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ORIGINAL PAPER

A regularization method for Caputo fractional derivatives in the Banach space $L^\infty[0, T]$

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Abstract
This work is dedicated to the investigation of a regularization method for the problem of determining Caputo fractional derivatives of a function in the Banach space $L^\infty[0, T]$. This regularization method is based on the approximation of the first-order derivative of the function by the solution of a well-posed problem depending on a regularization parameter. We then discuss the Hölder type stability results for the method according to two choice rules for the regularization parameter, which are an a priori parameter choice rule and an a posteriori parameter choice rule. Some numerical examples are provided.

Keywords Caputo fractional derivative · Ill-posed problems · Regularization · A priori parameter choice rule · A posteriori parameter choice rules · Error estimates

1 Introduction

Introduced by M. Caputo [6] for the purpose of formulating viscoelastic problems satisfying the dissipation of energy, Caputo fractional derivatives as well as Caputo partial fractional derivatives have been applied in various fields such as physics, geohydrology [1, 10], and biomedical applications [9]. We also refer the reader to [2–5] and the references therein some recent works on the Caputo fractional derivative and Caputo fractional order differential equations.

It is well-known that the problem of determining Caputo fractional derivatives is ill-posed (see, e.g., [8]); in particular, it is unstable with respect to a small error of the data (e.g., consider a sequence of functions $f_n(t) = \frac{t^{2n}}{n^\alpha}$, $0 < \alpha < 1$, $t \in [0, 1]$,

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$1 \leq n \in \mathbb{Z}$, and a function $f(t) = 0$, then $\|f_n - f\|_\infty = \sup_{t\in[0,1]} |f_n(t) - f(t)| = \sup_{t\in[0,1]} \left|\frac{t^{2n}}{n^\alpha}\right| = \frac{1}{n^\alpha} \to 0$ as $n \to \infty$, thus f_n converges to f in $L^\infty[0, 1]$ as $n \to \infty$; but $|D^\alpha f_n(1) - D^\alpha f(1)| = \frac{1}{\Gamma(1-\alpha)} \frac{2n}{n^\alpha} \int_0^1 \frac{s^{2n-1}}{(1-s)^\alpha} ds \geq \frac{1}{\Gamma(1-\alpha)} \frac{2}{2+\alpha}$ for all n, which implies $D^\alpha f_n(t)$ does not converge to $D^\alpha f(t)$ in $L^\infty[0, 1]$). Therefore, the search for an efficient regularization method to this problem is of great interest, notably the Tikhonov regularization [7] and the mollification method [8]. Inspired by the fact that noisy data are usually given randomly and measured by the L^∞ norm, we are interested in the investigation of a regularization method for such data. The method that we propose to investigate in this paper is based on approximating the first order derivative by the solution of a well-posed problem. Details will be described hereafter. We consider the problem of determining the Caputo fractional derivative of order α with $0 < \alpha < 1$. For each differentiable function q defined in $[0, T]$, the Caputo fractional derivative of order α of q is given by

$$\left(D^{(\alpha)}q\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{q'(s)}{(t-s)^\alpha} ds,\ 0 \leq t \leq T, \tag{1}$$

where Γ is the Gamma function.

Assume that the exact data, function q in (1), is unknown, but we know instead the noisy data, denoted by $q^\delta(t) \in L^\infty[0, T]$, satisfying

$$\|q^\delta - q\| \leq \delta, \tag{2}$$

with $\delta > 0$, denotes the noisy level, is given. Here, the notation $\|\cdot\|$ is to denote the *essential supremum*, i.e.,

$$\|f\| := \inf\left\{M \in \mathbb{R} \text{ s.t. } |f(t)| \leq M \text{ a.e. in } [0, T]\right\},$$

which is the regular norm in $L^\infty[0, T]$.

The objective of the work is to determine $D^{(\alpha)}q$ (approximately) from the knowledge of q^δ. Note that q^δ might not be differentiable. And, even if q^δ is differentiable, formula (1) with q is replaced by q^δ cannot be used to approximate $D^{(\alpha)}q$ due to the ill-posedness of the problem. To handle this issue, we develop a regularization method. That method is based on approximating the first order derivative $q'(t)$ using the solution $u_\beta(t)$ of the following equation:

$$\beta u_\beta(t) + \int_0^t u_\beta(s) ds = q^\delta(t),\ t \in [0, T]. \tag{3}$$

Here, $\beta > 0$ is called the regularization parameter. Later on, we will discuss the choice of β and as well as the Hölder type of estimation of errors. The well-posedness of the problem of determining u_β satisfying (3) and the explicit representation of its solution, that is,

$$u_\beta(t) = -\frac{1}{\beta^2}e^{-t/\beta}\int_0^t e^{s/\beta}q^\delta(s)ds + \frac{q^\delta(t)}{\beta},\ t \in [0, T], \tag{4}$$

are detailed in Appendix A.

The Caputo fractional derivative $D^{(\alpha)}q$ is then approximated by a function $u_{\alpha,\beta}$ as follows:

$$u_{\alpha,\beta}(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u_\beta(s)}{(t-s)^\alpha}ds,\ 0 \le t \le T. \tag{5}$$

To study the stability of the method with respect to the regularization parameter β, we assume further that the exact data $q(t)$ is of twice continuously differentiable function on $[0, T]$ satisfying $q(0) = q'(0) = 0$, and there exists a constant $E > \delta > 0$ (E is not required to be given explicitly) such that

$$\|q''\| \le E. \tag{6}$$

This paper is organized as follows: In Section 2, we briefly introduce some auxiliary results and some assumptions for later use. In Section 3, we present the main results on the stability corresponding to two choice rules, an a priori parameter choice rule and an a posteriori parameter choice rule, of the regularization parameter β. The numerical part is discussed in Section 4.

2 Some auxiliary results

Before discussing the stability results, we provide the following auxiliary result for later use.

Theorem 2.1 *Let $v_\beta(t)$ be the solution to the equation*

$$\beta v_\beta(t) + \int_0^t v_\beta(s)ds = q(t),\ \beta > 0,\ t \in [0, T], \tag{7}$$

and $v_{\alpha,\beta}(t)$ be defined by

$$v_{\alpha,\beta}(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{v_\beta(s)}{(t-s)^\alpha}ds,\ 0 \le t \le T, 0 < \alpha < 1. \tag{8}$$

Then, the following estimates hold

a) $\|u_\beta - v_\beta\| \le \dfrac{2 - e^{-T/\beta}}{\beta}\delta,$

b) $\|u_{\alpha,\beta} - v_{\alpha,\beta}\| \le \dfrac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\dfrac{2 - e^{-T/\beta}}{\beta}\delta.$

where $u_\beta(t)$ and $u_{\alpha,\beta}(t)$ defined in (4) and (5) respectively.

