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Research Article

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Regularization of backward parabolic equations in Banach spaces by generalized Sobolev equations

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Abstract: Let X be a Banach space with norm $\|\cdot\|$. Let $A : D(A) \subset X \to X$ be an (possibly unbounded) operator that generates a uniformly bounded holomorphic semigroup. Suppose that $\varepsilon > 0$ and $T > 0$ are two given constants. The backward parabolic equation of finding a function $u : [0, T] \to X$ satisfying

$$u_t + Au = 0, \quad 0 < t < T, \ \|u(T) - \varphi\| \leqslant \varepsilon,$$

for φ in X, is regularized by the generalized Sobolev equation

$$u_{\alpha t} + A_\alpha u_\alpha = 0, \quad 0 < t < T, \ u_\alpha(T) = \varphi,$$

where $0 < \alpha < 1$ and $A_\alpha = A(I + \alpha A^b)^{-1}$ with $b \geqslant 1$. Error estimates of the method with respect to the noise level are proved.

Keywords: Backward parabolic equations, ill-posed problems, regularization, Sobolev equation

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1 Introduction

Let X be a Banach space with norm $\|\cdot\|$. Let $A : D(A) \subset X \to X$ be an (possibly unbounded) operator that generates a uniformly bounded holomorphic semigroup. Suppose that $\varepsilon > 0$ and $T > 0$ are two given constants. Consider the backward parabolic equation of finding a function $u : [0, T] \to X$ such that

$$\begin{cases} u_t + Au = 0, \quad 0 < t < T, \\ \|u(T) - \varphi\| \leqslant \varepsilon, \end{cases} \tag{1.1}$$

for φ in X and the given positive noise level ε. This problem is well known to be severely ill-posed [21] and regularization methods for it are required. As we noted in [14], although there have been many papers devoted to backward parabolic equations in Hilbert spaces, there are very few ones devoted to those in Banach spaces. Among them we list the first work on this problem by Krein and Prozorovskaja [20], and then by Agmon and Nirenberg [1] and by Miller [23]. Regularization methods have been proposed in [2, 3, 8–10, 17–19, 22]. However, in these papers no convergence rate with respect to the noise level has been given. Some convergence rate of Hölder type has been established in [12] by a mollification method, and in [14] by the Tikhonov regularization method and by the non-local boundary value problem method, see also our related papers [6, 13, 15, 16]. Finally,

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we note the paper by Chen, Hofmann and Zou [4] where the authors presented an approach to investigate the convergence and stability of a class of regularized solutions for ill-posed backward evolution equations associated with sectorial or half-strip operators. They introduced a concept of qualification pairs and index functions to characterize the explicit convergence rates of the concerned regularized solutions. Some convergence rates of logarithmic type can be found there.

In this paper, we regularize problem (1.1) by the problem

$$\begin{cases} u_{\alpha t} + A_\alpha u_\alpha = 0, & 0 < t < T, \\ u_\alpha(T) = \varphi, \end{cases} \tag{1.2}$$

where $0 < \alpha < 1$ and $A_\alpha = A(I + \alpha A^b)^{-1}$ with $b \geqslant 1$. We note that Ewing [7] and Showalter [25] (see also [11]) regularized problem (1.1) by the Sobolev equation

$$\begin{cases} u_{\alpha t} + Au_\alpha + \alpha A u_{\alpha t} = 0, & 0 < t < T, \\ u_\alpha(T) = \varphi, \end{cases}$$

which can be rewritten in the form

$$\begin{cases} (I + \alpha A)u_{\alpha t} + Au_\alpha = 0, & 0 < t < T, \\ u_\alpha(T) = \varphi. \end{cases} \tag{1.3}$$

Applying $(I + \alpha A)^{-1}$ to the both sides of the first equation of (1.3), we get $u_{\alpha t} + (I + \alpha A)^{-1}Au_\alpha = 0$. Formally, we have $(I + \alpha A)^{-1}A = A(I + \alpha A)^{-1}$. Thus, we arrive at the equation $u_{\alpha t} + A(I + \alpha A)^{-1}u_\alpha = 0$. In this paper we slightly modify this method. Namely, we use $A_\alpha = A(I + \alpha A^b)^{-1}$ with $b \geqslant 1$ instead of $A(I + \alpha A)^{-1}$. That is why we call $u_{\alpha t} + A_\alpha u_\alpha = 0$ by *generalized Sobolev equation*. We note that our paper is close to those by Huang and his co-author [17, 18]. However, our method is different from theirs and we establish error estimates for our method, but they did not.

In the next section we will introduce some auxiliary results concerning the operator A_α. In the last section we present the main results on the well-posedness of problem (1.2) and convergence rates of u_α to the exact solution when the noise level ε tends to zero and the regularization parameter α is properly chosen.

2 Analytic semigroups generated by $A_\alpha = A(I + \alpha A^b)^{-1}$

Definition 2.1 ([5, p. 93]). We will call the (possibly unbounded) operator A, a generator if A generates a uniformly bounded strongly continuous holomorphic semigroup $\{e^{-zA}\}_{\operatorname{Re} z \geqslant 0}$. By switching to equivalent norm

$$|||x||| = \sup_{\operatorname{Re} z \geqslant 0} \|e^{-zA}x\|,$$

if necessary, we may assume that $\|e^{-zA}\| \leqslant 1$, whenever $\operatorname{Re} z \geqslant 0$. For $s \geqslant 0$, define

$$G(s, A) = \int_{\mathbb{R}} \frac{1 - \cos(sr)}{r^2} e^{irA} \frac{dr}{\pi}.$$

Remark 2.2 (see [5, Proposition 10]). The following inequality holds:

$$\|G(s, A)\| \leqslant s \quad \text{for all } s \geqslant 0.$$

Remark 2.3 (see [5, Theorem 12]). The functional calculus

$$f(A) = \Big(\lim_{t\to\infty} f(t)\Big)I + \int_0^\infty f''(s)G(s, A)\,ds$$

for

$$f \in \mathrm{AC}^1[0, \infty) := \{h \circ g : h \in \mathrm{AC}^1[0, 1]\},$$

where $g(t) = (1 + t)^{-1}$ and

$$\mathrm{AC}^1[0, 1] = \{f : f' \text{ exists and is absolutely continuous on } [0, 1]\}.$$

