b} \|g^\varepsilon\|.$$

Proof. Similar to (2.6), the solution of problem (1.4) exists and is determined by the formula

$$v(t) = \sum_{n=1}^{\infty} \left(\int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda^\beta (t-s)^\gamma) h(s) \langle f_\alpha, \phi_n \rangle \, ds \right) \phi_n.$$

Similar to Lemma 6, we have

$$\begin{aligned} &\langle g^\varepsilon, \phi_n \rangle \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) \left\langle (I + \alpha A^b) f_\alpha, \phi_n \right\rangle h(s) ds \\ &\quad = (1 + \alpha \lambda_n^b) \langle f_\alpha, \phi_n \rangle \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds. \end{aligned}$$

Therefore, f_α in problem (1.4) is given by

$$f_\alpha = \sum_{n=1}^{\infty} \frac{\langle g^\varepsilon, \phi_n \rangle \phi_n}{(1 + \alpha \lambda_n^b) \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds}, \tag{5.1}$$

and

$$v(t) = \sum_{n=1}^{\infty} \frac{\int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda^\beta (t-s)^\gamma) h(s) ds \, \langle g^\varepsilon, \phi_n \rangle \phi_n}{(1 + \alpha \lambda_n^b) \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds}.$$

Furthermore, by (3.8) we have

$$\begin{aligned}
\|f_\alpha\|_{b-\beta}^2 &= \sum_{n=1}^{\infty} \frac{\lambda_n^{2(b-\beta)} \langle g^\varepsilon, \phi_n\rangle^2}{\left((1+\alpha\lambda_n^b)\int\limits_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s)ds\right)^2} \\
&\le \sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta}\lambda_n^{2(b-\beta)} \langle g^\varepsilon, \phi_n\rangle^2}{C_4^2\left(1+\alpha\lambda_n^b\right)^2} \\
&\le \frac{\|g^\varepsilon\|^2}{\alpha^2 C_4^2} < +\infty.
\end{aligned}$$

This proves that $f_\alpha \in D(A^{b-\beta})$. Similarly, we have the following evaluation

$$\begin{aligned}
\|v(t)\|_b^2 &\le \sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta}\lambda_n^{2b}\left(\int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda^\beta (t-s)^\gamma) h(s)ds\right)^2 \langle g^\varepsilon, \phi_n\rangle^2}{C_4^2\left(1+\alpha\lambda_n^b\right)^2} \\
&\le C_6^2 \sum_{n=1}^{\infty} \frac{\lambda_n^{2\beta}\left(\int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda^\beta (t-s)^\gamma) ds\right)^2 \langle g^\varepsilon, \phi_n\rangle^2}{C_4^2\alpha^2} \\
&= C_6^2 \sum_{n=1}^{\infty} \frac{\left(1-E_{\gamma,1}(-\lambda^\beta t^\gamma)\right)^2 \langle g^\varepsilon, \phi_n\rangle^2}{C_4^2\alpha^2} \\
&\le \frac{C_6^2\|g^\varepsilon\|^2}{\alpha^2 C_4^2} < +\infty
\end{aligned}$$

where C_6 is given in Remark 4. Therefore $v(t) \in D(A^b)$. On the other hand

$$\|f_\alpha\|^2 = \sum_{n=1}^{\infty} \frac{\langle g^\varepsilon, \phi_n\rangle^2}{\left((1+\alpha\lambda_n^b)\int\limits_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s)ds\right)^2}.$$

For $b > \beta$, using Young's inequality, we have

$$1+\alpha\lambda_n^b \ge \frac{b-\beta}{b}.1^{\frac{b}{b-\beta}} + \frac{\beta}{b}\left(\alpha^{\beta/b}\lambda_n^\beta\right)^{b/\beta} \ge \alpha^{\beta/b}\lambda_n^\beta,$$

or

$$1+\alpha\lambda_n^b \ge \alpha^{\beta/b}\lambda_n^\beta \text{ for all } b \ge \beta.$$

It follows that

$$\|f_\alpha\|^2 \le \sum_{n=1}^{\infty} \frac{\alpha^{\frac{-2\beta}{b}} \langle g^\varepsilon, \phi_n\rangle^2}{\left(\lambda_n^\beta\int\limits_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s)ds\right)^2}. \tag{5.2}$$

From Lemma 7 and (5.2), there exists a constant C_7 such that

$$\|f_\alpha\| \le C_7 \alpha^{-\beta/b} \|g^\varepsilon\|.$$

The lemma is proved. □

In the following, we denote by $f_{1\alpha}$ the solution of problem

$$\begin{cases} \dfrac{\partial^\gamma w}{\partial t^\gamma} + A^\beta w = (I + \alpha A^b) f_{1\alpha} h(t), \quad 0 < t < T, \\ w(0) = 0 \\ w(T) = g. \end{cases} \tag{5.3}$$

The solution of (5.3) is given by

$$f_{1\alpha} = \sum_{n=1}^{\infty} \frac{\langle g, \phi_n \rangle \phi_n}{(1 + \alpha \lambda_n^b) \int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds}. \tag{5.4}$$

Lemma 9. *If $f_{1\alpha}$ is the solution of problem (5.3) and f_α is the solution of problem (1.4), then*

$$\|f_\alpha - f_{1\alpha}\| \le C_7 \alpha^{-\beta/b} \varepsilon.$$

Proof. We see that $f_\alpha - f_{1\alpha}$ solves problem (1.4) with g^ε being replaced by $g^\varepsilon - g$. Using Lemma 8, we have

$$\|f_\alpha - f_{1\alpha}\| \le C_7 \alpha^{-\beta/b} \|g^\varepsilon - g\| \le C_7 \alpha^{-\beta/b} \varepsilon.$$

