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On Riemann–Liouville type operators, bounded mean oscillation, gradient estimates and approximation on the Wiener space

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ABSTRACT

We discuss in a stochastic framework the interplay between Riemann–Liouville type operators applied to stochastic processes, bounded mean oscillation, real interpolation, and approximation. In particular, we investigate the singularity of gradient processes on the Wiener space arising from parabolic PDEs via the Feynman–Kac theory. The singularity is measured in terms of bmo-conditions on the fractional integrated gradient. As an application we treat an approximation problem for stochastic integrals on the Wiener space. In particular, we provide a discrete time hedging strategy for the binary option with a uniform local control of the hedging error under a shortfall constraint.

1. Introduction

We deal with the interplay between Riemann–Liouville type operators applied to adapted càdlàg processes, weighted bounded mean oscillation (weighted bmo), real interpolation, and approximation theory. Given a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$, satisfying the usual conditions, there are various applications in which stochastic processes $L = (L_t)_{t\in[0,T)}$ appear that have a singularity when $t \uparrow T$. Examples are gradient processes obtained in connection to semi-linear parabolic backward PDEs or (non-local) operators, that occur as integrands in stochastic integral representations on the Wiener- and Lévy-Itô space, or trading strategies in stochastic finance for non-smooth pay-offs. We investigate quantitative properties of $(L_t)_{t\in[0,T)}$, as the degree of blow-up and distributional properties, and apply the results to approximation problems for stochastic processes, in particular to the discrete time hedging in the Black–Scholes model.

To be more precise, we discover and exploit the interplay between the following topics and think that the methodology behind might be of wider interest:

(a) *Self-similarity and bmo:* There is a self-similar structure behind in the sense that, given $a \in (0, T)$ and $A \in \mathcal{F}_a$ of positive measure, then $(L_t)_{t\in[a,T)}$ restricted to A has similar properties as $(L_t)_{t\in[0,T)}$ has. If one seeks for good distributional estimates for $(L_t)_{t\in[0,T)}$, then this suggests to consider $(L_t)_{t\in[0,T)}$ in bmo, and to exploit relations to the BMO-spaces to achieve better tail estimates for L than $\mathbf{L}_p$-estimates would imply.

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(b) *Polynomial blow-up, Riemann–Liouville type operators, weighted bmo:* If the process L has a singularity when $t \uparrow T$, then (a) might not work. We will have a polynomial blow-up of L in time with a rate $(T-t)^{-\alpha}$ for some $\alpha > 0$. So instead of measuring L in bmo, we will measure $\mathcal{I}^\alpha L$ in bmo, where $\mathcal{I}^\alpha$ is a Riemann–Liouville type operator given in (1.1) that resolves the singularity of the process L. However, classical bmo-spaces are not sufficient for our applications, we will need to use *weighted* bmo-spaces.

(c) *End-point estimates, interpolation, and Hölder-spaces:* The consideration of the bmo-setting follows a path known for singular integral operators or martingale transforms: $\mathbf{L}_p$-$\mathbf{L}_p$ estimates yield to bmo-$\mathbf{L}_\infty$ estimates when $p \uparrow \infty$. For us, the Hölder spaces will play the role of an $\mathbf{L}_\infty$-endpoint in the scale of Besov spaces on the Wiener space. Secondly, we again need to exploit *weighted* bmo-spaces instead of the non-weighted bmo-spaces. Only with these ingredients we obtain the desired bmo-Hölder estimates behind Corollary 1.2, Theorem 1.3, and Corollary 1.4.

The structure and background of the article are as follows: After the preliminaries in Section 2, we turn in Section 3 to Riemann–Liouville type operators in a more general context. Riemann–Liouville operators are a central object in fractional calculus. To extend them to probabilistic frameworks, for example to martingales $(L_t)_{t\in[0,T)}$, there are different options depending on the application:

(A) *Fractional martingales*, *i.e.* $t \mapsto \int_0^t (t-u)^{\alpha-1} L_u \mathrm{d}u$, $\alpha > 0$, were used in [1] for Gaussian processes. Here the martingale property gets lost.

We will not concentrate on approach (A), but develop further the following one:

(B) In [2, Definition 4.2] a path-wise approach was used for certain gradient processes on the Wiener space, but no systematic investigation was done. This approach is intended for the case when one has a singularity when $t \uparrow T$, it keeps the martingale property, and it commutes with horizontal and vertical derivatives from functional Itô-calculus [3] (see Remark 3.8 of this article). To be more precise: For $\alpha > 0$ and a càdlàg function $K : [0,T) \to \mathbb{R}$ we define the Riemann–Liouville type operator $\mathcal{I}^\alpha K := (\mathcal{I}_t^\alpha K)_{t\in[0,T)}$ by

$$\mathcal{I}_t^\alpha K := \frac{\alpha}{T^\alpha} \int_0^T (T-u)^{\alpha-1} K_{u\wedge t} \mathrm{d}u \quad \text{and} \quad \mathcal{I}_t^0 K := K_t. \tag{1.1}$$

Looking from the perspective of martingales, the idea behind is that we start with a non-closable martingale K and might obtain a closable martingale $\mathcal{I}^\alpha K$. In Proposition 3.6 we show that the approach via (1.1) yields to an extension of the *fractional martingale transform* (cf., for example, [4,5]) to all càdlàg processes. We also extend the family of operators $(\mathcal{I}^\alpha)_{\alpha>0}$ to $\alpha \leqslant 0$ to get the group structure

$$\mathcal{I}^\alpha(\mathcal{I}^\beta K) = \mathcal{I}^{\alpha+\beta} K \quad \text{for} \quad \alpha, \beta \in \mathbb{R} \quad \text{so that} \quad \mathcal{I}^{-\alpha}(\mathcal{I}^\alpha K) = K.$$

A combination of the two approaches (A) and (B) can be found in [6].

Section 4 is about approximations of càdlàg processes. We define for a càdlàg process $L = (L_t)_{t\in[0,T)}$, $a \in [0,T]$, and a deterministic time-net $\tau = \{t_i\}_{i=0}^n$ with $0 = t_0 < t_1 < \cdots < t_n = T$, the $\mathbf{L}_2$-approximation of L along τ by

$$[L;\tau]_a^1 := \int_0^a \left| L_u - \sum_{i=1}^n L_{t_{i-1}} \mathbb{1}_{(t_{i-1},t_i]}(u) \right|^2 \mathrm{d}u.$$

If $L = (L_t)_{t\in[0,T)} \subseteq \mathbf{L}_2$ is a càdlàg martingale, then $\mathbb{E}[L;\tau]_T^1$ describes an $\mathbf{L}_2$-filtering problem where the filtration is kept constant on the time intervals of the net τ. In Theorem 4.9 we provide in the $\mathbf{L}_2$-context the connections between approximation theory, fractional closability, and real interpolation: for $\theta \in (0,1)$ we prove the equivalence of the three conditions

$$\sup_\tau \frac{\mathbb{E}[L;\tau]_T^1}{\|\tau\|_\theta} < \infty, \tag{1.2}$$

$$\mathcal{I}^{\frac{1-\theta}{2}} L \text{ is a martingale closable in } \mathbf{L}_2, \tag{1.3}$$

$$(L_{t_k})_{k=0}^\infty \in (\ell_2^{-\frac{1}{2}}(\mathbf{L}_2), \ell_\infty(\mathbf{L}_2))_{\theta,2} \quad \text{with} \quad t_k := T\left(1-2^{-k}\right). \tag{1.4}$$

Here, for $\theta \in (0,1]$, we use the adapted mesh-size

$$\|\tau\|_\theta := \sup_{i=1,\ldots,n} \frac{t_i - t_{i-1}}{(T-t_{i-1})^{1-\theta}}$$

to compensate a blow-up of $L = (L_t)_{t\in[0,T)}$ when $t \uparrow T$ as the mesh-size $\|\cdot\|_\theta$ assigns more weight to grid-points close to T when θ gets small. The prototype of nets such that $\|\tau_n\|_\theta \sim 1/n$ is defined as $\tau_n := \tau_n^\theta = \{t_{i,n}^\theta\}_{i=0}^n$ with

$$t_{i,n}^\theta := T - T\,(1-(i/n))^{1/\theta}. \tag{1.5}$$

This adapted mesh-size goes back (at least) to [7,8] and has been exploited in [2,9–11] in the diffusion setting and in [12,13] in the jump setting.

Condition (1.4), together with relations (4.13) and (4.14), says that L belongs to a space resulting from real interpolation between two end-points: the left end-point consists of martingales L with $\int_0^T \|L_t\|_{\mathbf{L}_2}^2 \mathrm{d}t < \infty$, a typical condition for integrands of stochastic integrals, the right end-point consists of martingales L with $\sup_{t\in[0,T)} \|L_t\|_{\mathbf{L}_2} < \infty$, *i.e.* martingales closable in $\mathbf{L}_2$.

Condition (1.3) says that after smoothing the martingale L with the operator $\mathcal{I}^{\frac{1-\theta}{2}}$ we get a martingale closable in $\mathbf{L}_2$. Giving a martingale closable in $\mathbf{L}_2$ the smoothness 1 and applying $\mathcal{I}^{\frac{\theta-1}{2}} = (\mathcal{I}^{\frac{1-\theta}{2}})^{-1}$, we interpret this as that L has a fractional smoothness of order $1 - \frac{1-\theta}{2} = \frac{1+\theta}{2}$ in $\mathbf{L}_2$.

Condition (1.2) concerns a discrete time approximation of L by the martingale $(\sum_{i=1}^n L_{t_{i-1}} \mathbb{1}_{(t_{i-1},t_i]}(t))_{t\in[0,T)}$. In Theorem 4.12 condition (1.2) also yields another approximation of a càdlàg martingale, a backward in time regularization with local Lipschitz trajectories.

Finally, one of the main results of Section 4 is Theorem 4.11, where we prove the equivalent to (1.2) ⇔ (1.3) in the bmo-context.

In Section 5 we apply the results to the Wiener space. We suppose a stochastic basis $(\Omega, \mathcal{F}_T, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ satisfying the usual conditions and generated by a standard one-dimensional Brownian motion $W = (W_t)_{t\in[0,T]}$ with continuous paths and starting in zero. We start with a diffusion

$$\mathrm{d}X_t = \hat{\sigma}(X_t)\mathrm{d}W_t + \hat{b}(X_t)\mathrm{d}t$$

with $X_0 \equiv x_0 \in \mathbb{R}$ with $0 < \varepsilon_0 \leqslant \hat{\sigma} \in C_b^\infty(\mathbb{R})$ and $\hat{b} \in C_b^\infty(\mathbb{R})$, and derive

$$\mathrm{d}Y_t = \sigma(Y_t)\mathrm{d}W_t \quad \text{with} \quad Y_0 \equiv y_0 \in \mathcal{R}_Y$$

in the two cases

(C1) $Y := X$ and $\mathcal{R}_Y := \mathbb{R}$ $\quad (\sigma \equiv \hat{\sigma},\ \hat{b} \equiv 0)$,
(C2) $Y := \mathrm{e}^X$ and $\mathcal{R}_Y := (0,\infty)$ $\quad (\sigma(y) := y\hat{\sigma}(\ln y),\ \hat{b}(x) := -\frac{1}{2}\hat{\sigma}^2(x))$.

We let C_Y be the set of all Borel functions $g : \mathcal{R}_Y \to \mathbb{R}$ satisfying the size condition (5.2) from Section 5.1. For $g \in C_Y$ we let

$$G(t,y) := \mathbb{E}[g(Y_T)|Y_t = y] \quad \text{for} \quad (t,y) \in [0,T] \times \mathcal{R}_Y,$$

so that

$$g(Y_T) - \mathbb{E}g(Y_T) = \int_{(0,T)} \varphi_t \mathrm{d}Y_t \quad \text{with} \quad \varphi_t := \partial_y G(t, Y_t). \tag{1.6}$$

In the Black–Scholes model (case (C2)) the process φ is the δ-hedging strategy for the pay-off $g(Y_T)$. For a deterministic net $\tau = \{0 = t_0 < \cdots < t_n = T\}$ we define the approximation error for the Riemann approximation of the stochastic integral in (1.6) as

$$E_t(g;\tau) := \int_{(0,t]} \varphi_s \mathrm{d}Y_s - \sum_{i=1}^n \varphi_{t_{i-1}}(Y_{t_i\wedge t} - Y_{t_{i-1}\wedge t}).$$

So, in the Black–Scholes model, $E_t(g;\tau)$ is the hedging error at time t when re-balancing the portfolio associated to an option with pay-off $g(Y_T)$ at the discrete times from τ only. Our starting point is [14, Theorems 7 and 8] from which, for $\sigma(y) = y$ and $y_0 = 1$ (geometric Brownian motion), it follows that

$$\begin{aligned} & g \text{ coincides a.e. with a Lipschitz function} \\ \iff & \sup_\tau \frac{\|E(g;\tau)\|_{\mathrm{bmo}_2^Y[0,T)}}{\sqrt{\|\tau\|_1}} < \infty. \end{aligned} \tag{1.7}$$

Here, for $p \in (0,\infty)$ and adapted càdlàg processes A and $\Phi \geqslant 0$ defined on $[0,T)$ we let $\|A\|_{\mathrm{bmo}_p^\Phi[0,T)} := \inf c$, where the infimum is taken over all $c \in [0,\infty)$ such that for all $0 \leqslant a \leqslant t < T$ one has

$$\mathbb{E}\left[|A_t - A_a|^p | \mathcal{F}_a\right] \leqslant c^p \Phi_a^p \text{ a.s.}$$

Hence the RHS of (1.7) is limited to Lipschitz functions. Looking instead at the binary option, one cannot expect an arbitrary small local error uniformly over $[0,T)$ when only finitely many times are used to re-balance the portfolio. In fact, given any deterministic time-net $0 = t_0 < \cdots < t_{n-1} < t_n = T$, on the last interval $[t_{n-1}, T]$ we always have a uniform lower bound independently of how small $T - t_{n-1}$ is:

Proposition 1.1. *Assume $\sigma(y) = y$ and $y_0 = 1$. There is a $T_0 \in [0,T)$ such that for all $a \in [T_0,T)$ there is a set $B_a \in \mathcal{F}_a$ of positive measure such that, for all $\mathcal{F}_a$-measurable $v_a, w_a : \Omega \to \mathbb{R}$,*

$$\int_{B_a} \left|\mathbb{1}_{[1,\infty)}(Y_T) - [v_a + w_a(Y_T - Y_a)]\right|^2 \frac{\mathrm{d}\mathbb{P}}{\mathbb{P}(B_a)} \geqslant \frac{1}{192}. \tag{1.8}$$

Proposition 1.1 is proven in Section 7.9. So there is a discrepancy between (1.7) and (1.8). We resolve this discrepancy in two steps:

Step 1: For $M = (M_t)_{t\in[0,T)}$ with

$$M_t := \int_0^t \left(\sigma^2 \partial_{yy}^2 G\right)(u, Y_u)\mathrm{d}W_u$$

we show in Theorem 5.8 for all deterministic nets $\tau = \{0 = t_0 < \cdots < t_n = T\}$ and all adapted càdlàg processes $\Phi > 0$ that

$$\frac{\|E(g;\tau)\|^2_{\mathrm{bmo}_2^{\Phi}[0,T)}}{\|\tau\|_\theta} \leqslant c_{(4.3)} \Bigg[4\|\mathcal{I}^{\frac{1-\theta}{2}} M\|^2_{\mathrm{bmo}_2^{\Phi}[0,T)} + \Bigg\| \sup_{k\in\{1,\ldots,n\}} \sup_{a\in[t_{k-1},t_k)} \frac{T-a}{(T-t_{k-1})^\theta} |\varphi_a - \varphi_{t_{k-1}}|^2 \frac{\sigma(Y_a)^2}{\Phi_a^2} \Bigg\|_{\mathbf{L}_\infty} \Bigg]. \tag{1.9}$$

Step 2 concerns an upper bound for the RHS of (1.9). For this we use a two parameter scale of Hölder spaces. Let $C_b^0(\mathbb{R})$ consist of the bounded continuous functions and $\mathrm{Höl}_1^0(\mathbb{R})$ of the Lipschitz functions, both defined on $\mathbb{R}$ and vanishing at zero. Then we define the scale of Hölder spaces

$$\mathrm{Höl}_{\theta,q}(\mathbb{R}) := (C_b^0(\mathbb{R}), \mathrm{Höl}_1^0(\mathbb{R}))_{\theta,q} \quad \text{for} \quad (\theta,q) \in (0,1)\times[1,\infty]$$

by real interpolation [15,16], where the fine-index $q = \infty$ recovers the θ-Hölder functions. The inclusions $\mathrm{Höl}_{\theta,q_0}(\mathbb{R}) \subsetneq \mathrm{Höl}_{\theta,q_1}(\mathbb{R})$ for $1 \leqslant q_0 < q_1 \leqslant \infty$ are strict which follows from [17, Theorem 3.1]. For $0 \leqslant s \leqslant a < T$ we will prove in Theorem 5.9 that

$$\|\mathcal{I}^{\frac{1-\theta}{2}} M\|_{\mathrm{bmo}_2^{\sigma(Y)^\theta}[0,T)} \leqslant c_{(1.10)} \|g\|_{\mathrm{Höl}_{\theta,2}(\mathbb{R})} \qquad \text{for} \quad \theta \in (0,1), \tag{1.10}$$

$$\frac{T-a}{(T-s)^\theta} |\varphi_a - \varphi_s|^2 \leqslant c^2_{(1.11)} |g|^2_\theta \left(\sigma(Y_a)^{2(\theta-1)} + \sigma(Y_s)^{2(\theta-1)}\right) \tag{1.11}$$

for $\theta \in [0,1]$, where $|g|_\theta := \sup_{x\neq y} |g(x)-g(y)|/|x-y|^\theta$ is the θ-Hölder semi-norm. In Section 7.4 we show that (1.9), (1.10), and (1.11) imply the following:

Corollary 1.2. *Let $\theta \in (0,1)$ and $g = g^{(\theta)} + g^{(1)} \in \mathrm{Höl}_{\theta,2}(\mathbb{R}) + \mathrm{Höl}_1^0(\mathbb{R})$. For a deterministic time-net $\tau = \{0 = t_0 < \cdots < t_n = T\}$ and $a \in [0,T)$ we define the weight process $\Phi = (\Phi_a)_{a\in[0,T)}$ by*

$$\Phi_a(\tau,\theta) := \sigma(Y_a)^\theta + \sigma(Y_{t_{k-1}})^{\theta-1}\sigma(Y_a) \qquad \textit{if} \quad a \in [t_{k-1},t_k),$$
$$\Phi_a := \|g^{(\theta)}\|_{\mathrm{Höl}_{\theta,2}(\mathbb{R})} \Phi_a(\tau,\theta) + |g^{(1)}|_1 \sigma(Y_a).$$

Then there is a $c = c(T,\theta,\sigma) > 0$ such that

$$\|E(g;\tau)\|_{\mathrm{bmo}_2^{\Phi}[0,T)} \leqslant c\sqrt{\|\tau\|_\theta}.$$

Moreover, for $\eta \in (\theta,1)$ and an η-Hölder function $g : \mathbb{R} \to \mathbb{R}$ one has

$$\|E(g;\tau)\|_{\mathrm{bmo}_2^{\Phi(\tau,\theta)+\sigma(Y)}[0,T)} \leqslant d\, |g|_\eta \sqrt{\|\tau\|_\theta}$$

for some $d = d(T,\theta,\eta,\sigma) > 0$.

In case (C2) Corollary 1.2 applies to powered call and put options (see [18]). Moreover, it plays the key role to provide a solution to overcome the discrepancy between (1.7) and (1.8): For $K > 0$ we first replace the pay-off $g = \mathbb{1}_{[K,\infty)}$ by

$$g_\varepsilon(y) := \frac{1}{\varepsilon}\int_{y-\varepsilon}^{y} \mathbb{1}_{[K,\infty)}(z)\mathrm{d}z \leqslant \mathbb{1}_{[K,\infty)}(y) \quad \text{for some} \quad \varepsilon > 0.$$

The Lipschitz constant blows up like ε^{-1}. The trick is to use the Lipschitz function g_ε, however *to measure this Lipschitz function in the* $\mathrm{Höl}_{\theta,2}(\mathbb{R})$ *space*. With inequality Eq. (7.11) we will show that

$$\|g_\varepsilon\|_{\mathrm{Höl}_{\theta,2}(\mathbb{R})} \leqslant c_\theta \varepsilon^{-\theta},$$

so that the blow-up is $\varepsilon^{-\theta}$ instead of ε^{-1}. This effect can be made arbitrary strong by $\theta \downarrow 0$ and yields to a significant improvement of the approximation of the binary option. Specializing the results from Section 5 to the function $g(y) = \mathbb{1}_{[K,\infty)}(y)$, which is the pay-off of the binary option in the case (C2) introduces above, we prove in Section 7.8:

Theorem 1.3. *For $\theta \in (0,1)$, $D \geqslant 1$, $\varepsilon := 2D^{-\frac{1}{\theta}}(\sup_{y\in\mathbb{R}} p_T(y))^{-1}$, where p_T is the continuous density of Y_T, $K > 0$, and a $c = c(T,\theta,\sigma) > 0$ one has*

(1) $\mathbb{E}\mathbb{1}_{[K,\infty)}(Y_T) - \mathbb{E}g_\varepsilon(Y_T) \leqslant D^{-\frac{1}{\theta}}$ and $\mathbb{P}(g_\varepsilon(Y_T) < \mathbb{1}_{[K,\infty)}(Y_T)) \leqslant 2D^{-\frac{1}{\theta}}$,
(2) $\|E(g_\varepsilon;\tau_n^\theta)\|_{\mathrm{bmo}_2^{\Phi}[0,T)} \leqslant c\frac{D}{\sqrt{n}}$ for $\Phi := \Phi(\tau_n^\theta,\theta)$,[3]
(3) $|\varphi^\varepsilon| \leqslant cD^{\frac{1}{\theta}}$ on $[0,T)\times\Omega$, where φ^ε is defined as in (1.6) for g_ε.

In the case (C2) the interpretation is as follows: In (1) we estimate the difference of the option prices for $\mathbb{1}_{[K,\infty)}(Y_T)$ and $g_\varepsilon(Y_T)$ and the shortfall probability, respectively, (3) is a size-constraint on the trading strategy, and (2) bounds the uniform local hedging error with the optimal rate $1/\sqrt{n}$. One essential point of Theorem 1.3 is that in (1) the constant $1/D$ is raised to the exponent $1/\theta$,

[3] τ_n^θ is given in (1.5) and $\Phi(\tau,\theta)$ in *Corollary* 1.2.

so that for $D > 1$ the constant $D^{-\frac{1}{\theta}}$ gets arbitrary small for small θ, whereas in (2) we have the exponent 1 for D. To illustrate this further we couple the cardinality n of the time-net to D by $D := n^\delta$ for $\delta \in (0, 1/4)$ to get a balance between the *hedging error* and the *shortfall probability*:

Corollary 1.4. *For $\delta \in (0, \frac{1}{4})$, $\theta := \frac{2\delta}{1-2\delta}$, $n \in \mathbb{N}$, $\varepsilon_n := 2n^{-\frac{\delta}{\theta}}(\sup_{y\in\mathbb{R}} p_T(y))^{-1}$, and $K > 0$ one has:*

(1) $\mathbb{E}\mathbb{1}_{[K,\infty)}(Y_T) - \mathbb{E}g_{\varepsilon_n}(Y_T) \leqslant n^{\delta-\frac{1}{2}}$ and $\mathbb{P}(g_{\varepsilon_n}(Y_T) < \mathbb{1}_{[K,\infty)}(Y_T)) \leqslant 2n^{\delta-\frac{1}{2}}$.