Proof Let $w_\beta := u_\beta - v_\beta$ and $w_{\alpha,\beta} := u_{\alpha,\beta} - v_{\alpha,\beta}$ in $[0, T]$. Then, $w_\beta(t)$ solves the equation

$$\beta w_\beta(t) + \int_0^t w_\beta(s)ds = q^\delta(t) - q(t),\ t \in [0, T] \tag{9}$$

and

$$w_{\alpha,\beta}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{w_\beta(s)}{(t-s)^\alpha} ds,\ 0 \le t \le T, 0 < \alpha < 1. \tag{10}$$

It is known that (9) has a unique solution (see Appendix A, Theorem A.1):

$$\begin{aligned} w_\beta(t) &= -\frac{1}{\beta^2} e^{-t/\beta} \int_0^t e^{s/\alpha} \left(q^\delta(s) - q(s)\right) ds + \frac{q^\delta(t) - q(t)}{\beta} \\ &= g_\beta(t) + \frac{q^\delta(t) - q(t)}{\beta},\ t \in [0, T], \end{aligned} \tag{11}$$

with $g_\beta(t) := -\frac{1}{\beta^2} e^{-t/\beta} \int_0^t e^{s/\alpha} \left(q^\delta(s) - q(s)\right) ds$. Thus,

$$\|w_\beta\| \le \|g_\beta\| + \frac{1}{\beta} \|q^\delta - q\|. \tag{12}$$

In addition, using the analytical formula of $g_\beta(t)$, we estimate

$$\begin{aligned} |g_\beta(t)| = \left| \frac{1}{\beta^2} e^{-t/\beta} \int_0^t e^{s/\beta} \left(q^\delta(s) - q(s)\right) ds \right| &\le \frac{1}{\beta^2} e^{-t/\beta} \int_0^t e^{s/\beta} \|q^\delta - q\| ds \\ &= \frac{\|q^\delta - q\|}{\beta} \left(1 - e^{-t/\beta}\right) \\ &\le \left(1 - e^{-T/\beta}\right) \frac{\|q^\delta - q\|}{\beta},\ \forall t \in [0, T], \end{aligned}$$

which implies

$$\|g_\beta\| \le \left(1 - e^{-T/\beta}\right) \frac{\|q^\delta - q\|}{\beta}. \tag{13}$$

Combining (12) and (13), we obtain

$$\|w_\beta\| \le \frac{2 - e^{-T/\beta}}{\beta} \|q^\delta - q\| \le \frac{2 - e^{-T/\beta}}{\beta} \delta,$$

which proves the first estimation.
In addition, from (10), we have

$$\left|w_{\alpha,\beta}(t)\right| = \left|\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{w_\beta(s)}{(t-s)^\alpha}ds\right| \leq \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{|w_\beta(s)|}{(t-s)^\alpha}ds$$
$$\leq \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\|w_\beta\|}{(t-s)^\alpha}ds \leq \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\|w_\beta\|.$$

Combine with the first estimation, we end up with

$$\|w_{\alpha,\beta}\| \leq \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\frac{2-e^{-T/\beta}}{\beta}\delta,$$

that proves the theorem. □

As mentioned in the introduction, the Caputo fractional derivative of the noisy data, $D^{(\alpha)}q^\delta$, is approximated by $u_{\alpha,\beta}$, which is determined by (4)-(5). In the next part, we will discuss the stability of this approximation, i.e., estimate the error between $u_{\alpha,\beta}$ and $D^{(\alpha)}q$ knowing (2).

3 Stability results

The stability estimates of the method depend on how to choose the regularization parameter β. We here propose to discuss two ways to choose the parameter, that are an a prior parameter choice rule and an a posteriori parameter choice rule.

3.1 A prior parameter choice rule

For this rule, the choice of parameter β is independent on the measured data, but it depends on the noisy level. The following theorem will summary the error estimates for an arbitrary choice and a special choice of β.

Theorem 3.1 *Let $u_\beta(t)$ and $u_{\alpha,\beta}(t)$ be defined by (4) and (5) (respectively). The following estimate holds for all $\beta > 0$*

$$\left\|u_{\alpha,\beta} - \left(D^{(\alpha)}q\right)\right\| \leq \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\left(\frac{2-e^{-T/\beta}}{\beta}\delta + (1-e^{-T/\beta})\beta E\right).$$

In particular, if the constant E is known, choose $\beta = \sqrt{\frac{2\delta}{E}}$, the error estimate is

$$\|u_{\alpha,\beta} - \left(D^{(\alpha)}q\right)\| \leq \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}2\sqrt{2\delta E};$$

otherwise, choose $\beta = \sqrt{\delta}$*, the error estimate is*

$$\|u_{\alpha,\beta} - \left(D^{(\alpha)}q\right)\| \le \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\sqrt{\delta}(2+E).$$

Proof Apply the triangle inequality

$$\|u_\beta - q'\| \le \|u_\beta - v_\beta\| + \|v_\beta - q'\|, \tag{14}$$

where $\|u_\beta - v_\beta\| \le \dfrac{2-e^{-T/\beta}}{\beta}\delta$ is obtained by Theorem 2.1. It remains to prove that

$$\|v_\beta - q'\| \le \left(1 - e^{-T/\beta}\right)\beta E.$$

Denote by

$$w_\beta(t) := q'(t) - v_\beta(t),\ t \in [0, T],$$

then w_β verifies

$$\begin{aligned} \beta w_\beta(t) + \int_0^t w_\beta(z)dz &= \beta(q'(t) - v_\beta(t)) + \int_0^t \left(q'(s) - v_\beta(s)\right) ds \\ &= \beta q'(t) - \left(\beta v_\beta(t) + \int_0^t v_\beta(s)ds\right) + q(s)\Big|_0^t \\ &= \beta q'(t) - q(t) + q(t) - q(0) = \beta q'(t) - q(0) \ = \ \beta q'(t). \end{aligned} \tag{15}$$

That deduces, w_β is the solution of (3) with the right hand side is replaced by $\beta q'(t)$. Applying Theorem A.1 yields,

$$w_\beta(t) = -\frac{1}{\beta}e^{-t/\beta}\int_0^t e^{s/\beta}q'(s)ds + q'(t),\ t \in [0, T]. \tag{16}$$

Integrating by part, we can simplify (16) as

$$\begin{aligned} w_\beta(t) &= -\frac{1}{\beta}e^{-t/\beta}\int_0^t e^{s/\beta}q'(s)ds + q'(t) \\ &= -e^{-t/\beta}\int_0^t q'(s)d\left(e^{s/\beta}\right) + q'(t) \\ &= -e^{-t/\beta}\left(\left(q'(s)e^{s/\beta}\right)\Big|_0^t - \int_0^t e^{s/\beta}d(q'(s))\right) + q'(t) \\ &= -e^{-t/\beta}\left(q'(t)e^{t/\beta} - q'(0) - \int_0^t e^{s/\beta}q''(s)ds\right) + q'(t) \\ &= e^{-t/\beta}\int_0^t e^{s/\beta}q''(s)ds,\ \forall t \in [0, T]. \end{aligned} \tag{17}$$

Using the boundedness of q'' by a constant E, we obtain from (17) that $\forall t \in [0, T]$,

$$|w_\beta(t)| \leq e^{-t/\beta} E \int_0^t e^{s/\beta} ds = E e^{-t/\beta} \left(\beta e^{s/\beta}\right)\Big|_0^t = \beta E \left(1 - e^{-t/\beta}\right)$$
$$\leq \left(1 - e^{-T/\beta}\right) \beta E, \qquad (18)$$

which implies

$$\|w_\beta\| \leq \left(1 - e^{-T/\beta}\right) \beta E. \qquad (19)$$

Combining (14), (19), and the first item of Theorem 2.1, we have

$$\|u_\beta - q'\| \leq \left(\frac{2 - e^{-T/\beta}}{\beta}\delta + \left(1 - e^{-T/\beta}\right) \beta E\right). \qquad (20)$$

