STABILITY ESTIMATE FOR THE HEAT EQUATION BACKWARD IN TIME WITH NEUMANN AND INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT

In this paper, we first prove that the heat equation backward in time with Neumann and integral boundary conditions is an ill-posed problem. Then, we establish a stability estimate of Hölder type for this ill-posed problem.

Keywords: Heat equation backward; ill-posed problem; stability estimate.

1. Introduction

Consider the problem of determining $u(x,t)$ *satisfying*

$$\begin{cases} u_t - u_{xx} = 0, 0 < x < 1, 0 < t < T \\ u_x(0,t) = 0, \ \int_0^1 u(x,t)dx = 0, \ \ 0 \le t \le T \end{cases} \tag{1.1}$$

with measurement data at $t = T$:

$$u(x,T) = \varphi(x), x \in [0,1] \tag{1.2}$$

where φ is a given function.

Problem (1.1)-(1.2) is an ill-posed problem in the sense of Hadamard (see Theorem 2.1). Therefore, stability estimates and regularization methods are desired.

Although there have been a number of research works on various inverse problems of parabolic equations with integral conditions ([1], [2], [3], [4], [5], [6]), to my knowledge, so far, there have not been any results on stability estimates of heat equation backward in time with Neumann and integral boundary conditions (1.1)-(1.2).

The purpose of this paper is to prove that problem (1.1)-(1.2) is an ill-posed problem and propose a stability estimate result of the Hölder type for this problem. These results are presented in Section 2.

2 Main results

For simplicity of notation, in this section we denote $\|\cdot\|_{L^2(0,1)}$ by $\|\cdot\|$.

Theorem 2.1. (ill-posedness) *Problem* (1.1)-(1.2) *is an ill-posed problem.*

Proof. Set

$$\begin{aligned}
&u^n(x,t) = \frac{1}{n}e^{4\pi^2n^2(T-t)}\cos(2\pi nx),\ 0\leqslant x\leqslant 1,\ 0\leqslant t\leqslant T,\\
&\varphi^n(x) = \frac{1}{n}\cos(2\pi nx),\ 0\leqslant x\leqslant 1,\\
&u(x,t) = 0,\ 0\leqslant x\leqslant 1,\ 0\leqslant t\leqslant T,\\
&\varphi(x) = 0,\ 0\leqslant x\leqslant 1.
\end{aligned}$$

We have

$$\begin{aligned}
&u_t^n(x,t) = -4\pi^2n^2\frac{1}{n}e^{4\pi^2n^2(T-t)}\cos(2\pi nx),\ 0\leqslant x\leqslant 1,\ 0\leqslant t\leqslant T,\\
&u_x^n(x,t) = -2\pi n\frac{1}{n}e^{4\pi^2n^2(T-t)}\sin(2\pi nx),\ 0\leqslant x\leqslant 1,\ 0\leqslant t\leqslant T,\\
&u_x^n(0,t) = -2\pi n\frac{1}{n}e^{4\pi^2n^2(T-t)}\sin(2\pi n0) = -2\pi n\frac{1}{n}e^{4\pi^2n^2(T-t)}\sin 0 = 0,\ 0\leqslant t\leqslant T,\\
&u_{xx}^n(x,t) = -4\pi^2n^2\frac{1}{n}e^{4\pi^2n^2(T-t)}\cos(2\pi nx),\ 0\leqslant x\leqslant 1,\ 0\leqslant t\leqslant T,
\end{aligned}$$

$$\begin{aligned}
\int_0^1 u^n(x,t)dx &= \int_0^1 \frac{1}{n}e^{4\pi^2n^2(T-t)}\cos(2\pi nx)dx\\
&= \frac{1}{n}e^{4\pi^2n^2(T-t)}\int_0^1\cos(2\pi nx)dx\\
&= \frac{1}{n}e^{4\pi^2n^2(T-t)}\frac{\sin(2\pi nx)}{2\pi n}\Big|_0^1\\
&= \frac{1}{n}e^{4\pi^2n^2(T-t)}\frac{\sin(2\pi n)-\sin 0}{2\pi n} = 0,\ 0\leqslant t\leqslant T,\\
u^n(x,T) &= \frac{1}{n}\cos(2\pi nx) = \varphi_n(x),\ 0\leqslant x\leqslant 1,
\end{aligned}$$