(2) If $Q \in \mathrm{RH}_q(\mathbb{P})$ for some $q \in (1, \infty)$, then one has, for $\lambda \geqslant 1$ and $a \in [0, T)$,

$$Q\left(\sup_{t\in[a,T)} \left| E_t(g_{\varepsilon_n}; \tau) - E_a(g_{\varepsilon_n}; \tau) \right| > c n^{\delta-\frac{1}{2}} \lambda \Phi_a \,\middle|\, \mathcal{F}_a \right) \leqslant c \begin{cases} \mathrm{e}^{-\frac{\lambda}{c}} & : \text{(C1)} \\ \mathrm{e}^{-\frac{|\ln \lambda|^2}{c}} & : \text{(C2)} \end{cases}$$

where $c = c(T, \theta, \sigma, q, \|\mathrm{d}Q/\mathrm{d}\mathbb{P}\|_{\mathrm{RH}_q(\mathbb{P})}) > 0$ in case (C1), and in case (C2) we additionally assume $Q \in \mathrm{RH}^\xi_\infty(\mathbb{P})$ and get $c = c(T, \theta, \sigma, q, \xi) > 0$.

Corollary 1.4 is proven in Section 7.8. Here $Q \in \mathrm{RH}_q(\mathbb{P})$ and $Q \in \mathrm{RH}^\xi_\infty(\mathbb{P})$ mean that the Radon–Nikodym derivative $\mathrm{d}Q/\mathrm{d}\mathbb{P}$ satisfies a reverse Hölder inequality (see Definitions 2.4 and 5.5).

Remark 1.5. In Corollary 1.4 we see that $\frac{1}{4} > \delta \downarrow 0$ implies $1 > \theta \downarrow 0$, so that Theorem 1.3 becomes essential for small θ. Without Corollary 1.2 and only using [14], which considers equidistant time-nets for Lipschitz terminal conditions, would necessary yield to $\theta = 1$ and therefore to $\delta = 1/4$ in the proof of Corollary 1.4. So the rate would be only $n^{-\frac{1}{4}}$, while we get $n^{\delta-\frac{1}{2}}$ for any $\delta \in (0, 1/4)$ in this article.

Sections 6 and 7 contain proofs of the results presented before. In Section 8 we provide some auxiliary results, in particular with Section 8.1 the necessary connections to [14] regarding the BMO-spaces and the weights from $\mathcal{SM}_p$.

2. Preliminaries

2.1. General notation

We let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $a, b \in \mathbb{R}$ we use $a \vee b := \max\{a, b\}$, $a \wedge b := \min\{a, b\}$, $a^+ := a \vee 0$, $a^- := (-a) \vee 0$, and for $A, B \geqslant 0$ and $c \geqslant 1$ the notation $A \sim_c B$ for $\frac{1}{c} B \leqslant A \leqslant cB$. The corresponding one-sided inequalities are abbreviated by $A \succeq_c B$ and $A \preceq_c B$ and we agree about $0^0 := 1$. Given a metric space M, $\mathcal{B}(M)$ denotes the Borel σ-algebra generated by the open sets. For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable map $X : \Omega \to \mathbb{R}^d$, where $\mathbb{R}^d$ is equipped with $\mathcal{B}(\mathbb{R}^d)$, the law of X is denoted by $\mathbb{P}_X$. For $p \in (0, \infty]$ and a measure space $(\Omega, \mathcal{F}, \mu)$ we use the standard Lebesgue spaces $\mathbf{L}_p(\Omega, \mathcal{F}, \mu)$ and omit parts of $(\Omega, \mathcal{F}, \mu)$ in the notation if there is no risk of confusion. For a set $A \in \mathcal{F}$ with $\mu(A) \in (0, \infty)$ we let μ_A be the normalized restriction of μ to the trace σ-algebra $\mathcal{F}|_A$.

2.2. Interpolation spaces

Let (E_0, E_1) be a couple of Banach spaces over $\mathbb{R}$ such that E_0 and E_1 are continuously embedding into some topological Hausdorff space X ((E_0, E_1) is called an interpolation couple). We equip $E_0 + E_1 := \{x = x_0 + x_1 : x_i \in E_i\}$ with the norm $\|x\|_{E_0+E_1} := \inf\{\|x_0\|_{E_0} + \|x_1\|_{E_1} : x_i \in E_i, x = x_0 + x_1\}$ and $E_0 \cap E_1$ with the norm $\|x\|_{E_0\cap E_1} := \max\{\|x\|_{E_0}, \|x\|_{E_1}\}$ to get Banach spaces $E_0 \cap E_1 \subseteq E_0 + E_1$. For $x \in E_0 + E_1$ and $v \in (0, \infty)$ we define the K-functional

$$K(v, x; E_0, E_1) := \inf\{\|x_0\|_{E_0} + v\|x_1\|_{E_1} : x = x_0 + x_1\},$$

which is continuous in v. Given $(\theta, q) \in (0, 1) \times [1, \infty]$ we set

$$(E_0, E_1)_{\theta,q} := \left\{ x \in E_0 + E_1 : \left\| v \mapsto v^{-\theta} K(v, x; E_0, E_1) \right\|_{\mathbf{L}_q\left((0,\infty), \frac{\mathrm{d}v}{v}\right)} < \infty \right\} \tag{2.1}$$

and $\|x\|_{(E_0,E_1)_{\theta,q}} := \|v \mapsto v^{-\theta} K(v, x; E_0, E_1)\|_{\mathbf{L}_q\left((0,\infty), \frac{\mathrm{d}v}{v}\right)}$. We obtain a family of Banach spaces $\left((E_0, E_1)_{\theta,q}, \|\cdot\|_{(E_0,E_1)_{\theta,q}}\right)$ with the lexicographical ordering

$$(E_0, E_1)_{\theta,q_0} \subseteq (E_0, E_1)_{\theta,q_1} \quad \text{for all} \quad \theta \in (0, 1) \text{ and } 1 \leqslant q_0 < q_1 \leqslant \infty, \tag{2.2}$$

with

$$\|x\|_{(E_0,E_1)_{\theta,q_1}} \leqslant c_{(2.3)} \|x\|_{(E_0,E_1)_{\theta,q_0}} \tag{2.3}$$

where $c_{(2.3)} = c_{(2.3)}(\theta, q_0, q_1) > 0$, and, under the additional assumption that $E_1 \subseteq E_0$ with $\|x\|_{E_0} \leqslant c\|x\|_{E_1}$ for some $c > 0$,

$$(E_0, E_1)_{\theta_0,q_0} \subseteq (E_0, E_1)_{\theta_1,q_1} \quad \text{for all} \quad 0 < \theta_1 < \theta_0 < 1 \text{ and } q_0, q_1 \in [1, \infty].$$

For more information about real interpolation the reader is referred to [15,16,19]. Given a Banach space E and $(q,s) \in [1,\infty] \times \mathbb{R}$, we will use the Banach spaces

$$\ell_q^s(E) := \{(x_k)_{k=0}^\infty \subseteq E : \|(2^{ks}\|x_k\|_E)_{k=0}^\infty\|_{\ell_q} < \infty\}$$

with $\|(x_k)_{k=0}^\infty\|_{\ell_q^s(E)} := \|(2^{ks}\|x_k\|_E)_{k=0}^\infty\|_{\ell_q}$ and ℓ_q being the standard Lorentz sequence space, and the notation $\ell_q(E) := \ell_q^0(E)$. For $q_0, q_1, q \in [1,\infty]$ and $s_0, s_1 \in \mathbb{R}$ with $s_0 \neq s_1$, and $\theta \in (0,1)$, one has according to [15, Theorem 5.6.1] that

$$(\ell_{q_0}^{s_0}(E), \ell_{q_1}^{s_1}(E))_{\theta,q} = \ell_q^s(E) \quad \text{where} \quad s := (1-\theta)s_0 + \theta s_1 \tag{2.4}$$

and there is a $c_{(2.5)} \geqslant 1$ that depends at most on $(s_0, s_1, q_0, q_1, \theta, q)$ such that

$$\|\cdot\|_{\ell_q^s(E)} \sim_{c_{(2.5)}} \|\cdot\|_{(\ell_{q_0}^{s_0}(E), \ell_{q_1}^{s_1}(E))_{\theta,q}}. \tag{2.5}$$

2.3. Function spaces

Given $\emptyset \neq A \in \mathcal{B}(\mathbb{R})$, we let $B_b(A)$ be the Banach space of bounded Borel functions $f : A \to \mathbb{R}$ with $\|f\|_{B_b(A)} := \sup_{x\in A} |f(x)|$, $C_b^0(\mathbb{R})$ be the closed subspace of $B_b(\mathbb{R})$ of continuous functions vanishing at zero, and $C_b^\infty(\mathbb{R}) \subseteq B_b(\mathbb{R})$ the infinitely often differentiable functions such that the derivatives satisfy $f^{(k)} \in B_b(\mathbb{R})$, $k \geqslant 1$. The space $C^1(\mathbb{R})$ consists of differentiable functions with continuous derivative and $C^\infty(\mathbb{R})$ of the functions that are infinitely often differentiable. For $\theta \in [0,1]$ we use the Hölder spaces

$$\begin{aligned}
\text{Höl}_\theta(\mathbb{R}) &:= \left\{ f : \mathbb{R} \to \mathbb{R} \text{ Borel function} : |f|_\theta := \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\theta} < \infty \right\}, \\
\text{Höl}_\theta^0(\mathbb{R}) &:= \{ f \in \text{Höl}_\theta(\mathbb{R}) : f(0) = 0 \}, \\
\text{Höl}_{\theta,q}(\mathbb{R}) &:= (C_b^0(\mathbb{R}), \text{Höl}_1^0(\mathbb{R}))_{\theta,q} \quad \text{for} \quad (\theta,q) \in (0,1) \times [1,\infty].
\end{aligned}$$

The space $\text{Höl}_0(\mathbb{R})$ is the space of bounded Borel functions, but equipped with the semi-norm $|f|_0 := \sup_{x,y\in\mathbb{R}} |f(x)-f(y)|$. This norm is the correct one for Theorem 5.9. To shorten the notation we will use

$$|f|_{\theta,q} := \|f\|_{\text{Höl}_{\theta,q}(\mathbb{R})}.$$

Note that if we use the Banach space $C_b^0(\mathbb{R}) + \text{Höl}_1^0(\mathbb{R})$, then we see that $(C_b^0(\mathbb{R}), \text{Höl}_1^0(\mathbb{R}))$ forms an interpolation pair. Moreover, by the above definitions we obtain Banach spaces $(\text{Höl}_\theta^0(\mathbb{R}), |\cdot|_\theta)$ and for $\theta \in (0,1)$ we have that $\text{Höl}_{\theta,\infty}(\mathbb{R}) = \text{Höl}_\theta^0(\mathbb{R})$ with equivalent norms up to a multiplicative constant (a direct proof can be obtained by an adaptation of [20, Lemma A.3], see also [19, Theorem 2.7.2/1]). The fine line between different fine-indices q is illustrated in Section 8.4. The following inclusions regarding different indices for the Hölder continuity will be used:

Remark 2.1. For $\theta \in (0,1)$ and $1 \leqslant q_0 \leqslant q_1 \leqslant \infty$, by (2.2), it holds

$$\text{Höl}_{\theta,q_0}(\mathbb{R}) \subseteq \text{Höl}_{\theta,q_1}(\mathbb{R}).$$

Moreover, for $0 < \theta_0 < \theta_1 < 1$ and $q \in [1,\infty]$ one has that

$$\text{Höl}_{\theta_1,\infty}(\mathbb{R}) \subseteq \text{Höl}_{\theta_0,1}(\mathbb{R}) + \text{Höl}_1^0(\mathbb{R}) \subseteq \text{Höl}_{\theta_0,q}(\mathbb{R}) + \text{Höl}_1^0(\mathbb{R}). \tag{2.6}$$

Although (2.6) should be folklore, we include its proof in Section 8.5.

2.4. Stochastic basis

We fix a time horizon $T \in (0,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a right continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ such that $\mathcal{F}_0$ is generated by the $\mathbb{P}$-null sets and $\mathcal{F} = \mathcal{F}_T$. For $\mathbb{I} = [0,T]$ or $\mathbb{I} = [0,T)$ we denote by $\text{CL}(\mathbb{I})$ the set of $\mathbb{F}$-adapted *càdlàg* (right continuous with left limits) processes $A = (A_t)_{t\in\mathbb{I}}$, by $\text{CL}^+(\mathbb{I})$ the subset of $A \in \text{CL}(\mathbb{I})$ with $A_t(\omega) \geqslant 0$ on $\mathbb{I} \times \Omega$, and by $\text{CL}_0(\mathbb{I})$ the subset of $A \in \text{CL}(\mathbb{I})$ with $A_0 \equiv 0$. For $A \in \text{CL}(\mathbb{I})$ we use

(1) $A^* = (A_t^*)_{t\in\mathbb{I}}$ with $A_t^* := \sup_{s\in[0,t]} |A_s|$,
(2) $\Delta A = (\Delta A_t)_{t\in\mathbb{I}}$ with $\Delta A_t := A_t - A_{t-}$, where $A_{0-} := A_0$ and $A_{t-} := \lim_{s<t, s\uparrow t} A_s$ for $t > 0$.

We write $\mathbb{E}^{\mathcal{G}}[\xi]$ for the conditional expectation of ξ given $\mathcal{G} \subseteq \mathcal{F}$ (where we exploit extended conditional expectations if ξ is non-negative) and use $\mathbb{P}_{\mathcal{G}}(B) := \mathbb{E}^{\mathcal{G}}\left[\mathbb{1}_B\right]$ for $B \in \mathcal{F}$. The usual conditions imposed on $\mathbb{F}$ allow us to assume that every martingale adapted to this filtration is càdlàg. Given a càdlàg $\mathbf{L}_2$-martingale $A = (A_t)_{t\in\mathbb{I}}$ with $A_0 \equiv 0$, the sharp bracket process is denoted by $\langle A \rangle = (\langle A \rangle_t)_{t\in\mathbb{I}}$ and the square bracket process by $[A] = ([A]_t)_{t\in\mathbb{I}}$ (see [21, Chapter VII]). Both processes are assumed to be path-wise non-negative, càdlàg, and non-decreasing on $\mathbb{I}$, and such that $\langle A \rangle_0 \equiv [A]_0 \equiv 0$. In particular, the process $\langle A \rangle = (\langle A \rangle_t)_{t\in\mathbb{I}}$ is the unique (up to indistinguishability) non-decreasing, predictable, càdlàg process with $\langle A \rangle_0 \equiv 0$ such that $(A_t^2 - \langle A \rangle_t)_{t\in\mathbb{I}}$ is a martingale.

Given $q \in (1,\infty)$, we say that an $(\mathcal{F}_t)_{t\in[0,T)}$-martingale $A = (A_t)_{t\in[0,T)}$ is closable in $\mathbf{L}_q$ provided that $(A_t)_{t\in[0,T)}$ converges in $\mathbf{L}_q$ as $t \uparrow T$. This is equivalent to $\sup_{t\in[0,T)} \|A_t\|_{\mathbf{L}_q} < \infty$, see [22, Corollary 7.22].

2.5. Bounded mean oscillation and regular weights

We use the following weighted bmo-spaces, where we agree about $\inf \emptyset := \infty$ in this subsection.

Definition 2.2. For $p \in (0,\infty)$, $A \in \mathrm{CL}_0([0,T))$, and $\Phi \in \mathrm{CL}^+([0,T))$ we let

$$\|A\|_{\mathrm{bmo}_p^{\Phi}[0,T)} := \inf \left\{ c \in [0,\infty) : \begin{array}{c} \mathbb{E}^{\mathcal{F}_a}\left[|A_t - A_a|^p\right] \leqslant c^p \Phi_a^p \quad \text{a.s.} \\ \text{for all } 0 \leqslant a \leqslant t < T \end{array} \right\}.$$

If $\|A\|_{\mathrm{bmo}_p^{\Phi}[0,T)} < \infty$., then we write $A \in \mathrm{bmo}_p^{\Phi}[0,T)$. In particular, for $\Phi \equiv 1$ we use the notation $\mathrm{bmo}_p[0,T)$. To normalize a process to start at zero to be measured in bmo, we use $A - A_0$ to denote the process $(A_t - A_0)_{t\in[0,T)}$.

In stochastic process theory there are two classes of spaces of bounded mean oscillation which are frequently used, namely, bmo and BMO. In their definitions bmo uses the increments $A_t - A_\rho$, whereas BMO uses $A_t - A_{\rho-}$, where $\rho : \Omega \to [0,t]$ is a stopping time (cf. Definition 8.2 and Proposition 8.3 for their relations in our framework). In general, the spaces bmo and BMO are significantly different, even in discrete time. In fact, the bmo-spaces are more convenient to work with, however, to obtain good distributional properties of the process via a John–Nirenberg type theorem one needs to use the BMO-spaces in general. Therefore, in this article we mainly work with the bmo-norms, and in the applications when $A = (A_t)_{t\in[0,T)}$ has continuous trajectories, we can achieve the good distributional properties for A because both bmo- and BMO-norm coincide as $A_{\rho-} = A_\rho$. For the theory of classical non-weighted BMO-martingales and applications we refer exemplary to [21, Ch.VII], [23, Ch.IV], and [24, Section X.1]. Non-weighted bmo-martingales were mentioned in [21, Ch.VII, Remark 87] and used after that in [25,26]. The BMO_p^{Φ}-spaces, which we will exploit later to obtain tail-estimates and which coincide with the bmo_p^{Φ}-spaces for continuous processes, were introduced and discussed in [14].

Next we recall (and adapt) the class $\mathcal{SM}_p$, introduced in [14, Definition 3]:

Definition 2.3. For $p \in (0,\infty)$ and $\Phi \in \mathrm{CL}^+([0,T))$ we let $\|\Phi\|_{\mathcal{SM}_p([0,T))} := \inf c$, where the infimum is taken over all $c \in [1,\infty)$ such that for all $a \in [0,T)$ one has

$$\mathbb{E}^{\mathcal{F}_a}\left[\sup_{t\in[a,T)} \Phi_t^p\right] \leqslant c^p \Phi_a^p \quad \text{a.s.}$$

If $\|\Phi\|_{\mathcal{SM}_p([0,T))} < \infty$, then we write $\Phi \in \mathcal{SM}_p([0,T))$.

By choosing $a = 0$, $\Phi \in \mathcal{SM}_p([0,T))$ implies $\mathbb{E}\sup_{t\in[0,T)} \Phi_t^p < \infty$. Moreover, it follows directly from the definition that $\mathcal{SM}_p([0,T)) \subseteq \mathcal{SM}_r([0,T))$ whenever $0 < r \leqslant p < \infty$. If $p \in (1,\infty)$ and Φ is a martingale, then $\Phi \in \mathcal{SM}_p([0,T))$ is equivalent to the standard reverse Hölder condition $\mathbb{E}^{\mathcal{F}_a}\left[\Phi_t^p\right] \leqslant d^p \Phi_a^p$ a.s. for $0 \leqslant a \leqslant t < T$. For this article we need to make the connection from Definitions 2.2 and 2.3 to the setting of [14], which is done in Section 8.1.

2.6. Reverse Hölder condition and bmo

Definition 2.4. A probability measure $\mathbb{Q}$ on $(\Omega,\mathcal{F},\mathbb{P})$ satisfies a reverse Hölder inequality with exponent $q \in (1,\infty)$ if $\mathrm{d}\mathbb{Q} = \mathcal{E}\mathrm{d}\mathbb{P}$ with $\mathcal{E} > 0$ on Ω and $\mathcal{E} \in \mathbf{L}_q(\mathbb{P})$, and if there is a $c > 0$ such that

$$\sqrt[q]{\mathbb{E}^{\mathcal{F}_a}\left[\mathcal{E}^q\right]} \leqslant c\,\mathbb{E}^{\mathcal{F}_a}\left[\mathcal{E}\right] \text{ a.s.} \quad \text{for all} \quad a \in [0,T].$$

In this case we let $\|\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}\|_{\mathrm{RH}_q(\mathbb{P})} := \inf c$ where the infimum is taken over all $c > 0$ as above and we write $\mathbb{Q} \in \mathrm{RH}_q(\mathbb{P})$.

If we define $\|\cdot\|_{\mathrm{bmo}_r^{\Phi,\mathbb{Q}}[0,T)}$ as in Definition 2.2, but under the measure $\mathbb{Q}$, then, for $0 < r < p < \infty$ a direct application of the conditional Hölder inequality yields

$$\|\cdot\|_{\mathrm{bmo}_r^{\Phi,\mathbb{Q}}[0,T)} \leqslant \sqrt[r]{\|\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}\|_{\mathrm{RH}_{\frac{p}{p-r}}(\mathbb{P})}}\,\|\cdot\|_{\mathrm{bmo}_p^{\Phi}[0,T)}. \tag{2.7}$$

2.7. Uniform quantization and time-nets

For $\theta \in (0,1]$ and $n \in \mathbb{N}$ we introduce the non-uniform time-nets $\tau_n^\theta = \{t_{i,n}^\theta\}_{i=0}^n$ with

$$t_{i,n}^\theta := T - T\,(1-(i/n))^{1/\theta} \tag{2.8}$$

for $i = 0,\dots,n$, that are characterized by the uniform quantization property

$$\frac{\theta}{T^\theta}\int_{t_{i-1,n}^\theta}^{t_{i,n}^\theta} (T-u)^{\theta-1}\mathrm{d}u = \frac{1}{n} \quad \text{for} \quad i = 1,\dots,n.$$

We define the set of all *deterministic* time-nets

$$\mathcal{T} := \{\tau = \{t_i\}_{i=0}^n : 0 = t_0 < t_1 < \cdots < t_n = T,\ n \in \mathbb{N}\}$$

and, for $\theta \in (0,1]$ and $\tau = \{t_i\}_{i=0}^n \in \mathcal{T}$,

$$\|\tau\|_\theta := \sup_{i=1,\dots,n} \frac{t_i - t_{i-1}}{(T - t_{i-1})^{1-\theta}}.$$

Note that

$$\|\tau_n^\theta\|_1 \leqslant \frac{T}{\theta n} \quad \text{and} \quad \|\tau_n^\theta\|_\theta \leqslant \frac{T^\theta}{\theta n}. \tag{2.9}$$

3. Riemann–Liouville type operators

As described in the introduction we shall not use the concept of fractional martingales, instead the following different approach:

Definition 3.1. For $\alpha > 0$ and a càdlàg function $K : [0,T) \to \mathbb{R}$ we define $\mathcal{I}^\alpha K := (\mathcal{I}_t^\alpha K)_{t\in[0,T)}$ by

$$\mathcal{I}_t^\alpha K := \frac{\alpha}{T^\alpha} \int_0^T (T-u)^{\alpha-1} K_{u\wedge t} \mathrm{d}u. \tag{3.1}$$

Moreover, for $\alpha = 0$ we let $\mathcal{I}_t^0 K := K_t$.