Next, we have that

$$\begin{aligned}
\left|u_{\alpha,\beta}(t) - \left(D^{(\alpha)} q\right)(t)\right| &= \left|\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_\beta(s) - q'(s)}{(t-s)^\alpha} ds\right| \\
&\leq \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{|u_\beta(s) - q'(s)|}{(t-s)^\alpha} ds \\
&\leq \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\|u_\beta - q'\|}{(t-s)^\alpha} ds \\
&\leq \left(\frac{2 - e^{-T/\beta}}{\beta}\delta + \left(1 - e^{-T/\beta}\right) \beta E\right) \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} ds \\
&= \left(\frac{2 - e^{-T/\beta}}{\beta}\delta + \left(1 - e^{-T/\beta}\right) \beta E\right) \frac{t^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \\
&\leq \left(\frac{2 - e^{-T/\beta}}{\beta}\delta + \left(1 - e^{-T/\beta}\right) \beta E\right) \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}, \ \forall t \in [0, T],
\end{aligned}$$

which implies,

$$\left\|u_{\alpha,\beta} - \left(D^{(\alpha)} q\right)\right\| \leq \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \left(\frac{2 - e^{-T/\beta}}{\beta}\delta + (1 - e^{-T/\beta})\beta E\right). \qquad (21)$$

Substitute $\beta = \sqrt{\frac{2\delta}{E}}$ and $\beta = \sqrt{\delta}$ into the inequality (21), we obtain the results that stated in the theorem. □

Based on the Theorem 3.1, we establish the following stability estimates.

Theorem 3.2 *(stability estimates) Suppose that $q_i(t)$, $i = 1, 2$ are twice continuously differentiable functions in $L^\infty[0, T]$ satisfying $q_i(0) = q_i'(0) = 0$, $\|q_i''\| \le E$, $i = 1, 2$, and*

$$\|q_1 - q_2\| \le \delta. \tag{22}$$

The following estimate holds

$$\|D^{(\alpha)}q_1 - D^{(\alpha)}q_2\| \le \frac{4T^{1-\alpha}\sqrt{\delta E}}{(1-\alpha)\Gamma(1-\alpha)}.$$

Proof Let $q^{\frac{\delta}{2}}(t) = \dfrac{q_1(t) + q_2(t)}{2}$, $\forall t \in [0, T]$. We have $q^{\frac{\delta}{2}} \in L^\infty[0, T]$ and

$$\|q_1 - q^{\frac{\delta}{2}}\| = \frac{1}{2}\|q_1 - q_2\| \le \frac{\delta}{2} \tag{23}$$

$$\|q_2 - q^{\frac{\delta}{2}}\| = \frac{1}{2}\|q_1 - q_2\| \le \frac{\delta}{2}. \tag{24}$$

Let denote by $f_\beta(t)$ the solution to equation

$$\beta f_\beta(t) + \int_0^t f_\beta(s)ds = q^{\frac{\delta}{2}}(t), \ \beta > 0, \ t \in [0, T], \tag{25}$$

and $f_{\alpha,\beta}(t)$ be defined by

$$f_{\alpha,\beta}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f_\beta(s)}{(t-s)^\alpha} ds, \ 0 \le t \le T, 0 < \alpha < 1. \tag{26}$$

From Theorem 3.1, we obtain

$$\left\| f_{\alpha,\beta} - D^{(\alpha)}q_1 \right\| \le \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \left(\frac{2 - e^{-T/\beta}}{\beta} \frac{\delta}{2} + (1 - e^{-T/\beta})\beta E \right),$$

$$\left\| f_{\alpha,\beta} - D^{(\alpha)}q_2 \right\| \le \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \left(\frac{2 - e^{-T/\beta}}{\beta} \frac{\delta}{2} + (1 - e^{-T/\beta})\beta E \right).$$

This implies that

$$\left\| f_{\alpha,\beta} - D^{(\alpha)}q_1 \right\| \le \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \left(\frac{\delta}{\beta} + \beta E \right),$$

$$\left\| f_{\alpha,\beta} - D^{(\alpha)}q_2 \right\| \le \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \left(\frac{\delta}{\beta} + \beta E \right).$$

By choosing $\beta = \sqrt{\frac{\delta}{E}}$, we get

$$\left\| f_{\alpha,\beta} - D^{(\alpha)} q_1 \right\| \leq \frac{2T^{1-\alpha}\sqrt{\delta E}}{(1-\alpha)\Gamma(1-\alpha)},$$
$$\left\| f_{\alpha,\beta} - D^{(\alpha)} q_2 \right\| \leq \frac{2T^{1-\alpha}\sqrt{\delta E}}{(1-\alpha)\Gamma(1-\alpha)}.$$

The triangle inequality yields

$$\|D^{(\alpha)} q_1 - D^{(\alpha)} q_2\| \leq \left\| f_{\alpha,\beta} - D^{(\alpha)} q_1 \right\| + \left\| f_{\alpha,\beta} - D^{(\alpha)} q_2 \right\| \leq \frac{4T^{1-\alpha}\sqrt{\delta E}}{(1-\alpha)\Gamma(1-\alpha)}.$$

The theorem is proved. □

3.2 A posteriori parameter choice rule

For this rule, the choice of the parameter β is in terms of the measured data. We assume that there exists a constant $\gamma \in (0, 1)$ such that $0 < \delta^\gamma < \|q^\delta\|$. Let $\tau > 1$ such that $0 < \tau\delta^\gamma < \|q^\delta\|$, and let $\beta_\delta > 0$ such that

$$\|\beta_\delta u_{\beta_\delta}\| = \tau\delta^\gamma, \tag{27}$$

where u_{β_δ} is determined by (4) with β is replaced by β_δ. We then choose $\beta = \beta_\delta$ as the regularization parameter. The existence of β_δ is guaranteed thanks to Lemma 3.3 below.

Lemma 3.3 *Let* $\rho(\beta) := \|\beta u_\beta\|, \ \beta > 0$. *Then,*

a. $\lim_{\beta \to +\infty} \rho(\beta) = \|q^\delta\|$,
b. $\rho(\beta) < \tau\delta^\gamma$ *with* δ, β *are small enough,*
c. $\rho(\beta)$ *is a continuous function in* $(0, +\infty)$.