Definition 2.4 ([24, p. 69] and [18, p. 42]). Let A be the generator of an holomorphic semigroup of angle θ (where $0 < \theta \leqslant \frac{\pi}{2}$), and let $0 \in \rho(A)$, the resolvent of A. For $b > 0$, the fractional power of A is defined by

$$A^{-b} = \frac{1}{2\pi i} \int_{\Gamma(\gamma)} z^{-b}(A - zI)^{-1}\, dz.$$

Here, z^b is taken as the principle branch, and the path $\Gamma(\gamma)$, $\frac{1}{2}\pi - \theta < \gamma < \pi$, connects the points $\infty e^{-i\gamma}$ and $\infty e^{i\gamma}$ in $\rho(A)$, while avoiding the negative real axis and the origin. Define $A^b = (A^{-b})^{-1}$ and $A^0 = I$.

Lemma 2.5 (see [18, p. 42]). *The following statements hold:*
(a) $A^{-b} \in B(X)$ *is injective for* $b > 0$,
(b) A^b *is a closed operator and* $D(A^b) \subset D(A^{b'})$ *for* $b > b' > 0$,
(c) $A^b x = A^{(b-n)} A^n x$ *for* $x \in D(A^n)$ *and* $n > b$, $n \in \mathbb{N}$,
(d) *if* $B \subset A^b$ *and* $D(B) = D(A^{b'})$ $(b' > b > 0)$ *then* B *is closable and* $\overline{B} = A^b$, *where* $\overline{B}$ *is the closure of* B.

Theorem 2.6. *The operator* $A_\alpha = A(I + \alpha A^b)^{-1}$ *generates a semigroup* $\{e^{tA_\alpha}\}_{t\geqslant 0}$ *and*

$$\|e^{tA_\alpha}\| \leqslant \begin{cases} 4e^{t\alpha^{-1}} & \text{if } b = 1, \\ 2b(b+1)\exp\Big(t\Big(\dfrac{b+1}{(b-1)\alpha}\Big)^{\frac{1}{b}}\Big) & \text{if } b > 1. \end{cases}$$

Proof. Let $f(s) = \frac{s}{1+\alpha s^b}$. We have

$$f'(s) = \frac{1 + (1-b)\alpha s^b}{(1+\alpha s^b)^2} \quad \text{and} \quad f''(s) = \frac{b\alpha s^{b-1}((b-1)\alpha s^b - b - 1)}{(1+\alpha s^b)^3}.$$

If $b = 1$, then

$$A_\alpha = f(A) = \frac{1}{\alpha} I + \int_0^\infty \frac{-2\alpha}{(1+\alpha s)^3} G(s, A)\, ds.$$

Therefore, we have

$$\|A_\alpha\| \leqslant \frac{1}{\alpha} + \int_0^\infty \frac{2\alpha s}{(1+\alpha s)^3}\, ds = \frac{2}{\alpha}.$$

Hence A_α is a bounded linear operator and it generates a semigroup $\{e^{tA_\alpha}\}_{t\geqslant 0}$. For $t > 0$, we have

$$e^{tA_\alpha} = e^{t\alpha^{-1}} I + \int_0^\infty \Big(\frac{t^2}{(1+\alpha s)^4} - \frac{2t\alpha}{(1+\alpha s)^3}\Big) e^{\frac{ts}{1+\alpha s}} G(s, A)\, ds.$$

This implies that

$$\|e^{tA_\alpha}\| \leqslant e^{t\alpha^{-1}} + \int_0^\infty \frac{st^2 e^{\frac{ts}{1+\alpha s}}}{(1+\alpha s)^4}\, ds + \int_0^\infty \frac{2ts\alpha e^{\frac{ts}{1+\alpha s}}}{(1+\alpha s)^3}\, ds. \tag{2.1}$$

Since

$$\begin{aligned} \int_0^\infty \frac{st^2 e^{\frac{ts}{1+\alpha s}}}{(1+\alpha s)^4}\, ds &= \frac{ste^{\frac{ts}{1+\alpha s}}}{(1+\alpha s)^2}\Big|_0^\infty - \int_0^\infty \Big(\frac{t}{(1+\alpha s)^2} - \frac{2t\alpha s}{(1+\alpha s)^3}\Big) e^{\frac{ts}{1+\alpha s}}\, ds \\ &= -e^{\frac{ts}{1+\alpha s}}\Big|_0^\infty + \int_0^\infty \frac{2t\alpha s e^{\frac{ts}{1+\alpha s}}}{(1+\alpha s)^3}\, ds \\ &= -e^{t\alpha^{-1}} + 1 + \int_0^\infty \frac{2t\alpha s e^{\frac{ts}{1+\alpha s}}}{(1+\alpha s)^3}\, ds \end{aligned}$$

and

$$\int_0^\infty \frac{t\alpha s e^{\frac{ts}{1+\alpha s}}}{(1+\alpha s)^3}\, ds \leqslant \int_0^\infty \frac{te^{\frac{ts}{1+\alpha s}}}{(1+\alpha s)^2}\, ds = e^{t\alpha^{-1}} - 1, \tag{2.2}$$

from (2.1)–(2.2) we obtain

$$\|e^{tA_\alpha}\| \leqslant 4e^{t\alpha^{-1}}.$$

If $b > 1$, then

$$A_\alpha = \int_0^\infty \frac{b\alpha s^{b-1}((b-1)\alpha s^b - b - 1)}{(1+\alpha s^b)^3} G(s, A)\,ds.$$