The càdlàg property implies the boundedness of K on any compact interval of $[0,T)$. Therefore, $\mathcal{I}^\alpha K$ is well-defined and càdlàg on $[0,T)$, *i.e.* $\mathcal{I}^\alpha$ operates from the space of càdlàg functions to the space of càdlàg functions on $[0,T)$. The above definition can be re-formulated in terms of the classical Riemann–Liouville operator $\mathcal{R}_a^\alpha(f) := \frac{1}{\Gamma(\alpha)} \int_0^a (a-u)^{\alpha-1} f(u)\mathrm{d}u$ by

$$\mathcal{R}_T^\alpha(K^{(t)}) = \frac{T^\alpha}{\Gamma(\alpha+1)} \mathcal{I}_t^\alpha K \quad \text{with} \quad K_u^{(t)} := K_{u\wedge t}$$

where we compute the Riemann–Liouville operator, applied to the function $u \mapsto K_u^{(t)}$, at $a = T$. We use a different normalization as we want to interpret the kernel in the Riemann–Liouville integral as density of a probability measure. The following statement is obvious, but useful and important to reveal the group structure of the Riemann–Liouville type operators:

Proposition 3.2. *For $\alpha \geqslant 0$ and $t \in [0,T)$ one has*

$$\mathcal{I}_t^\alpha K = \frac{\alpha}{T^\alpha} \int_0^t (T-u)^{\alpha-1} K_u \mathrm{d}u + \left(\frac{T-t}{T}\right)^\alpha K_t. \tag{3.2}$$

In the main part of the article we only need $\mathcal{I}^\alpha K$ for $\alpha \geqslant 0$. However, to derive an inversion formula we extend the definition by (3.2) to the case $\alpha < 0$ and prove that there is a group structure behind:

Proposition 3.3. *Define for $\alpha < 0$, a càdlàg function $K : [0,T) \to \mathbb{R}$, and $t \in [0,T)$, $\mathcal{I}_t^\alpha K$ by formula (3.2). Then*

(1) $\mathcal{I}_t^\alpha(\mathcal{I}^\beta K) = \mathcal{I}_t^{\alpha+\beta} K$ for all $\alpha, \beta \in \mathbb{R}$,
(2) $\mathcal{I}_t^{-\alpha}(\mathcal{I}^\alpha K) = K_t$ for all $\alpha \in \mathbb{R}$.

Proof. As (2) follows from (1), we only verify (1), which follows from

$$\begin{aligned}
\mathcal{I}_t^\alpha(\mathcal{I}^\beta K) &= \frac{\alpha}{T^\alpha} \int_0^t (T-u)^{\alpha-1} \mathcal{I}_u^\beta K \mathrm{d}u + \left(\frac{T-t}{T}\right)^\alpha \mathcal{I}_t^\beta K \\
&= \frac{\alpha}{T^\alpha} \int_0^t (T-u)^{\alpha-1} \left(\frac{\beta}{T^\beta} \int_0^u (T-v)^{\beta-1} K_v \mathrm{d}v + \left(\frac{T-u}{T}\right)^\beta K_u \right) \mathrm{d}u \\
&\quad + \left(\frac{T-t}{T}\right)^\alpha \left(\frac{\beta}{T^\beta} \int_0^t (T-u)^{\beta-1} K_u \mathrm{d}u + \left(\frac{T-t}{T}\right)^\beta K_t \right) \\
&= \frac{\beta}{T^{\alpha+\beta}} \int_0^t (T-v)^{\alpha+\beta-1} K_v \mathrm{d}v - \frac{\beta(T-t)^\alpha}{T^{\alpha+\beta}} \int_0^t (T-v)^{\beta-1} K_v \mathrm{d}v \\
&\quad + \frac{\alpha}{T^{\alpha+\beta}} \int_0^t (T-u)^{\alpha+\beta-1} K_u \mathrm{d}u \\
&\quad + \frac{\beta(T-t)^\alpha}{T^{\alpha+\beta}} \int_0^t (T-u)^{\beta-1} K_u \mathrm{d}u + \left(\frac{T-t}{T}\right)^{\alpha+\beta} K_t \\
&= \mathcal{I}_t^{\alpha+\beta} K. \quad \square
\end{aligned}$$

We continue with some more structural properties:

Proposition 3.4. *For a càdlàg function $K : [0,T) \to \mathbb{R}$ and $t \in [0,T)$ one has:*

(1) $\lim_{\alpha\downarrow 0} \mathcal{I}_t^\alpha K = K_t$.

(2) $\lim_{\alpha\uparrow\infty} \mathcal{I}_t^\alpha K = K_0$.
(3) $\Delta \mathcal{I}_t^\alpha K = \left(\frac{T-t}{T}\right)^\alpha \Delta K_t$ *for* $\alpha \in \mathbb{R}$.

Proof. (1) and (3) follow from Eq. (3.2). To verify (2) we use the probability measure $\mu_\alpha(\mathrm{d}u) := \frac{\alpha}{T^\alpha}(T-u)^{\alpha-1}\mathrm{d}u$ on $[0,T)$ and observe that $\lim_{\alpha\to\infty} \mu_\alpha([0,\varepsilon]) = 1$ for all $\varepsilon \in (0,T)$. As $\mathcal{I}_t^\alpha K = \int_0^T K_{u\wedge t}\mu_\alpha(\mathrm{d}u)$ the càdlàg property of K gives (2). □

With the next statement we derive properties of K from properties of $\mathcal{I}^\alpha K$:

Proposition 3.5. *For* $\alpha > 0$, *a càdlàg function* $K : [0,T) \to \mathbb{R}$, *and* $0 \leqslant s < t < T$ *we have*

$$K_t - K_s = \left(\frac{T}{T-t}\right)^\alpha \left(\mathcal{I}_t^\alpha K - \mathcal{I}_s^\alpha K\right) - \alpha T^\alpha \int_s^t (T-u)^{-\alpha-1}\left(\mathcal{I}_u^\alpha K - \mathcal{I}_s^\alpha K\right)\mathrm{d}u. \tag{3.3}$$

Consequently the following holds:

(1) $|K_t - K_s| \leqslant 2\left(\frac{T}{T-t}\right)^\alpha \sup_{u\in[s,t]} \left|\mathcal{I}_u^\alpha K - \mathcal{I}_s^\alpha K\right|$.
(2) If $\mathcal{I}_T^\alpha K := \lim_{t\uparrow T} \mathcal{I}_t^\alpha K \in \mathbb{R}$ *does exist, then* $\lim_{t\uparrow T}(T-t)^\alpha K_t = 0$.

Proof. To verify relation (3.3) we define $L_t := \mathcal{I}_t^\alpha K$ for $t \in [0,T)$, express $K_t - K_s$ as $\mathcal{I}_t^{-\alpha} L - \mathcal{I}_s^{-\alpha} L$, and use (3.2) to get

$$\begin{aligned}
\mathcal{I}_t^{-\alpha} L - \mathcal{I}_s^{-\alpha} L &= \left(\frac{T}{T-t}\right)^\alpha L_t - \left(\frac{T}{T-s}\right)^\alpha L_s \\
&\quad -\alpha T^\alpha \int_0^t (T-u)^{-\alpha-1} L_u \mathrm{d}u + \alpha T^\alpha \int_0^s (T-u)^{-\alpha-1} L_u \mathrm{d}u \\
&= \left(\frac{T}{T-t}\right)^\alpha L_t - \left(\frac{T}{T-s}\right)^\alpha L_s - \alpha T^\alpha \int_s^t (T-u)^{-\alpha-1} L_u \mathrm{d}u \\
&= \left(\frac{T}{T-t}\right)^\alpha (L_t - L_s) - \alpha T^\alpha \int_s^t (T-u)^{-\alpha-1}(L_u - L_s)\mathrm{d}u.
\end{aligned}$$

Now claim (1) follows from (3.3). For claim (2) we use

$$\left(\frac{T-t}{T}\right)^\alpha K_t = \mathcal{I}_t^\alpha K - \int_0^t (\mathcal{I}_u^\alpha K)\left(\alpha \frac{(T-t)^\alpha}{(T-u)^{\alpha+1}}\right)\mathrm{d}u,$$

$\lim_{t\uparrow T} \int_0^t \left(\alpha \frac{(T-t)^\alpha}{(T-u)^{\alpha+1}}\right)\mathrm{d}u = 1$, and that $\lim_{t\uparrow T} \sup_{u\in[0,v]} \frac{(T-t)^\alpha}{(T-u)^{\alpha+1}} = 0$ for all $v \in [0,T)$. □

If the function K is a path of a càdlàg semi-martingale L (see [24, Chapter VIII]) we obtain a fractional semi-martingale transform:

Proposition 3.6. *For* $\alpha \geqslant 0$, *a càdlàg semi-martingale* $L = (L_t)_{t\in[0,T)}$, *and* $0 \leqslant a < t < T$ *one has*

$$\mathcal{I}_t^\alpha L = L_0 + \int_{(0,t]} \left(\frac{T-u}{T}\right)^\alpha \mathrm{d}L_u \ \text{a.s.} \tag{3.4}$$

Proof. As the case $\alpha = 0$ is evident we assume $\alpha > 0$. We apply integration by parts [24, Corollary 9.34] to $\left(\left(\frac{T-t}{T}\right)^\alpha L_t\right)_{t\in[0,T)}$ and obtain, for $t \in [0,T)$, that

$$\left(\frac{T-t}{T}\right)^\alpha L_t = L_0 + \int_{(0,t]} \left(\frac{T-u}{T}\right)^\alpha \mathrm{d}L_u - \frac{\alpha}{T^\alpha}\int_{(0,t]} (T-u)^{\alpha-1} L_{u-}\mathrm{d}u \ \text{a.s.}$$

Because $L = (L_t)_{t\in[0,T)}$ has ω-wise only countably many jumps, we can replace L_{u-} by L_u in the last term. Then, taking this term to the left side and using (3.2), we obtain (3.4). □

Now Proposition 3.3 (1) for $\alpha, \beta \geqslant 0$ can be understood from Eq. (3.4) in the semi-martingale setting. The operator $\mathcal{I}^\alpha$ preserves the (super-, sub-) martingale property, which can be checked directly from its definition so that we leave out the proof:

Proposition 3.7. *If* $\alpha \geqslant 0$ *and* $L = (L_t)_{t\in[0,T)}$ *is a càdlàg martingale (càdlàg super-, or sub-martingale), then* $(\mathcal{I}_t^\alpha L)_{t\in[0,T)}$ *is a càdlàg martingale (càdlàg super-, or sub-martingale).*

It might be of future interest, that the Riemann–Liouville type operator $\mathcal{I}^\alpha$ turns into a multiplier when commuting with the horizontal and vertical derivative in the path-dependent setting:

Remark 3.8. Let D be the space of all càdlàg functions $x = (x(u))_{u\in[0,T)} : [0,T) \to \mathbb{R}$. For $x \in D$ and $t \in [0,T)$ we define the horizontal modification $\vec{x}_t$ by $\vec{x}_t(u) := x(u\wedge t)$ and the vertical modification x_t^h by $x_t^h(u) := x(u) + \mathbb{1}_{[t,T)}(u)h$. Let $Z : [0,T)\times D \to \mathbb{R}$ such that $Z(\cdot,x) \in D$ for $x \in D$ and such that Z is non-anticipating, *i.e.* $Z(t,x) = Z(t,\vec{x}_t)$. Then the horizontal and vertical derivatives may be defined as $(\partial Z/\partial t)(t,x) := \lim_{h\downarrow 0, h<T-t} \frac{Z(t+h,\vec{x}_t)-Z(t,x)}{h}$ and $(\partial Z/\partial x)(t,x) := \lim_{h\to 0} \frac{Z(t,x_t^h)-Z(t,x)}{h}$ if the corresponding limit exist (see [3]). Then one has

$$\frac{\partial}{\partial t}(\mathcal{I}^\alpha Z) = \left(\frac{T-t}{T}\right)^\alpha \left(\frac{\partial Z}{\partial t}\right) \quad \text{and} \quad \frac{\partial}{\partial x}(\mathcal{I}^\alpha Z) = \left(\frac{T-t}{T}\right)^\alpha \left(\frac{\partial Z}{\partial x}\right), \tag{3.5}$$

provided that all the corresponding limits exist. To check this one observes that

$$(\mathcal{I}^\alpha Z)(t,x) = \int_{[0,t)} Z(u,x)\frac{\alpha}{T^\alpha}(T-u)^{\alpha-1}\mathrm{d}u + \left(\frac{T-t}{T}\right)^\alpha Z(t,x).$$

Then the second relation in (3.5) is obvious and the first one follows from

$$\begin{aligned} & Z(t,x)\frac{\alpha}{T^\alpha}(T-t)^{\alpha-1} - \frac{\alpha}{T^\alpha}(T-t)^{\alpha-1}Z(t,x) + \left(\frac{T-t}{T}\right)^\alpha \frac{\partial Z}{\partial t}(t,x) \\ &= \left(\frac{T-t}{T}\right)^\alpha \frac{\partial Z}{\partial t}(t,x). \end{aligned}$$

For later use in the article we need the following quantitative relations:

Proposition 3.9. *For $\alpha > 0$, a càdlàg martingale $L = (L_t)_{t\in[0,T)} \subseteq \mathbf{L}_2$ and $0 \leqslant a < t < T$ one has, a.s.,*

$$\mathbb{E}^{\mathcal{F}_a}\left[\left|\mathcal{I}_t^\alpha L - \mathcal{I}_a^\alpha L\right|^2\right] = 2\alpha \mathbb{E}^{\mathcal{F}_a}\left[\int_a^T |L_{u\wedge t} - L_a|^2 \left(\frac{T-u}{T}\right)^{2\alpha-1}\frac{\mathrm{d}u}{T}\right], \tag{3.6}$$

$$\begin{aligned} & \mathbb{E}^{\mathcal{F}_a}\left[\left|\mathcal{I}_t^\alpha L - \mathcal{I}_a^\alpha L\right|^2\right] \\ & + \left(\frac{T-a}{T}\right)^{2\alpha}|L_a|^2 = 2\alpha \mathbb{E}^{\mathcal{F}_a}\left[\int_a^T |L_{u\wedge t}|^2 \left(\frac{T-u}{T}\right)^{2\alpha-1}\frac{\mathrm{d}u}{T}\right]. \end{aligned} \tag{3.7}$$

Proof. For (3.6) we exploit (3.4) and Itô's isometry to get, a.s.,

$$\begin{aligned} \mathbb{E}^{\mathcal{F}_a}\left[\left|\mathcal{I}_t^\alpha L - \mathcal{I}_a^\alpha L\right|^2\right] &= \mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,t]}\left(\frac{T-u}{T}\right)^{2\alpha}\mathrm{d}[L]_u\right] \\ &= \frac{2\alpha}{T^{2\alpha}}\mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,t]}\int_{[u,T)}(T-v)^{2\alpha-1}\mathrm{d}v\mathrm{d}[L]_u\right] \\ &= \frac{2\alpha}{T^{2\alpha}}\mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,T)}\int_{(a,v\wedge t]}\mathrm{d}[L]_u(T-v)^{2\alpha-1}\mathrm{d}v\right] \\ &= \frac{2\alpha}{T^{2\alpha}}\mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,T)}|L_{v\wedge t} - L_a|^2(T-v)^{2\alpha-1}\mathrm{d}v\right]. \end{aligned} \tag{3.8}$$

Relation (3.7) follows directly from (3.6) and the orthogonality of $L_{u\wedge t} - L_a$ and L_a. □

4. Riemann–Liouville type operators and approximation of càdlàg martingales

Various $\mathbf{L}_p$-approximation problems in stochastic integration theory can be translated by the Burkholder–Davis–Gundy inequalities into problems about quadratic variation processes. In the special case of $\mathbf{L}_2$-approximations this is particularly useful as there is a chance to turn the approximation problem into – in a sense – more deterministic problem by Fubini's theorem when the interchange of the integration in time and in ω is possible. When $p \neq 2$ this does not work (at least) in this straight way, see for example [11]. However, passing from global $\mathbf{L}_2$-estimates to weighted local $\mathbf{L}_2$-estimates, *i.e.* weighted bounded mean oscillation estimates, and exploiting a weighted John–Nirenberg type theorem, gives an approach to $\mathbf{L}_p$- and exponential estimates.

The plan of this section is as follows:

(A) Theorems 4.3 and 4.4 are the key to exploit the local $\mathbf{L}_2$-estimates in the sequel. It turned out that one can formulate these theorems in the general setting of random measures (Π, Y). For this we need relation (4.1), which is a general form of the identity (4.12), the latter based on the conditional orthogonality of increments of an $\mathbf{L}_2$-martingale.

(B) By Assumption 4.5 we specialize the setting given in Assumption 4.1 so that the measure Π will describe the quadratic variation of the driving process of the stochastic integral to be approximated and Y will describe some kind of *curvature* of the stochastic integral. As results we obtain Theorem 4.6 and Corollary 4.7.

(C) As an application we provide two approximation results for general càdlàg $\mathbf{L}_2$-martingales: Theorem 4.9 describes a discrete time martingale approximation which relates to real interpolation, while Theorem 4.12 provides a regularization, based on adapted backward smoothing, leading to local Lipschitz trajectories.

4.1. The general result in terms of random measures

First we introduce the random measures and the quasi-orthogonality (4.1) where we use extended conditional expectations for non-negative random variables.

Assumption 4.1. We assume random measures

$$\Pi, Y : \Omega \times \mathcal{B}((0,T)) \to [0,\infty],$$

a progressively measurable process $(\varphi_t)_{t\in[0,T)}$, and a constant $\kappa \geqslant 1$, such that

$$\Pi(\omega,(0,b]) + Y(\omega,(0,b]) + \sup_{t\in[0,b]} |\varphi_t(\omega)| < \infty$$

for $(\omega,b) \in \Omega\times(0,T)$ and such that, for $0 \leqslant s \leqslant a < b < T$,

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,b]} \left|\varphi_u - \varphi_s\right|^2 \Pi(\mathrm{d}u)\right]\\ &\sim_\kappa \mathbb{E}^{\mathcal{F}_a}\left[\left|\varphi_a - \varphi_s\right|^2 \Pi((a,b]) + \int_{(a,b]} (b-u) Y(\mathrm{d}u)\right] \text{ a.s.}\end{aligned} \tag{4.1}$$

When (4.1) holds with $\preceq_\kappa$, then we denote the inequality by $(4.1)^{\leqslant}$, in case of $\succeq_\kappa$, by $(4.1)^{\geqslant}$.

To simplify the notation in some situations we extend Π and Y to $\Pi, Y : \Omega\times\mathcal{B}((0,T]) \to [0,\infty]$ by $\Pi(\omega,\{T\}) = Y(\omega,\{T\}) = 0$ for all $\omega\in\Omega$.

Definition 4.2. For a random measure $\Pi : \Omega\times\mathcal{B}((0,T)) \to [0,\infty]$ and a progressively measurable process $(\varphi_t)_{t\in[0,T)}$ such that $\Pi(\omega,(0,b]) + \sup_{t\in[0,b]}|\varphi_t(\omega)| < \infty$ for $(\omega,b)\in\Omega\times(0,T)$ we define for $\tau = \{t_i\}_{i=0}^n \in \mathcal{T}$ the càdlàg process $[\varphi;\tau]^\pi = ([\varphi;\tau]^\pi_a)_{a\in[0,T)}$ by

$$[\varphi;\tau]^\pi_a := \int_{(0,a]} \left|\varphi_u - \sum_{i=1}^n \varphi_{t_{i-1}} \mathbb{1}_{(t_{i-1},t_i]}(u)\right|^2 \Pi(\mathrm{d}u)$$

and $[\varphi;\tau]^\pi_T := \lim_{a\uparrow T}[\varphi;\tau]^\pi_a \in [0,\infty]$. We also use the notation

$$[\varphi;\tau]^\pi_{a,b} := [\varphi;\tau]^\pi_b - [\varphi;\tau]^\pi_a \quad \text{for} \quad 0\leqslant a<b\leqslant T$$

so that, in particular, $[\varphi;\tau]^\pi_{0,b} = [\varphi;\tau]^\pi_b$.

The next two statements, Theorems 4.3 and 4.4, develop further ideas from [10, Lemma 3.8] and [11, Lemma 5.6] to a general conditional setting using random measures we exploit in the sequel:

Theorem 4.3 (*Upper Bounds*). *Suppose Assumption 4.1 with* $(4.1)^{\leqslant}$. *If* $\theta\in(0,1]$, $\tau = \{t_i\}_{i=0}^n \in\mathcal{T}$, *and* $a\in[t_{k-1},t_k)$, *then one has, a.s.,*

$$\begin{aligned}&\frac{\mathbb{E}^{\mathcal{F}_a}\left[[\varphi;\tau]^\pi_{a,T}\right]}{\|\tau\|_\theta} \leqslant \kappa\\ &\mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,T)} (T-u)^{1-\theta} Y(\mathrm{d}u) + \frac{(T-t_{k-1})^{1-\theta}}{t_k-t_{k-1}} |\varphi_a - \varphi_{t_{k-1}}|^2 \Pi((a,t_k])\right].\end{aligned}$$

Theorem 4.4 (*Lower Bounds*). *Suppose Assumption 4.1 with* $(4.1)^{\geqslant}$ *and assume* $\theta\in(0,1]$.

(1) If $\tau = \{t_i\}_{i=0}^n\in\mathcal{T}$, $a\in[t_{k-1},t_k)$, *and* $\|\tau\|_\theta = \frac{t_k-t_{k-1}}{(T-t_{k-1})^{1-\theta}}$, *then*

$$\frac{\mathbb{E}^{\mathcal{F}_a}\left[[\varphi;\tau]^\pi_{a,t_k}\right]}{\|\tau\|_\theta} \geqslant \frac{1}{\kappa}\mathbb{E}^{\mathcal{F}_a}\left[\frac{(T-t_{k-1})^{1-\theta}}{t_k-t_{k-1}}|\varphi_a-\varphi_{t_{k-1}}|^2\Pi((a,t_k])\right] \text{ a.s.}$$

(2) For any $a\in[0,T)$ *there exist* $\tau_n\in\mathcal{T}$, $n\in\mathbb{N}$, *with* $a\in\tau_n$ *and* $\lim_n\|\tau_n\|_\theta = 0$ *such that, a.s.,*

$$\liminf_n \frac{\mathbb{E}^{\mathcal{F}_a}\left[[\varphi;\tau_n]^\pi_{a,T}\right]}{\|\tau_n\|_\theta} \geqslant \frac{1}{c_{(4.2)}}\mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,T)}(T-u)^{1-\theta}Y(\mathrm{d}u)\right] \tag{4.2}$$

with $c_{(4.2)} := 4\kappa 2^{\frac{1}{\theta}}$.

Theorems 4.3 and 4.4 are proven in Section 6. Now we specialize Assumption 4.1 to the settings that will be used in Section 5:

Assumption 4.5. We assume that there are

(1) a positive, càdlàg, and adapted process $(\sigma_t)_{t\in[0,T]}$ such that $\sigma^*_T\in\mathbf{L}_2$ and such that there is a $c_\sigma\geqslant 1$ with

$$\mathbb{E}^{\mathcal{F}_a}\left[\frac{1}{b-a}\int_a^b \sigma_u^2\mathrm{d}u\right] \sim_{c_\sigma} \sigma_a^2 \text{ a.s.} \quad \text{for all} \quad 0\leqslant a<b\leqslant T,$$

(2) a càdlàg square integrable martingale $M = (M_t)_{t\in[0,T)}$ with $M_0\equiv 0$,

(3) let $\Pi(\omega,\mathrm{d}u) := \sigma_u^2(\omega)\mathrm{d}u$ and $Y(\omega,\mathrm{d}u) := \mathrm{d}\langle M\rangle_u(\omega)$ for $u\in[0,T)$, where $\langle M\rangle$ is the conditional square-function (see Section 2.4),

(4) a $\varphi\in\mathrm{CL}([0,T))$ such that Eq. (4.1) is satisfied and let $[\varphi;\tau]^\sigma := [\varphi;\tau]^\pi$.