The lemma is proved. □

Lemma 10. *If $\|f\|_p \leqslant E$ for some positive constants p, $E > 0$, then there exists a constant $C_8 > 0$ such that*

$$\|f - f_{1\alpha}\| \leqslant \begin{cases} \alpha^{p/b} E & \text{if} \quad p < b, \\ C_8 \alpha E & \text{if} \quad p \ge b. \end{cases}$$

Proof. We have

$$\begin{aligned} \|f - f_{1\alpha}\|^2 &= \sum_{n=1}^{\infty} \langle f - f_{1\alpha}, \phi_n \rangle^2 \\ &= \sum_{n=1}^{\infty} \Big\{ \frac{\langle g, \phi_n \rangle}{\int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds} \end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{\langle g, \phi_n \rangle}{(1+\alpha\lambda_n^b)\int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds} \Big\}^2 \\
&= \sum_{n=1}^{\infty} \frac{\alpha^2 \lambda_n^{2b} \langle g, \phi_n \rangle^2}{\left((1+\alpha\lambda_n^b)\int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds\right)^2} \\
&= \sum_{n=1}^{\infty} \left(\frac{\alpha\lambda_n^{b-p}}{1+\alpha\lambda_n^b}\right)^2 \lambda_n^{2p} \langle f, \phi_n \rangle^2 . \qquad (5.5)
\end{aligned}
$$

If $p < b$, using Young's inequality, we have

$$
1 + \alpha\lambda_n^b \geq \frac{p}{b}.1^{\frac{b}{p}} + \frac{b-p}{b}\left(\alpha^{(b-p)/b}\lambda_n^{b-p}\right)^{b/(b-p)} \geq \alpha^{(b-p)/b}\lambda_n^{b-p}.
$$

Hence,

$$
\|f - f_{1\alpha}\|^2 \leq \sum_{n=1}^{\infty} \alpha^{2p/b}\lambda_n^{2p} \langle f, \phi_n \rangle^2 \leq \alpha^{2p/b} E^2.
$$

If $p \geq b$, then

$$
\|f - f_{1\alpha}\|^2 \leq \sum_{n=1}^{\infty} \alpha^2 \lambda_1^{2(b-p)} \lambda_n^{2p} \langle f, \phi_n \rangle^2 \leq \lambda_1^{2(b-p)} \alpha^2 E^2.
$$

The lemma is proved. □

Now we are in a position to prove Theorem 3.

Proof of part (i) of Theorem 3.
If $p < b$, from Lemma 9 and Lemma 10 we have

$$
\begin{aligned}
\|f - f_\alpha\| &\leq \|f - f_{1\alpha}\| + \|f_\alpha - f_{1\alpha}\| \\
&\leqslant \alpha^{p/b} E + C_7 \alpha^{-\beta/b} \varepsilon.
\end{aligned}
$$

Choosing $\alpha = \left(\frac{\varepsilon}{E}\right)^{\frac{b}{p+\beta}}$, there exists a constant $\overline{C}_2 > 0$ such that

$$
\|f - f_\alpha\| \leqslant \overline{C}_2 \varepsilon^{\frac{p}{p+\beta}} E^{\frac{\beta}{p+\beta}}.
$$

Part (i) of Theorem 3 is proved.

Proof of part (ii) of Theorem 3.
If $p \geq b$, we have

$$\begin{aligned}\|f - f_\alpha\| &\leq \|f - f_{1\alpha}\| + \|f_\alpha - f_{1\alpha}\| \\ &\leqslant C_8 \alpha E + C_7 \alpha^{-\beta/b} \varepsilon.\end{aligned}$$

Choosing $\alpha = \left(\frac{\varepsilon}{E}\right)^{\frac{b}{b+\beta}}$, there exists a constant $\overline{C}_3 > 0$ such that

$$\|f - f_\alpha\| \leqslant \overline{C}_3 \varepsilon^{\frac{b}{b+\beta}} E^{\frac{\beta}{b+\beta}}.$$

Part (ii) of Theorem 3 is proved. Therefore, the proof of Theorem 3 is complete.

5.2. *Proof of Theorem 4*

First, we need the following lemmas.

Lemma 11. ([7]) *Set $\rho(\alpha) := \|B_\alpha g^\varepsilon - g^\varepsilon\|$ and suppose that $g^\varepsilon \neq 0$. Then*
a) ρ is a continuous function,
b) $\lim\limits_{\alpha \to 0^+} \rho(\alpha) = 0$,
c) $\lim\limits_{\alpha \to +\infty} \rho(\alpha) = \|g^\varepsilon\|$,
d) ρ is a strictly increasing function.