This implies that

$$\begin{aligned}\|A_\alpha\| &\leqslant \int_0^\infty \frac{b\alpha s^{b-1}|(b-1)\alpha s^b - b - 1|}{(1+\alpha s^b)^3} s\,ds\\ &= -\int_0^a \frac{b\alpha s^{b-1}((b-1)\alpha s^b - b - 1)}{(1+\alpha s^b)^3} s\,ds + \int_a^\infty \frac{b\alpha s^{b-1}((b-1)\alpha s^b - b - 1)}{(1+\alpha s^b)^3} s\,ds,\end{aligned} \tag{2.3}$$

where $a = (\frac{b+1}{(b-1)\alpha})^{\frac{1}{b}}$. Furthermore,

$$\begin{aligned}-\int_0^a \frac{b\alpha s^{b-1}((b-1)\alpha s^b - b - 1)}{(1+\alpha s^b)^3} s\,ds &= -s\frac{1+(1-b)\alpha s^b}{(1+\alpha s^b)^2}\Big|_0^a + \int_0^a \frac{1+(1-b)\alpha s^b}{(1+\alpha s^b)^2}\,ds\\ &= -\Big(\frac{b+1}{(b-1)\alpha}\Big)^{\frac{1}{b}}.\frac{-b}{(1+\frac{b+1}{b-1})^2} + \frac{s}{1+\alpha s^b}\Big|_0^a\\ &= \frac{b-1}{2}\Big(\frac{b+1}{(b-1)\alpha}\Big)^{\frac{1}{b}},\end{aligned}$$

and

$$\int_a^\infty \frac{b\alpha s^{b-1}((b-1)\alpha s^b - b - 1)}{(1+\alpha s^b)^3} s\,ds = \frac{b-1}{2}\Big(\frac{b+1}{(b-1)\alpha}\Big)^{\frac{1}{b}}. \tag{2.4}$$

Hence, from (2.3)–(2.4) we attain

$$\|A_\alpha\| \leqslant (b-1)\Big(\frac{b+1}{(b-1)\alpha}\Big)^{\frac{1}{b}}.$$

Therefore, A_α is a bounded linear operator and it generates a semigroup $\{e^{tA_\alpha}\}_{t\geqslant 0}$.

For $t > 0$, let $f_1(s) = e^{\frac{ts}{1+\alpha s^b}}$. We have

$$\begin{aligned}f_1'(s) &= \frac{t(1+(1-b)\alpha s^b)}{(1+\alpha s^b)^2} e^{\frac{ts}{1+\alpha s^b}},\\ f_1''(s) &= \Big(\frac{t^2(1+(1-b)\alpha s^b)^2}{(1+\alpha s^b)^4} + \frac{tb\alpha s^{b-1}((b-1)\alpha s^b - b - 1)}{(1+\alpha s^b)^3}\Big) e^{\frac{ts}{1+\alpha s^b}}.\end{aligned} \tag{2.5}$$

Therefore,

$$e^{tA_\alpha} = \int_0^\infty f_1''(s) G(s, A)\,ds. \tag{2.6}$$

From (2.5) and (2.6) we have

$$\|e^{tA_\alpha}\| \leqslant \int_0^\infty \frac{st^2(1+(1-b)\alpha s^b)^2}{(1+\alpha s^b)^4} e^{\frac{ts}{1+\alpha s^b}}\,ds + \int_0^\infty \frac{tb\alpha s^b|(b-1)\alpha s^b - b - 1|}{(1+\alpha s^b)^3} e^{\frac{ts}{1+\alpha s^b}}\,ds. \tag{2.7}$$

On the other hand, we have

$$\begin{aligned}\int_0^\infty \frac{st^2(1+(1-b)\alpha s^b)^2}{(1+\alpha s^b)^4} e^{\frac{ts}{1+\alpha s^b}}\,ds &= \frac{ste^{\frac{ts}{1+\alpha s^b}}}{(1+\alpha s^b)^2}\Big|_0^\infty - \int_0^\infty \Big(\frac{t(1+(1-b)\alpha s^b)}{(1+\alpha s^b)^2} + \frac{tb\alpha s^b((b-1)\alpha s^b - b - 1)}{(1+\alpha s^b)^3}\Big) e^{\frac{ts}{1+\alpha s^b}}\,ds\\ &= -e^{\frac{ts}{1+\alpha s^b}} - \int_0^\infty \frac{tb\alpha s^b((b-1)\alpha s^b - b - 1)}{(1+\alpha s^b)^3} e^{\frac{ts}{1+\alpha s^b}}\,ds.\end{aligned} \tag{2.8}$$

From (2.7) and (2.8) we have

$$\begin{aligned}\|e^{tA_\alpha}\| &\leqslant -2\int_0^a \frac{tbas^b((b-1)as^b-b-1)}{(1+as^b)^3}e^{\frac{ts}{1+as^b}}\,ds\\ &\leqslant 2\int_0^a \frac{tb(b+1)as^b}{(1+as^b)^3}e^{\frac{ts}{1+as^b}}\,ds\\ &\leqslant 2b(b+1)\int_0^a te^{ts}\,ds\\ &\leqslant 2b(b+1)\exp\Big(t\Big(\frac{b+1}{(b-1)\alpha}\Big)^{\frac{1}{b}}\Big).\end{aligned}$$

The theorem is proved. □

3 Regularization and convergence rate

In this section, we suppose that the operator A satisfies the conditions in Definitions 2.1 and 2.4. We will prove that problem (1.2) is well posed and if we choose the regularization parameter α properly, then we get a convergence rate of u_α to u as the noise level tends to zero. In doing so we first prove some auxiliary results.