Now we transfer Theorem 4.3 and Theorem 4.4 into the setting of Assumption 4.5, where $(4.1)^{\leqslant}$ and $(4.1)^{\geqslant}$ simultaneously hold (see, for example Lemma 4.8 and Proposition 5.4). In this case the upper and lower bounds of Theorem 4.3 and Theorem 4.4 hold simultaneously and are therefore sharp in general:

Theorem 4.6. *Suppose $\theta \in (0,1]$, $\alpha := \frac{1-\theta}{2}$, that Assumption 4.5 holds, $a \in [0,T)$, and define $c_{(4.3)} := \kappa(4T^{1-\theta} \vee c_\sigma)$, $c_{(4.4)} := \kappa c_\sigma$, $c_{(4.5)} := T^{\theta-1} 16\kappa 2^{\frac{1}{\theta}}$, and $c_{(4.6)} := 16\kappa 2^{\frac{1}{\theta}}$. Then the following statements are true:*

(1) For all $\tau = \{t_i\}_{i=0}^n \in \mathcal{T}$ and $a \in [t_{k-1}, t_k)$, one has, a.s.,

$$\frac{\mathbb{E}^{\mathcal{F}_a}\left[[\varphi;\tau]^\sigma_{a,T}\right]}{\|\tau\|_\theta} \leqslant c_{(4.3)} \left(\mathbb{E}^{\mathcal{F}_a}\left[\sup_{t\in[a,T)} \left|\mathcal{I}^\alpha_t M - \mathcal{I}^\alpha_a M\right|^2\right] + \frac{T-a}{(T-t_{k-1})^\theta} |\varphi_a - \varphi_{t_{k-1}}|^2 \sigma_a^2\right). \tag{4.3}$$

(2) For all $s \in [0,a]$ there is a $\tau = \{t_i\}_{i=0}^n \in \mathcal{T}$ with $s = t_{k-1} \leqslant a < t_k$ and

$$\frac{\mathbb{E}^{\mathcal{F}_a}\left[[\varphi;\tau]^\sigma_{a,T}\right]}{\|\tau\|_\theta} \geqslant \frac{1}{c_{(4.4)}} \frac{T-a}{(T-t_{k-1})^\theta} |\varphi_a - \varphi_{t_{k-1}}|^2 \sigma_a^2 \text{ a.s.} \tag{4.4}$$

(3) There are $(\tau_n)_{n\in\mathbb{N}} \subset \mathcal{T}$ with $a \in \tau_n$ and $\lim_n \|\tau_n\|_\theta = 0$, such that, for all σ-algebras $\mathcal{G} \subseteq \mathcal{F}_a$, one has, a.s.,

$$\liminf_n \frac{\mathbb{E}^{\mathcal{G}}\left[[\varphi;\tau_n]^\sigma_{a,T}\right]}{\|\tau_n\|_\theta} \geqslant \frac{1}{c_{(4.5)}} \mathbb{E}^{\mathcal{G}}\left[\sup_{t\in[a,T)} \left|\mathcal{I}^\alpha_t M - \mathcal{I}^\alpha_a M\right|^2\right]. \tag{4.5}$$

(4) One has

$$\sup_{\tau\in\mathcal{T}} \frac{\|[\varphi;\tau]^\sigma_T\|_{\mathbf{L}_1}}{\|\tau\|_\theta} \sim_{c_{(4.6)}} T^{1-\theta} \left\|\sup_{t\in[0,T)} \left|\mathcal{I}^\alpha_t M\right|\right\|^2_{\mathbf{L}_2}. \tag{4.6}$$

Proof. (1) For $a \in [t_{k-1}, t_k)$, Assumption 4.5 implies that

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_a}\left[\frac{(T-t_{k-1})^{1-\theta}}{t_k - t_{k-1}} |\varphi_a - \varphi_{t_{k-1}}|^2 \Pi((a,t_k])\right] \\
&\overset{a.s.}{\sim}_{c_\sigma} |\varphi_a - \varphi_{t_{k-1}}|^2 \frac{(T-t_{k-1})^{1-\theta}}{t_k - t_{k-1}} \sigma_a^2 (t_k - a) \\
&\leqslant \frac{T-a}{(T-t_{k-1})^\theta} |\varphi_a - \varphi_{t_{k-1}}|^2 \sigma_a^2.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,T)} \left(\frac{T-u}{T}\right)^{1-\theta} Y(\mathrm{d}u)\right] &= \mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,T)} \left(\frac{T-u}{T}\right)^{1-\theta} \mathrm{d}\langle M\rangle_u\right] \\
&= \mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,T)} \left(\frac{T-u}{T}\right)^{1-\theta} \mathrm{d}[M]_u\right] \\
&\sim_4 \mathbb{E}^{\mathcal{F}_a}\left[\sup_{t\in[a,T)} \left|\mathcal{I}^\alpha_t M - \mathcal{I}^\alpha_a M\right|^2\right] \text{ a.s.}
\end{aligned} \tag{4.7}$$

by Doob's maximal inequality and (3.4). Theorem 4.3 implies, a.s.,

$$\begin{aligned}
&\frac{\mathbb{E}^{\mathcal{F}_a}\left[[\varphi;\tau]^\sigma_{a,T}\right]}{\|\tau\|_\theta} \\
&\leqslant \kappa \mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,T)} (T-u)^{1-\theta} Y(\mathrm{d}u) + \frac{(T-t_{k-1})^{1-\theta}}{t_k - t_{k-1}} |\varphi_a - \varphi_{t_{k-1}}|^2 \Pi((a,t_k])\right] \\
&\leqslant \kappa \left(4T^{1-\theta} \mathbb{E}^{\mathcal{F}_a}\left[\sup_{t\in[a,T)} \left|\mathcal{I}^\alpha_t M - \mathcal{I}^\alpha_a M\right|^2\right] + c_\sigma \frac{T-a}{(T-t_{k-1})^\theta} |\varphi_a - \varphi_{t_{k-1}}|^2 \sigma_a^2\right).
\end{aligned}$$

(2) We choose a net $\tau = \{t_i\}_{i=0}^n \in \mathcal{T}$ such that $s = t_{n-1} \leqslant a < t_n = T$ and $\|\tau\|_\theta = \frac{T-t_{n-1}}{(T-t_{n-1})^{1-\theta}}$, so that Theorem 4.4 (1) yields to

$$\begin{aligned}
\frac{\mathbb{E}^{\mathcal{F}_a}\left[[\varphi;\tau]^\sigma_{a,T}\right]}{\|\tau\|_\theta} &\geqslant \frac{1}{\kappa c_\sigma} \frac{(T-t_{n-1})^{1-\theta}}{T-t_{n-1}} |\varphi_a - \varphi_{t_{n-1}}|^2 (T-a)\sigma_a^2 \\
&= \frac{1}{\kappa c_\sigma} \frac{T-a}{(T-t_{n-1})^\theta} |\varphi_a - \varphi_{t_{n-1}}|^2 \sigma_a^2 \text{ a.s.}
\end{aligned}$$

(3) As we have $\mathcal{G} \subseteq \mathcal{F}_a$ we use Fatou's lemma, (4.2), and (4.7) to get a.s. that

$$\mathbb{E}^{\mathcal{G}}\left[\sup_{t\in[a,T)} \left|\mathcal{I}^\alpha_t M - \mathcal{I}^\alpha_a M\right|^2\right] \leqslant 4c_{(4.2)} T^{\theta-1} \mathbb{E}^{\mathcal{G}}\left[\liminf_n \mathbb{E}^{\mathcal{F}_a}\left[\frac{[\varphi;\tau_n]^\sigma_{a,T}}{\|\tau_n\|_\theta}\right]\right]$$

$$\leqslant 4c_{(4.2)} T^{\theta-1} \liminf_n \mathbb{E}^{\mathcal{G}} \left[\mathbb{E}^{\mathcal{F}_a} \left[\frac{[\varphi;\tau_n]^\sigma_{a,T}}{\|\tau_n\|_\theta} \right] \right]$$
$$= 4c_{(4.2)} T^{\theta-1} \liminf_n \mathbb{E}^{\mathcal{G}} \left[\frac{[\varphi;\tau_n]^\sigma_{a,T}}{\|\tau_n\|_\theta} \right].$$

(4) We let $a=0$ and $\mathcal{G} := \{\emptyset, \Omega\}$. The equivalence (4.6) follows from Eq. (4.3) (where in this case only the first term appears which gives the constant $\kappa 4 T^{1-\theta}$) and from (4.5). □

By Theorem 4.6 we characterize $\|[\varphi;\tau]^\sigma\|_{\mathrm{bmo}_1^{\Phi^2}[0,T)} \leqslant c^2 \|\tau\|_\theta$:

Corollary 4.7. *Assume that Assumption 4.5 is satisfied. Then for $\theta \in (0,1]$, $\alpha := \frac{1-\theta}{2}$, and $\Phi \in \mathrm{CL}^+([0,T))$ the following assertions are equivalent:*

(1) One has $\mathcal{I}^\alpha M \in \mathrm{bmo}_2^\Phi[0,T)$ and there is a $c_{(4.8)} > 0$ such that

$$|\varphi_a - \varphi_s| \sigma_a \leqslant c_{(4.8)} \frac{(T-s)^{\frac{\theta}{2}}}{(T-a)^{\frac{1}{2}}} \Phi_a \quad \textit{for} \quad 0 \leqslant s < a < T \textit{ a.s.} \tag{4.8}$$

(2) There is a constant $c_{(4.9)} > 0$ such that, for all time-nets $\tau \in \mathcal{T}$,

$$\|[\varphi;\tau]^\sigma\|_{\mathrm{bmo}_1^{\Phi^2}[0,T)} \leqslant c^2_{(4.9)} \|\tau\|_\theta. \tag{4.9}$$

If $\Phi = (\sigma_t \Psi_t)_{t\in[0,T)}$, where $\Psi \in \mathrm{CL}^+([0,T))$ is non-decreasing, then (4.8) is equivalent to the existence of $c_{(4.10)}, c_{(4.11)} > 0$ such that the following holds:

$$\theta \in (0,1): \quad |\varphi_a - \varphi_0| \leqslant c_{(4.10)} (T-a)^{-\alpha} \Psi_a \quad \textit{for} \quad 0 \leqslant a < T \textit{ a.s.}, \tag{4.10}$$

$$\theta = 1: \quad |\varphi_a - \varphi_s| \leqslant c_{(4.11)} \left(1 + \ln \frac{T-s}{T-a}\right) \Psi_a \quad \textit{for} \quad 0 \leqslant s < a < T \textit{ a.s.} \tag{4.11}$$

Proof. The equivalence between (1) and (2) follows directly from Theorem 4.6 and Doob's maximal inequality applied to $\mathcal{I}^\alpha M$. The equivalence between (4.8) and (4.10)–(4.11) follows from Lemma 8.6 below. □

4.2. $\mathbf{L}_2$ - and bmo-*approximation of càdlàg martingales*

First we show that Assumption 4.5 allows for the investigation of general càdlàg martingales:

Lemma 4.8. *Assume a càdlàg martingale $L = (L_t)_{t\in[0,T)} \subseteq \mathbf{L}_2$, $\sigma :\equiv 1$, $M :\equiv L - L_0$, and $\varphi :\equiv L$. Then Assumption 4.5 is satisfied with $\kappa = 1$.*

Proof. We have that $\Pi(\omega, \mathrm{d}u) = \mathrm{d}u$ and only need to show Eq. (4.1), *i.e.*

$$\mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,b]} |L_u - L_s|^2 \mathrm{d}u\right] = \mathbb{E}^{\mathcal{F}_a}\left[(b-a)|L_a - L_s|^2 + \int_{(a,b]} (b-u)\mathrm{d}\langle M\rangle_u\right] \text{ a.s.} \tag{4.12}$$

One can replace in the formula L by M and write on the LHS $M_u - M_s = (M_u - M_a) + (M_a - M_s)$. Using the conditional orthogonality of these terms, we reduce the above equation to

$$\mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,b]} |M_u - M_a|^2 \mathrm{d}u\right] = \mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,b]} (b-u)\mathrm{d}\langle M\rangle_u\right] \text{ a.s.}$$

This equality follows from, a.s.,

$$\mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,b]} |M_u - M_a|^2 \mathrm{d}u\right] = \mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,b]} \int_{(a,u]} d\langle M\rangle_v \mathrm{d}u\right]$$
$$= \mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,b]} (b-u)\mathrm{d}\langle M\rangle_u\right]. \quad \square$$

If $\sigma \equiv 1$, then the functional $[L;\tau]^\sigma = [L;\tau]^1$ measures by

$$[L;\tau]^1_t = \mathbb{E} \int_0^t \left| L_u - \sum_{i=1}^n L_{t_{i-1}} \mathbb{1}_{(t_{i-1},t_i]}(u) \right|^2 \mathrm{d}u$$

the approximation of L by the martingale $(\sum_{i=1}^n L_{t_{i-1}} \mathbb{1}_{(t_{i-1},t_i]}(u))_{u\in[0,T)}$ up to time $t \in [0,T)$. To characterize the behaviour of $[L;\tau]^1$ in terms of real interpolation, we replace a martingale $L = (L_t)_{t\in[0,T)}$ by its discrete time variant

$$L^d := (L_{t_k})_{k=0}^\infty \quad \text{with} \quad t_k := T\left(1 - 2^{-k}\right).$$

For the interpolation couple we use the sequence spaces $\ell_2^{-\frac{1}{2}}(H)$ and $\ell_\infty(H)$ as introduced in Section 2.2, where $H := \mathbf{L}_2(\Omega, \mathcal{F}, \mathbb{P})$. Since L is an $\mathbf{L}_2$-martingale, it turns out that

$$L^d \in \ell_2^{-\frac{1}{2}}(H) \Longleftrightarrow \int_0^T \|L_t\|_{\mathbf{L}_2}^2 \mathrm{d}t < \infty, \tag{4.13}$$

$$L^d \in \ell_\infty(H) \Longleftrightarrow \|L^d\|_{\ell_\infty(H)} = \sup_{t\in[0,T)} \|L_t\|_{\mathbf{L}_2} < \infty, \tag{4.14}$$

where (4.13) follows from (4.15) below. The first condition, $\int_0^T \|L_t\|_{\mathbf{L}_2}^2 \mathrm{d}t < \infty$, is a typical condition on martingales that appear as gradient processes. The other condition, $\sup_{t\in[0,T)} \|L_t\|_{\mathbf{L}_2} < \infty$, means that the martingale L is closable in $\mathbf{L}_2$. Here the interpolation is done with the spaces $\ell_2^{-\frac{1}{2}}(H)$ and $\ell_\infty(H)$ as these end-point spaces and the resulting interpolation spaces are technically convenient to handle. An alternative approach might be to investigate the usage of $\mathbf{L}_q([0,T); H)$, $q \in \{2, \infty\}$, as end-points along with the results from [19, Section 1.18]. However, here Bochner integration and related measurability issues have to be addressed.

Theorem 4.9. *For $\theta \in (0,1)$, $\alpha := \frac{1-\theta}{2}$, and an $\mathbf{L}_2$-càdlàg martingale $L = (L_t)_{t\in[0,T)}$ with $L_0 \equiv 0$ the following assertions are equivalent:*

(1) $L^d \in (\ell_2^{-\frac{1}{2}}(H), \ell_\infty(H))_{\theta,2}$.
(2) $(\mathcal{I}_t^\alpha L)_{t\in[0,T)}$ is closable in $\mathbf{L}_2$.
(3) There is a $c > 0$ such that $\mathbb{E}[L;\tau]_T^1 \leqslant c\|\tau\|_\theta$ for all $\tau \in \mathcal{T}$.

Proof. Because $(\|L_{t_k}\|_H)_{k=0}^\infty$ is non-decreasing we observe for $s > -1/2$ that

$$\begin{aligned} \frac{\|(L_{t_k})_{k=0}^\infty\|_{\ell_2^s(H)}^2}{2T^{2s}} &= \sum_{k=0}^\infty (T-t_k)^{-1-2s}(t_{k+1}-t_k)\|L_{t_k}\|_H^2 \\ &\sim_{c_{T,s}} \int_0^T (T-t)^{-1-2s}\|L_t\|_H^2 \mathrm{d}t \end{aligned} \tag{4.15}$$

for some $c_{T,s} \geqslant 1$. The inequality $\geqslant$ in (4.15) follows from

$$\begin{aligned} \int_0^T (T-t)^{-1-2s}\|L_t\|_H^2 \mathrm{d}t &\leqslant \sum_{k=0}^\infty (t_{k+1}-t_k)(T-t_{k+1})^{-1-2s}\|L_{t_{k+1}}\|_H^2 \\ &= 2\sum_{k=0}^\infty (t_{k+2}-t_{k+1})(T-t_{k+1})^{-1-2s}\|L_{t_{k+1}}\|_H^2 \\ &\leqslant 2\sum_{k=0}^\infty (t_{k+1}-t_k)(T-t_k)^{-1-2s}\|L_{t_k}\|_H^2. \end{aligned}$$

The proof of the inequality $\leqslant$ is analogous. Now for $s := (1-\theta)\left(-\frac{1}{2}\right) + \theta 0$ (so that $-1-2s = -\theta$) we use Proposition 3.9 (Eq. (3.7)) with $a = 0$ to get

$$\int_0^T (T-t)^{-\theta}\|L_t\|_H^2 \mathrm{d}t = \sup_{t\in[0,T)} \frac{T^{2\alpha}}{2\alpha}\mathbb{E}|\mathcal{I}_t^\alpha L|^2.$$

Now the equivalence (1) ⇔ (2) follows from (2.4) and (4.15). The equivalence (2) ⇔ (3) follows from Eq. (4.6) applied to $M := L$, $\varphi := L$, and $\sigma \equiv 1$. □

Remark 4.10. From Theorem 4.9 (2) we get for all $\varepsilon > 0$ a $t(\varepsilon) \in [0,T)$ such that for $s \in [t(\varepsilon), T)$ one has

$$\mathbb{E} \sup_{t\in[s,T)} \left| \int_s^T (L_{u\wedge t} - L_s)(T-u)^{\alpha-1}\frac{\alpha}{T^\alpha}\mathrm{d}u \right|^2 < \varepsilon. \tag{4.16}$$

Without the supremum the left-hand side is equal to $\mathbb{E}|\mathcal{I}_t^\alpha L - \mathcal{I}_s^\alpha L|^2$, the statement including the supremum follows from Doob's maximal inequality. The behaviour in (4.16) when $s \uparrow T$ might be seen as a replacement of the $\mathbf{L}_2$- and a.s. convergence of L in the case L would be closable in $\mathbf{L}_2$.

The counterpart to Theorem 4.9((2) ⇔ (3)) for the bmo-setting follows directly from Lemma 4.8 and Corollary 4.7:

Theorem 4.11 (*Approximation vs. Fractional Integration in bmo*)**.** *For a càdlàg martingale $L = (L_t)_{t\in[0,T)} \subseteq \mathbf{L}_2$, $\theta \in (0,1]$, and $\alpha := \frac{1-\theta}{2}$ the following is equivalent:*

(1) $\sup_{\tau\in\mathcal{T}} \frac{\|[L;\tau]^1\|_{\mathrm{bmo}_1[0,T)}}{\|\tau\|_\theta} < \infty$.
(2) $\mathcal{I}^\alpha L - L_0 \in \mathrm{bmo}_2[0,T)$ and $\left\| \sup_{0\leqslant s<a<T} \frac{(T-a)^{\frac{1}{2}}}{(T-s)^{\frac{\theta}{2}}} |L_a - L_s| \right\|_{\mathbf{L}_\infty} < \infty$.

4.3. Backward in time regularization of càdlàg martingales

Theorem 4.9 and Theorem 4.11 describe discrete time martingale approximations. Another possible approximation is obtained by a backwards smoothing in time, which yields to local Lipschitz trajectories. For this we define $H_\theta : (-\infty, T] \to [0, T]$ by

$$H_\theta(s) := T - T\sqrt[\theta]{1 - \max\{(s/T), 0\}}.$$

For $t \in (0, T]$ and $n \in \mathbb{N}$ let $\nu_n^\theta(t, \cdot)$ be the image measure of the uniform distribution $(n/T)\lambda_1|_{[s-\frac{T}{n},s]}$ with respect to H_θ where λ_1 is the 1-dimensional Lebesgue measure and $t = H_\theta(s)$. We define the time adapted $\mathbf{L}_2$-oscillation of a càdlàg process $L = (L_t)_{t\in[0,T)}$ at $t \in [0, T)$ by

$$\mathrm{Osc}_n^\theta(L, t) := \sqrt{\int_{[0,t]} (L_t - L_v)^2 \nu_n^\theta(t, \mathrm{d}v)}$$

and set

$$L_t^{n,\theta} := \int_{[0,t]} L_v \nu_n^\theta(t, \mathrm{d}v) \quad \text{for} \quad t \in [0, T).$$

Theorem 4.12. *For a càdlàg martingale $(L_t)_{t\in[0,T)} \subseteq \mathbf{L}_2$ and $\theta \in (0, 1]$ the following is equivalent:*

(1) There is a $c > 0$ such that $\mathbb{E}\int_0^T |\mathrm{Osc}_n^\theta(L, t)|^2 \mathrm{d}t \leqslant \frac{c^2}{n}$ for all $n \in \mathbb{N}$.
(2) $I^{\frac{1-\theta}{2}} L$ is closable in $\mathbf{L}_2$.