Proof To prove the first item, we reformulate (4) as

$$\beta u_\beta(t) - q^\delta(t) = -\frac{1}{\beta} e^{-t/\beta} \int_0^t e^{z/\beta} q^\delta(z) dz, \ t \in [0, T] \tag{28}$$

and see that the right hand side is upper bounded by a linear quantity of $\frac{1}{\beta}$ for all $t \in [0, T]$ as follows:

$$\begin{aligned} \left| -\frac{1}{\beta} e^{-t/\beta} \int_0^t e^{z/\beta} q^\delta(z) dz \right| &\leq \frac{1}{\beta} e^{-t/\beta} \int_0^t e^{z/\beta} |q^\delta(z)| dz \\ &\leq \frac{1}{\beta} e^{-t/\beta} \int_0^t e^{z/\beta} \|q^\delta\| dz \\ &= \frac{1}{\beta} \|q^\delta\| \int_0^t e^{(z-t)/\beta} dz \leq \frac{1}{\beta} \|q^\delta\| T. \end{aligned} \tag{29}$$

From (28) and (29), we have

$$\|q^\delta\| - \frac{1}{\beta}\|q^\delta\|T \le \rho(\beta) \le \|q^\delta\| + \frac{1}{\beta}\|q^\delta\|T, \tag{30}$$

which implies $\lim_{\beta\to+\infty} \rho(\beta) = \|q^\delta\|$ since $\lim_{\beta\to+\infty} \frac{1}{\beta}\|q^\delta\|T = 0$.
Next part, we prove the second item. We have that

$$\begin{aligned}\beta u_\beta(t) &= -\frac{1}{\beta}e^{-t/\beta}\int_0^t e^{z/\beta}q^\delta(z)dz + q^\delta(t)\\ &= -\frac{1}{\beta}e^{-t/\beta}\int_0^t e^{z/\beta}\big(q^\delta(z)-q(z)\big)dz + \big(q^\delta(t)-q(t)\big)\\ &\quad + \left(q(t) - \frac{1}{\beta}e^{-t/\beta}\int_0^t e^{z/\beta}q(z)dz\right).\end{aligned}$$

Therefore, for all $t \in [0, T]$

$$\begin{aligned}|\beta u_\beta(t)| \le &\left|-\frac{1}{\beta}e^{-t/\beta}\int_0^t e^{z/\beta}(q^\delta(z)-q(z))dz\right| + \|q^\delta - q\|\\ &+ \left|q(t) - \frac{1}{\beta}e^{-t/\beta}\int_0^t e^{z/\beta}q(z)dz\right|.\end{aligned} \tag{31}$$

Here,

$$\begin{aligned}\left|-\frac{1}{\beta}e^{-t/\beta}\int_0^t e^{z/\beta}(q^\delta(z)-q(z))dz\right| &\le \frac{1}{\beta}e^{-t/\beta}\int_0^t e^{z/\beta}|q^\delta(z)-q(z)|dz\\ &\le \frac{1}{\beta}e^{-t/\beta}\int_0^t e^{z/\beta}\|q^\delta - q\|dz\\ &= \|q^\delta - q\|(1-e^{-t/\beta}) \le \|q^\delta - q\|.\end{aligned} \tag{32}$$

In addition, integrating by part the last term of (31) and using $q(0) = 0$, we have

$$q(t) - \frac{1}{\beta}e^{-t/\beta}\int_0^t e^{z/\beta}q(z)dz = e^{-t/\beta}\int_0^t e^{z/\beta}q'(z)dz. \tag{33}$$

From $q'(0) = 0$ and $\|q''\| \le E$, it follows that for all $z \in [0, T]$

$$\begin{aligned}|q'(z)| = |q'(z) - q'(0)| &= \left|\int_0^z q''(s)ds\right|\\ &\le \int_0^z |q''(s)|ds \le \int_0^T |q''(s)|ds\\ &\le \int_0^T \|q''\|ds \le E\int_0^T ds = ET.\end{aligned} \tag{34}$$

We obtain from (33) and (34) that

$$\left|q(t)-\frac{1}{\beta}e^{-t/\beta}\int_0^t e^{z/\beta}q(z)dz\right| \leq ETe^{-t/\beta}\int_0^t e^{z/\beta}dz \leq \beta ET(1-e^{-t/\beta}) \leq \beta ET. \tag{35}$$

From (31), (32), and (35), we have

$$|\beta u_\beta(t)| \leq 2\|q^\delta - q\| + \beta ET \leq 2\delta + \beta ET,\ \forall t \in [0, T]. \tag{36}$$

Thus,

$$\rho(\beta) = \|\beta u_\beta\| \leq 2\delta + \beta ET \leq 3\delta \tag{37}$$

for all $\beta \in \left(0, \dfrac{\delta}{ET}\right)$. We note that if $0 < \delta < \left(\frac{\tau}{3}\right)^{\frac{1}{1-\gamma}}$ then

$$\rho(\beta) \leq 3\delta < \tau\delta^\gamma.$$

Therefore, $\rho(\beta) < \tau\delta^\gamma$ with δ, β are small enough.
For the proof of the last item, we first see that function $\beta \mapsto \beta u_\beta$ in $(0, +\infty)$ is continuous due to the continuitivity function $\beta \mapsto \dfrac{1}{\beta}e^{-t/\beta}\displaystyle\int_0^t e^{z/\beta}q^\delta(z)dz$ on $(0, +\infty)$. As a result,

$$\left|\rho(\beta) - \rho(\beta_1)\right| = \left|\|\beta u_\beta\| - \|\beta_1 u_{\beta_1}\|\right| \leq \|\beta u_\beta - \beta_1 u_{\beta_1}\| \to 0,\ \text{as } \beta \to \beta_1,\ \forall \beta_1 \in (1, +\infty)$$

which implies $\rho(\beta)$ is continuous with respect to β on $(0, +\infty)$. □

Theorem 3.4 below summarizes the error estimate according to the a posteriori choice of the regularization parameter.

Theorem 3.4 *If $u_{\beta_\delta}(t)$ is the solution of problem (4) and $u_{\alpha,\beta_\delta}(t)$ is determined by (5) with β is replaced by β_δ, the following estimate holds for δ that is small enough*

$$\left\|u_{\alpha,\beta_\delta} - \left(D^{(\alpha)}q\right)\right\| \leq \frac{2T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\left(2\sqrt{(\tau\delta^\gamma + 2\delta)E} + \delta^{1-\gamma}TE\right).$$

Proof The proof of this theorem is directly obtained from Lemma 3.5 and Lemma 3.6 below and the triangular inequality of norms. □

Lemma 3.5 *If $v_{\beta_\delta}(t)$ is the solution of problem (7) and $v_{\alpha,\beta_\delta}(t)$ is determined by (8) with β is replaced by β_δ, the following estimate holds*

$$\left\|v_{\alpha,\beta_\delta} - \left(D^{(\alpha)}q\right)\right\| \leq \frac{4T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\sqrt{(\tau\delta^\gamma + 2\delta)E}.$$

Proof Denote by $u_{\beta_\delta}(t)$ the solution to (3) with $\beta_\delta > 0$ is solution to (27). From Theorem 2.1 a), we have

$$\|u_{\beta_\delta} - v_{\beta_\delta}\| \leq \frac{1 - e^{-T/\beta_\delta}}{\beta_\delta}\delta < \frac{2}{\beta_\delta}\delta,$$

which implies

$$\|\beta_\delta u_{\beta_\delta} - \beta_\delta v_{\beta_\delta}\| < 2\delta. \tag{38}$$