Lemma 12. *Suppose that f is the solution of problem (1.1) satisfying $\|f\|_p \leq E$ and $f_{1\alpha}$ is the solution of problem (5.3) with $b > \beta$. If α_ε satisfies (4.2) then there exists a constant $C_9 > 0$ such that*

$$\|f_{1\alpha_\varepsilon} - f\| \leq C_9 \varepsilon^{p/(p+\beta)} E^{\beta/(p+\beta)}.$$

Proof. Let $z(t) = B_\alpha w(t)$ with $w(t)$ satisfying problem (5.3). Then $z(t)$ solves problem

$$\begin{cases} \dfrac{\partial^\gamma z}{\partial t^\gamma} + Az = f_{1\alpha} h(t), \quad 0 < t < T, \\ z(0) = 0 \\ z(T) = B_\alpha g. \end{cases} \tag{5.6}$$

We have

$$\begin{aligned}\|f - f_{1\alpha_\varepsilon}\|_p &= \sum_{n=1}^{\infty} \lambda_n^{2p} \Big(\langle f, \phi_n \rangle - \frac{\langle g, \phi_n \rangle}{(1 + \alpha_\varepsilon \lambda_n^b) \int\limits_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds} \Big)^2 \\ &= \sum_{n=1}^{\infty} \lambda_n^{2p} \left(\langle f, \phi_n \rangle - \frac{\langle f, \phi_n \rangle}{(1 + \alpha_\varepsilon \lambda_n^b)} \right)^2\end{aligned}$$

$$= \sum_{n=1}^{\infty} \lambda_n^{2p} \left(\frac{\alpha \lambda_n^b \langle f, \phi_n \rangle}{1 + \alpha_\varepsilon \lambda_n^b} \right)^2$$

$$\leq \sum_{n=1}^{\infty} \lambda_n^{2p} \langle f, \phi_n \rangle^2 \leq E^2 \tag{5.7}$$

and

$$\begin{aligned} \|u(T) - z(T)\| &= \|B_{\alpha_\varepsilon} g - g\| \\ &\leq \|B_{\alpha_\varepsilon} g - B_{\alpha_\varepsilon} g^\varepsilon\| + \|B_{\alpha_\varepsilon} g^\varepsilon - g^\varepsilon\| + \|g^\varepsilon - g\| \\ &\leq \left\| \frac{\alpha_\varepsilon \lambda_n^b \langle g - g^\varepsilon, \phi_n \rangle \phi_n}{1 + \alpha_\varepsilon \lambda_n^b} \right\| + \tau\varepsilon + \varepsilon \\ &\leq \|g - g^\varepsilon\| + (\tau + 1)\varepsilon \leq (\tau + 2)\varepsilon. \end{aligned} \tag{5.8}$$

From (5.7), (5.8) and Theorem 1, there exists a constant $C_9 > 0$ such that

$$\|f_{1\alpha_\varepsilon} - f\| \leq C_9 \varepsilon^{p/(p+\beta)} E^{\beta/(p+\beta)}.$$

The lemma is proved. □

Lemma 13. *Suppose f is a solution of problem (1.1) satisfying $\|f\|_p \leq E$ and $f_{1\alpha}$ is the solution of problem (5.3) with $b = \beta$. If α_ε satisfies (4.3) with ε is sufficiently small then there exists a constant $C_{10} > 0$ such that*

$$\|f_{1\alpha_\varepsilon} - f\| \leq C_{10} \varepsilon^{p\sigma/(p+\beta)} E^{\beta/(p+\beta)}.$$

Proof. From the proof of Lemma 12, we have

$$\|f - f_{1\alpha_\varepsilon}\|_p \leq E.$$

Similar to (5.8), with ε is sufficiently small, we obtain

$$\begin{aligned} \|u(T) - z(T)\| &\leq \|B_{\alpha_\varepsilon} g - B_{\alpha_\varepsilon} g^\varepsilon\| + \|B_{\alpha_\varepsilon} g^\varepsilon - g^\varepsilon\| + \|g - g^\varepsilon\| \\ &\leq \varepsilon + \tau\varepsilon^\sigma + \varepsilon \leq (\tau + 2)\varepsilon^\sigma. \end{aligned}$$

Using Theorem 1, there exists a constant $C_{10} > 0$ such that

$$\|f_{1\alpha_\varepsilon} - f\| \leq C_{10} \varepsilon^{p\sigma/(p+\beta)} E^{\beta/(p+\beta)}.$$

The lemma is proved. □

Now we are in a position to prove Theorem 4.

Proof of part (i) of Theorem 4.

It follows from Lemma 11 that there exists a unique number $\alpha_\varepsilon > 0$ satisfying (4.2). From Lemma 9 and Lemma 12, we have

$$\begin{aligned}\|f - f_{\alpha_\varepsilon}\| &\le \|f - f_{1\alpha_\varepsilon}\| + \|f_{1\alpha_\varepsilon} - f_{\alpha_\varepsilon}\| \\ &\le C_9 \varepsilon^{p/(p+\beta)} E^{\beta/(p+\beta)} + C_7 \alpha_\varepsilon^{-\beta/b} \varepsilon. \end{aligned} \tag{5.9}$$

On the other hand,

$$\begin{aligned}\tau\varepsilon = \|B_{\alpha_\varepsilon} g^\varepsilon - g^\varepsilon\| &= \left\| \sum_{n=1}^{\infty} \frac{\alpha_\varepsilon \lambda_n^b \langle g^\varepsilon, \phi_n\rangle \phi_n}{1+\alpha_\varepsilon \lambda_n^b} \right\| \\ &= \left\| \sum_{n=1}^{\infty} \frac{\alpha_\varepsilon \lambda_n^b \langle g, \phi_n\rangle \phi_n}{1+\alpha_\varepsilon \lambda_n^b} - \sum_{n=1}^{\infty} \frac{\alpha_\varepsilon \lambda_n^b \langle g - g^\varepsilon, \phi_n\rangle \phi_n}{1+\alpha_\varepsilon \lambda_n^b} \right\| \\ &\le \left\| \sum_{n=1}^{\infty} \frac{\alpha_\varepsilon \lambda_n^b \Big(\int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s) ds \Big) \langle f, \phi_n\rangle \phi_n}{1+\alpha_\varepsilon \lambda_n^b} \right\| \\ &\quad + \left\| \sum_{n=1}^{\infty} \langle g - g^\varepsilon, \phi_n\rangle \phi_n \right\|. \end{aligned}$$