If (1) or (2) is satisfied, then

$$|L^{n,\theta}|_{\mathrm{Höl}_1([0,t])} \leqslant 2n\theta \left(1 - \frac{t}{T}\right)^{\theta-1} \sup_{r\in[0,t]} |L_r|$$

and

$$\left\| L - L^{n,\theta} \right\|_{\mathbf{L}_2([0,T)\times\Omega)} \leqslant \frac{c}{\sqrt{n}}.$$

Proof. (2)⇒(1) For $\xi, \theta \in (0, 1)$,

$$\tau_n^1(\xi) := \left\{ (i + \xi)\frac{T}{n}, i = 0, \ldots, n - 1 \right\} \cup \{0, T\} \in \mathcal{T},$$

and $\tau_n^\theta(\xi) := H_\theta(\tau_n^1(\xi))$ we get $\|\tau_n^\theta(\xi)\|_\theta \leqslant \frac{T^\theta}{\theta n}$, where we use $\frac{(1-a)^{1/\theta} - (1-b)^{1/\theta}}{(1-a)^{(1-\theta)/\theta}} \leqslant (b - a)/\theta$ for $0 \leqslant a < b \leqslant 1$. Because

$$\int_0^1 |L_t - L_{h(\tau_n^\theta(\xi),t)}|^2 \mathrm{d}\xi = |\mathrm{Osc}_n^\theta(L, t)|^2 \quad \text{for} \quad t \in [0, T)$$

with $h(\tau, t) := s_{j-1}$ if $t \in [s_{j-1}, s_j)$ and $\tau = \{s_j\}_{j=0}^m \in \mathcal{T}$, Theorem 4.9((2) ⇒ (3)) implies

$$\mathbb{E}\int_0^T |\mathrm{Osc}_n^\theta(L, t)|^2 \mathrm{d}t = \int_0^1 \mathbb{E}[L; \tau_n^\theta(\xi)]_T^1 \mathrm{d}\xi \leqslant c\frac{T^\theta}{\theta n}.$$

(1)⇒(2) Directly from the definition of $\mathrm{Osc}_n^\theta(L, t)$ we get that

$$\begin{aligned}
&n\mathbb{E}\int_{(0,T)} |\mathrm{Osc}_n^\theta(L, t)|^2 \mathrm{d}t \\
&= \frac{n^2}{T}\int_{(0,T)} \left(\int_{[H_\theta^{-1}(t)-\frac{T}{n}, H_\theta^{-1}(t)]} \mathbb{E}(L_t - L_{H_\theta(u)})^2 \mathrm{d}u \right) \mathrm{d}t \\
&= \frac{n^2}{T}\int_{(0,T)} \left(\int_{[H_\theta^{-1}(t)-\frac{T}{n}, H_\theta^{-1}(t)]} \mathbb{E}\int_{(H_\theta(u),t]} \mathrm{d}[L]_r \right) \mathrm{d}t \\
&= \mathbb{E}\int_{(0,T)} \left(\frac{n^2}{T} \int_{\substack{\{H_\theta^{-1}(r) \leqslant H_\theta^{-1}(t) < H_\theta^{-1}(r) + \frac{T}{n}\} \\ \cap (0,T)}} \left(H_\theta^{-1}(r) + \frac{T}{n} - H_\theta^{-1}(t) \right) \mathrm{d}t \right) \mathrm{d}[L]_r.
\end{aligned}$$

For $\varepsilon \in \left(0, \frac{T}{2}\right)$ we choose $n(\varepsilon) \in \mathbb{N}$ such that one has

$$0 < H_\theta^{-1}(\varepsilon) < H_\theta^{-1}(T - \varepsilon) + \frac{T}{n(\varepsilon)} < T.$$

For $r \in [\varepsilon, T - \varepsilon]$ and $n \geqslant n(\varepsilon)$ we get that

$$\frac{n^2}{T}\int_{H_\theta^{-1}(r) \leqslant H_\theta^{-1}(t) < H_\theta^{-1}(r) + \frac{T}{n}} \left(H_\theta^{-1}(r) + \frac{T}{n} - H_\theta^{-1}(t) \right) \mathrm{d}t$$

$$= \frac{T}{2}\left(\frac{T^2}{2n^2}\right)^{-1}\int_{H_\theta^{-1}(r)}^{H_\theta^{-1}(r)+\frac{T}{n}}\left(H_\theta^{-1}(r)+\frac{T}{n}-s\right)H_\theta'(s)\mathrm{d}s$$
$$\to \frac{T}{2}H_\theta'(H_\theta^{-1}(r)) = \frac{T}{2\theta}\left(1-\frac{r}{T}\right)^{1-\theta} \quad \text{as} \quad n\to\infty.$$

Therefore the Lemma of Fatou implies that

$$\liminf_{n\to\infty} n\mathbb{E}\int_{(0,T)}|\mathrm{Osc}_n^\theta(L,t)|^2\mathrm{d}t \geqslant \frac{T}{2\theta}\mathbb{E}\int_{[\varepsilon,T-\varepsilon]}\left(1-\frac{r}{T}\right)^{1-\theta}\mathrm{d}[L]_r.$$

As this is true for all $\varepsilon\in\left(0,\frac{T}{2}\right)$ we can replace the range of integration $[\varepsilon,T-\varepsilon]$ by $(0,T)$. By Proposition 3.6 this implies

$$\liminf_{n\to\infty} n\mathbb{E}\int_{(0,T)}|\mathrm{Osc}_n^\theta(L,t)|^2\mathrm{d}t \geqslant \frac{T}{2\theta}\sup_{t\in[0,T)}\mathbb{E}|\mathcal{I}_t^{\frac{1-\theta}{2}}L-L_0|^2.$$

Finally, $\|L-L^{n,\theta}\|_{\mathbf{L}_2([0,T)\times\Omega)} \leqslant \|\mathrm{Osc}_n^\theta(L,\cdot)\|_{\mathbf{L}_2([0,T)\times\Omega)}$ and the standard computation $|L^{n,\theta}(\omega)|_{\mathrm{Höl}_1([0,t])} \leqslant 2n\theta\,(1-(t/T))^{\theta-1}\sup_{r\in[0,t]}|L_r(\omega)|$ imply the remaining part of Theorem 4.12. □

5. Gradient estimates and approximation on the Wiener space

One background of this section is the problem from stochastic finance to estimate and control the error while discrete time hedging a continuous time portfolio for a European option. Estimates in the $\mathbf{L}_2$-sense for irregular pay-offs have been obtained in [8–10,27,28]. Although the $\mathbf{L}_2$-estimates have the advantage that one can exploit arguments based on orthogonality, the corresponding tail-estimates are usually far from being optimal - although the $\mathbf{L}_2$-estimates itself are optimal. To obtain better tail-estimates is significantly more difficult, and led to and inspired results that went far beyond the original problem. In the context we are concerned with there are two approaches: One can consider $\mathbf{L}_p$-estimates as in [11], where one has to give up the orthogonality, or one can consider bmo-estimates as in [14], where we keep some sort of orthogonality but use concepts similar as in harmonic analysis, spaces of bounded mean oscillation and reverse Hölder inequalities. In this section we develop further the approach from [14] and provide weighted bmo-Hölder estimates for certain gradient processes on the Wiener space to deduce from them approximation results.

5.1. Setting

We suppose additionally that $\mathcal{F}=\mathcal{F}_T$ and that $(\mathcal{F}_t)_{t\in[0,T]}$ is the augmentation of the natural filtration of a standard one-dimensional Brownian motion $W=(W_t)_{t\in[0,T]}$ with continuous paths and starting in zero for all $\omega\in\Omega$. We recall the setting from [9] and start with the stochastic differential equation (SDE)

$$\mathrm{d}X_t = \hat\sigma(X_t)\mathrm{d}W_t + \hat b(X_t)\mathrm{d}t \quad \text{with} \quad X_0\equiv x_0\in\mathbb{R} \tag{5.1}$$

where $0<\varepsilon_0\leqslant\hat\sigma\in C_b^\infty(\mathbb{R})$ for some constant ε_0 and $\hat b\in C_b^\infty(\mathbb{R})$ and where all paths of X are assumed to be continuous. From this equation we derive the SDE

$$\mathrm{d}Y_t = \sigma(Y_t)\mathrm{d}W_t \quad \text{with} \quad Y_0\equiv y_0\in\mathcal{R}_Y$$

where two settings are used simultaneously:

Case (C1): $Y:=X$ with $\sigma\equiv\hat\sigma$, $\hat b\equiv 0$, and $\mathcal{R}_Y:=\mathbb{R}$.

Case (C2): $Y:=\mathrm{e}^X$ with $\sigma(y):=y\hat\sigma(\ln y)$, $\hat b(x):=-\frac{1}{2}\hat\sigma^2(x)$, and $\mathcal{R}_Y:=(0,\infty)$.

In the context of stochastic finance, (C1) describes the generalized Bachelier setting and (C2) the generalized Black–Scholes setting. In both cases, we let C_Y be the set of all Borel functions $g:\mathcal{R}_Y\to\mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}}\mathrm{e}^{-m|x|}\int_{\mathbb{R}}|g(\alpha(x+ty))|^2\,\mathrm{e}^{-y^2}\,\mathrm{d}y<\infty \quad \text{for all} \quad t>0 \tag{5.2}$$

for some $m>0$, where $\alpha(x)=x$ in the case (C1) and $\alpha(x)=\mathrm{e}^x$ in the case (C2). Under (C1) and (C2) any polynomially bounded g belongs to C_Y. Let us denote by $(Y_s^{t,y})_{s\in[t,T]}$ the diffusion Y started at time $t\in[0,T]$ in $y\in\mathcal{R}_Y$ and let us define, for $g\in C_Y$,

$$G(t,y):=\mathbb{E}g(Y_T^{t,y}) \quad \text{for} \quad (t,y)\in[0,T]\times\mathcal{R}_Y.$$

Remark 5.1. We collect some facts we shall use and that hold in both cases, (C1) and (C2):

(A) $\|\sigma'\|_{B_b(\mathcal{R}_Y)}+\|\sigma\sigma''\|_{B_b(\mathcal{R}_Y)}<\infty$.
(B) In the case (C2) we have $\sigma(y)\sim_c y$ for $y\in\mathcal{R}_Y$ and some $c\geqslant 1$.
(C) One has $G\in C^\infty([0,T)\times\mathcal{R}_Y)$ and $\partial_t G+\frac{1}{2}\sigma^2\partial_{yy}^2 G=0$ on $[0,T)\times\mathcal{R}_Y$.
(D) $\mathbb{E}\left[|g(Y_T)|^2+\sup_{t\in[0,b]}\left|\left(\sigma\partial_y G\right)(t,Y_t)\right|^2\right]<\infty$ for all $b\in[0,T)$.
(E) The process $\left(\left(\sigma^2\partial_{yy}^2 G\right)(t,Y_t)\right)_{t\in[0,T)}$ is an $\mathbf{L}_2$-martingale.

(F) The process X has a transition density Γ_X in the sense of Theorem 8.5.

Items (A) and (B) are obvious, (C) is contained in [9, Preliminaries], (D) follows from the definition of C_Y, Theorem 8.5, and [9, Lemma 5.2], and (E) is [9, Lemma 5.3].

This yields to the following setting:

Setting 5.2. *In the notation of Assumption 4.5 we set*

(1) $\sigma = (\sigma_t)_{t\in[0,T]} := (\sigma(Y_t))_{t\in[0,T]}$,
(2) $M := \left(\int_0^t \left(\sigma^2 \partial^2_{yy} G\right)(u, Y_u)\mathrm{d}W_u\right)_{t\in[0,T)}$ *(with* $M_0 \equiv 0$ *and the continuity of all paths),*
(3) $\varphi := \left(\partial_y G(t, Y_t)\right)_{t\in[0,T)}$.

Denote by $E(g;\tau) = (E_t(g;\tau))_{t\in[0,T]}$ the path-wise continuous error process resulting from the difference between the stochastic integral and its Riemann approximation associated with the time-net $\tau = \{t_i\}_{i=0}^n \in \mathcal{T}$, *i.e.*

$$E_t(g;\tau) := \int_{(0,t]} \varphi_s \mathrm{d}Y_s - \sum_{i=1}^n \varphi_{t_{i-1}}(Y_{t_i\wedge t} - Y_{t_{i-1}\wedge t}) \quad \text{for } t \in [0,T].$$

We also denote the approximation error on $(a,t]$ by

$$E_{a,t}(g;\tau) := E_t(g;\tau) - E_a(g;\tau) \quad \text{for} \quad 0 \leqslant a < t < T$$

and remark that $E_{0,t}(g;\tau) = E_t(g;\tau)$. Then, for any $0 \leqslant a \leqslant t \leqslant T$, we apply the conditional Itô's isometry to obtain that, a.s.,

$$\begin{aligned}\mathbb{E}^{\mathcal{F}_a}\left[|E_{a,t}(g;\tau)|^2\right] &= \mathbb{E}^{\mathcal{F}_a}\left[\int_a^t \left|\varphi_u - \sum_{i=1}^n \varphi_{t_{i-1}} \mathbb{1}_{(t_{i-1},t_i]}(u)\right|^2 \sigma_u^2 \mathrm{d}u\right] \\ &= \mathbb{E}^{\mathcal{F}_a}\left[[\varphi;\tau]^\sigma_{a,t}\right].\end{aligned} \tag{5.3}$$

For $\Phi \in \mathrm{CL}^+([0,T))$ this implies that

$$\|E(g;\tau)\|^2_{\mathrm{bmo}_2^\Phi[0,T)} = \|[\varphi;\tau]^\sigma\|_{\mathrm{bmo}_1^{\Phi^2}[0,T)}. \tag{5.4}$$

To shorten the notation at some places we use, for $(t,y) \in [0,T) \times \mathcal{R}_Y$,

$$Z_t := \sigma_t\varphi_t, \quad \varphi(t,y) := \partial_y G(t,y), \quad H_t := \sigma_t^2 \partial^2_{yy} G(t, Y_t).$$

Next we verify Assumption 4.5. For this reason and later usage, we have the following lemma:

Lemma 5.3. *The following assertions hold true:*

(1) In the case (C2) *one has* $(Y_t^{\beta_0}(Y^{\beta_1})^*_t)_{t\in[0,T)} \in \mathcal{SM}_p([0,T))$ *for* $p \in (0,\infty)$ *and* $\beta_0, \beta_1 \in \mathbb{R}$.
(2) There is a constant $c_{(5.5)} > 0$ *such that, for all* $0 \leqslant a < b \leqslant T$,

$$\mathbb{E}^{\mathcal{F}_a}\left[\frac{1}{b-a}\int_a^b \sigma_u^2 \mathrm{d}u\right] \sim_{c^2_{(5.5)}} \sigma_a^2 \quad \textit{a.s.} \tag{5.5}$$

(3) For $g \in C_Y$ *one has* $\mathbb{E}\sup_{u\in[a,T]} |\varphi_a\sigma_u|^2 < \infty$ *for* $a \in [0,T)$.

Proof. (1) Because $\hat{\sigma} \in B_b(\mathbb{R})$, for all $\alpha \in \mathbb{R}$ there is a constant $c_{(5.6)} = c_{(5.6)}(\alpha, T, \hat{\sigma}) > 0$ such that

$$\mathbb{E}^{\mathcal{F}_a}\left[\sup_{t\in[a,T]} \mathrm{e}^{\alpha\int_{(a,t]}\hat{\sigma}(X_s)\mathrm{d}W_s}\right] \leqslant c_{(5.6)} \text{ a.s.} \tag{5.6}$$

for $a \in [0,T]$. Because $\hat{b}$ is bounded this implies that $(Y_t^\beta)_{t\in[0,T)} \in \mathcal{SM}_p([0,T))$ for all $p \in (0,\infty)$ and $\beta \in \mathbb{R}$. Therefore we may conclude by Items (2)and (3) of Proposition 8.1.

(2) We only need to check the case (C2) where we replace σ by Y due to (B). As Y is a martingale we get $\mathbb{E}^{\mathcal{F}_a}\left[\int_a^b Y_u^2 \mathrm{d}u\right] \geqslant (b-a)Y_a^2$ a.s., otherwise $\mathbb{E}^{\mathcal{F}_a}\left[\int_a^b Y_u^2\mathrm{d}u\right] \leqslant \|Y\|^2_{\mathcal{SM}_2([0,T))}(b-a)Y_a^2$ a.s.

(3) Because of (D) we only need to check (C2), use again (B) to replace σ by Y, and obtain

$$\mathbb{E}\sup_{u\in[a,T]}|\varphi_a Y_u|^2 = \mathbb{E}\left[|\varphi_a|^2\mathbb{E}^{\mathcal{F}_a}\left[\sup_{u\in[a,T)} Y_u^2\right]\right] \leqslant \|Y\|^2_{\mathcal{SM}_2([0,T))}\mathbb{E}|\varphi_a Y_a|^2 < \infty. \quad \square$$

Proposition 5.4. *The Property 5.2 under the assumptions made before in Section 5.1 guarantee that Assumption 4.5 is satisfied.*

Proof. The statement follows from Lemma 5.3 and [8, Corollary 3.3]. $\square$

For $\alpha > 0$ and $0 \leqslant a \leqslant t < T$ we get by (3.4) and (3.8), a.s.,

$$\mathcal{I}_t^\alpha M = \int_{(0,t]} \left(\frac{T-u}{T}\right)^\alpha H_u \mathrm{d}W_u$$
$$\text{and} \quad \mathbb{E}^{\mathcal{F}_a}\left[\left|\mathcal{I}_t^\alpha M - \mathcal{I}_a^\alpha M\right|^2\right] = \mathbb{E}^{\mathcal{F}_a}\left[\int_a^t \left(\frac{T-u}{T}\right)^{2\alpha} H_u^2 \mathrm{d}u\right]. \tag{5.7}$$

Finally, we will use the following condition on a change of measure which implies the reverse Hölder condition $\mathbb{Q} \in \mathrm{RH}_q(\mathbb{P})$ from Definition 2.4 for all $q \in (1,\infty)$:

Definition 5.5. Given a measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, we say that $\mathbb{Q} \in \mathrm{RH}_\infty^\xi(\mathbb{P})$ if there is a progressively measurable process $\xi = (\xi_t)_{t\in[0,T]}$ such that

$$\left\| \int_0^T |\xi_u|^2 \mathrm{d}u \right\|_{\mathbf{L}_\infty(\mathbb{P})} < \infty \quad \text{and} \quad \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathrm{e}^{\int_{(0,T]} \xi_u \mathrm{d}W_u - \frac{1}{2}\int_0^T |\xi_u|^2 \mathrm{d}u} \quad \text{a.s.}$$

If $\mathbb{Q} \in \mathrm{RH}_\infty^\xi(\mathbb{P})$ as in Definition 5.5, then for all $q \in (1,\infty)$ one has that $\mathbb{Q} \in \mathrm{RH}_q(\mathbb{P})$ with $\|\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}\|_{\mathrm{RH}_q(\mathbb{P})} \leqslant \exp\left(\frac{q-1}{2}\left\|\int_0^T |\xi_u|^2 \mathrm{d}u\right\|_{\mathbf{L}_\infty(\mathbb{P})}\right)$.

5.2. Convergence and closure properties

In this section we explain that the process $(\mathcal{I}_t^\alpha Z)_{t\in[0,T)}$ inherits its limit behaviour from the martingale $(M_t)_{t\in[0,T)}$. The relevance of the process Z is that $Z_t = (\sigma\partial_y G)(t, Y_t)$ is accessible as Markovian functional using the underlying PDE, has a direct interpretation in option pricing models, and relates to the Malliavin derivative of $g(Y_T)$ by the Clark–Ocone formula.

Theorem 5.6. *For $(\alpha, q) \in (0,\infty) \times [2,\infty)$, $\Phi \in \mathcal{SM}_2([0,T))$, $g \in C_Y$, and under the a priori estimate for the process $Z = (Z_t)_{t\in[0,T)}$,*

$$(T-t)^{\frac{1}{2}} |Z_t| \leqslant c_\Phi \Phi_t \text{ a.s.} \quad \text{for} \quad t \in [0,T) \tag{C_Φ}$$

for some $c_\Phi > 0$, one has:

1. *$(\mathcal{I}_t^\alpha Z - Z_0)_{t\in[0,T)} \in \mathrm{bmo}_2^\Phi[0,T) \iff \mathcal{I}^\alpha M \in \mathrm{bmo}_2^\Phi[0,T)$.*
2. *If $\sup_{t\in[0,T)} \Phi_t \in \mathbf{L}_q$, then $\mathcal{I}^\alpha Z$ converges (is bounded) in $\mathbf{L}_q \iff \mathcal{I}^\alpha M$ converges (is bounded) in $\mathbf{L}_q$ as $t \uparrow T$.*[4]
3. *$\mathcal{I}^\alpha Z$ converges a.s. $\iff \mathcal{I}^\alpha M$ converges a.s. if $t \uparrow T$.*
4. *If $\sup_{t\in[0,T)} \Phi_t \in \mathbf{L}_q$ and $\mathcal{I}^\alpha Z - Z_0 \in \mathrm{bmo}_2^\Phi[0,T)$, then $\lim_{t\uparrow T} \mathcal{I}_t^\alpha Z$ exists in $\mathbf{L}_q$ and a.s.*

We prove this statement in Section 7.1. In the cases, which we are interested in, condition (C_Φ) will be satisfied and corresponds to a known a priori condition from the theory of parabolic PDEs. For $q = 2$ in item (4) the condition $\sup_{t\in[0,T)} \Phi_t \in \mathbf{L}_q$ follows from $\Phi \in \mathcal{SM}_2([0,T))$.

5.3. Approximation results for Hölder spaces and Hölder interpolation spaces

In this section we formulate the results related to the approximation under Hölder conditions. These can be seen as a counterpart to Theorem 4.9((2) ⇔ (3)) and Theorem 4.11 in the context of weighted bmo-spaces. We start with the case $\theta = 1$ in which we extend [14, Theorem 8] from the geometric Brownian motion to the process Y and which we prove in Section 7.2:

Theorem 5.7. *For $g \in C_Y$ and $\Phi = \sigma$ the following assertions are equivalent:*

1. *There exists a Lipschitz function $\tilde{g} : \mathcal{R}_Y \to \mathbb{R}$ such that $g = \tilde{g}$ a.e. on $\mathcal{R}_Y$ with respect to the Lebesgue measure.*
2. *There is a $c \geqslant 0$ such that $\|E(g;\tau)\|_{\mathrm{bmo}_2^\Phi[0,T)} \leqslant c\sqrt{\|\tau\|_1}$ for all $\tau \in \mathcal{T}$.*

In (1) ⇒ (2) one can choose $c = d|\tilde{g}|_1$, where $d = d(\sigma, T) > 0$ is independent from g and $\tilde{g}$.

We have the following counterpart to Theorem 5.7:

Theorem 5.8. *Let $g \in C_Y$, $\Phi \in \mathrm{CL}^+([0,T))$ with $\Phi > 0$, and $\theta \in (0,1]$.*

1. *If $\tau = \{t_i\}_{i=0}^n \in \mathcal{T}$, then*

$$\frac{\|E(g;\tau)\|_{\mathrm{bmo}_2^\Phi[0,T)}^2}{\|\tau\|_\theta} \leqslant c_{(4.3)} \left[4\|\mathcal{I}^{\frac{1-\theta}{2}} M\|_{\mathrm{bmo}_2^\Phi[0,T)}^2 \right.$$

[4] For $\mathcal{I}^\alpha M$ the $\mathbf{L}_q$-boundedness and the convergence in $\mathbf{L}_q$ is equivalent because of the martingale property.

$$+\left\| \sup_{k\in\{1,\ldots,n\}} \sup_{a\in[t_{k-1},t_k)} \frac{T-a}{(T-t_{k-1})^\theta} |\varphi_a - \varphi_{t_{k-1}}|^2 \frac{\sigma_a^2}{\Phi_a^2} \right\|_{\mathbf{L}_\infty} \Bigg],$$

where $c_{(4.3)} > 0$ *is the constant from inequality* (4.3).

(2) *There is a* $c \geqslant 0$ *such that*

$$\|E(g;\tau)\|_{\mathrm{bmo}_2^{\Phi}[0,T)} \leqslant c\sqrt{\|\tau\|_\theta} \quad \textit{for all} \quad \tau \in \mathcal{T}$$

if and only if $\mathcal{I}^{\frac{1-\theta}{2}} M \in \mathrm{bmo}_2^{\Phi}[0,T)$ *and there is a* $d \geqslant 0$ *such that*

$$|\varphi_a - \varphi_s|\sigma_a \leqslant d\frac{(T-s)^{\frac{\theta}{2}}}{(T-a)^{\frac{1}{2}}}\Phi_a \quad \textit{for} \quad 0 \leqslant s < a < T \ \textit{a.s.}$$

Proof. Item (1) follows from Theorem 4.6 (1), where Doob's maximal inequality gives the additional factor 4. Corollary 4.7 and Eq. (5.4) imply item (2). □

In the next result, whose proof can be found in Section 7.3, we finally use Hölder terminal conditions to apply Theorem 5.8:

Theorem 5.9. *For* $(\theta,p) \in [0,1]\times(0,\infty)$ *and* $g \in \mathrm{Höl}_\theta(\mathbb{R})$ *we have:*

(1) $g|_{\mathcal{R}_Y} \in \mathcal{C}_Y$.
(2) $\sigma^\theta \in \mathcal{SM}_p([0,T))$.
(3) *For* $0 \leqslant s \leqslant a < T$ *and* $c_{(7.1)} = c_{(7.1)}(\sigma,T) > 0$ *from* (7.1) *it holds that* $(T-s)^{\frac{1-\theta}{2}}|\varphi_s| \leqslant c_{(7.1)}|g|_\theta \sigma_s^{\theta-1}$ *so that*

$$\frac{T-a}{(T-s)^\theta}\left|\varphi_a - \varphi_s\right|^2 \leqslant 2c_{(7.1)}^2 |g|_\theta^2 \ \left(\sigma_a^{2(\theta-1)} + \sigma_s^{2(\theta-1)}\right).$$

(4) *If* $\theta \in (0,1)$, *then there is a* $c_{(5.8)} = c_{(5.8)}(\sigma,T,\theta) > 0$ *such that*

$$\|\mathcal{I}^{\frac{1-\theta}{2}} M\|_{\mathrm{bmo}_2^{\sigma^\theta}[0,T)} \leqslant c_{(5.8)}|g|_{\theta,2} \quad \textit{for} \quad g \in \mathrm{Höl}_{\theta,2}(\mathbb{R}). \tag{5.8}$$

(5) *Let* $\theta \in (0,1)$ *and* $A = Z - Z_0$ *or* $A = M$. *If* $g \in \mathrm{Höl}_{\theta,2}(\mathbb{R})$, *then*

$$\mathcal{I}^{\frac{1-\theta}{2}} A \in \mathrm{bmo}_p^{\sigma^\theta}[0,T) \quad \textit{and} \quad \lim_{t\uparrow T} \mathcal{I}_t^{\frac{1-\theta}{2}} A \ \textit{exists in } \mathbf{L}_p \textit{ and a.s.}$$

Remark 5.10. The statement Theorem 5.9 (4) does not hold for $g \in \mathrm{Höl}_{\theta,q}(\mathbb{R})$ for $q \in (2,\infty]$, so that the condition $g \in \mathrm{Höl}_{\theta,2}(\mathbb{R})$ is sharp (in [29, Theorem 5.1] we consider an example for the case $\sigma \equiv 1$ and $T = 1$).