Lemma 3.1. *The following inequality holds:*

$$\|e^{-TA}e^{TA_\alpha}\| \leqslant \begin{cases} 8+2T+\dfrac{16}{T} & \text{if } b=1,\\ \dfrac{4b^2}{b-1} & \text{if } b>1.\end{cases}$$

Proof. Let $f_2(s)=e^{-Ts}e^{\frac{Ts}{1+as^b}}=e^{-\frac{Tas^{b+1}}{1+as^b}}$. Then

$$f_2''(s)=\Big(T^2\Big(\frac{1+(1-b)as^b}{(1+as^b)^2}-1\Big)^2+\frac{Tbas^{b-1}((b-1)as^b-b-1)}{(1+as^b)^3}\Big)e^{-\frac{Tas^{b+1}}{1+as^b}}.$$

Therefore, we have

$$f_2(A)=e^{-TA}e^{TA_\alpha}=\int_0^\infty f_2''(s)G(s,A)\,ds.$$

This implies that

$$\begin{aligned}\|f_2(A)\| &\leqslant \int_0^\infty s|f''(s)|\,ds\\ &\leqslant \int_0^\infty s\Big(T\frac{1+(1-b)as^b}{(1+as^b)^2}-T\Big)^2e^{-\frac{Tas^{b+1}}{1+as^b}}\,ds+\int_0^\infty \frac{bas^b|(b-1)as^b-b-1|}{(1+as^b)^3}e^{-\frac{Tas^{b+1}}{1+as^b}}\,ds.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_0^\infty s\Big(T\frac{1+(1-b)as^b}{(1+as^b)^2}-T\Big)^2e^{-\frac{Tas^{b+1}}{1+as^b}}\,ds\\ &\quad= s\Big(T\frac{1+(1-b)as^b}{(1+as^b)^2}-T\Big)e^{-\frac{Tas^{b+1}}{1+as^b}}\Big|_0^\infty-\int_0^\infty\Big(\Big(T\frac{1+(1-b)as^b}{(1+as^b)^2}-T\Big)+\frac{bas^b((b-1)as^b-b-1)}{(1+as^b)^3}\Big)e^{-\frac{Tas^{b+1}}{1+as^b}}\,ds\\ &\quad= -e^{-\frac{Tas^{b+1}}{1+as^b}}\Big|_0^\infty-\int_0^\infty\frac{Tbas^b((b-1)as^b-b-1)}{(1+as^b)^3}e^{-\frac{Tas^{b+1}}{1+as^b}}\,ds.\end{aligned}$$

Hence

$$\begin{aligned}
\|f_2(A)\| &\leqslant 4\int_0^\infty \frac{T\alpha s}{(1+\alpha s)^3}e^{-\frac{T\alpha s^2}{1+\alpha s}}\,ds\\
&= 4\int_0^{\alpha^{-1/2}} \frac{T\alpha s}{(1+\alpha s)^3}e^{-\frac{T\alpha s^2}{1+\alpha s}}\,ds + 4\int_{\alpha^{-1/2}}^\infty \frac{T\alpha s}{(1+\alpha s)^3}e^{-\frac{T\alpha s^2}{1+\alpha s}}\,ds\\
&\leqslant 4\int_0^{\alpha^{-1/2}} T\alpha s\,ds + 4\int_{\alpha^{-1/2}}^\infty T\alpha s e^{-\frac{T\alpha^{1/2}s}{(1+\alpha^{1/2})}}\,ds \quad \Big(\text{since } -\tfrac{\alpha s}{1+\alpha s} \leqslant -\tfrac{\alpha^{1/2}}{1+\alpha^{-1/2}} \text{ for all } s \geqslant \alpha^{-1/2}\Big)\\
&\leqslant 2T + 4\int_{\alpha^{-1/2}}^\infty T\alpha s e^{-\frac{T\alpha^{1/2}s}{2}}\,ds\\
&= 2T + \Big(8 + \frac{16}{T}\Big)e^{-\frac{T}{2}}\\
&\leqslant 8 + 2T + \frac{16}{T}.
\end{aligned}$$

If $b > 1$, then

$$\|f_2(A)\| \leqslant -2\int_0^a \frac{Tb\alpha s^b((b-1)\alpha s^b - b - 1)}{(1+\alpha s^b)^3}e^{-\frac{T\alpha s^{b+1}}{1+\alpha s^b}}\,ds,$$

where $a = (\frac{b+1}{(b-1)\alpha})^{\frac{1}{b}}$. This implies that

$$\|f_2(A)\| \leqslant 2\int_0^a \frac{Tb(b+1)\alpha s^b}{(1+\alpha s^b)^3}e^{-\frac{T\alpha s^{b+1}}{1+\alpha s^b}}\,ds.$$

We have

$$-\frac{T\alpha s^{b+1}}{1+\alpha s^b} \leqslant -\frac{(b-1)T\alpha s^{b+1}}{2b} \quad \text{for all } s < \Big(\frac{b+1}{(b-1)\alpha}\Big)^{\frac{1}{b}}.$$

Therefore, we obtain

$$\|f_2(A)\| \leqslant 2\int_0^a Tb(b+1)\alpha s^b e^{-\frac{(b-1)T\alpha s^{b+1}}{2b}}\,ds \leqslant \frac{4b^2}{b-1}.$$

The lemma is proved. □

Lemma 3.2. *There exists a constant C which only depends on T and b such that*

$$\|e^{-\beta A}(I - e^{-\gamma A}e^{\gamma A_\alpha})\| \leqslant C(\beta^{3-k} + \beta^{2-k} + \beta^{1-k})\alpha,$$

where $\beta > 0$, $\gamma \geq 0$. *Here, k is an integer satisfying* $1 \leq k < b + 3$.