$$\begin{aligned}
&u_t(x,t) = 0 = u_{xx}(x,t),\ 0\leqslant x\leqslant 1,\ 0\leqslant t\leqslant T,\\
&u_x(0,t) = 0,\ \int_0^1 u(x,t)dx = 0,\ 0\leqslant t\leqslant T,\\
&u(x,T) = 0 = \varphi(x),\ 0\leqslant x\leqslant 1.
\end{aligned}$$

Therefore $u_n(x,t)$ satisfies the following conditions

$$\begin{cases} u^n_t - u^n_{xx} = 0,\ 0 < x < 1, 0 < t < T \\ u^n_x(0,t) = 0,\ \int_0^1 u^n(x,t)dx = 0, 0 \le t \le T \\ u^n(x,T) = \varphi_n(x),\ 0 \leqslant x \leqslant 1 \end{cases}$$

and $u(x,t)$ satisfies the following conditions

$$\begin{cases} u_t - u_{xx} = 0,\ 0 < x < 1, 0 < t < T \\ u_x(0,t) = 0,\ \int_0^1 u(x,t)dx = 0, 0 \le t \le T \\ u(x,T) = \varphi(x),\ 0 \leqslant x \leqslant 1. \end{cases}$$

We have

$$\begin{aligned} \|\varphi^n - \varphi\| &= \left(\int_0^1 (\varphi^n)^2(x)dx\right)^{\frac{1}{2}} = \left(\int_0^1 \left(\frac{1}{n}\cos(2\pi nx)\right)^2 dx\right)^{\frac{1}{2}} \\ &= \frac{1}{n}\left(\int_0^1 \cos^2(2\pi nx)dx\right)^{\frac{1}{2}} = \frac{1}{n}\left(\int_0^1 \frac{1+\cos(4\pi nx)}{2}dx\right)^{\frac{1}{2}} \\ &= \frac{1}{n}\left(\frac{1}{2}x\Big|_0^1 + \frac{\sin(4\pi nx)}{8\pi n}\Big|_0^1\right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}n} \to 0 \text{ as } n \to +\infty. \end{aligned}$$

However, for $0 \leqslant t < T$ we have

$$\begin{aligned} \|u^n(\cdot,t) - u(\cdot,t)\| &= \left(\int_0^1 u_n(x,t)^2(x)dx\right)^{\frac{1}{2}} = \left(\int_0^1 \left(\frac{1}{n}e^{4\pi^2n^2(T-t)}\cos(2\pi nx)\right)^2 (x)dx\right)^{\frac{1}{2}} \\ &= \frac{1}{n}e^{4\pi^2n^2(T-t)}\left(\int_0^1 \left(\cos(2\pi nx)\right)^2(x)dx\right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}n}e^{4\pi^2n^2(T-t)} \to +\infty \text{ as } n \to +\infty. \end{aligned}$$

This proves that problem (1.1)-(1.2) is an ill-posed problem.
The theorem is proved. □

Theorem 2.2. (Stability estimate) *Let $u_1(x,t)$ and $u_2(x,t)$ be solutions of problem* (1.1). *If $u_1(x,t)$ and $u_2(x,t)$ satisfy*

$$\|u_i(\cdot,0)\| \leqslant E,\ \|u_{ix}(\cdot,t)\| \leqslant E,\ i = 1,2, t \in [0,T] \tag{2.3}$$

and $\|u_1(\cdot,T) - u_2(\cdot,T)\| \leqslant \delta$ where δ and E are some positive constants then

$$\|u_1(\cdot,t) - u_2(\cdot,t)\| \leqslant \sqrt[4]{2}2^{\frac{1}{2}\left(1-\frac{t}{T}\right)}\delta^{\frac{t}{2T}}E^{1-\frac{t}{2T}},\ \forall t \in [0,T].$$