The above Theorems 5.8 and 5.9 will allow us to deduce Corollary 1.2 in Section 7.4. Moreover, in order to prove Corollary 1.4 we provide the following general tail-estimate, which might be of independent interest and which is verified in Section 7.5:

Theorem 5.11. *Let* $(\theta,p,q) \in (0,1)\times(0,\infty)\times(1,\infty)$, $\tau \in \mathcal{T}$, $\Phi := A\Phi(\tau,\theta) + B\sigma$ *with constants* $A, B \geqslant 0$, $A + B > 0$, $Q \in \mathrm{RH}_q(\mathbb{P})$, *and let* $R \in \mathrm{bmo}_p^{\Phi}[0,T)$ *be continuous. Then one has for* $\lambda \geqslant 1$ *and* $a \in [0,T)$ *that, a.s.,*

$$Q\left(\sup_{t\in[a,T)} \frac{|R_t - R_a|}{\Phi_a} > c\|R\|_{\mathrm{bmo}_p^{\Phi}[0,T)}\lambda \middle| \mathcal{F}_a\right) \leqslant c \begin{cases} \mathrm{e}^{-\frac{\lambda}{c}} & : \text{(C1)} \\ \mathrm{e}^{-\frac{|\ln\lambda|^2}{c}} & : \text{(C2)}, Q \in \mathrm{RH}_\infty^{\xi}(\mathbb{P}) \end{cases}$$

where $c > 0$ *depends at most on* $(\sigma,p,q,\|\mathrm{d}Q/\mathrm{d}\mathbb{P}\|_{\mathrm{RH}_q(\mathbb{P})})$ *in the case* (C1), *and on* $(T,\sigma,p,q,\left\|\int_0^T |\xi_u|^2 \mathrm{d}u\right\|_{\mathbf{L}_\infty(\mathbb{P})})$ *in the case* (C2).

Finally we include terminal conditions g of bounded variation in our considerations. This is the key to obtain the results about the binary option in Theorem 1.3 and Corollary 1.4. To simplify the formulation of the following theorem we do this for a special class of g:

Theorem 5.12. *For* $\theta \in (0,1)$, $D \geqslant 1$, $\varepsilon := 2D^{-\frac{1}{\theta}}(\sup_{y\in\mathbb{R}} p_T(y))^{-1}$, *where* p_T *is the continuous density of* Y_T, *and a right-continuous distribution function* g *of a probability measure on* $\mathcal{B}(\mathbb{R})$ *with* $g(0) = 0$, *one has*

(1) $\mathbb{E}g(Y_T) - \mathbb{E}g_\varepsilon(Y_T) \leqslant D^{-\frac{1}{\theta}}$ *with* $g_\varepsilon(y) := \frac{1}{\varepsilon}\int_{y-\varepsilon}^y g(z)\mathrm{d}z \leqslant g(y)$,
(2) $\|E(g_\varepsilon;\tau_n^\theta)\|_{\mathrm{bmo}_2^{\Phi}[0,T)} \leqslant c_{5.12}\frac{D}{\sqrt{n}}$ *for* $\Phi := \Phi(\tau_n^\theta,\theta)$ *with* τ_n^θ *from* (2.8),
(3) $|\varphi^\varepsilon| \leqslant c_{5.12} D^{\frac{1}{\theta}}$ *on* $[0,T)\times\Omega$ *with* φ^ε *defined as in Property* 5.2 (3) *for* g_ε,

where $c_{5.12} = c_{5.12}(T,\theta,\sigma) > 0$.

Balancing the corresponding errors from items (1) and (2) in Theorem 5.12 we arrive at the following result:

Corollary 5.13. *For $\delta \in (0,1/4)$, $\theta := (2\delta)/(1-2\delta) \in (0,1)$, $n \in \mathbb{N}$, (g, g_{ε_n}) as in Theorem 5.12 with $\varepsilon_n := 2n^{-\frac{\delta}{\theta}}(\sup_{y\in\mathbb{R}} p_T(y))^{-1}$, and for $\Phi = \Phi(\tau_n^\theta, \theta)$ one has*

$$\|E(g_{\varepsilon_n};\tau_n^\theta)\|_{\mathrm{bmo}_2^\Phi[0,T)} + (\mathbb{E}g(Y_T) - \mathbb{E}g_{\varepsilon_n}(Y_T)) \leqslant (c_{5.12}+1)\, n^{\delta-\frac{1}{2}},$$
$$|\varphi^{\varepsilon_n}| \leqslant c_{5.12} n^{\frac{1}{2}-\delta} \quad \text{on} \quad [0,T)\times\Omega,$$

where φ^{ε_n} is defined for g_{ε_n} as in Property 5.2 (3).

6. Proof of results of Section 4

6.1. Proof of Theorem 4.3

To simplify the notation we set $\varphi_T := 0$. For $\tau = \{t_i\}_{i=0}^n \in \mathcal{T}$, $a \in [t_{k-1}, t_k)$, and $s_i := t_i \vee a$ one has, a.s.,

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_a}\left[[\varphi;\tau]^\pi_{a,T}\right] &= \mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,T]}\left|\varphi_u - \sum_{i=1}^n \varphi_{t_{i-1}}\mathbb{1}_{(t_{i-1},t_i]}(u)\right|^2 \Pi(\mathrm{d}u)\right] \\
&= \mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,t_k]}\left|\varphi_u - \varphi_{t_{k-1}}\right|^2 \Pi(\mathrm{d}u) + \sum_{i=k+1}^n \int_{(t_{i-1},t_i]}\left|\varphi_u - \varphi_{t_{i-1}}\right|^2 \Pi(\mathrm{d}u)\right] \\
&\leqslant \kappa\mathbb{E}^{\mathcal{F}_a}\left[\left|\varphi_a - \varphi_{t_{k-1}}\right|^2 \Pi((a,t_k]) + \sum_{i=k}^n \int_{(s_{i-1},s_i]}(s_i-u)Y(\mathrm{d}u)\right] \\
&\leqslant \kappa\mathbb{E}^{\mathcal{F}_a}\Bigg[\frac{t_k-t_{k-1}}{(T-t_{k-1})^{1-\theta}}\left|\varphi_a - \varphi_{t_{k-1}}\right|^2 \frac{(T-t_{k-1})^{1-\theta}}{t_k-t_{k-1}}\Pi((a,t_k]) \\
&\qquad + \sum_{i=k}^n \int_{(s_{i-1},s_i]}\frac{s_i-u}{(T-u)^{1-\theta}}(T-u)^{1-\theta}Y(\mathrm{d}u)\Bigg] \\
&\leqslant \kappa\|\tau\|_\theta \\
&\qquad \mathbb{E}^{\mathcal{F}_a}\left[\left|\varphi_a - \varphi_{t_{k-1}}\right|^2 \frac{(T-t_{k-1})^{1-\theta}}{t_k-t_{k-1}}\Pi((a,t_k]) + \int_{(a,T)}(T-u)^{1-\theta}Y(\mathrm{d}u)\right]
\end{aligned}$$

where we use $\frac{s_i-u}{(T-u)^{1-\theta}} = \frac{t_i-u}{(T-u)^{1-\theta}} \leqslant \frac{t_i-t_{i-1}}{(T-t_{i-1})^{1-\theta}} \leqslant \|\tau\|_\theta$ for $u \in (s_{i-1},s_i]\cap[0,T)$ as $s_{i-1} < s_i$ implies $t_i = s_i$. □

6.2. Proof of Theorem 4.4

(1) Beginning the proof as for Theorem 4.3 with $a \in [t_{k-1}, t_k)$, we get, a.s.,

$$\begin{aligned}
\mathbb{E}^{\mathcal{F}_a}\left[[\varphi;\tau]^\pi_{a,t_k}\right] &= \mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,t_k]}\left|\varphi_u - \sum_{i=1}^n \varphi_{t_{i-1}}\mathbb{1}_{(t_{i-1},t_i]}(u)\right|^2 \Pi(\mathrm{d}u)\right] \\
&= \mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,t_k]}\left|\varphi_u - \varphi_{t_{k-1}}\right|^2 \Pi(\mathrm{d}u)\right] \\
&\geqslant \frac{1}{\kappa}\mathbb{E}^{\mathcal{F}_a}\left[\left|\varphi_a - \varphi_{t_{k-1}}\right|^2 \Pi((a,t_k])\right].
\end{aligned}$$

Dividing by $\|\tau\|_\theta = \frac{t_k-t_{k-1}}{(T-t_{k-1})^{1-\theta}}$ we obtain the desired statement.

(2) We partition the interval $[a,T]$ with $(u_{i,n}^{\theta,a})_{i=0}^n$ and $(r_{i,n}^{\theta,a})_{i=1}^n$. As n, a, θ remain fixed in the following estimates we write for ease of notation and readability simply u_i and r_i defined as

$$u_i := a + (T-a)\left[1 - \left(1-\tfrac{i}{n}\right)^{\frac{1}{\theta}}\right], \quad i = 0,\dots,n,$$
$$r_i := a + (T-a)\left[1 - \left(1-\tfrac{2i-1}{2n}\right)^{\frac{1}{\theta}}\right], \quad i = 1,\dots,n,$$

and add $r_0 := a$ and $r_{n+1} := T$. Choosing for both nets the remaining time-knots on $[0,a]$ fine enough, we obtain nets $\tau_n^\theta(a)$ and $\widetilde{\tau}_n^\theta(a)$ satisfying

$$\|\tau_n^\theta(a)\|_\theta = \sup_{i=1,\dots,n}\frac{u_i-u_{i-1}}{(T-u_{i-1})^{1-\theta}} \quad \text{and} \quad \|\widetilde{\tau}_n^\theta(a)\|_\theta = \sup_{i=0,1,\dots,n}\frac{r_{i+1}-r_i}{(T-r_i)^{1-\theta}}.$$

By a computation, we have for $i = 1,\dots,n$ and $u \in (u_{i-1}, r_i]$ that

$$\frac{(T-a)^\theta}{\theta 2^{\frac{1}{\theta}+1} n} \leqslant \frac{u_i-r_i}{(T-r_i)^{1-\theta}} \leqslant \frac{u_i-u}{(T-u)^{1-\theta}} \leqslant \frac{u_i-u_{i-1}}{(T-u_{i-1})^{1-\theta}} \leqslant \frac{(T-a)^\theta}{\theta n}, \tag{6.1}$$

and for $i = 1, \dots, n-1$ and $u \in (r_i, u_i]$ that

$$\frac{(T-a)^\theta}{\theta 2^{\frac{1}{\theta}+1} n} \leqslant \frac{r_{i+1}-u_i}{(T-u_i)^{1-\theta}} \leqslant \frac{r_{i+1}-u}{(T-u)^{1-\theta}} \leqslant \frac{r_{i+1}-r_i}{(T-r_i)^{1-\theta}} \leqslant \frac{(T-a)^\theta}{\theta n}, \tag{6.2}$$

where the last inequality in the chain of inequalities in (6.2) holds for $i \in \{0, n\}$ as well. This implies

$$\|\tau_n^\theta(a)\|_\theta \leqslant \frac{(T-a)^\theta}{\theta n} \quad \text{and} \quad \|\widetilde{\tau}_n^\theta(a)\|_\theta \leqslant \frac{(T-a)^\theta}{\theta n}. \tag{6.3}$$

Next we obtain, a.s.,

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,r_n]} (T-u)^{1-\theta} Y(\mathrm{d}u)\right] \\
&= \sum_{i=1}^n \mathbb{E}^{\mathcal{F}_a}\left[\int_{(u_{i-1}\, r_i]} (T-u)^{1-\theta} Y(\mathrm{d}u)\right] + \sum_{i=1}^{n-1} \mathbb{E}^{\mathcal{F}_a}\left[\int_{(r_i, u_i]} (T-u)^{1-\theta} Y(\mathrm{d}u)\right] \\
&\leqslant \frac{\theta 2^{\frac{1}{\theta}+1} n}{(T-a)^\theta}\left[\sum_{i=1}^n \mathbb{E}^{\mathcal{F}_a}\left[\int_{(u_{i-1}\, r_i]} (u_{i,n}^{\theta,a} - u) Y(\mathrm{d}u)\right]\right. \\
&\qquad\qquad\qquad\qquad\qquad \left. + \sum_{i=1}^{n-1} \mathbb{E}^{\mathcal{F}_a}\left[\int_{(r_i\, u_i]} (r_{i+1} - u) Y(\mathrm{d}u)\right]\right] \\
&\leqslant (\kappa 2^{\frac{1}{\theta}+1}) \mathbb{E}^{\mathcal{F}_a}\left[\frac{[\varphi; \tau_n^\theta(a)]_{a,T}^\pi}{\|\tau_n^\theta(a)\|_\theta} + \frac{[\varphi; \widetilde{\tau}_n^\theta(a)]_{a,T}^\pi}{\|\widetilde{\tau}_n^\theta(a)\|_\theta}\right],
\end{aligned}$$

where for the last inequality we first use (4.1), that gives the factor κ. For each n we choose the time-net that gives the larger quotient and obtain the desired nets as we have (6.3) and $r_n = a + (T-a)\left[1 - \left(1 - \frac{2n-1}{2n}\right)^{\frac{1}{\theta}}\right] \uparrow T$ as $n \to \infty$. $\square$

7. Proof of results of Section 5 and Section 1

Lemma 7.1. *Assume that $\theta \in (0,1]$, $g \in C_Y$, $\tau \in \mathcal{T}$, and $\Phi \in \mathrm{CL}^+([0,T))$ such that, for all $a \in [0,T)$,*

$$\mathbb{E}^{\mathcal{F}_a}\left[\sup_{t\in[a,T)}\left|\mathcal{I}_t^{\frac{1-\theta}{2}} M - \mathcal{I}_a^{\frac{1-\theta}{2}} M\right|^2\right] + \frac{T-a}{(T-t_{k-1})^\theta}\left|\varphi_a - \varphi_{t_{k-1}}\right|^2 \sigma_a^2 \leqslant \Phi_a^2 \quad a.s.$$

when $a \in [t_{k-1}, t_k)$. Then $\|E(g;\tau)\|_{\mathrm{bmo}_2^\Phi[0,T)} \leqslant \sqrt{c_{(4.3)}}\sqrt{\|\tau\|_\theta}$, where $c_{(4.3)} > 0$ is taken from inequality (4.3).

Proof. The statement follows directly from (5.4) and inequality (4.3). $\square$

Lemma 7.2. *For $\alpha \geqslant 0$ and $t \in [0,T)$ one has, a.s.,*

$$\begin{aligned}
(T-t)^\alpha Z_t = T^\alpha Z_0 &+ \int_{(0,t]} (T-u)^\alpha H_u \mathrm{d}W_u + \int_{(0,t]} (T-u)^\alpha \sigma'(Y_u) Z_u \mathrm{d}W_u \\
&- \alpha \int_{(0,t]} (T-u)^{\alpha-1} Z_u \mathrm{d}u + \frac{1}{2}\int_{(0,t]} (T-u)^\alpha (\sigma\sigma'')(Y_u) Z_u \mathrm{d}u.
\end{aligned}$$

Proof. The assertion follows by Itô's formula applied to the function $(t,y) \mapsto (T-t)^\alpha \left(\sigma \partial_y G\right)(t,y)$ with Y_t inserted into the y-component, where we use the PDE from (C). $\square$

Lemma 7.3. *There exists a constant $c_{(7.1)} = c_{(7.1)}(c_{(8.3)}, T) > 0$ such that for $\theta \in [0,1]$ and $g \in \mathrm{H\ddot{o}l}_\theta(\mathbb{R})$ one has*

$$\left|\partial_y G(u,y)\right| \leqslant c_{(7.1)}\, |g|_\theta\, \sigma(y)^{\theta-1} (T-u)^{\frac{\theta-1}{2}} \quad \textit{for} \ (u,y) \in [0,T) \times \mathcal{R}_Y. \tag{7.1}$$

Proof. Set $f := g$ and $F := G$ in case (C1) and $f(x) := g(\mathrm{e}^x)$ and $F(u,x) := G(u, \mathrm{e}^x)$ for $(u,x) \in [0,T) \times \mathbb{R}$ in case (C2). Let us fix $u \in [0,T)$. In both cases, (C1) and (C2), we have

$$\partial_x F(u,x) = \int_{\mathbb{R}} \partial_x \Gamma_X(T-u,x,\xi) f(\xi) \mathrm{d}\xi = \int_{\mathbb{R}} \partial_x \Gamma_X(T-u,x,\xi)(f(\xi) - f(x)) \mathrm{d}\xi$$

where we use (F) with the transition density Γ_X from Theorem 8.5. For $t > 0$ denote $\gamma_t(x) := \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{x^2}{2t}}$. In the case (C1) we derive for $y = x$ that

$$\left|\partial_y G(u,y)\right| = \left|\partial_x F(u,x)\right| \leqslant |g|_\theta \int_{\mathbb{R}} \left|\partial_x \Gamma_X(T-u,x,\xi)\right| |\xi - x|^\theta \mathrm{d}\xi$$

$$\leqslant |g|_\theta \int_{\mathbb{R}} c_{(8.3)}(T-u)^{-\frac{1}{2}} \gamma_{c_{(8.3)}(T-u)}(x-\xi)|\xi - x|^\theta \mathrm{d}\xi$$
$$= |g|_\theta (T-u)^{\frac{\theta-1}{2}} \int_{\mathbb{R}} c_{(8.3)} \gamma_{c_{(8.3)}}(\eta)|\eta|^\theta \mathrm{d}\eta$$
$$\leqslant |g|_\theta (T-u)^{\frac{\theta-1}{2}} \int_{\mathbb{R}} c_{(8.3)} \gamma_{c_{(8.3)}}(\eta)(1+|\eta|)\mathrm{d}\eta$$

where we use $\int_{\mathbb{R}} \partial_x \Gamma_X(T-u,x,\xi)\mathrm{d}\xi = \frac{\partial}{\partial x}\int_{\mathbb{R}} \Gamma_X(T-u,x,\xi)\mathrm{d}\xi = 0$. For $y = \mathrm{e}^x$ we get for (C2) that

$$\left| y\partial_y G(u,y)\right| = \left|\partial_x F(u,x)\right|$$
$$\leqslant |g|_\theta \int_{\mathbb{R}} \left|\partial_x \Gamma_X(T-u,x,\xi)\right| |\, \mathrm{e}^\xi - \mathrm{e}^x \,|^\theta \mathrm{d}\xi$$
$$= |g|_\theta \, \mathrm{e}^{x\theta} \int_{\mathbb{R}} \left|\partial_x \Gamma_X(T-u,x,\xi)\right| |\, \mathrm{e}^{\xi - x} - 1|^\theta \mathrm{d}\xi$$
$$\leqslant |g|_\theta \, \mathrm{e}^{x\theta} \int_{\mathbb{R}} c_{(8.3)}(T-u)^{-\frac{1}{2}} \gamma_{c_{(8.3)}(T-u)}(x-\xi)|\, \mathrm{e}^{\xi-x} - 1|^\theta \mathrm{d}\xi.$$

We conclude by

$$\int_{\mathbb{R}} \gamma_{c_{(8.3)}(T-u)}(x-\xi)|\, \mathrm{e}^{\xi-x} - 1|^\theta \mathrm{d}\xi \leqslant \int_{\mathbb{R}} \gamma_{c_{(8.3)}(T-u)}(\xi)|\xi|^\theta \, \mathrm{e}^{\theta|\xi|}\, \mathrm{d}\xi$$
$$\leqslant (T-u)^{\frac{\theta}{2}} \int_{\mathbb{R}} \gamma_{c_{(8.3)}}(\eta)|\eta|^\theta \, \mathrm{e}^{\theta\sqrt{T}|\eta|}\, \mathrm{d}\eta$$
$$\leqslant (T-u)^{\frac{\theta}{2}} \int_{\mathbb{R}} \gamma_{c_{(8.3)}}(\eta)(1+|\eta|)\, \mathrm{e}^{\sqrt{T}|\eta|}\, \mathrm{d}\eta$$
$$< \infty. \qquad \square$$

Lemma 7.4. *Let* $\mathrm{d}\hat{\mathbb{P}} := L\mathrm{d}\mathbb{P}$ *with* $L := \mathrm{e}^{\int_{(0,T]} \sigma'(Y_t)\mathrm{d}W_t - \frac{1}{2}\int_{(0,T]} |\sigma'(Y_t)|^2 \mathrm{d}t}$ *and* $g \in C_Y$. *Then the process* $(\varphi(t,Y_t))_{t\in[0,T)} = (\partial_y G(t,Y_t))_{t\in[0,T)}$ *is a* $\hat{\mathbb{P}}$*-martingale.*

Proof. Applying the PDE from (C) we get that

$$\partial_t \varphi(t,y) + (\sigma\sigma')(y)\partial_y \varphi(t,y) + \frac{1}{2}\sigma(y)^2 \partial^2_{yy}\varphi(t,y)$$
$$= \frac{\partial}{\partial y}\left[\partial_t G(t,y) + \frac{1}{2}\sigma(y)^2\partial^2_{yy} G(t,y)\right] = 0$$

on $[0,T)\times \mathcal{R}_Y$. By Itô's formula this implies that

$$\varphi(t,Y_t) = \varphi(0,y_0) + \int_{(0,t]} \left(\sigma\partial_y\varphi\right)(u,Y_u)\left[\mathrm{d}W_u - \sigma'(Y_u)\mathrm{d}u\right] \text{ a.s.}$$

for $t \in [0,T)$. Because of (A) and Girsanov's theorem we obtain a $\hat{\mathbb{P}}$ standard Brownian motion $\hat{W}_t := W_t - \int_{(0,t]} \sigma'(Y_u)\mathrm{d}u$, $t \in [0,T]$. Moreover, for $t \in [0,T)$, $p,q \in (1,\infty)$, $1 = \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'}$, and with $p'q' = 2$ we have that

$$\mathbb{E}^{\hat{\mathbb{P}}}\left(\int_0^t \left|\left(\sigma\partial_y\varphi\right)(u,Y_u)\right|^2 \mathrm{d}u\right)^{\frac{1}{2}}$$
$$\leqslant (\mathbb{E}^{\mathbb{P}} L^p)^{\frac{1}{p}} \left(\mathbb{E}^{\mathbb{P}}\left(\int_0^t \left|\left(\sigma\partial_y\varphi\right)(u,Y_u)\right|^2 \mathrm{d}u\right)^{\frac{p'}{2}}\right)^{\frac{1}{p'}}$$
$$\leqslant (\mathbb{E}^{\mathbb{P}} L^p)^{\frac{1}{p}} \left(\mathbb{E}^{\mathbb{P}}\left(\sup_{u\in[0,T]}(\sigma_u^{-p'})\right)\left(\int_0^t \left|\left(\sigma^2\partial_y\varphi\right)(u,Y_u)\right|^2 \mathrm{d}u\right)^{\frac{p'}{2}}\right)^{\frac{1}{p'}}$$
$$\leqslant (\mathbb{E}^{\mathbb{P}} L^p)^{\frac{1}{p}} \left(\mathbb{E}^{\mathbb{P}}\left(\sup_{u\in[0,T]}(\sigma_u^{-p'})\right)^q\right)^{\frac{1}{p'q}} \left(\mathbb{E}^{\mathbb{P}}\int_0^t \left|\left(\sigma^2\partial_y\varphi\right)(u,Y_u)\right|^2 \mathrm{d}u\right)^{\frac{1}{2}}.$$