Proof. Let

$$f_3(s) = e^{-Ts}(1 - e^{-Ts}e^{\frac{Ts}{1+\alpha s^b}}).$$

Then we have

$$\begin{aligned}
f_3'(s) &= \beta^2 e^{-\beta s}(1 - e^{-\gamma s}e^{\frac{\gamma s}{1+\alpha s^b}}) - \frac{\gamma b\alpha s^{b-1}((b-1)\alpha s^b - b - 1)}{(1+\alpha s^b)}e^{-(\beta+\gamma)s}e^{\frac{\gamma s}{1+\alpha s^b}}\\
&\quad - \Big(2\beta\gamma\Big(\frac{1+(1-b)\alpha s^b}{(1+\alpha s)^2} - 1\Big) + \gamma^2\Big(\frac{1+(1-b)\alpha s^b}{(1+\alpha s)^2} - 1\Big)^2\Big)e^{-(\beta+\gamma)s}e^{\frac{\gamma s}{1+\alpha s^b}}.
\end{aligned}$$

On the other hand, since

$$f_3(A) = e^{-TA}(I - e^{-TA}e^{TA_\alpha}) = \int_0^\infty f_3''(s)G(s,A)\,ds,$$

we get

$$\begin{aligned}\|f_3(A)\| &\leqslant \int_0^\infty s|f_3''(s)|\,ds\\ &\leqslant \int_0^\infty \beta^2 s e^{-\beta s}(1-e^{-\gamma s}e^{\frac{\gamma s}{1+\alpha s^b}})\,ds + \int_0^\infty 2\beta\gamma s\Big|\frac{1+(1-b)\alpha s^b}{(1+\alpha s)^2}-1\Big|e^{-(\beta+\gamma)s}e^{\frac{\gamma s}{1+\alpha s^b}}\,ds\\ &\quad+\int_0^\infty \gamma^2 s(\frac{1+(1-b)\alpha s^b}{(1+\alpha s)^2}-1)^2 e^{-(\beta+\gamma)s}e^{\frac{\gamma s}{1+\alpha s^b}}\,ds\\ &\quad+\int_0^\infty \frac{\gamma b\alpha s^b|(b-1)\alpha s^b-b-1|}{(1+\alpha s^b)}e^{-(\beta+\gamma)s}e^{\frac{\gamma s}{1+\alpha s^b}}\,ds. \end{aligned}\tag{3.1}$$

Now we estimate the right-hand side of this inequality. We have

$$\int_0^\infty \beta^2 s e^{-\beta s}(1-e^{-\frac{\gamma\alpha s^{b+1}}{1+\alpha s^b}})\,ds \leqslant \int_0^\eta \beta^2 s e^{-\beta s}(1-e^{-\frac{\gamma\alpha s^{b+1}}{1+\alpha s^b}})\,ds + \int_\eta^\infty \beta^2 s e^{-\beta s}\,ds,\tag{3.2}$$

where $\eta = ((\gamma+1)\alpha)^{-\frac{1}{b+1}}$. Since $s \leqslant \eta$, we have $\frac{\gamma\alpha s^{b+1}}{1+\alpha s^b} \leqslant 1$. This implies that

$$e^{-\frac{\gamma\alpha s^{b+1}}{1+\alpha s^b}} \geqslant \frac{1}{1+\frac{3\gamma\alpha s^{b+1}}{1+\alpha s^b}}.$$

Therefore, we obtain

$$\int_0^\eta \beta^2 s e^{-\beta s}(1-e^{\frac{-\gamma\alpha s^{b+1}}{1+\alpha s^b}})\,ds \leqslant \int_0^\eta 3\beta^2\gamma s^{b+2}\alpha e^{-\beta s}\,ds.$$

Taking integration by parts the right-hand side of this inequality $k-1$ times, we get

$$\begin{aligned}\int_0^\eta \beta^2 s e^{-\beta s}(1-e^{\frac{-\gamma\alpha s^{b+1}}{1+\alpha s^b}})\,ds &\leqslant \int_0^\eta 3\beta^2\gamma s^{b+2}\alpha e^{-\beta s}\,ds \leqslant -3\beta\gamma\alpha s^{b+2}e^{-\beta s}\Big|_0^\eta + \int_0^\eta 3\beta(b+2)\gamma s^{b+1}\alpha e^{-\beta s}\,ds\\ &\leqslant \int_0^\eta 3\beta(b+2)\gamma s^{b+1}\alpha e^{-\beta s}\,ds\\ &\leqslant \int_0^\eta 3\beta^{3-k}(b+2)(b+1)\cdots(b+3-k)\gamma s^{b+3-k}\alpha e^{-\beta s}\,ds.\end{aligned}\tag{3.3}$$

Set $C_1 = (b+2)(b+1)b\cdots(b+3-k)$. We have

$$\begin{aligned}\int_0^\eta \beta^2 s e^{-\beta s}(1-e^{\frac{-\gamma\alpha s^{b+1}}{1+\alpha s^b}})\,ds &\leqslant 3C_1\int_0^{\frac{1}{\gamma+1}} \beta^{3-k}\gamma s^{b+3-k}\alpha e^{-\beta s}\,ds + 3C_1\int_{\frac{1}{\gamma+1}}^\eta \beta^{3-k}\gamma s^{b+3-k}\alpha e^{-\beta s}\,ds\\ &\leqslant 3\gamma C_1\beta^{3-k}\alpha + 3C_1\int_{\frac{1}{\gamma+1}}^\eta \beta^{3-k}\gamma s^{b+3-k}\alpha e^{-\beta s}\,ds\\ &= 3\gamma C_1\beta^{3-k}\alpha - 3C_1\alpha\beta^{3-k}\gamma s^{b+3-k}e^{-\beta s}\Big|_{\frac{1}{\gamma+1}}^\eta + 3C_1\beta^{2-k}\alpha\int_{\frac{1}{\gamma+1}}^\eta \gamma s^{b+2-k}e^{-\beta s}\,ds\\ &\leqslant 6\gamma C_1\beta^{3-k}\alpha + 3C_1\beta^{2-k}\alpha\gamma(1+\gamma)^{k-b-2}\int_{\frac{1}{\gamma+1}}^\eta e^{-\beta s}\,ds\\ &\leqslant 6\gamma C_1\beta^{3-k}\alpha + 3C_1\beta^{1-k}\alpha\gamma(1+\gamma)^{k-b-2}.\end{aligned}$$