Proof. Set

$$
\begin{aligned}
u(x,t) &= u_1(x,t) - u_2(x,t),\ (x,t) \in [0,1] \times [0,T],\\
v(x,t) &= \int_0^x u(y,t)dy,\ (x,t) \in [0,1] \times [0,T].
\end{aligned}
$$

We have

$$
\begin{aligned}
v(0,t) &= \int_0^0 u(y,t)dy = 0,\ t \in [0,T],\\
v(1,t) &= \int_0^1 u(y,t)dy = \int_0^1 u_1(y,t)dy - \int_0^1 u_2(y,t)dy = 0,\ t \in [0,T],\\
v_t(x,t) &= \int_0^x u_t(y,t)dy = \int_0^x u_{xx}(y,t)dy\\
&= u_x(x,t) - u_x(0,t) = u_x(x,t) \text{ (since } u_x(0,t) = u_{1x}(0,t) - u_{2x}(0,t) = 0),\\
v_x(x,t) &= u(x,t),\ (x,t) \in [0,1] \times [0,T],\ v_{xx}(x,t) = u_x(x,t),\ 0 < x < 1, 0 < t < T.
\end{aligned}
$$

Thus, the function $v(x,t)$ satisfies the following conditions

$$
\begin{cases}
v_t - v_{xx} = 0,\ 0 < x < 1, 0 < t < T\\
v(0,t) = v(1,t) = 0, 0 \le t \le T.
\end{cases}
\tag{2.4}
$$

Set $h(t) = \int_0^1 v^2(x,t)dx,\ t \in [0,T]$. By the integration by part, we have

$$
\begin{aligned}
h'(t) &= 2\int_0^1 v(x,t)v_t(x,t)dx\\
&= 2\int_0^1 v(x,t)v_{xx}(x,t)dx\\
&= 2\int_0^1 v(x,t)d(v_x(x,t)) = 2v(x,t)v_x(x,t)\Big|_0^1 - 2\int_0^1 v_x^2(x,t)dx
\end{aligned}
$$

$$
\begin{aligned}
&= 2v(1,t)v_x(1,t) - 2v(0,t)v_x(0,t) - 2\int_0^1 v_x^2(x,t)dx \\
&= -2\int_0^1 v_x^2(x,t)dx \text{ (since } v(1,t)=v(0,t)=0) \\
h''(t) &= -4\int_0^1 v_x(x,t)v_{xt}(x,t)dx \\
&= -4\int_0^1 v_x(x,t)d(v_t(x,t)) \\
&= -4v_x(x,t)v_t(x,t)\Big|_0^1 + 4\int_0^1 v_t(x,t)d(v_x(x,t)) \\
&= -4v_x(1,t)v_t(1,t) + 4v_x(0,t)v_t(0,t) + 4\int_0^1 v_t(x,t)v_{xx}(x,t)dx \\
&= 4\int_0^1 v_t^2(x,t)dx \text{ } (v(0,t)=v(1,t)=0 \text{ implies } v_t(0,t)=v_t(1,t)=0).
\end{aligned}
$$

Due to the Cauchy-Schwarz inequality, we have

$$
\left(\int_0^1 v(x,t)v_t(x,t)dx\right)^2 \leqslant \int_0^1 v^2(x,t)dx \int_0^1 v_t^2(x,t)dx.
$$

This implies that

$$
\begin{aligned}
&h(t)h''(t) - (h'(t))^2 \\
&\quad = 4\left(\int_0^1 v^2(x,t)dx\int_0^1 v_t^2(x,t)dx - \left(\int_0^1 v(x,t)v_t(x,t)dx\right)^2\right) \geqslant 0,\ \forall t\in[0,T].
\end{aligned}
$$

Consider the case $h(t) > 0,\ \forall t \in [0,T]$. Set $f(t) = \ln h(t),\ \forall t \in [0,T]$. We have

$$
\begin{aligned}
f'(t) &= \frac{h'(t)}{h(t)},\ \forall t\in[0,T] \\
f''(t) &= \frac{h(t)h''(t) - (h'(t))^2}{h^2(t)} \geqslant 0,\ \forall t\in[0,T].
\end{aligned}
$$