Combining (38) with $\|\beta_\delta u_{\beta_\delta}\| = \tau\delta^\gamma$ and the triangular inequality of norms, we have

$$\|\beta_\delta v_{\beta_\delta}\| = \|\beta_\delta v_{\beta_\delta} - \beta_\delta u_{\beta_\delta} + \beta_\delta u_{\beta_\delta}\| \leq \|\beta_\delta v_{\beta_\delta} - \beta_\delta u_{\beta_\delta}\| + \|\beta_\delta u_{\beta_\delta}\| < 2\delta + \tau\delta^\gamma.$$

Set $\widetilde{q}(t) = \int_0^t v_{\beta_\delta}(z)dz,\ t \in [0, T]$. Clearly, $\widetilde{q}(0) = 0$. Moreover, v_{β_δ} solves equation

$$\beta_\delta v_{\beta_\delta}(t) + \int_0^t v_{\beta_\delta}(s)ds = q(t),$$

which yields

$$\|q - \widetilde{q}\| = \left\|q(t) - \int_0^t v_{\beta_\delta}(z)dz\right\| = \|\beta_\delta v_{\beta_\delta}\| < \tau\delta^\gamma + 2\delta. \tag{39}$$

Similar to (4), v_{β_δ} is represented by

$$v_{\beta_\delta}(t) = -\frac{1}{{\beta_\delta}^2}e^{-t/\beta_\delta}\int_0^t e^{z/\beta_\delta}q(z)dz + \frac{q(t)}{\beta_\delta},\ t \in [0, T]. \tag{40}$$

The assumption $q(0) = 0$ and (40) implies $\widetilde{q}'(0) = v_{\beta_\delta}(0) = 0$. Differentiating both side of (40) with respect to t obtains

$$\begin{aligned}
\widetilde{q}''(t) = v'_{\beta_\delta}(t) &= -\frac{1}{{\beta_\delta}^2}\left(-\frac{1}{\beta_\delta}\right)e^{-t/\beta_\delta}\int_0^t e^{z/\beta_\delta}q(z)dz - \frac{1}{{\beta_\delta}^2}e^{-t/\beta_\delta}e^{t/\beta_\delta}q(t) + \frac{q'(t)}{\beta_\delta}\\
&= \frac{1}{{\beta_\delta}^3}e^{-t/\beta_\delta}\int_0^t e^{z/\beta_\delta}q(z)dz - \frac{1}{{\beta_\delta}^2}q(t) + \frac{q'(t)}{\beta_\delta}\\
&= \frac{1}{{\beta_\delta}^2}e^{-t/\beta_\delta}\int_0^t q(z)d\left(e^{z/\beta_\delta}\right) - \frac{1}{{\beta_\delta}^2}q(t) + \frac{q'(t)}{\beta_\delta}\\
&= \frac{1}{{\beta_\delta}^2}e^{-t/\beta_\delta}q(z)e^{z/\beta_\delta}\Big|_0^t - \frac{1}{{\beta_\delta}^2}e^{-t/\beta_\delta}\int_0^t e^{z/\beta_\delta}q'(z)dz - \frac{1}{{\beta_\delta}^2}q(t) + \frac{q'(t)}{\beta_\delta}\\
&= -\frac{1}{{\beta_\delta}^2}e^{-t/\beta_\delta}\int_0^t e^{z/\beta_\delta}q'(z)dz + \frac{q'(t)}{\beta_\delta}
\end{aligned}$$

$$= -\frac{1}{\beta_\delta} e^{-t/\beta_\delta} \int_0^t q'(z) d\left(e^{z/\beta_\delta}\right) + \frac{q'(t)}{\beta_\delta}$$
$$= -\frac{1}{\beta_\delta} e^{-t/\beta_\delta} e^{z/\beta_\delta} q'(z)\Big|_0^t + \frac{1}{\beta_\delta} e^{-t/\beta_\delta} \int_0^t e^{z/\beta_\delta} q''(z) dz + \frac{q'(t)}{\beta_\delta}$$
$$= \frac{1}{\beta_\delta} e^{-t/\beta_\delta} \int_0^t e^{z/\beta_\delta} q''(z) dz.$$

By assumption

$$\|q''\| \le E \tag{41}$$

we can estimate

$$\begin{aligned} |\widetilde{q}''(t)| = |v'_{\beta_\delta}(t)| &= \left| \frac{1}{\beta_\delta} e^{-t/\beta_\delta} \int_0^t e^{z/\beta_\delta} q''(z) dz \right| \\ &\le \frac{1}{\beta_\delta} e^{-t/\beta_\delta} \int_0^t e^{z/\beta_\delta} |q''(z)| dz \\ &\le \frac{1}{\beta_\delta} e^{-t/\beta_\delta} \int_0^t e^{z/\beta_\delta} \|q''\| dz \\ &\le \|q''\| e^{-t/\beta_\delta} e^{z/\beta_\delta}\Big|_0^t = \|q''\|(1 - e^{-t/\beta_\delta}) \\ &\le \|q''\|(1 - e^{-T/\beta_\delta}) \le \|q''\| \le E, \ \forall t \in [0, T], \end{aligned} \tag{42}$$

which implies

$$\|\widetilde{q}''\| = \|v'_{\beta_\delta}\| \le E. \tag{43}$$

Therefore, $\widetilde{q}$ and q satisfy the hypothesis of Theorem 3.2, that is, $q(0) = q'(0) = 0$, $\widetilde{q}(0) = \widetilde{q}'(0) = 0$, $\|q''\| \le E$, $\|\widetilde{q}''\| \le E$, and

$$\|\widetilde{q} - q\| \le \tau\delta^\gamma + 2\delta.$$

Thus, by Theorem 3.2:

$$\left\| v_{\alpha,\beta_\delta} - \left(D^{(\alpha)} q\right) \right\| = \left\| D^{(\alpha)} \widetilde{q} - D^{(\alpha)} q \right\| \le \frac{4T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \sqrt{(\tau\delta^\gamma + 2\delta)E}.$$

The lemma is proved. □

Lemma 3.6 *If $u_{\beta_\delta}(t)$, $u_{\alpha,\beta_\delta}(t)$, $v_{\beta_\delta}(t)$, $v_{\alpha,\beta_\delta}(t)$ are determined by (3), (5), (7) and (8) respectively with $\beta_\delta > 0$ is solution to (27) and δ is small enough, the following estimates hold*

$$\|u_{\alpha,\beta_\delta} - v_{\alpha,\beta_\delta}\| \le \frac{2\delta^{1-\gamma} T^{2-\alpha} E}{(1-\alpha)\Gamma(1-\alpha)}.$$

Proof Recall from (27) and (38) that $\|\beta_\delta u_{\beta_\delta}\| = \tau\delta^\gamma$ and $\|\beta_\delta v_{\beta_\delta} - \beta_\delta u_{\beta_\delta}\| < 2\delta$ (respectively), so we have

$$\|\beta_\delta v_{\beta_\delta}\| = \|\beta_\delta v_{\beta_\delta} - \beta_\delta u_{\beta_\delta} + \beta_\delta u_{\beta_\delta}\| \geq \|\beta_\delta u_{\beta_\delta}\| - \|\beta_\delta v_{\beta_\delta} - \beta_\delta u_{\beta_\delta}\| > \tau\delta^\gamma - 2\delta.$$

On the other hand, $\lim_{\delta\to 0^+} 2\delta^{1-\gamma} = 0$, hence for δ is small enough $(\tau-1) - 2\delta^{1-\gamma} > 0$. This implies

$$\tau\delta^\gamma - 2\delta = \delta^\gamma + \left((\tau-1) - 2\delta^{1-\gamma}\right)\delta^\gamma > \delta^\gamma$$

and therefore

$$\|\beta_\delta v_{\beta_\delta}\| > \delta^\gamma \tag{44}$$

with δ is small enough.