From (3.5), we obtain

$$\begin{aligned}\tau\varepsilon &\le C_6 \left\| \sum_{n=1}^{\infty} \frac{\alpha_\varepsilon \lambda_n^b \langle f, \phi_n\rangle \phi_n}{\lambda_n^\beta (1+\alpha_\varepsilon \lambda_n^b)} \right\| + \varepsilon \\ &\le C_6 \left\| \sum_{n=1}^{\infty} \frac{\alpha_\varepsilon \lambda_n^{b-\beta} \langle f, \phi_n\rangle \phi_n}{1+\alpha_\varepsilon \lambda_n^b} \right\| + \varepsilon. \end{aligned} \tag{5.10}$$

If $0 < p < b - \beta$, it follows from Lemma 4 that

$$\begin{aligned}\alpha_\varepsilon \lambda_n^b + 1 &\ge \frac{b-p-\beta}{b} \left((\alpha_\varepsilon \lambda_n^b)^{\frac{b-p-\beta}{b}} \right)^{\frac{b}{b-p-\beta}} + \frac{p+\beta}{b} . 1^{\frac{b}{p+\beta}} \\ &\ge (\alpha_\varepsilon \lambda_n^b)^{\frac{b-p-\beta}{b}}. \end{aligned} \tag{5.11}$$

From (5.10) and (5.11), we have

$$(\tau - 1)\varepsilon \le C_6 \alpha_\varepsilon^{\frac{p+\beta}{b}} \left\| \sum_{n=1}^{\infty} \lambda_n^p \langle f, \phi_n\rangle \phi_n \right\| \le C_6 \alpha_\varepsilon^{\frac{p+\beta}{b}} E. \tag{5.12}$$

Hence, from (5.9) and (5.12), there exists a constant $\overline{C}_4 > 0$ such that

$$\|f - f_{\alpha_\varepsilon}\| \le \overline{C}_4 \varepsilon^{p/(p+\beta)} E^{\beta/(p+\beta)}.$$

If $p \ge b - \beta$, then from (5.10) we have

$$\begin{aligned}(\tau - 1)\varepsilon &\le C_4 \alpha_\varepsilon \left\| \sum_{n=1}^{\infty} \lambda_n^{b-\beta} \langle f, \phi_n \rangle \phi_n \right\| \\ &\le C_6 \lambda_1^{b-\beta-p} \alpha_\varepsilon \left\| \sum_{n=1}^{\infty} \lambda_n^{p} \langle f, \phi_n \rangle \phi_n \right\| \\ &\le C_6 \lambda_1^{b-\beta-p} \alpha_\varepsilon E. \end{aligned} \tag{5.13}$$

From (5.9) and (5.13), there exists a constant $\overline{C}_5 > 0$ such that

$$\|f - f_{\alpha_\varepsilon}\| \le \overline{C}_5 \left(\varepsilon^{p/(p+\beta)} E^{\beta/(p+\beta)} + \varepsilon^{(b-\beta)/b} E^{\beta/b} \right).$$

The Part (i) of Theorem 4 is proved.

Proof of part (ii) of Theorem 4.

It follows from Lemma 11 that there exists a unique number $\alpha_\varepsilon > 0$ satisfying (4.3). From Lemma 9 and Lemma 13, we have

$$\begin{aligned}\|f - f_{\alpha_\varepsilon}\| &\le \|f - f_{1\alpha_\varepsilon}\| + \|f_{1\alpha_\varepsilon} - f_{\alpha_\varepsilon}\| \\ &\le C_{10} \varepsilon^{p\sigma/(p+\beta)} E^{\beta/(p+\beta)} + C_7 \alpha_\varepsilon^{-\beta/b} \varepsilon. \end{aligned} \tag{5.14}$$

Similar to (5.10), with $b = \beta$ we obtain

$$\begin{aligned}\tau \varepsilon^{\sigma} &\le C_6 \left\| \sum_{n=1}^{\infty} \frac{\alpha_\varepsilon \langle f, \phi_n \rangle \phi_n}{1 + \alpha_\varepsilon \lambda_n^{\beta}} \right\| + \varepsilon \\ &= C_6 \left\| \sum_{n=1}^{\infty} \frac{\alpha_\varepsilon \lambda_n^{-p} \lambda_n^{p} \langle f, \phi_n \rangle \phi_n}{1 + \alpha_\varepsilon \lambda_n^{\beta}} \right\| + \varepsilon \\ &\le C_6 \alpha_\varepsilon \lambda_1^{-p} \left\| \sum_{n=1}^{\infty} \lambda_n^{p} \langle f, \phi_n \rangle \phi_n \right\| + \varepsilon \\ &\le C_6 \alpha_\varepsilon \lambda_1^{-p} E + \varepsilon. \end{aligned} \tag{5.15}$$

If ε is sufficiently small then

$$(\tau - 1)\varepsilon^{\sigma} \le C_6 \alpha_\varepsilon \lambda_1^{-p} E. \tag{5.16}$$

Hence, from (5.14) and (5.16) we arrive at the conclusion of part (ii) of Theorem 4. Theorem 4 is proved.