This proves that f is a convex function. Therefore, we have

$$
f(t) = f\left(\frac{t}{T}T + \left(1-\frac{t}{T}\right)0\right) \leqslant \frac{t}{T}f(T) + \left(1-\frac{t}{T}\right)f(0),\ \forall t\in[0,T].
$$

This implies that

$$
h(t) \leqslant h(T)^{\frac{t}{T}} h(0)^{1-\frac{t}{T}},\ \forall t\in[0,T] \tag{2.5}
$$

or

$$\|v(\cdot,t)\| \leqslant \|v(\cdot,T)\|^{\frac{t}{T}} \|v(\cdot,0)\|^{1-\frac{t}{T}}, \ \forall t \in [0,T]. \tag{2.6}$$

Now we consider the case when $h(t)$ can vanish. Since $h'(t) = -2\int_0^1 v_x^2(x,t)dx \leqslant 0, \ \forall t \in [0,T]$, h is a decreasing function on $[0,T]$. If $h(0) = 0$, then $h(t) = 0, \ \forall t \in [0,T]$. Therefore, if $h(0) = 0$ then the inequality (2.5) is obvious. If $h(0) > 0$, then $h(t) > 0, \ \forall t \in [0,T]$. In fact, supposing the contrary, let t_0 be the first point where $h(t) = 0$. By continuity, $h(t) > 0$ for $0 \leqslant t < t_0$. Therefore $h(t) > 0$ for $0 \leqslant t \leqslant s < t_0$. Using the stability estimate (2.5) with T replacing by $s < t_0$ and by letting $s \uparrow t_0$ we obtain a contradiction. Therefore, when $h(t)$ can vanish, the inequality (2.5) is still correct. So, in all cases, inequality (2.6) holds.

We have

$$\begin{aligned}
\|u(\cdot,t)\|^2 &= \int_0^1 u^2(x,t)dx = \int_0^1 v_x^2(x,t)dx \\
&= \int_0^1 v_x(x,t)d(v(x,t)) \\
&= v_x(x,t)v(x,t)\Big|_0^1 - \int_0^1 v(x,t)v_{xx}(x,t)dx \\
&= v_x(1,t)v(1,t) - v_x(0,t)v(0,t) - \int_0^1 v(x,t)v_{xx}(x,t)dx \\
&= -\int_0^1 v(x,t)v_{xx}(x,t)dx \text{ (since } v(1,t) = v(0,t) = 0) \\
&= -\int_0^1 v(x,t)u_x(x,t)dx \text{ (since } v_{xx}(x,t) = u_x(x,t)).
\end{aligned}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\|u(\cdot,t)\|^4 &= \left(\int_0^1 v(x,t)u_x(x,t)dx\right)^2 \\
&\leqslant \int_0^1 v^2(x,t)dx \int_0^1 u_x^2(x,t)dx \\
&= \|v(\cdot,t)\|^2 \|u_x(\cdot,t)\|^2, \ \forall t \in [0,T].
\end{aligned}$$

This implies that

$$\|u(\cdot,t)\|^2 \leqslant \|v(\cdot,t)\| \|u_x(\cdot,t)\|, \ \forall t \in [0,T]. \tag{2.7}$$

On the other hand, we have

$$\begin{aligned}\|v(\cdot,T)\|^2 &= \int_0^1 v^2(x,T)dx = \int_0^1 \left(\int_0^x u(y,T)dy\right)^2 dx\\ &\leqslant \int_0^1 \left(\int_0^x dy \int_0^x u^2(y,T)dy\right) dx\\ &\leqslant \int_0^1 \left(\int_0^x dy \int_0^1 u^2(y,T)dy\right) dx\\ &= \int_0^1 u^2(y,T)dy \int_0^1 \left(\int_0^x dy\right) dx\\ &= \frac{1}{2}\|u(\cdot,T)\|^2.\end{aligned}$$