From the proof of Lemma 3.5, we see that $\|v'_{\beta_\delta}\| \leq E$. It implies from (40) and $q(0) = 0$ that $v_{\beta_\delta}(0) = 0$. For $t \in (0, T]$, by Lagrange's Theorem there exists $\widetilde{t} \in (0, t)$ such that

$$|v_{\beta_\delta}(t)| = |v_{\beta_\delta}(t) - v_{\beta_\delta}(0)| = |t-0||v'_{\beta_\delta}(\widetilde{t})| = t|v'_{\beta_\delta}(\widetilde{t})| \leq T\|v'_{\beta_\delta}\| \leq TE.$$

This implies that

$$\|v_{\alpha_\delta}\| \leq TE. \tag{45}$$

From (44) and (45), we have the estimate

$$\delta^\gamma < \|\beta_\delta v_{\beta_\delta}\| = \beta_\delta\|v_{\beta_\delta}\| \leq \beta_\delta TE \tag{46}$$

with δ is small enough. Therefore, $\beta_\delta > \dfrac{\delta^\gamma}{TE}$ with δ is small enough. From Theorem 2.1, with δ is small enough

$$\|u_{\beta_\delta} - v_{\beta_\delta}\| \leq \frac{2}{\beta_\delta}\delta < \frac{2}{\frac{\delta^\gamma}{TE}}\delta = 2\delta^{1-\gamma}TE.$$

Next, we prove for δ is small enough

$$\begin{aligned}\left|u_{\alpha,\beta_\delta}(t) - v_{\alpha,\beta_\delta}(t)\right| &= \left|\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u_{\beta_\delta}(s) - v_{\beta_\delta}(s)}{(t-s)^\alpha}ds\right| \\ &\leq \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{|u_{\beta_\delta}(s) - v_{\beta_\delta}(s)|}{(t-s)^\alpha}ds \\ &\leq \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\|u_{\beta_\delta} - v_{\beta_\delta}\|}{(t-s)^\alpha}ds\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2\delta^{1-\gamma} T E}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} ds \\
&= 2\delta^{1-\gamma} T E \frac{t^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \\
&\leq \frac{2\delta^{1-\gamma} T^{2-\alpha} E}{(1-\alpha)\Gamma(1-\alpha)}, \ \forall t \in [0, T],
\end{aligned}
$$

which implies

$$
\|u_{\alpha,\beta_\delta} - v_{\alpha,\beta_\delta}\| \leq \frac{2\delta^{1-\gamma} T^{2-\alpha} E}{(1-\alpha)\Gamma(1-\alpha)}.
$$

The lemma is proved. □

4 Numerical implementation

In this section, we provide some numerical examples to illustrate the error estimates of the method, discussed in Theorem 3.1 and Theorem 3.4, associated with the a prior parameter and the a posteriori parameter choice rules. Recall that, the method aims to find an approximation of the Caputo fractional derivative of a differentiable function $q \in L^\infty[0, T]$ where the (unknown) data q is perturbed by some noise. The relation between the exact data q and the measured data q^δ is given by (2). In each of the following examples, the exact data q and its Caputo fractional derivative $D^\alpha q$ are given explicitly, and the noisy data q^δ is taken from $C[0, T]$. In particular, we construct $q^\delta \in C[0, T]$ as follows: We first discretize the interval $[0, T]$ by N points $t_n = (n-1)\Delta t$, $n = 1, \ldots, N$, where $\Delta t := T/(N-1)$ is the step-size. The noisy data q^δ is then defined by the following:

$$
\begin{cases}
q^\delta(t_n) = q(t_n) + \delta q_n, \ n = 1, \ldots, N, \\
q^\delta \ \text{ is linear in each interval } \ [t_n, t_{n+1}], \ n = 1, \ldots, N-1,
\end{cases}
\tag{47}
$$

where q_n, $n = 1, \ldots, N$ are uniformly distributed random numbers in $(-1, 1)$. The proposed regularization method is summarized in two steps:

Step 1: Define u_β by (4) as a regularized approximation of q' with noisy data q^δ.

Step 2: $D^\alpha q$ is approximated by $u_{\alpha,\beta}$ defined by (5).

The integrals that appear in (4) and (5) are calculated numerically using the trapezoidal rule. For the prior parameter choice rule, we choose $\beta = \sqrt{2\delta/E}$ (the value of E will be provided in each example). For the a posteriori parameter choice, the value of β_δ is chosen such as relation (27) holds (which depends on two parameters τ and γ). The values of τ and γ will be specified in each example. The relative error will be calculated using the l^2-norm.

Example 1 The exact data is given by $q(t) = t^2 e^{\lambda t}$ with $\lambda < 0$. Note that, the Caputo derivatives of the exponential function $e^{\lambda t}$ is given explicitly by

$$D^\alpha e^{\lambda t} = \lambda t^{1-\alpha} E_{1,2-\alpha}(\lambda t), \quad \forall\, 0 < \alpha < 1,$$

where $E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)}$ is the two - parameter function of Mittag - Leffeler type. Therefore, the Caputo fractional derivative $D^\alpha q$ can be calculated explicitly (using integrating by part), which is

$$D^\alpha q(t) = \lambda u_2 - (2\lambda t + 2)u_1 + \frac{1}{\lambda}(2t + \lambda t^2) D^\alpha e^{\lambda t},$$

where

$$u_1 = \frac{1}{\Gamma(1-\alpha)} \frac{-1}{\lambda} t^{1-\alpha} + \frac{1-\alpha}{\lambda^2} D^\alpha e^{\lambda t}$$
$$u_2 = \frac{1}{\Gamma(1-\alpha)} \frac{-1}{\lambda} t^{2-\alpha} + \frac{2-\alpha}{\lambda} \left(\frac{1}{\Gamma(1-\alpha)} \frac{-1}{\lambda} t^{1-\alpha} + \frac{1-\alpha}{\lambda^2} D^\alpha e^{\lambda t} \right).$$

We here choose particularly $\lambda = -1$. The noisy data q^δ is defined from q by (47), where $q_n, n = 1, \ldots, N$ are generated numerically. Figure 1 plots the data for Example 1 where the measured data are perturbed by 5% of noise. Figure 2 plots the analytical