6. Numerical algorithm and examples

In this section we analyze the effectiveness of the proposed regularization method using numerical examples with simulated data. The noniterative numerical algorithm used in the following tests is based on the representation (5.1) of the source function f_α. Assuming that the eigenvalues and eigenfunctions of operator A are known, the algorithm approximates the regularized solution f_α using the following partial sum of the series (5.1).

$$f_\alpha \approx \sum_{n=1}^{N_i} \frac{\langle g^\varepsilon, \phi_n \rangle \phi_n}{(1+\alpha\lambda_n^b)\int_0^T (T-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma) h(s)ds}, \tag{6.1}$$

where N_i is an integer. We recall that the regularization parameter is calculated using either the a priori or a posteriori parameter choice rules presented in Theorems 3 and 4. In the latter, the operator B_α in (2.5) should also be approximated using a similar truncated sum.

The computation of f_α using (6.1) requires the evaluation of the Mittag-Leffler function $E_{\gamma,\gamma}$ and the weakly singular integrals in the denominator. For the former, we use an implementation in Matlab by Garrappa using an optimal parabolic contour algorithm [6]. The Matlab code is available for download at https://www.mathworks.com/matlabcentral/fileexchange/48154-the-mittag-leffler-function. For the latter, we use the following approximation method, see also in [33]. Denote by $w_n(s) := E_{\gamma,\gamma}(-\lambda_n^\beta (T-s)^\gamma)h(s)$, $n = 1, 2, \dots$ Note that for each n, $w_n(s)$ is a continuous function on $[0, T]$. Divide $[0, T]$ into k equal intervals by the grid points $0 = t_0 < t_1 < \dots < t_k = T$ with step size $\Delta t = \dfrac{T}{k}$. In each interval (t_i, t_{i+1}), $w_n(s)$ is approximated by the linear function:

$$w_n(s) \approx \frac{1}{\Delta t}\big[w_n(t_i)(t_{i+1}-s) + w_n(t_{i+1})(s-t_i)\big], \quad t_i \le s \le t_{i+1}. \tag{6.2}$$

The integrals in the denominator of (6.1) are approximated by

$$\int_0^T (T-s)^{\gamma-1} w_n(s)ds = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} (T-s)^{\gamma-1} w_n(s)ds$$

$$\approx \frac{1}{\Delta t}\sum_{i=0}^{k-1}\Big[w_n(t_i)\int_{t_i}^{t_{i+1}} (t_{i+1}-s)(T-s)^{\gamma-1}ds + w_n(t_{i+1})\int_{t_i}^{t_{i+1}} (s-t_i)(T-s)^{\gamma-1}ds\Big].$$

The last integrals are elementary. Their values are given by

$$\int_{t_i}^{t_{i+1}} (t_{i+1}-s)(T-s)^{\gamma-1}ds = \frac{t_{i+1}-T}{\gamma}\left[(T-t_{i+1})^\gamma - (T-t_i)^\gamma\right]$$

$$+ \frac{1}{\gamma+1}\left[(T-t_{i+1})^{\gamma+1} - (T-t_i)^{\gamma+1}\right],$$

$$\int_{t_i}^{t_{i+1}} (s-t_i)(T-s)^{\gamma-1} ds = \frac{T-t_i}{\gamma}\left[(T-t_{i+1})^{\gamma} - (T-t_i)^{\gamma}\right] - \frac{1}{\gamma+1}\left[(T-t_{i+1})^{\gamma+1} - (T-t_i)^{\gamma+1}\right].$$

To generate simulated data, we solve the forward problem (i.e., the first two equations of (1.1)) with the exact source function f using expansion (2.6). More precisely, we approximate the infinite series in (2.6) by the following partial sum

$$u(t) \approx \sum_{n=1}^{N_p} \langle f, \phi_n \rangle \phi_n \int_0^t (t-s)^{\gamma-1} E_{\gamma,\gamma}(-\lambda_n^{\beta}(t-s)^{\gamma}) h(s) ds, \tag{6.3}$$

where N_p represents the number of eigenvalues taken in the partial sum. Here N_p is chosen different from N_i to avoid the inverse crime. We have also observed numerically that N_i should be chosen relatively small in order to enhance the stability of the inverse algorithm. Additive uniformly distributed random noise of L^2-norm ε is added to $u(T)$ to obtain noisy measured data g^{ε}. Although ε is the absolute noise level, in the following discussion, we will use relative noise as it represents the signal-to-noise ratio.

In the following we discuss the performance of the proposed algorithm for some fractional equations in one and two spatial dimensions. In the following examples, we chose operator A whose eigenvalues and eigenfunctions are available in closed forms.

Example 1. Consider the following one-dimensional initial boundary value problem as the forward problem:

$$\begin{aligned} \frac{\partial^{\gamma} u(x,t)}{\partial t^{\gamma}} &= \Delta^{\beta} u(x,t) + f(x)h(t), \quad x \in (0,\pi),\ t \in (0,T), \\ u(0,t) &= u(\pi,t) = 0, \quad t \in (0,T), \\ u(x,0) &= 0, \quad x \in (0,\pi), \end{aligned} \tag{6.4}$$

where Δ is the Laplacian with respect to the spatial variable x. The eigenvalues and orthonormal eigenfunctions of the Laplacian with the Dirichlet boundary conditions are given by $\lambda_n = n^2$ and $\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$, $n = 1, 2, \ldots$.