Furthermore, we estimate the second term in the right-hand side of (3.2):

$$\int_\eta^\infty \beta^2 s e^{-\beta s}\,ds = -\beta s e^{-\beta s}\Big|_\eta^\infty + \int_\eta^\infty \beta e^{-\beta s}\,ds = \beta\eta e^{-\beta\eta} + e^{-\beta\eta}.$$

From

$$\beta\eta e^{-\beta\eta} \leqslant \frac{\beta\eta}{1 + \frac{(\beta\eta)^k}{k!}} \leqslant k!(\beta\eta)^{1-k} \leqslant k!\beta^{1-k}\alpha$$

and

$$e^{-\beta\eta} \leqslant \frac{1}{1 + \frac{(\beta\eta)^{k-1}}{(k-1)!}} \leqslant (k-1)!(\beta\eta)^{1-k} \leqslant (k-1)!\beta^{1-k}\alpha$$

there exists a constant C_2 such that

$$\int_\eta^\infty \beta^2 s e^{-\beta s}\,ds \leqslant C_2\beta^{1-k}\alpha. \tag{3.4}$$

From (3.2), (3.3) and (3.4) we conclude that there exists a constant C_3 such that

$$\int_0^\infty T^2 s e^{-Ts}\big(1 - e^{-\frac{T\alpha s^{b+1}}{1+\alpha s^b}}\big)\,ds \leqslant C_3(\beta^{3-k} + \beta^{1-k})\alpha.$$

On the other hand, we have

$$\begin{aligned}2\beta\gamma\int_0^\infty s\Big|\frac{1+(1-b)\alpha s^b}{(1+\alpha s^b)^2} - 1\Big| e^{-(\beta+\gamma)s} e^{\frac{\gamma s}{1+\alpha s^b}}\,ds &= 2\beta\gamma\int_0^\infty \frac{(b+1)\alpha s^{b+1} + \alpha^2 s^{2b+1}}{(1+\alpha s^b)^2} e^{-\beta s} e^{-\frac{\gamma\alpha s^{b+1}}{1+\alpha s^b}}\,ds\\ &\leqslant 2\beta\gamma(b+2)\alpha\int_0^\infty s^{b+1}e^{-\beta s}\,ds.\end{aligned}$$

Taking integration by parts $k-2$ times, we get

$$\begin{aligned}2\beta\gamma\int_0^\infty s\Big|\frac{1+(1-b)\alpha s^b}{(1+\alpha s^b)^2} - 1\Big| e^{-(\beta+\gamma)s} e^{\frac{\gamma s}{1+\alpha s^b}}\,ds &\leqslant 2\beta\gamma(b+2)\alpha\int_0^\infty s^{b+1}e^{-\beta s}\,ds\\ &= -2\gamma(b+2)\alpha s^{b+1}e^{-\beta s}\Big|_0^\infty + 2\gamma(b+2)(b+1)\alpha\int_0^\infty s^b e^{-\beta s}\,ds\\ &= 2\gamma(b+2)(b+1)\alpha\int_0^\infty s^b e^{-\beta s}\,ds\\ &= 2\gamma\beta^{3-k}(b+2)(b+1)b\cdots(b+4-k)\alpha\int_0^\infty s^{b+3-k}e^{-\beta s}\,ds\\ &= 2\gamma\beta^{3-k}C_4\alpha\int_0^1 s^{b+3-k}e^{-\beta s}\,ds + 2\gamma\beta^{3-k}C_4\alpha\int_1^\infty s^{b+3-k}e^{-\beta s}\,ds\\ &\leq 2\gamma\beta^{3-k}C_4\alpha - 2\gamma\beta^{2-k}C_4\alpha s^{b+3-k}e^{-\beta s}\Big|_1^\infty\\ &\qquad + 2\gamma\beta^{2-k}C_4(b+3-k)\alpha\int_1^\infty s^{b+2-k}e^{-\beta s}\,ds\\ &\leq 2\gamma\beta^{3-k}C_4\alpha + 2\gamma\beta^{2-k}C_4\alpha + 2\gamma\beta^{2-k}C_4(b+3-k)\alpha\int_1^\infty e^{-\beta s}\,ds\\ &\leq 2\gamma\beta^{3-k}C_4\alpha + 2\gamma\beta^{2-k}C_4\alpha + 2\gamma\beta^{1-k}C_4(b+3-k)\alpha,\end{aligned}$$

where $C_4 = (b+2)(b+1)b\cdots(b+4-k)$. Therefore

$$2\beta\gamma\int_0^\infty s\Big|\frac{1+(1-b)\alpha s^b}{(1+\alpha s^b)^2}-1\Big|e^{-(\beta+\gamma)s}e^{\frac{\gamma s}{1+\alpha s^b}}\,ds \leqslant 2C_4\gamma(\beta^{3-k}+\beta^{2-k}+\beta^{1-k}(b+3-k))\alpha. \tag{3.5}$$

By a similar argument, we claim that there exist constants C_5 and C_6 such that

$$\begin{aligned}\int_0^\infty \gamma^2 s\Big(\frac{1+(1-b)\alpha s^b}{(1+\alpha s^b)^2}-1\Big)^2 e^{-(\beta+\gamma)s}e^{\frac{\gamma s}{1+\alpha s^b}}\,ds &= \int_0^\infty \gamma^2 s\Big(\frac{(1+b)\alpha s^b+\alpha^2 s^{2b}}{(1+\alpha s^b)^2}\Big)^2 e^{-(\beta+\gamma)s}e^{\frac{\gamma s}{1+\alpha s^b}}\,ds\\ &\le 2\int_0^\infty \gamma^2 s\Big\{\Big(\frac{(1+b)\alpha s^b}{(1+\alpha s^b)^2}\Big)^2+\Big(\frac{\alpha^2 s^{2b}}{(1+\alpha s^b)^2}\Big)^2\Big\}e^{-\beta s}e^{\frac{-\gamma s^{b+1}}{1+\alpha s^b}}\,ds\\ &\le 2\int_0^\infty \gamma^2 s\Big\{\frac{(1+b)^2\alpha s^b}{(1+\alpha s^b)^2}+\frac{\alpha^2 s^{2b}}{(1+\alpha s^b)^2}\Big\}e^{-\beta s}e^{\frac{-\gamma s^{b+1}}{1+\alpha s^b}}\,ds\\ &\le 4(b+1)^2\int_0^\infty \gamma^2 s^{b+1}e^{-\beta s}\,ds\\ &\le C_5\gamma^2(\beta^{3-k}+\beta^{2-k}+\beta^{1-k}(b+3-k))\alpha \end{aligned}\tag{3.6}$$