This implies that

$$\|v(\cdot,T)\| \leqslant \frac{1}{\sqrt{2}}\|u(\cdot,T)\|. \tag{2.8}$$

Similarly, we have

$$\begin{aligned}\|v(\cdot,0)\|^2 &= \int_0^1 v^2(x,0)dx = \int_0^1 \left(\int_0^x u(y,0)dy\right)^2 dx\\ &\leqslant \int_0^1 \left(\int_0^x dy \int_0^x u^2(y,0)dy\right) dx\\ &\leqslant \int_0^1 \left(\int_0^x dy \int_0^1 u^2(y,0)dy\right) dx\\ &= \int_0^1 u^2(y,0)dy \int_0^1 \left(\int_0^x dy\right) dx\\ &= \frac{1}{2}\|u(\cdot,0)\|^2.\end{aligned}$$

This implies that

$$\|v(\cdot,0)\| \leqslant \frac{1}{\sqrt{2}}\|u(\cdot,0)\|. \tag{2.9}$$

From (2.6), (2.7), (2.8) and (2.9), we obtain

$$\begin{aligned}\|u(\cdot,t)\|^2 &\leqslant \|v(\cdot,t)\|\|u_x(\cdot,t)\|\\ &\leqslant \|v(\cdot,T)\|^{\frac{t}{T}}\|v(\cdot,0)\|^{1-\frac{t}{T}}\|u_x(\cdot,t)\|\\ &\leqslant \left(\frac{1}{\sqrt{2}}\|u(\cdot,T)\|\right)^{\frac{t}{T}}\left(\frac{1}{\sqrt{2}}\|u(\cdot,0)\|\right)^{1-\frac{t}{T}}\|u_x(\cdot,t)\|\\ &= \frac{1}{\sqrt{2}}\|u(\cdot,T)\|^{\frac{t}{T}}\|u(\cdot,0)\|^{1-\frac{t}{T}}\|u_x(\cdot,t)\|,\ \forall t\in[0,T].\end{aligned} \tag{2.10}$$

We have

$$\|u(\cdot,T)\| = \|u_1(\cdot,T) - u_2(\cdot,T)\| \leqslant \delta \tag{2.11}$$

$$\|u(\cdot,0)\| = \|u_1(\cdot,0) - u_2(\cdot,0)\| \leqslant \|u_1(\cdot,0)\| + \|u_2(\cdot,0)\| \leqslant E + E = 2E \tag{2.12}$$

$$\|u_x(\cdot,t)\| = \|u_{1x}(\cdot,t) - u_{2x}(\cdot,t)\| \leqslant \|u_{1x}(\cdot,t)\| + \|u_{2x}(\cdot,t)\| \leqslant E + E = 2E. \tag{2.13}$$

From (2.10), (2.11), (2.12) and (2.13), we obtain

$$\begin{aligned}\|u(\cdot,t)\|^2 &\leqslant \frac{1}{\sqrt{2}}\delta^{\frac{t}{T}}(2E)^{1-\frac{t}{T}}2E \\ &= \sqrt{2}2^{1-\frac{t}{T}}\delta^{\frac{t}{T}}E^{2-\frac{t}{T}},\ \forall t \in [0,T]. \end{aligned}\tag{2.14}$$

This implies that

$$\|u_1(\cdot,t) - u_2(\cdot,t)\| = \|u(\cdot,t)\| \leqslant \sqrt[4]{2}2^{\frac{1}{2}\left(1-\frac{t}{T}\right)}\delta^{\frac{t}{2T}}E^{1-\frac{t}{2T}},\ \forall t \in [0,T]. \tag{2.15}$$

The theorem is proved. □

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TÓM TẮT

ĐÁNH GIÁ ỔN ĐỊNH CHO PHƯƠNG TRÌNH TRUYỀN NHIỆT NGƯỢC THỜI GIAN VỚI CÁC ĐIỀU KIỆN BIÊN NEUMANN VÀ TÍCH PHÂN

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Ngày nhận bài 03/5/2024, ngày nhận đăng 23/7/2024

Trong bài báo này, đầu tiên chúng tôi chứng minh phương trình truyền nhiệt ngược thời gian với các điều kiện biên Neumann và tích phân là một bài toán đặt không chỉnh. Sau đó, chúng tôi thành lập kết quả đánh giá ổn định kiểu Hölder cho bài toán này.

Từ khóa: Phương trình truyệt nhiệt ngược; bài toán đặt không chỉnh; đánh giá ổn định.