The time-dependent function $h(t)$ was chosen as $h(t) = e^{-t}$ and the final time was chosen as $T = 1$. To approximate the integrals in the denominator of (6.1), we divided the time interval $(0, 1)$ into 50 equal subintervals. The fractional orders were chosen as $\gamma = \beta = 1/2$. In solving the forward problem, 100 eigenvalues and eigenfunctions were used in (6.3), i.e., $N_p = 100$.

The exact source function $f(x)$ was chosen as

$$f(x) := 5e^{-8(x-1)^2} + 2e^{-8(x-2)^2}.$$

The inner products in (6.1) and (6.3) were approximated by the trapezoidal rule using 101 uniform grid points on $[0, \pi]$. In the inverse problem, the parameters were chosen as follows: $N_i = 10$, $p = 2$, $E = 10^6$, and $b = 3$. We considered 4 noise levels of 1%, 2%, 5%, and 10%.

Table 1
Relative error $\|f - f_\alpha\|_{L_2}/\|f\|_{L_2}$ between the exact and reconstructed source functions in Example 1. In the a priori parameter choice rule, $E = 10^6$, $p = 2$. In the a posteriori parameter choice rule, α was chosen according to (4.2) with $\tau = 1.01$. The relative error of f is between 2% and 13.57%. The errors were obtained as the average of 100 runs.

Relative noise level (%)	1.00	2.00	5.00	10.00
Relative error, *a priori* choice rule (%)	2.08	2.88	5.81	11.38
Relative error, *a posteriori* choice rule (%)	2.44	3.81	7.45	13.57

Table 2
Relative error $\|f - f_\alpha\|_{L_2}/\|f\|_{L_2}$ between the exact and reconstructed source functions in Example 1 for four values of b. The effect of b on the reconstruction is insignificant.

Relative noise level (%)	1.00	2.00	5.00	10.00
Relative error, $b = 1$	2.07	2.88	5.76	10.92
Relative error, $b = 2$	2.08	2.85	6.06	11.01
Relative error, $b = 3$	2.08	2.88	5.81	11.38
Relative error, $b = 4$	2.05	2.78	5.75	11.47

Table 1 shows the relative L^2-norm errors of the reconstructed source function for both a priori and a posteriori regularization parameter choice rules. To avoid the effect of a particular set of additive noise, the error was averaged over 100 runs with random noise regenerated in each run. As the table shows, the errors reduce when the data error reduces. We also can see that the a priori parameter choice rule provides slightly better accuracy than the a posteriori parameter choice rule. However, the former depends on the value of E. In this test, E was chosen larger than the exact H_2 norm of the source function.

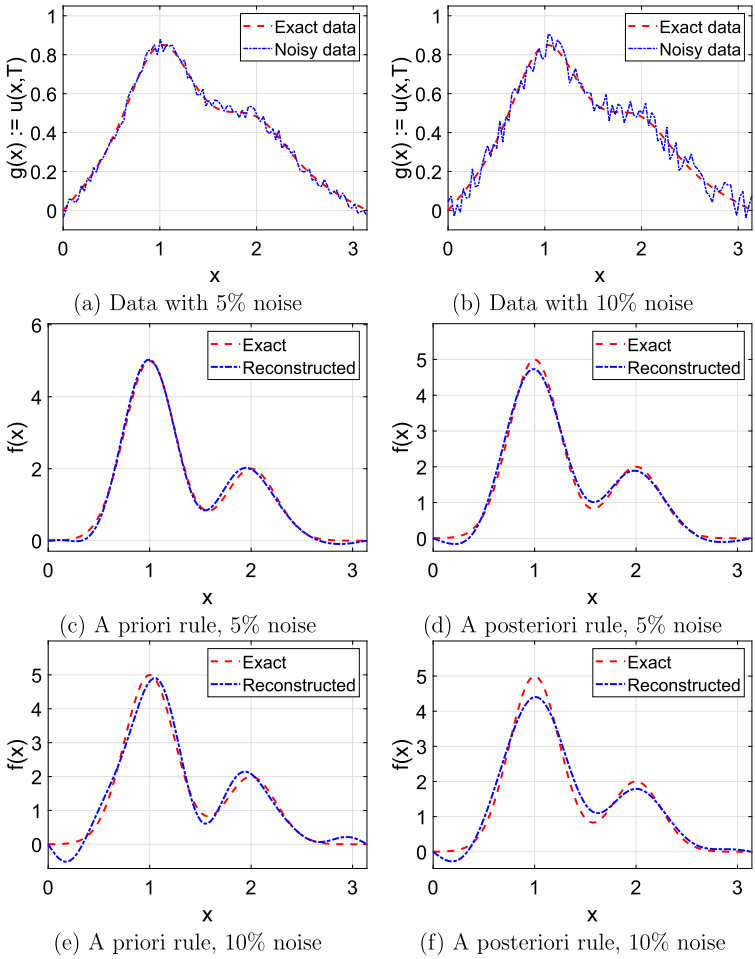
The reconstruction accuracy is also illustrated in Fig. 1 for data corrupted with 5% and 10% noise. We can see that the reconstructed source function follows well the behavior of the exact one, even at 10% noise.

Next, we analyzed the effect of the parameter b on the reconstruction result. For this purpose, we considered four values of b: 1, 2, 3, 4. For each value of b we also took the average error of 100 runs. Table 2 shows that the difference in the reconstruction errors is insignificant. Therefore, in the following examples fixed b at $b = 3$.