and

$$\begin{aligned}\int_0^\infty \frac{\gamma b\alpha s^b|(b-1)\alpha s^b-b-1|}{(1+\alpha s^b)^3}e^{-(\beta+\gamma)s}e^{\frac{\gamma s}{1+\alpha}}\,ds &\leqslant 2\gamma b^2\alpha\int_0^\infty s^b e^{-\beta s}e^{\frac{-\gamma s^{b+1}}{1+\alpha}}\,ds \leqslant 2\gamma b^2\alpha\int_0^\infty s^b e^{-\beta s}\,ds\\ &\leqslant 2C_6\beta^{3-k}\gamma\alpha\int_0^\infty s^{b+3-k}e^{-\beta s}\,ds\\ &\leqslant 2C_6\gamma(\beta^{3-k}+\beta^{2-k}+\beta^{1-k})\alpha.\end{aligned}\tag{3.7}$$

From (3.1), (3.5), (3.6) and (3.7) we conclude that there exists a constant C such that

$$\|f_3(A)\| = \|e^{-\beta A}(I-e^{-\gamma A}e^{\gamma A_\alpha})\| \leqslant C(\beta^{3-k}+\beta^{2-k}+\beta^{1-k})\alpha.$$

The lemma is proved. □

Theorem 3.3. *Problem* (1.2) *is well posed.*

Proof. The solution of problem (1.2) is determined by the formula

$$u_\alpha(t) = e^{(T-t)A_\alpha}\varphi.$$

Assume that $u_{1\alpha}$ and $u_{2\alpha}$ are solutions of problem (1.2) corresponding the data φ_1 and φ_2, respectively:

$$u_{i\alpha}(t) = e^{(T-t)A_\alpha}\varphi_i, \quad i = 1,2.$$

This implies that

$$\|u_{1\alpha}(t)-u_{2\alpha}(t)\| = \|e^{(T-t)A_\alpha}(\varphi_1-\varphi_2)\| \leqslant \|e^{(T-t)A_\alpha}\|\cdot\|(\varphi_1-\varphi_2)\|. \tag{3.8}$$

We consider the following two cases:

Case 1: $b = 1$. From Theorem 2.6 and (3.8) we obtain

$$\|u_{1\alpha}(t)-u_{2\alpha}(t)\| \leqslant 4e^{\frac{T-t}{\alpha}}\|\varphi_1-\varphi_2\|. \tag{3.9}$$

Case 2: $b > 1$. From Theorem 2.6 and (3.8) we obtain

$$\|u_{1\alpha}(t)-u_{2\alpha}(t)\| \leqslant 2b(b+1)\exp\Big((T-t)\Big(\frac{b+1}{(b-1)\alpha}\Big)^{\frac{1}{b}}\Big)\|\varphi_1-\varphi_2\|. \tag{3.10}$$

From (3.9) and (3.10) we conclude that for all $t \in [0, T]$,

$$\|u_{1\alpha}(t) - u_{2\alpha}(t)\| \leqslant \begin{cases} 4e^{\frac{T-t}{\alpha}} \|\varphi_1 - \varphi_2\| & \text{if } b = 1, \\ 2b(b+1)\exp\Big((T-t)\Big(\dfrac{b+1}{(b-1)\alpha}\Big)^{\frac{1}{b}}\Big)\|\varphi_1 - \varphi_2\| & \text{if } b > 1. \end{cases} \tag{3.11}$$

Inequality (3.11) shows the continuous dependence of u_α on the data φ.

The theorem is proved. □

Theorem 3.4. *If $u(t)$ is a solution of problem* (1.1) *satisfying the condition*

$$\|u(0)\| \leqslant E, \tag{3.12}$$

and $u_\alpha(t)$ is the solution of problem (1.2), *then there exists a constant C such that for any $t \in (0, T]$,*

$$\|u(t) - u_\alpha(t)\| \leqslant \begin{cases} 4e^{\frac{T-t}{\alpha}} \varepsilon + C(t^{3-k} + t^{2-k} + t^{1-k})E\alpha & \text{if } b = 1, \\ 2b(b+1)\exp\Big((T-t)\Big(\dfrac{b+1}{(b-1)\alpha}\Big)^{\frac{1}{b}}\Big)\varepsilon + C(t^{3-k} + t^{2-k} + t^{1-k})E\alpha & \text{if } b > 1. \end{cases}$$

By choosing,

$$\alpha = \begin{cases} T\Big(\ln \dfrac{E}{\varepsilon}\Big)^{-1} & \text{if } b = 1 \\ T\Big(\dfrac{b+1}{b-1}\Big(\ln \dfrac{E}{\varepsilon}\Big)^{-1}\Big)^{b} & \text{if } b > 1, \end{cases} \tag{3.13}$$

we obtain, for any $t \in (0, T]$,

$$\|u(t) - u_\alpha(t)\| \leqslant \begin{cases} 4\varepsilon^{\frac{t}{T}} E^{1-\frac{t}{T}} + CT(t^{3-k} + t^{2-k} + t^{1-k})E\Big(\ln \dfrac{E}{\varepsilon}\Big)^{-1} & \text{if } b = 1, \\ 2b(b+1)\varepsilon^{\frac{t}{T}} E^{1-\frac{t}{T}} + CT(t^{3-k} + t^{2-k} + t^{1-k})E\Big(\dfrac{b+1}{b-1}\Big(\ln \dfrac{E}{\varepsilon}\Big)^{-1}\Big)^{b} & \text{if } b > 1. \end{cases}$$