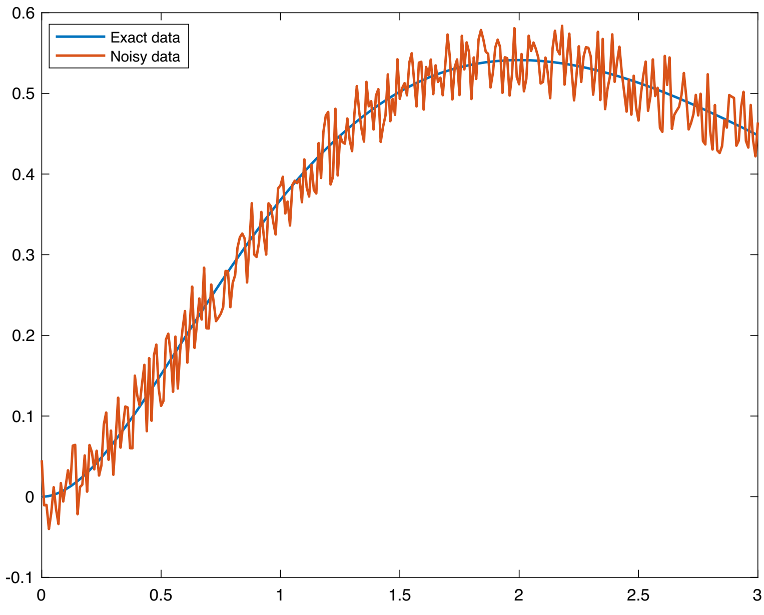

Fig. 1 Data for Example 1. The exact data (the smooth blue curve) and the measured data with 5% of noise (the zigzag orange curve)

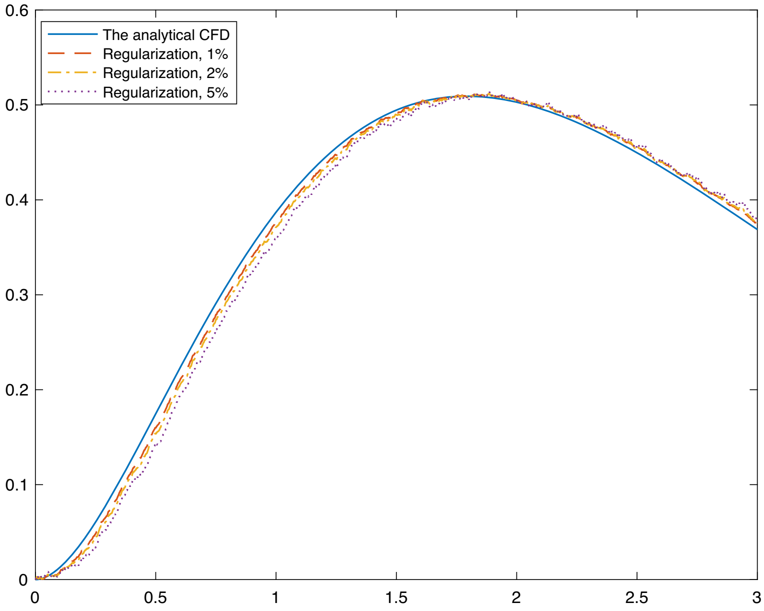

Fig. 2 The analytical CFD and the regularization with 1%, 2%, and 5% for Example 1 using the a priori parameter choice

Caputo fractional derivative (CFD) with $\alpha = 0.1$ and the numerical approximation for Example 1 using the prior parameter choice rule with the data contain 1%, 2%, and 5% of noise. The relative error for the prior parameter choice rule is shown in Table 1. Here, $T = 3$, $\Delta t = 10^{-3}$, $E = 23$.
For the a posteriori parameter choice, parameter $\gamma = 0.95$ should be taken close to 1 in order to ensure the existence of τ such that $\tau > 1$ in practice. Here, we fix $\gamma = 0.95$ for all examples. In Fig. 3 (left), we plot the regularization results for Example 1 using the a posteriori parameter choice with $\alpha = 0.1$, and the data are perturbed by 1%, 2%, and 5% of noise (similar to the setup for the numerical results that are shown in Fig. 2). The case of 5% of noise is also plotted separately (in comparing with the analytical CFD) in Fig. 3 (right). In the tests, the parameter τ and the regularization parameter β_δ are chosen numerically. In particular, $\tau = 1.6353, 1.2995, 1.1346$, $\beta_\delta = 0.0210, 0.0210, 0.0260$ corresponding with $\delta = 0.01, 0.02, 0.05$. The relative errors

Table 1 The relative error for Example 1 using the a priori parameter choice

	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$
$\delta = 0.01$	0.0201	0.0244	0.0419
$\delta = 0.02$	0.0281	0.0335	0.0560
$\delta = 0.05$	0.0443	0.0525	0.0851
$\delta = 0.1$	0.0628	0.0739	0.1177

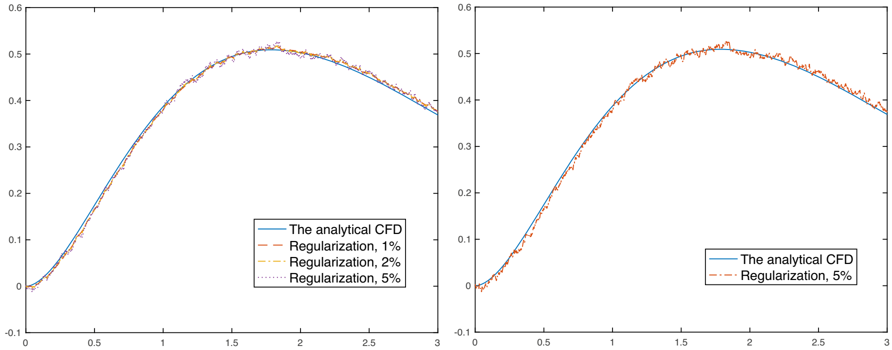

Fig. 3 The analytical CFD and the regularization for Example 1 using the a posteriori parameter choice. Left: with 1%, 2%, and 5% of noise. Right: with 5% of noise

Table 2 The relative error for Example 1 using the a priori parameter choice

	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$
$\delta = 0.01$	0.0163	0.0212	0.0406
$\delta = 0.02$	0.0171	0.0246	0.0598
$\delta = 0.05$	0.0228	0.0369	0.1081
$\delta = 0.1$	0.0337	0.0548	0.2441

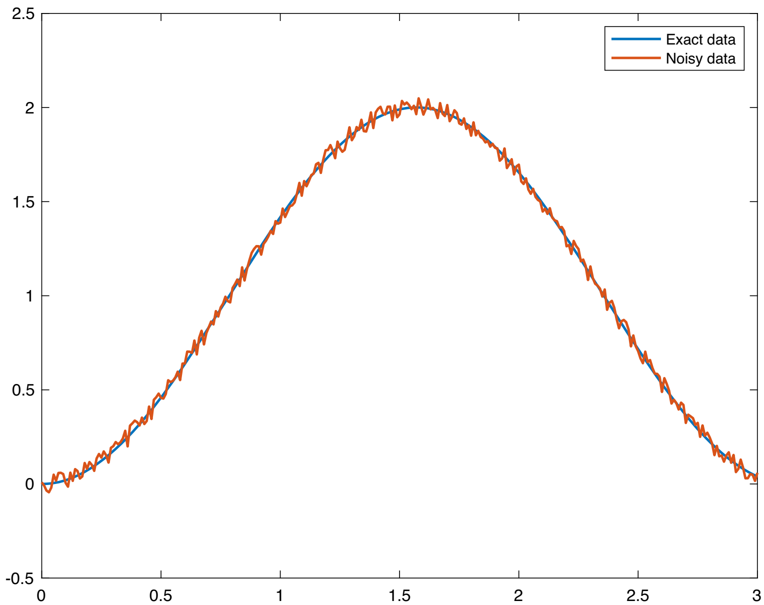

Fig. 4 Data for Example 2. The exact data (the smooth blue curve) and the measured data with 5% of noise (the zigzag orange curve)