Example 2. In the second example, we also considered the same equation as in Example 1. However, the source term was chosen to be the piecewise linear function given by $f(x) = 4f_1(x) + f_2(x)$, where

$$f_1(x) = \begin{cases} 4(x-1.2)/0.3, & 1.2 \le x \le 1.5 \\ 4(1.8-x)/0.3, & 1.5 < x \le 1.8 \\ 0, & \text{otherwise.} \end{cases}$$

$$f_2(x) = \begin{cases} (x-2.2)/0.3, & 2.2 \le x \le 2.5 \\ (2.8-x)/0.3, & 2.5 < x \le 2.8 \\ 0, & \text{otherwise.} \end{cases}$$



(a) Data with 5% noise (b) Data with 10% noise

(c) A priori rule, 5% noise (d) A posteriori rule, 5% noise

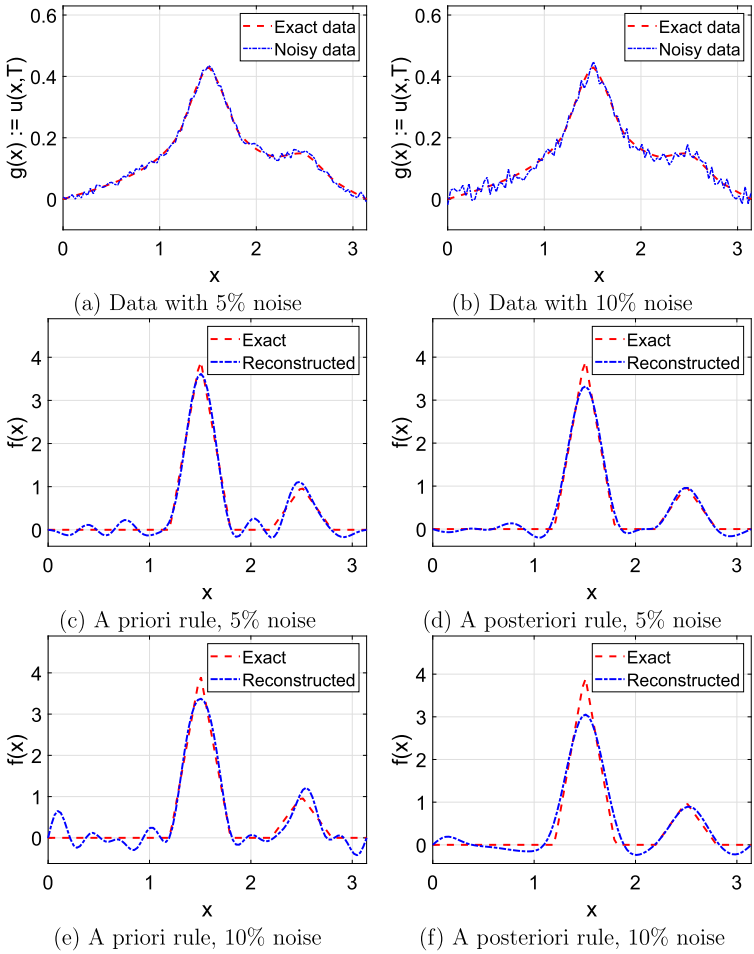
(e) A priori rule, 10% noise (f) A posteriori rule, 10% noise

Fig. 1. Measured data (first row) and the reconstruction results of the source function in Example 1 with 5% noise (second row) and 10% noise (third row).

Table 3
Relative error $\|f - f_\alpha\|_{L_2}/\|f\|_{L_2}$ between the exact and reconstructed source functions in Example 2. In the a priori parameter choice rule, $E = 10^3$, $p = 1$. In the a posteriori parameter choice rule, α was chosen according to (4.2) with $\tau = 1.01$. The errors were obtained as the average of 100 runs.

Relative noise level (%)	1.00	2.00	5.00	10.00
Relative error, *a priori* choice rule (%)	4.79	8.31	16.77	26.70
Relative error, *a posteriori* choice rule (%)	5.07	6.71	11.87	19.37

We note that the source function $f(x)$ in this example has a bounded H_1 norm. Therefore we chose $p = 1$ in the inverse algorithm. Since $f(x)$ is not smooth, the number of eigenfunctions in (6.3) should be chosen large enough. Our numerical experiments have indicated that $N_i = 20$



(a) Data with 5% noise (b) Data with 10% noise

(c) A priori rule, 5% noise (d) A posteriori rule, 5% noise

(e) A priori rule, 10% noise (f) A posteriori rule, 10% noise

Fig. 2. Measured data (first row) and the reconstruction results of the source function in Example 2 with 5% noise (second row) and 10% noise (third row).