Remark 3.5. To obtain the result of error estimate between the regularized solution and the exact solution, Chen, Hofmann and Zou [4, Theorem 3.15] assume that $u(0) \in X_\varphi$ and $\|u(0)\|_\varphi \leqslant Q$. This means

$$u(0) \in D(\varphi(A)^{-1}), \tag{3.14}$$

$$\|\varphi(A)^{-1}u(0)\| \leqslant Q. \tag{3.15}$$

It is clear that $D(\varphi(A)^{-1}) \subset X$. Hence our condition $u(0) \in X$ is weaker than condition (3.14). Further, since $\varphi(A)$ is bounded, if $u(0)$ satisfies condition (3.15), then

$$\|u(0)\| = \|\varphi(A)\varphi(A)^{-1}u(0)\| \leqslant \|\varphi(A)\|\|\varphi(A)^{-1}u(0)\| \leqslant \|\varphi(A)\|Q.$$

Hence, our condition $\|u(0)\| \leqslant E$, with $E \geqslant \|\varphi(A)\|Q$, is weaker than condition (3.15). The above mentioned authors also show a specific case where the logarithmic convergence rate (see [4, p. 3548]) is

$$\|R_{\alpha(\delta),t}f^\delta - u(t)\| = O\Big(\Big[\frac{1}{\ln(1/\delta)}\Big]^{\xi}\Big) \quad \text{as } \delta \to 0^+ \tag{3.16}$$

for some exponent $\xi > 0$. We want to emphasize that the convergence rate in Theorem 3.4 is better than (3.16) if $b > \max\{1, \xi\}$.

Proof. Let w_α denote the solution of the problem

$$\begin{cases} u_{\alpha t} + A_\alpha u_\alpha = 0, & 0 < t < T \\ u_\alpha(T) = u(T). \end{cases}$$

We have

$$\|u(t) - u_\alpha(t)\| = \|u(t) - w_\alpha(t) + w_\alpha(t) - u_\alpha(t)\| \leqslant \|w_\alpha(t) - u_\alpha(t)\| + \|u(t) - w_\alpha(t)\|. \tag{3.17}$$

We evaluate the first term on the right-hand side of inequality (3.17). Similar to inequality (3.11), we have

$$\|w_\alpha(t) - u_\alpha(t)\| \leqslant \begin{cases} 4e^{\frac{T-t}{\alpha}}\|u(T) - \varphi\| & \text{if } b = 1, \\ 2b(b+1)\exp\Big((T-t)\Big(\dfrac{b+1}{(b-1)\alpha}\Big)^{\frac{1}{b}}\Big)\|u(T) - \varphi\| & \text{if } b > 1. \end{cases}$$

Since $\|u(T) - \varphi\| \leqslant \varepsilon$, it follows that

$$\|w_\alpha(t) - u_\alpha(t)\| \leqslant \begin{cases} 4e^{\frac{T-t}{\alpha}}\varepsilon & \text{if } b = 1, \\ 2b(b+1)\exp\Big((T-t)\Big(\dfrac{b+1}{(b-1)\alpha}\Big)^{\frac{1}{b}}\Big)\varepsilon & \text{if } b > 1. \end{cases} \tag{3.18}$$

Next, we evaluate the second term on the right-hand side of inequality (3.17). We have

$$w_\alpha(t) = e^{(T-t)A_\alpha}u(T) \quad \text{for all } t \in [0, T] \tag{3.19}$$

and

$$u(t) = e^{-tA}u(0) \quad \text{for all } t \in [0, T]. \tag{3.20}$$

This implies that

$$u(T) = e^{-TA}u(0) \quad \text{for all } t \in [0, T]. \tag{3.21}$$

From (3.19) and (3.21), we conclude that

$$w_\alpha(t) = e^{(T-t)A_\alpha}(e^{-TA}u(0)) = e^{(T-t)A_\alpha}e^{-TA}u(0) \quad \text{for all } t \in [0, T]. \tag{3.22}$$

From (3.20) and (3.22), we obtain

$$u(t) - w_\alpha(t) = (e^{-tA} - e^{(T-t)A_\alpha}e^{-TA})u(0) \quad \text{for all } t \in [0, T]. \tag{3.23}$$

It follows from (3.23) that

$$\|u(t) - w_\alpha(t)\| \leqslant \|e^{-tA} - e^{(T-t)A_\alpha}e^{-TA}\|\|u(0)\| \quad \text{for all } t \in [0, T]. \tag{3.24}$$

Using Lemma 3.2, we conclude that there exists a constant C such that

$$\|e^{-tA} - e^{(T-t)A_\alpha}e^{-TA}\| \leqslant C(t^{3-k} + t^{2-k} + t^{1-k})\alpha, 0 < t \leqslant T. \tag{3.25}$$

From (3.12), (3.24) and (3.25) we obtain

$$\|u(t) - w_\alpha(t)\| \leqslant C(t^{3-k} + t^{2-k} + t^{1-k})E\alpha \quad \text{for all } t \in (0, T]. \tag{3.26}$$

It follows from (3.17), (3.18) and (3.26) that, for any $t \in (0, T]$,

$$\|u(t) - u_\alpha(t)\| \leqslant \begin{cases} 4e^{\frac{T-t}{\alpha}}\varepsilon + C(t^{3-k} + t^{2-k} + t^{1-k})E\alpha & \text{if } b = 1, \\ 2b(b+1)\exp\Big((T-t)\Big(\dfrac{b+1}{(b-1)\alpha}\Big)^{\frac{1}{b}}\Big)\varepsilon + C(t^{3-k} + t^{2-k} + t^{1-k})E\alpha & \text{if } b > 1. \end{cases}$$

By choosing α according to formula (3.13), we obtain the assertion as in the theorem.

The theorem is proved. □

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