The last term is finite because of (E), the first term is finite as σ' is bounded, the second term is finite in the case $(C1)$, but also finite in the case $(C2)$ because of $\sigma(y) \sim y$ and Lemma 5.3 (1). As by the Burkholder–Davis–Gundy inequalities applied to continuous local martingales we also have

$$\mathbb{E}^{\hat{\mathbb{P}}}\left|\int_{(0,t]} \left|\left(\sigma\partial_y\varphi\right)(u,Y_u)\right|^2 \mathrm{d}u\right|^{\frac{1}{2}} \sim_c \mathbb{E}^{\hat{\mathbb{P}}} \sup_{s\in[0,t]}\left|\int_{(0,s]} \left(\sigma\partial_y\varphi\right)(u,Y_u)\mathrm{d}\hat{W}_u\right|$$

for some absolute constant $c \geqslant 1$ and $t \in [0,T)$, we get that $(\varphi(t,Y_t))_{t\in[0,T)}$ is a $\hat{\mathbb{P}}$-martingale. $\square$

7.1. Proof of Theorem 5.6

(1) For $t \in [0,T)$ the relation

$$\alpha \int_0^T (T-u)^{\alpha-1} Z_{u\wedge t} \mathrm{d}u = \alpha \int_0^t (T-u)^{\alpha-1} Z_u \mathrm{d}u + (T-t)^\alpha Z_t$$

and Lemma 7.2 imply that

$$\begin{aligned}&\alpha \int_{(0,T]} (T-u)^{\alpha-1} Z_{u\wedge t} \mathrm{d}u \\ &= T^\alpha Z_0 + \int_{(0,t]} (T-u)^\alpha H_u \mathrm{d}W_u + \int_{(0,t]} (T-u)^\alpha \sigma'(Y_u) Z_u \mathrm{d}W_u \\ &\quad + \frac{1}{2}\int_{(0,t]} (T-u)^\alpha (\sigma\sigma'')(Y_u) Z_u \mathrm{d}u \text{ a.s.}\end{aligned}$$

Denote $b_u(\omega) := \frac{1}{2}(\sigma\sigma'')(Y_u(\omega))$ and $B := \frac{1}{2}\|\sigma\sigma''\|_{B_b(\mathcal{R}_Y)} < \infty$. Dividing both sides of the equality above by T^α and using (5.7), gives

$$(\mathcal{I}_t^\alpha Z - Z_0) - \mathcal{I}_t^\alpha M = \int_{(0,t]} \left(\frac{T-u}{T}\right)^\alpha Z_u (\sigma'(Y_u)\mathrm{d}W_u + b_u \mathrm{d}u) \quad \text{a.s.} \tag{7.2}$$

Next we observe that, for $0 \leqslant a < t < T$, a.s.,

$$\begin{aligned}&\left(\mathbb{E}^{\mathcal{F}_a}\left[\left|\int_{(a,t]} \left(\frac{T-u}{T}\right)^\alpha Z_u \sigma'(Y_u) \mathrm{d}W_u\right|^2\right]\right)^{\frac{1}{2}} \\ &+ \left(\mathbb{E}^{\mathcal{F}_a}\left[\left|\int_{(a,t]} \left(\frac{T-u}{T}\right)^\alpha |Z_u b_u| \mathrm{d}u\right|^2\right]\right)^{\frac{1}{2}} \\ &\leqslant (\|\sigma'\|_{B_b(\mathcal{R}_Y)} + B\sqrt{T}) \left(\mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,t]} \left(\frac{T-u}{T}\right)^{2\alpha} Z_u^2 \mathrm{d}u\right]\right)^{\frac{1}{2}} \\ &\leqslant c_\Phi (\|\sigma'\|_{B_b(\mathcal{R}_Y)} + B\sqrt{T}) \left(\mathbb{E}^{\mathcal{F}_a}\left[\sup_{u\in[a,T)} \Phi_u^2 \int_{(a,t]} \left(\frac{T-u}{T}\right)^{2\alpha} \frac{\mathrm{d}u}{T-u}\right]\right)^{\frac{1}{2}} \\ &\leqslant \frac{c_\Phi (\|\sigma'\|_{B_b(\mathcal{R}_Y)} + B\sqrt{T})}{\sqrt{2\alpha}} \|\Phi\|_{\mathcal{SM}_2([0,T))} \left(\frac{T-a}{T}\right)^\alpha \Phi_a.\end{aligned}$$

Because of (7.2) Item (1) follows.

(2) Also, the martingale $\left(\int_{(0,t]} \left(\frac{T-u}{T}\right)^\alpha Z_u \sigma'(Y_u) \mathrm{d}W_u\right)_{t\in[0,T)}$ converges in $\mathbf{L}_q$ and a.s. because of $\Phi \in \mathcal{SM}_2([0,T))$, $\sup_{t\in[0,T)} \Phi_t \in \mathbf{L}_q$, and (8.2) of Theorem 8.4. Again by (8.2), the non-negative and non-decreasing process

$$\left(\int_0^t \left(\frac{T-u}{T}\right)^\alpha |Z_u b_u| \mathrm{d}u\right)_{t\in[0,T)}$$

converges in $\mathbf{L}_q$ and a.s. For this reason $\left(\int_0^t \left(\frac{T-u}{T}\right)^\alpha Z_u b_u \mathrm{d}u\right)_{t\in[0,T)}$ converges in $\mathbf{L}_q$ and a.s. as well. So, again using (7.2), Item (2) follows.

(3) This part follows from the proof of (2) for $q = 2$ as $\Phi \in \mathcal{SM}_2([0,T))$ gives $\sup_{t\in[0,T)} \Phi_t \in \mathbf{L}_2$.

(4) From $\sup_{t\in[0,T)} \Phi_t \in \mathbf{L}_q$, $\mathcal{I}^\alpha Z - Z_0 \in \mathrm{bmo}_2^\Phi([0,T))$, and (8.2) of Theorem 8.4 we deduce that $\sup_{t\in[0,T)} |\mathcal{I}_t^\alpha Z| \in \mathbf{L}_q$. By (2) we conclude that $\sup_{t\in[0,T)} \left\|\mathcal{I}_t^\alpha M\right\|_{\mathbf{L}_q} < \infty$ and obtain from the martingale property the $\mathbf{L}_q$- and a.s. convergence of $\mathcal{I}^\alpha M$. Now we can use (2) and (3) to obtain the $\mathbf{L}_q$- and a.s. convergence of $(\mathcal{I}_t^\alpha Z)_{t\in[0,T)}$. $\square$

7.2. Proof of Theorem 5.7

(1) ⇒ (2) We may assume that $g : \mathcal{R}_Y \to \mathbb{R}$ is Lipschitz. By Lemma 7.3 we have

$$\left|\partial_y G(u,y)\right| \leqslant c_{(7.1)} |g|_1 \quad \text{and} \quad \left|Z_u\right| \leqslant c_{(7.1)} |g|_1 \sigma_u \quad \text{for} \quad (u,y) \in [0,T) \times \mathcal{R}_Y.$$

Let $0 \leqslant a < t < T$. From Lemma 7.2 with $\alpha = 0$ we get that

$$Z_t = Z_a + \int_{(a,t]} H_u \mathrm{d}W_u + \int_{(a,t]} \sigma'(Y_u) Z_u \mathrm{d}W_u + \frac{1}{2}\int_{(a,t]} (\sigma\sigma'')(Y_u) Z_u \mathrm{d}u \quad \text{a.s.}$$

Then one has, a.s.,

$$\sqrt{\mathbb{E}^{\mathcal{F}_a}\left[\int_a^t H_u^2 \mathrm{d}u\right]}$$

$$
\begin{aligned}
&\leqslant \sqrt{\mathbb{E}^{\mathcal{F}_a}\left[|Z_t - Z_a|^2\right]} + \|\sigma'\|_{B_b(\mathcal{R}_Y)}\sqrt{\mathbb{E}^{\mathcal{F}_a}\left[\int_a^t Z_u^2 \mathrm{d}u\right]} \\
&+ \frac{1}{2}\|\sigma\sigma''\|_{B_b(\mathcal{R}_Y)}\sqrt{\mathbb{E}^{\mathcal{F}_a}\left[\left|\int_a^t |Z_u|\mathrm{d}u\right|^2\right]} \\
&\leqslant \sqrt{\mathbb{E}^{\mathcal{F}_a}\left[|Z_t - Z_a|^2\right]} + \left[\|\sigma'\|_{B_b(\mathcal{R}_Y)} + \frac{\sqrt{T}}{2}\|\sigma\sigma''\|_{B_b(\mathcal{R}_Y)}\right]\sqrt{\mathbb{E}^{\mathcal{F}_a}\left[\int_a^t Z_u^2 \mathrm{d}u\right]} \\
&\leqslant c_{(7.1)}|g|_1\left[\sqrt{\mathbb{E}^{\mathcal{F}_a}\left[\sigma_t^2\right]} + \sigma_a\right] \\
&+ c_{(7.1)}|g|_1\left[\|\sigma'\|_{B_b(\mathcal{R}_Y)} + \frac{\sqrt{T}}{2}\|\sigma\sigma''\|_{B_b(\mathcal{R}_Y)}\right]\sqrt{T}\sqrt{\mathbb{E}^{\mathcal{F}_a}\left[\sup_{u\in[a,T)}\sigma_u^2\right]} \\
&\leqslant c_{(7.1)}|g|_1\left[2 + \sqrt{T}\|\sigma'\|_{B_b(\mathcal{R}_Y)} + \frac{T}{2}\|\sigma\sigma''\|_{B_b(\mathcal{R}_Y)}\right]\sqrt{\mathbb{E}^{\mathcal{F}_a}\left[\sup_{u\in[a,T)}\sigma_u^2\right]} \\
&\leqslant c_{(7.1)}|g|_1\left[2 + \sqrt{T}\|\sigma'\|_{B_b(\mathcal{R}_Y)} + \frac{T}{2}\|\sigma\sigma''\|_{B_b(\mathcal{R}_Y)}\right]\|\sigma\|_{\mathcal{SM}_2([0,T))}\sigma_a
\end{aligned}
$$

and hence

$$
\sqrt{\mathbb{E}^{\mathcal{F}_a}\left[|M_t - M_a|^2\right]} = \sqrt{\mathbb{E}^{\mathcal{F}_a}\left[\int_{(a,t]} H_u^2 \mathrm{d}u\right]} \leqslant c_{(7.3)}|g|_1\|\sigma\|_{\mathcal{SM}_2([0,T))}\sigma_a \text{ a.s.,} \tag{7.3}
$$

for some $c_{(7.3)}(\sigma, T) > 0$. Applying Lemma 7.3 for $\theta = 1$ yields $\left|\partial_y G(u,y)\right| \leqslant c_{(7.1)}|g|_1$ for $(u,y) \in [0,T) \times \mathcal{R}_Y$. Therefore, by applying Doob's maximal inequality in (7.3) and assuming $a \in [t_{k-1}, t_k)$ we have

$$
\begin{aligned}
&\mathbb{E}^{\mathcal{F}_a}\left[\sup_{t\in[a,T)}\left|M_t - M_a\right|^2\right] + \frac{T-a}{T-t_{k-1}}\left|\varphi_a - \varphi_{t_{k-1}}\right|^2\sigma_a^2 \\
&\leqslant 4\left[c_{(7.3)}|g|_1\|\sigma\|_{\mathcal{SM}_2([0,T))}\right]^2\sigma_a^2 + 4\left[c_{(7.1)}|g|_1\right]^2\sigma_a^2 \\
&= 4\left[c_{(7.3)}^2\|\sigma\|_{\mathcal{SM}_2([0,T))}^2 + c_{(7.1)}^2\right]|g|_1^2\sigma_a^2
\end{aligned}
$$

where we used $\frac{T-a}{T-t_{k-1}} \leqslant 1$. Now Lemma 7.1 implies (2).

(2) ⇒ (1) Given $a \in (0,T)$, exploiting (4.4) and (5.3) give

$$
\sup_{s\in[0,a]}\frac{T-a}{T-s}|\varphi_a - \varphi_s|^2 \leqslant c_{(7.4)}^2 \text{ a.s.} \tag{7.4}
$$

For $a \in \left(\frac{T}{2}, T\right)$ we choose $s \in (0,a)$ such that $\frac{T-a}{T-s} = \frac{1}{2}$. Therefore we may continue to

$$
\left|\partial_y G(a, y_a)\right| \leqslant \left|\partial_y G(s, y_s)\right| + \sqrt{2}c_{(7.4)} \quad \text{for all} \quad y_a, y_s \in \mathcal{R}_Y
$$

where we use the positivity and continuity of the transition density Γ_Y with

$$
\Gamma_Y(t-s; y_1, y_2) = \frac{1}{y_2}\Gamma_X(t-s; \ln(y_1), \ln(y_2))
$$

in the case (C2) and $\Gamma_Y = \Gamma_X$ in the case (C1) (Γ_X is taken from Theorem 8.5), that the support of the law of (Y_s, Y_a) is $\mathcal{R}_Y \times \mathcal{R}_Y$, and the continuity of $\partial_y G(t,\cdot) : \mathcal{R}_Y \to \mathbb{R}$ for $t \in [0,T)$. Applying Lemma 7.4, we have $\mathbb{E}^{\hat{\mathbb{P}}}\varphi(s, Y_s) = \varphi(0, Y_0)$ for $s \in [0,T)$. Therefore, for each $s \in [0,T)$ there are $\omega_s^0, \omega_s^1 \in \Omega$ such that for $y_s^i := Y_s(\omega_s^i) \in \mathcal{R}_Y$ we have $\varphi(s, y_s^0) \leqslant \varphi(0, Y_0) \leqslant \varphi(s, y_s^1)$. Because $y \mapsto \partial_y G(s,y)$ is continuous on $\mathcal{R}_Y$ we find a $y_s \in \mathcal{R}_Y$ such that $\varphi(s, y_s) = \varphi(0, y_0)$. Therefore,

$$
\left|\partial_y G(a,y)\right| \leqslant \left|\partial_y G(0, y_0)\right| + \sqrt{2}c_{(7.4)} =: c_{(7.5)} \quad \text{for all} \quad (a,y) \in \left(\frac{T}{2}, T\right) \times \mathcal{R}_Y. \tag{7.5}
$$

Let $\Omega_g \in \mathcal{F}$ be of measure one such that for all $\omega \in \Omega_g$ one has

$$
\lim_{t\uparrow T} G(t, Y_t(\omega)) = g(Y_T(\omega)).
$$

Let $I_g := Y_T(\Omega_g) \subseteq \mathcal{R}_Y$. Then g is Lipschitz on I_g with Lipschitz constant $c_{(7.5)}$, and since I_g is dense in $\mathcal{R}_Y$ (as Y_T has a positive density on $\mathcal{R}_Y$), the function $g|_{I_g}$ can be extended to $\tilde{g} : \mathcal{R}_Y \to \mathbb{R}$ as a Lipschitz function on $\mathcal{R}_Y$. Moreover, $\mathbb{P}(g(Y_T) = \tilde{g}(Y_T)) \geqslant \mathbb{P}(\Omega_g) = 1$. □

7.3. Proof of Theorem 5.9

As (1) is obvious we start with (2). We only need to check the case (C2) and in this case we have $\sigma(y) \sim y$ so that we can use Lemma 5.3 (1). (3) follows directly from Lemma 7.3.

(4) By Proposition 5.4 we have that Assumption 4.5 (and therefore relation (4.1)) is satisfied. To derive (5.8), our idea is to use the Stein–Weiss interpolation theorem [15, Theorem 5.4.1]. To do that, we need to establish the respective end-point estimates in the cases $\theta = 0$ and $\theta = 1$. We fix $a \in [0,T)$, a set $A \in \mathcal{F}_a$ of positive measure. First we observe that by (4.1) (applied to $s = a$ and with $b \uparrow T$), Lemma 7.3 for $\theta = 0$, and Lemma 5.3,

$$
\begin{aligned}
&\frac{1}{\sqrt{\kappa}}\sqrt{\int_A\int_a^T (T-u)H_u^2 \mathrm{d}u\mathrm{d}\mathbb{P}} \\
&\leqslant \sqrt{\int_A\int_a^T |\varphi_u - \varphi_a|^2\, \sigma_u^2 \mathrm{d}u\mathrm{d}\mathbb{P}} \\
&\leqslant \sqrt{\int_A\int_a^T Z_u^2 \mathrm{d}u\mathrm{d}\mathbb{P}} + \sqrt{\int_A \varphi_a^2 \int_a^T \sigma_u^2 \mathrm{d}u\mathrm{d}\mathbb{P}} \\
&\leqslant \sqrt{\int_A g(Y_T)^2 \mathrm{d}\mathbb{P}} + \sqrt{\int_A \left[c_{(7.1)}^2 |g|_0^2 \sigma_a^{-2}(T-a)^{-1}\right]\left[c_{(5.5)}^2 (T-a)\sigma_a^2\right] \mathrm{d}\mathbb{P}} \\
&\leqslant c_0 \|g\|_{B_b(\mathcal{R}_Y)} \sqrt{\mathbb{P}(A)}.
\end{aligned}
$$

On the other hand, the end-point estimate for $\theta = 1$ follows from (7.3) as

$$
\sqrt{\int_A\int_a^T H_u^2 \mathrm{d}u\mathrm{d}\mathbb{P}} \leqslant c_{(7.3)} |g|_1 \|\sigma\|_{\mathcal{SM}_2([0,T))} \sqrt{\int_A \sigma_a^2 \mathrm{d}\mathbb{P}}.
$$

For the linear map $T : g \mapsto \left(H_u\right)_{u\in[a,T)}$ and λ_1 being the Lebesgue measure we get

$$
\left\| T : C_b^0(\mathbb{R}) \to \mathbf{L}_2([a,T)\times A, ((T-\cdot)\lambda_1)\otimes \mathbb{P}_A)\right\| \leqslant c_0\sqrt{\kappa}, \tag{7.6}
$$

$$
\left\| T : \text{Höl}_1^0(\mathbb{R}) \to \mathbf{L}_2([a,T)\times A, \lambda_1 \otimes \mathbb{P}_A)\right\| \leqslant c_1 \sqrt{\int_A \sigma_a^2 \mathrm{d}\mathbb{P}_A}, \tag{7.7}
$$

where $\mathbb{P}_A$ is the normalized restriction of $\mathbb{P}$ to A. Applying the Stein–Weiss interpolation theorem [15, Theorem 5.4.1] to (7.6) and (7.7) yields

$$
\begin{aligned}
&\left\| T : (C_b^0(\mathbb{R}), \text{Höl}_1^0(\mathbb{R}))_{\theta,2} \to \mathbf{L}_2([a,T)\times A, ((T-\cdot)^{1-\theta}\lambda_1)\otimes \mathbb{P}_A)\right\| \\
&\leqslant c_{(7.8)} \left(\int_A \sigma_a^2 \mathrm{d}\mathbb{P}_A\right)^{\frac{\theta}{2}},
\end{aligned} \tag{7.8}
$$

with $c_{(7.8)} := C(c_0\sqrt{\kappa})^{1-\theta} c_1^\theta$. In other words, we did prove

$$
\left(\int_A\int_a^T (T-u)^{1-\theta} H_u^2 \mathrm{d}u\mathrm{d}\mathbb{P}_A\right)^{\frac{1}{2}} \leqslant c_{(7.8)} \left(\int_A \sigma_a^2 \mathrm{d}\mathbb{P}_A\right)^{\frac{\theta}{2}} |g|_{\theta,2}.
$$

For $\delta \in (0,1)$ and $l \in \mathbb{Z}$ define $A_l := \{\delta^{l+1} < \sigma_a^2 \leqslant \delta^l\}$. Then

$$
\begin{aligned}
&\int_A\int_a^T (T-u)^{1-\theta} H_u^2 \mathrm{d}u\mathrm{d}\mathbb{P}_A \\
&= \sum_{\mathbb{P}(A\cap A_l)>0} \left(\int_{A\cap A_l}\int_a^T (T-u)^{1-\theta} H_u^2 \mathrm{d}u\mathrm{d}\mathbb{P}_{A\cap A_l}\right) \mathbb{P}_A(A\cap A_l) \\
&\leqslant c_{(7.8)}^2 \sum_{\mathbb{P}(A\cap A_l)>0} \left(\int_{A\cap A_l} \sigma_a^2 \mathrm{d}\mathbb{P}_{A\cap A_l}\right)^\theta \mathbb{P}_A(A\cap A_l)\; |g|_{\theta,2}^2 \\
&\leqslant c_{(7.8)}^2 \sum_{\mathbb{P}(A\cap A_l)>0} \delta^{l\theta} \mathbb{P}_A(A\cap A_l)\; |g|_{\theta,2}^2 \\
&\leqslant c_{(7.8)}^2 \delta^{-\theta} \int_A \sigma_a^{2\theta} \mathrm{d}\mathbb{P}_A\; |g|_{\theta,2}^2.
\end{aligned}
$$

As $\delta \in (0,1)$ was arbitrary, we derive

$$
\int_A\int_a^T (T-u)^{1-\theta} H_u^2 \mathrm{d}u\mathrm{d}\mathbb{P}_A \leqslant c_{(7.8)}^2 \int_A \sigma_a^{2\theta} \mathrm{d}\mathbb{P}_A\; |g|_{\theta,2}^2
$$

and by (5.7), for $0 \leqslant a \leqslant t < T$,

$$
\begin{aligned}
T^{1-\theta}\mathbb{E}^{\mathcal{F}_a}\left[\left|\mathcal{I}_t^{\frac{1-\theta}{2}} M - \mathcal{I}_a^{\frac{1-\theta}{2}} M\right|^2\right] &= \mathbb{E}^{\mathcal{F}_a}\left[\int_a^T (T-u)^{1-\theta} H_u^2 \mathrm{d}u\right] \\
&\leqslant c_{(7.8)}^2 \sigma_a^{2\theta} |g|_{\theta,2}^2 \text{ a.s.}
\end{aligned}
$$