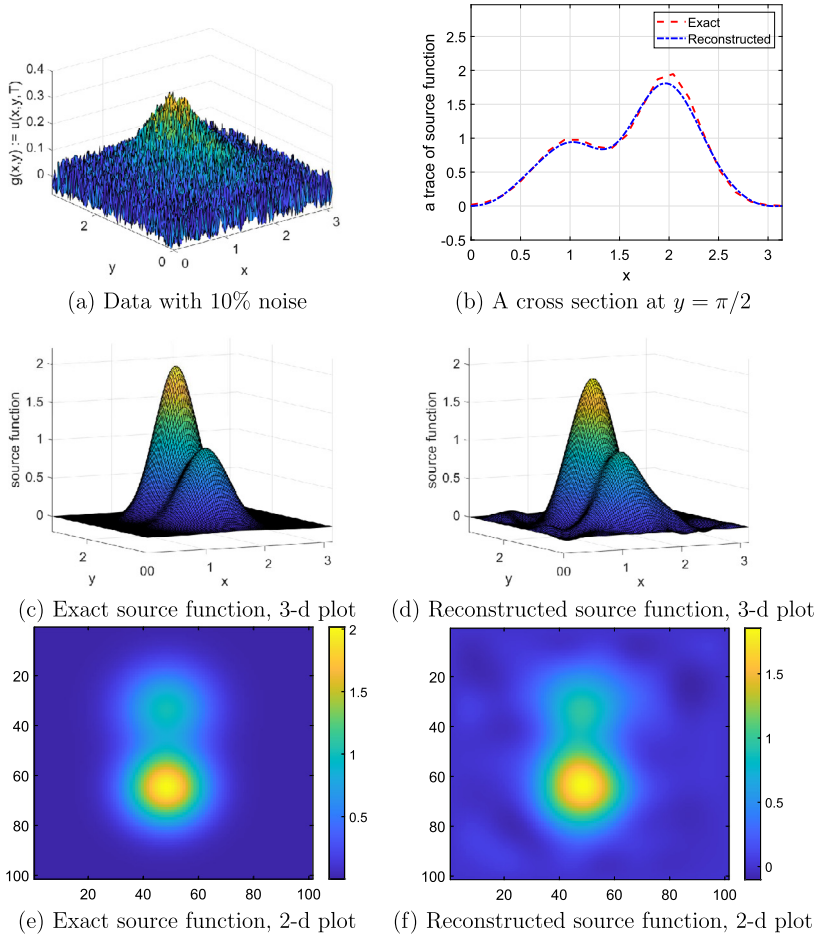
resulted in good approximation. The parameter E was chosen to be $E = 10^3$ since the H_1 norm of $f(x)$ was approximately 200. Other parameters were chosen the same as in Example 1.

Table 3 and Fig. 2 show the results of this example for both a priori and a posteriori parameter choice rules. As can be seen from Fig. 2, the reconstruction accuracy is still very good at 5% of measurement noise. The result at 10% is still reasonably accurate, except at the largest peak of $f(x)$. Note that since the eigenfunctions are smooth, a large number of terms in (6.1) must be required for an accurate approximation of $f(x)$. However, when the number of terms is too large, the inverse problem becomes less stable. This is a trade-off between the accuracy and stability of the proposed inverse algorithm.

Example 3. As the last example, we considered a two-dimensional problem. For the clarity of notation, in this example we use $f(x, y)$ to denote the source function instead of $f(x)$. The forward problem reads:

Table 4
Relative error $\|f - f_\alpha\|_{L_2}/\|f\|_{L_2}$ between the exact and reconstructed source functions in Example 3. In the a priori parameter choice rule, $E = 10^3$, $p = 2$. In the a posteriori parameter choice rule, α was chosen according to (4.2) with $\tau = 1.01$. The errors were obtained as the average of 100 runs.

Relative noise level (%)	1.00	2.00	5.00	10.00
Relative error, *a priori* choice rule (%)	0.75	1.72	3.76	6.08
Relative error, *a posteriori* choice rule (%)	1.01	1.81	4.20	7.51



(a) Data with 10% noise (b) A cross section at $y = \pi/2$

(c) Exact source function, 3-d plot (d) Reconstructed source function, 3-d plot

(e) Exact source function, 2-d plot (f) Reconstructed source function, 2-d plot

Fig. 3. Result of Example 3: First row: measured data with 10% noise and a cross section of the source function at $y = \pi/2$. Second row: 3-d plots of the exact and reconstructed source function. Third row: 2-d plots of the exact and reconstructed source function. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$\frac{\partial^\gamma u(x,y,t)}{\partial t^\gamma} = \Delta^\beta u(x,y,t) + f(x,y)h(t), \quad (x,y) \in (0,\pi)^2,\ t \in (0,T),$$
$$u(0,y,t) = u(\pi,y,t) = 0, \quad t \in (0,T),\ y \in (0,\pi),$$

$$u(x,0,t) = u(x,\pi,t) = 0, \quad t \in (0,T),\ x \in (0,\pi),$$
$$u(x,y,0) = 0, \quad (x,y) \in (0,\pi)^2. \tag{6.5}$$

For this problem, the eigenvalues and eigenfunctions are given as follows:

$$\lambda_{nm} = n^2 + m^2, \quad \phi_{nm}(x,y) = \frac{2\sin(nx)\sin(my)}{\pi}, \quad n,m = 1,2,\dots$$

In this example, we reconstructed the source function $f(x,y)$ of the form:

$$f(x,y) := e^{-4(x-1)^2-4(y-1.5)^2} + 2e^{-5(x-2)^2-5(y-1.5)^2}.$$

Since the source function is smooth, we again chose $p = 2$ as in Example 1. The parameters were chosen as $E = 10^3$, $N_p = 400$, $N_i = 100$. All other parameters were chosen the same as in the previous examples.

The reconstruction results are summarized in Table 4. Fig. 3 shows the reconstructed source function together with the exact one using the a priori parameter choice rule at 10% of noise. The corresponding result using the a posteriori method is very similar, so we do not show it here. As in the previous examples, the source function was accurately reconstructed even at 10% of noise added to the measured data.

7. Conclusions

In summary, we proved a stability estimate of optimal order for the inverse source problem (1.1) under Assumption H. We also proved Hölder-type error estimates for the regularized solution using the quasi-reversibility method. The numerical examples confirmed that the proposed regularization method provided accurate reconstruction results, especially for a smooth source function.

Data availability

No data was used for the research described in the article.

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