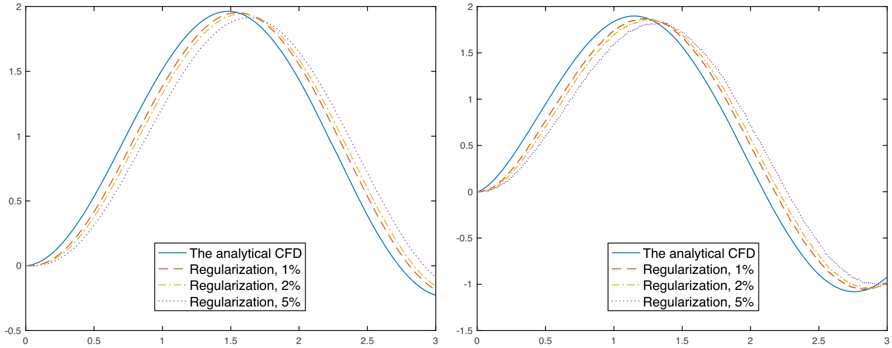

Fig. 5 The regularization results for Example 2 with 1%, 2%, and 5% of noise in the data, using the a priori parameter choice rule. Left: $\alpha = 0.1$. Right: $\alpha = 0.5$

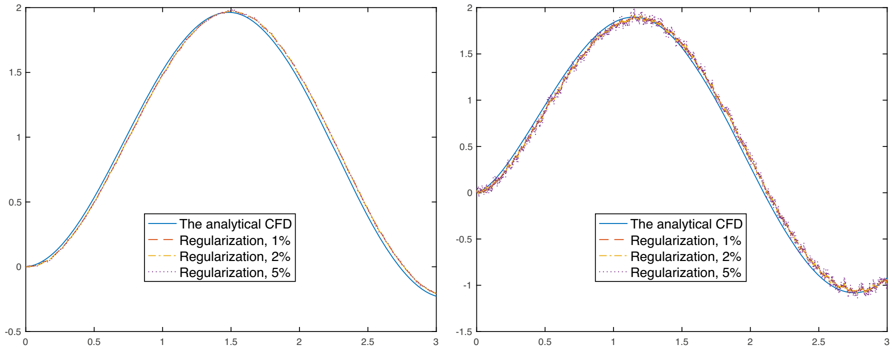

Fig. 6 The regularization results for Example 2 with 1%, 2%, and 5% of noise in the data, using the a posteriori parameter choice rule. Left: $\alpha = 0.1$. Right: $\alpha = 0.5$

Table 3 The relative error for Example 2 using the a priori parameter choice

	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$
$\delta = 0.01$	0.0887	0.1007	0.1226
$\delta = 0.02$	0.1240	0.1407	0.1671
$\delta = 0.05$	0.1904	0.2160	0.2506
$\delta = 0.1$	0.2578	0.2931	0.3357

Table 4 The relative error for Example 2 using the a posteriori parameter choice

	$\alpha = 0.1$	$\alpha = 0.25$	$\alpha = 0.5$
$\delta = 0.01$	0.0313	0.0354	0.0488
$\delta = 0.02$	0.0314	0.0355	0.0510
$\delta = 0.05$	0.0320	0.0373	0.0588
$\delta = 0.1$	0.0335	0.0417	0.0796

are shown in Table 2. The results above also shows that the a posteriori parameter choice rule is more accurate than the a priori parameter choice rule if $\alpha < 0.5$.

Example 2 In this example, we choose the exact data $q(t) = 1 - \cos(2t)$ and the noisy data q^δ is defined by (47), where q_n, $n = 1, \ldots, N$ are generated numerically. The exact Caputo derivatives is given by

$$D^\alpha q(t) = -D^\alpha \cos(t) = -it^{1-\alpha}\left(E_{1,2-\alpha}(i2t) - E_{1,2-\alpha}(-i2t)\right), \quad \forall\, 0 < \alpha < 1.$$

Figure 4 gives an illustration for the noisy data in comparing with the analytical formulation. Similar to Example 1, we here provide some numerical results for the regularization using the a priori and the a posteriori parameter choices. The results for the use of the a priori parameter choice are shown in Fig. 5. Here, $E = 4$, $T = 3$, $\Delta t = 0.001$. The value of the parameter β does not depend on α. In particular, $\beta = \sqrt{2\delta/E} = 0.0707, 0.1000, 0.1581$ for $\delta = 0.01, 0.02, 0.05$ (respectively). For the a posteriori parameter choice rule, the value of β_δ as well as the value of parameter τ are chosen numerically and they depend on the order α. In particular, for $\alpha = 0.1$ $\tau = 4.2282, 2.6495, 1.6996$ and $\beta_\delta = 0.0210, 0.0210, 0.0210$ corresponding with $\delta = 0.01, 0.02, 0.05$; and for $\alpha = 0.5$ $\tau = 4.2163, 2.6335, 1.6750$ and $\beta_\delta = 0.0210, 0.0220, 0.0220$ corresponding with $\delta = 0.01, 0.02, 0.05$. The numerical results are shown in Fig. 6

Finally, the relative results for both choice rules are given in Tables 3 and 4.

Appendix A: The uniqueness theorem

Theorem A.1 *(The uniqueness theorem) Problem (3) admits a unique solution in $L^\infty[0, T]$. In addition, the solution is given by*

$$u_\beta(t) = -\frac{1}{\beta^2}e^{-t/\beta}\int_0^t e^{z/\beta}q^\delta(z)dz + \frac{q^\delta(t)}{\beta}, \; t \in [0, T].$$

Proof Multiplying both side of (3) by $\dfrac{e^{t/\beta}}{\beta}$, we get

$$\frac{e^{t/\beta}}{\beta}\left(\beta u_\beta(t) + \int_0^t u_\beta(z)dz\right) = \frac{e^{t/\beta}}{\beta}q^\delta(t),$$

which is equivalent to

$$\frac{d}{dt}\left(e^{t/\beta}\int_0^t u_\beta(z)dz\right) = \frac{e^{t/\beta}}{\beta}q^\delta(t). \tag{48}$$

Integrating both sides of (48) from 0 to t, we get

$$e^{t/\beta}\int_0^t u_\beta(z)dz = \frac{1}{\beta}\int_0^t e^{z/\beta}q^\delta(z)dz$$

or

$$\int_0^t u_\beta(z)dz = \frac{1}{\beta} e^{-t/\beta} \int_0^t e^{z/\beta} q^\delta(z)dz. \tag{49}$$

Differentiating both sides of (49) with respect to t, we obtain (4). This formula also guarantees the uniqueness of the solution. ◻

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Data Availability Not applicable

Declarations

Ethical approval Not applicable

Conflict of interest The authors declare no competing interests.

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