(5) From (2) we know that $\sigma^\theta \in \mathcal{SM}_q([0,T))$ for all $q \in (0,\infty)$ which also implies that $\sup_{t\in[0,T)} \sigma_t^\theta \in \mathbf{L}_q$. For a continuous adapted process $(A_t)_{t\in[0,T)}$ with $A_0 \equiv 0$ we know from Proposition 8.3 (2) and Theorem 8.4 (1) that

$$\begin{aligned}\|A\|_{\mathrm{bmo}_p^{\sigma^\theta}[0,T)} &= |A|_{\mathrm{bmo}_p^{\sigma^\theta}[0,T)} = |A|_{\mathrm{BMO}_p^{\sigma^\theta}[0,T)} \\ &\sim_c |A|_{\mathrm{BMO}_2^{\sigma^\theta}[0,T)} = \|A\|_{\mathrm{bmo}_2^{\sigma^\theta}[0,T)}\end{aligned}$$

where $c > 0$ depends at most on $(p, \|\sigma^\theta\|_{\mathcal{SM}_2([0,T))}, \|\sigma^\theta\|_{\mathcal{SM}_p([0,T))})$. The a priori estimate (C_Φ) holds because $g \in \mathrm{H\ddot{o}l}_{\theta,2}(\mathbb{R}) \subseteq \mathrm{H\ddot{o}l}_{\theta,\infty}(\mathbb{R})$ and Lemma 7.3. So the statement follows from (4) and Theorem 5.6. $\square$

7.4. Proof of Corollary 1.2

By Theorem 5.7 we get some $c_1 = c_1(\sigma, T) > 0$ such that

$$\|E(g^{(1)};\tau)\|_{\mathrm{bmo}_2^\sigma[0,T)} \leqslant c_1 |g^{(1)}|_1 \sqrt{\|\tau\|_1} \quad \text{for} \quad \tau \in \mathcal{T}.$$

Combining Theorem 5.8 with Theorem 5.9 we get

$$\begin{aligned}\|E(g^{(\theta)};\tau)\|_{\mathrm{bmo}_2^{\Phi(\tau,\theta)}[0,T)} &\leqslant \sqrt{c_{(4.3)}\left[4c_{(5.8)}^2 |g^{(\theta)}|_{\theta,2}^2 + 2c_{(7.1)}^2 |g^{(\theta)}|_\theta^2\right]}\sqrt{\|\tau\|_\theta} \\ &\leqslant c_\theta |g^{(\theta)}|_{\theta,2}\sqrt{\|\tau\|_\theta}\end{aligned}$$

with $c_\theta = c_\theta(T,\sigma) > 0$ where we exploit (2.3). Finally, $\|\tau\|_1 \leqslant T^{1-\theta}\|\tau\|_\theta$, $\Phi = |g^{(\theta)}|_{\theta,2}\Phi(\tau,\theta) + |g^{(1)}|_1\sigma$ and $c := c_\theta \vee (T^{\frac{1-\theta}{2}} c_1)$ give

$$\|E(g;\tau)\|_{\mathrm{bmo}_2^\Phi[0,T)} \leqslant c\sqrt{\|\tau\|_\theta}.$$

Now we check the *moreover*-part: For an η-Hölder function g with $\eta \in (\theta,1)$ we apply Remark 2.1 with inequality (8.7) and get a decomposition $g = g^{(\theta)} + g^{(1)}$ with

$$|g^{(\theta)}|_{\theta,2} + |g^{(1)}|_1 \leqslant c_\theta |g^{(\theta)}|_{\theta,1} + |g^{(1)}|_1 \leqslant c_\theta c_{(8.7)} |g|_{\eta,\infty}$$

for some $c_\theta \geqslant 1$. This implies that

$$\Phi = |g^{(\theta)}|_{\theta,2}\Phi(\tau,\theta) + |g^{(1)}|_1\sigma \leqslant c_\theta c_{(8.7)}|g|_{\eta,\infty}[\Phi(\tau,\theta) + \sigma]$$

which proves this remaining part. $\square$

7.5. Proof of Theorem 5.11

We fix $\tau = \{t_i\}_{i=0}^n \in \mathcal{T}$ and will proceed in four steps.

(a) We choose $r \in (0,p)$ such that $q = \frac{p}{p-r}$ so that (2.7) implies

$$\|R\|_{\mathrm{bmo}_r^{\Phi,Q}[0,T)} \leqslant \sqrt[r]{\|\mathrm{d}Q/\mathrm{d}\mathbb{P}\|_{\mathrm{RH}_q(\mathbb{P})}}\|R\|_{\mathrm{bmo}_p^\Phi[0,T)} =: \rho < \infty.$$

For $a \in [0,T)$ and $F \in \mathcal{F}_a$ with $Q(F) > 0$ this implies

$$\left\|\left(\frac{R_t - R_a}{\Phi_a}\mathbb{1}_F\right)_{t\in[a,T)}\right\|_{\mathrm{bmo}_r^{\left(\frac{\Phi_t}{\Phi_a}\mathbb{1}_F\right)_{t\in[a,T)},Q}[a,T)} \leqslant \rho.$$

From Proposition 8.3 (2), the continuity of the process R, Proposition 8.3 (3), and Theorem 8.4 (2) we deduce for $\mu \geqslant 1$ and $\nu > 0$ that

$$Q_F\left(\sup_{t\in[a,T)}\left|\frac{R_t - R_a}{\Phi_a}\right| > b\rho\mu\nu\right) \leqslant \mathrm{e}^{1-\mu} + \beta Q_F\left(\sup_{t\in[a,T)}\frac{\Phi_t}{\Phi_a} > \nu\right) \tag{7.9}$$

where $b, \beta > 0$ depend at most on r.

(b) Let $0 \leqslant a \leqslant t < T$, $a \in [t_{k-1}, t_k)$, and $t \in [t_{j-1}, t_j)$. If $t_{j-1} = t_{k-1}$, then we get

$$\begin{aligned}\frac{\Phi_t}{\Phi_a} &= \frac{A\Phi_t(\tau,\theta) + B\sigma_t}{A\Phi_a(\tau,\theta) + B\sigma_a} \leqslant \frac{\sigma_t}{\sigma_a} \vee \frac{\Phi_t(\tau,\theta)}{\Phi_a(\tau,\theta)} \\ &= \frac{\sigma_t}{\sigma_a} \vee \frac{\sigma_t^\theta + \sigma_{t_{j-1}}^{\theta-1}\sigma_t}{\sigma_a^\theta + \sigma_{t_{k-1}}^{\theta-1}\sigma_a} \leqslant \frac{\sigma_t}{\sigma_a} \vee \frac{\sigma_t^\theta}{\sigma_a^\theta} \leqslant \sup_{u\in[a,t]}\left\{\left(\frac{\sigma_u}{\sigma_a}\right)^{\theta-1}\frac{\sigma_t}{\sigma_a}\right\}.\end{aligned}$$

Similarly, if $t_{j-1} > t_{k-1}$, then

$$\frac{\Phi_t}{\Phi_a} \leqslant \frac{\sigma_t}{\sigma_a} \vee \frac{\Phi_t(\tau,\theta)}{\Phi_a(\tau,\theta)} \leqslant \frac{\sigma_t}{\sigma_a} \vee \frac{\sigma_t^\theta + \sigma_{t_{j-1}}^{\theta-1}\sigma_t}{\sigma_a^\theta} \leqslant \sup_{u\in[a,t]}\left\{2\left(\frac{\sigma_u}{\sigma_a}\right)^{\theta-1}\frac{\sigma_t}{\sigma_a}\right\}.$$

Using $\sigma(y) \sim_{c_\sigma} 1$ for (C1) and $\sigma(y) \sim_{c_\sigma} y$ for (C2) we bound the last term with

$$2\left(\frac{\sigma_u}{\sigma_a}\right)^{\theta-1}\frac{\sigma_t}{\sigma_a} \leqslant 2c_\sigma^2 \begin{cases} 1 & (C1) \\ \sup_{u\in[a,t]}\left\{\left(\frac{Y_u}{Y_a}\right)^{\theta-1}\frac{Y_t}{Y_a}\right\} & (C2), \end{cases}$$

where the factor c_σ^2, and not a higher order, comes from a cancellation.

(c) Case (C1): If we set $v_\sigma := 2c_\sigma^2$, then (7.9) implies for $\mu \geqslant 1$ that

$$Q_F\left(\sup_{t\in[a,T)}\left|\frac{R_t - R_a}{\Phi_a}\right| > \left[b\sqrt[r]{\|\mathrm{d}Q/\mathrm{d}\mathbb{P}\|_{\mathrm{RH}_q(\mathbb{P})}}\, v_\sigma\right]\|R\|_{\mathrm{bmo}_p^\Phi[0,T)}\mu\right) \leqslant \mathrm{e}^{1-\mu}$$

which can be taken to the form used in Theorem 5.11.

(d) Case (C2): By assumption we have $\mathrm{d}Q = \mathrm{e}^{\int_{(0,T]}\xi_u \mathrm{d}W_u - \frac{1}{2}\int_0^T|\xi_u|^2\mathrm{d}u}\,\mathrm{d}\mathbb{P}$ with $\left\|\int_0^T|\xi_u|^2\mathrm{d}u\right\|_{\mathbf{L}_\infty(\mathbb{P})} < \infty$. Therefore, $\widetilde{W}_t := W_t - \int_0^t \xi_u \mathrm{d}u$ is a Q-Brownian motion by Girsanov's theorem. Using (b), $Y = \mathrm{e}^X$, and $\left\|\int_0^T|\xi_u|\mathrm{d}u\right\|_{\mathbf{L}_\infty(\mathbb{P})} \leqslant \sqrt{T}\left\|\int_0^T|\xi_u|^2\mathrm{d}u\right\|_{\mathbf{L}_\infty(\mathbb{P})}^{\frac{1}{2}}$ we get that, a.s.,

$$\begin{aligned}\sup_{t\in[a,T)}\frac{\Phi_t}{\Phi_a} &\leqslant c\,\mathrm{e}^{2\sup_{t\in[a,T]}\left|\int_{(a,t]}\hat{\sigma}(X_u)\mathrm{d}W_u\right|} \\ &\leqslant \left[c\,\mathrm{e}^{2\sqrt{T}\left\|\int_0^T|\xi_u|^2\mathrm{d}u\right\|_{\mathbf{L}_\infty(\mathbb{P})}^{\frac{1}{2}}\|\hat{\sigma}\|_{B_b(\mathbb{R})}}\right]\mathrm{e}^{2\sup_{t\in[a,T]}\left|\int_{(a,t]}\hat{\sigma}(X_u)\mathrm{d}\widetilde{W}_u\right|}\end{aligned}$$

with $c = c(T,\sigma) > 0$ independent of θ. With $\mu = \nu = \sqrt{\lambda}$ one can follow the arguments of [30, Lemma 6.2(ii)] to deduce from (7.9) the desired bound. □

7.6. Proof of Theorem 5.12

For $\varepsilon > 0$ we have $g, g_\varepsilon : \mathbb{R} \to [0,1]$ with $0 \leqslant g_\varepsilon \leqslant g$ and $g(y) = \int_{[0,\infty)} \mathbb{1}_{[K,\infty)}(y)\mu(\mathrm{d}K)$, *i.e.* g is an average over functions of type $\mathbb{1}_{[K,\infty)}(y)$ for $K \geqslant 0$. So g_ε is the corresponding average over $(\mathbb{1}_{[K,\infty)})_\varepsilon$. As $|(\mathbb{1}_{[K,\infty)})_\varepsilon|_1 = \frac{1}{\varepsilon}$, we get

$$|g_\varepsilon|_1 \leqslant \frac{1}{\varepsilon}.$$

Drawing $(\mathbb{1}_{[K,\infty)})_\varepsilon$ and $(\mathbb{1}_{[K,\infty)})_{t+\varepsilon}$ for $t > 0$, we see that

$$\|(\mathbb{1}_{[K,\infty)})_\varepsilon - (\mathbb{1}_{[K,\infty)})_{t+\varepsilon}\|_{C_b^0(\mathbb{R})} = 1 - \frac{\varepsilon}{t+\varepsilon},$$

whence $\|g_\varepsilon - g_{t+\varepsilon}\|_{C_b^0(\mathbb{R})} \leqslant 1 - \frac{\varepsilon}{t+\varepsilon}$. Next, we observe $\int_\mathbb{R}(g(y) - g_\varepsilon(y))\mathrm{d}y = \frac{\varepsilon}{2}$. If $p_T : \mathbb{R} \to [0,\infty)$ is the continuous density of the law of Y_T (see Theorem 8.5), then this implies

$$|\mathbb{E}g(Y_T) - \mathbb{E}g_\varepsilon(Y_T)| \leqslant \frac{\|p_T\|_{B_b(\mathbb{R})}}{2}\varepsilon.$$

Now we are in a position to verify

$$K(t, g_\varepsilon; C_b^0(\mathbb{R}), \mathrm{H\ddot{o}l}_1^0(\mathbb{R})) \leqslant \frac{2t}{t+\varepsilon} \quad \text{for} \quad t > 0. \tag{7.10}$$

This follows from the decomposition $g_\varepsilon = (g_\varepsilon - g_{t+\varepsilon}) + g_{t+\varepsilon}$ with $|g_{t+\varepsilon}|_1 \leqslant \frac{1}{t+\varepsilon}$ and $\|g_\varepsilon - g_{t+\varepsilon}\|_{C_b^0(\mathbb{R})} \leqslant 1 - \frac{\varepsilon}{t+\varepsilon} = \frac{t}{t+\varepsilon}$ which yields to (7.10). We deduce that

$$|g_\varepsilon|_{\theta,2} = \left(\int_0^\infty |t^{-\theta}K(t, g_\varepsilon; C_b^0(\mathbb{R}), \mathrm{H\ddot{o}l}_1^0(\mathbb{R}))|^2\frac{\mathrm{d}t}{t}\right)^{\frac{1}{2}} \leqslant c_\theta\varepsilon^{-\theta}. \tag{7.11}$$

Finally, with $\varepsilon := \frac{2D^{-\frac{1}{\theta}}}{\|p_T\|_{B_b(\mathbb{R})}}$ this implies,

$$|\mathbb{E}g(Y_T) - \mathbb{E}g_\varepsilon(Y_T)| \leqslant D^{-\frac{1}{\theta}}, \quad |g_\varepsilon|_{\theta,2} \leqslant \frac{c_\theta\|p_T\|_{B_b(\mathbb{R})}^\theta}{2^\theta}D, \quad |g_\varepsilon|_1 \leqslant \frac{\|p_T\|_{B_b(\mathbb{R})}}{2}D^{\frac{1}{\theta}}.$$

The first inequality is item (1). Using the second inequality, Corollary 1.2 ($g^{(1)} := 0$ and $g^{(\theta)} := g_\varepsilon$), and (2.9), we derive item (2), where an upper bound for $\|p_T\|_{B_b(\mathbb{R})}$ can be found in Theorem 8.5 (2). The third inequality and Lemma 7.3 with $\theta = 1$ imply item (3). □

7.7. Proof of Corollary 5.13

In Theorem 5.12 we choose $D_n := n^\delta$ so that $\frac{D_n}{\sqrt{n}} = D_n^{-\frac{1}{\theta}} = n^{\delta-\frac{1}{2}}$, and $\varepsilon_n := \frac{2D_n^{-\frac{1}{\theta}}}{\|p_T\|_{B_b(\mathbb{R})}} = \frac{2}{\|p_T\|_{B_b(\mathbb{R})}} n^{-\frac{\delta}{\theta}}$. We deduce $|\varphi^{\varepsilon_n}| \leqslant c_{5.12} D_n^{\frac{1}{\theta}} = c_{5.12} n^{\frac{1}{2}-\delta}$ and

$$\|E(g_{\varepsilon_n}; \tau_n^\theta)\|_{\mathrm{bmo}_2^\Phi[0,T)} + |\mathbb{E}g(Y_T) - \mathbb{E}g_\varepsilon(Y_T)| \leqslant c_{5.12} \frac{D_n}{\sqrt{n}} + D_n^{-\frac{1}{\theta}}$$
$$= \left(c_{5.12} + 1\right) n^{\delta-\frac{1}{2}}. \quad \square$$

7.8. Proof of Theorem 1.3 and Corollary 1.4

If $g(y) = \mathbb{1}_{[K,\infty)}(y)$ for $K > 0$, then Theorem 5.12 and Corollary 5.13 remain true with the term $\mathbb{P}(g_{\tilde\varepsilon}(Y_T) < \mathbb{1}_{[K,\infty)}(Y_T))/2$ instead of $\mathbb{E}\mathbb{1}_{[K,\infty)}(Y_T) - \mathbb{E}g_{\tilde\varepsilon}(Y_T)$, $\tilde\varepsilon \in \{\varepsilon, \varepsilon_n\}$. In fact, inspecting the proof of Theorem 5.12, we have $\mathbb{P}(g_\varepsilon(Y_T) < \mathbb{1}_{[K,\infty)}(Y_T)) \leqslant \|p_T\|_{B_b(\mathbb{R})}\varepsilon$ which implies this change for Theorem 5.12 and consequently for Corollary 5.13. By this observation Theorem 1.3 and Corollary 1.4 (1) follow. To check Corollary 1.4 (2), we note that Corollary 5.13 implies

$$\|E(g_{\varepsilon_n}; \tau_n^\theta)\|_{\mathrm{bmo}_2^\Phi[0,T)} \leqslant \left(c_{5.12} + 1\right) n^{\delta-\frac{1}{2}},$$

so that we may finish with Theorem 5.11. $\square$

7.9. Proof of Proposition 1.1

(a) We find an $\varepsilon_0 \in (0,T]$ and for all $\varepsilon \in (0,\varepsilon_0]$ an $\eta(\varepsilon) \in (0,1)$ such that

$$\min\left\{\mathbb{P}(yY_\varepsilon \geqslant 1), \mathbb{P}(yY_\varepsilon < 1)\right\} \geqslant \frac{5}{12}$$

for $\varepsilon \in (0,\varepsilon_0]$ and $y > 0$ with $|y-1| \leqslant \eta(\varepsilon)$. In fact, the condition is equivalent to

$$\min\left\{\mathbb{P}\left(W_1 \geqslant \frac{\sqrt{\varepsilon}}{2} + \frac{1}{\sqrt{\varepsilon}}\ln\frac{1}{y}\right), \mathbb{P}\left(W_1 < \frac{\sqrt{\varepsilon}}{2} + \frac{1}{\sqrt{\varepsilon}}\ln\frac{1}{y}\right)\right\} \geqslant \frac{5}{12}. \tag{7.12}$$

Now, choosing first $\varepsilon \in (0,T]$ small enough to bound $\frac{\sqrt{\varepsilon}}{2}$ and then $\eta(\varepsilon) \in (0,1)$ to bound $\frac{1}{\sqrt{\varepsilon}}\ln\frac{1}{y}$ we can arrange (7.12).

(b) Define $I_0 := [-1/4, 1/4]$, $I_1 := [3/4, 5/4]$, and $J := (1/4, 3/4)$. For $A, B \in \mathbb{R}$ the density of $Z := AY_\varepsilon + B$ is continuous and has exactly two monotonicity intervals when $A \neq 0$, otherwise Z is a constant. For this reason $\mathbb{P}(Z \in J) \geqslant \min\{\mathbb{P}(Z \in I_0), \mathbb{P}(Z \in I_1)\}$, which implies that there is an $i_0 \in \{0,1\}$ such that $\mathbb{P}(Z \in I_{i_0}) \leqslant 1/3$. Let $y > 0$. The random variable $\mathbb{1}_{[1,\infty)}(yY_\varepsilon)$ only takes the values 0 and 1, each of them with a probability larger than or equal to $5/12$. But $Z \in I_{i_0}$ holds only with probability less than or equal to $1/3$. Because the distance of $I_{i_0}^c$ to i_0 equals $1/4$, this implies

$$\mathbb{E}|\mathbb{1}_{[1,\infty)}(yY_\varepsilon) - Z|^2 \geqslant \frac{1}{4^2}\left(\frac{5}{12} - \frac{1}{3}\right) = \frac{1}{192}.$$

(c) Now let $T_0 := T - \varepsilon_0 \in (0,T]$ and define for $a \in [T_0, T)$ the set $B_a := \{|Y_a - 1| \leqslant \eta(T-a)\} \in \mathcal{F}_a$. We get from (a) and (b) that

$$\int_{B_a} \left|\mathbb{1}_{[1,\infty)}(Y_T) - [v_a + w_a(Y_T - Y_a)]\right|^2 \frac{d\mathbb{P}}{\mathbb{P}(B_a)}$$
$$= \int_{B_a} \left|\mathbb{1}_{[1,\infty)}\left(Y_a \frac{Y_T}{Y_a}\right) - \left[(v_a - w_a Y_a) + (w_a Y_a)\frac{Y_T}{Y_a}\right]\right|^2 \frac{d\mathbb{P}}{\mathbb{P}(B_a)} \geqslant \frac{1}{192}.$$

8. Auxiliary results

8.1. The class $\mathcal{SM}_p$ and BMO-spaces

We provide some facts about the class $\mathcal{SM}_p$ and the BMO-spaces that are required in this article, in particular to apply the results from [14]. We assume a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ with $T \in (0,\infty)$ such that $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, $\mathcal{F}_0$ contains all null-sets, and such that $\mathcal{F}_t = \bigcap_{s\in(t,T]} \mathcal{F}_s$ for all $t \in [0,T)$. Note that $\mathcal{F}_0$ is not necessarily generated by the null-sets only.

For this reason we add to Definition 2.3 in the definition of $\Phi \in \mathcal{SM}_p([0,T))$ that $\Phi_0 \in \mathbf{L}_p$.

Moreover, we again use $\inf \emptyset := \infty$. We start with some structural properties of the class $\mathcal{SM}_p([0,T))$:

Proposition 8.1. *For $0 < p, p_0, p_1 < \infty$ with $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$ and $\mathbb{I} = [0,T)$ the following holds:*

(1) $\mathcal{SM}_q(\mathbb{I}) \subseteq \mathcal{SM}_p(\mathbb{I})$ and $\|\Phi\|_{\mathcal{SM}_p(\mathbb{I})} \leqslant \|\Phi\|_{\mathcal{SM}_q(\mathbb{I})}$ whenever $0 < p < q < \infty$.

(2) If $\Phi \in \mathcal{SM}_p(\mathbb{I})$, then $\Phi^* \in \mathcal{SM}_p(\mathbb{I})$ and $\|\Phi^*\|_{\mathcal{SM}_p(\mathbb{I})} \leqslant \sqrt[p]{1+\|\Phi\|^p_{\mathcal{SM}_p(\mathbb{I})}}$.

(3) For $\Phi^i \in \mathcal{SM}_{p_i}(\mathbb{I})$, $i=0,1$, and $\Phi = (\Phi_a)_{a\in[0,T)}$ with $\Phi_a := \Phi^0_a\Phi^1_a$, one has

$$\|\Phi\|_{\mathcal{SM}_p(\mathbb{I})} \leqslant \|\Phi^0\|_{\mathcal{SM}_{p_0}(\mathbb{I})}\|\Phi^1\|_{\mathcal{SM}_{p_1}(\mathbb{I})}.$$

Proof. (1) follows from the definition. Now let $a \in \mathbb{I}$. To check (2) we observe $\Phi^*_0 = \Phi_0 \in \mathbf{L}_p$ and, a.s.,

$$\mathbb{E}^{\mathcal{F}_a}\left[\sup_{a\leqslant t\in\mathbb{I}} |\Phi^*_t|^p\right] = \mathbb{E}^{\mathcal{F}_a}\left[\sup_{t\in\mathbb{I}} \Phi^p_t\right] \leqslant |\Phi^*_a|^p + \|\Phi\|^p_{\mathcal{SM}_p(\mathbb{I})}\Phi^p_a \leqslant (1+\|\Phi\|^p_{\mathcal{SM}_p(\mathbb{I})})|\Phi^*_a|^p.$$

(3) We get $\Phi^0_0\Phi^1_0 \in \mathbf{L}_p$ and by the conditional Hölder inequality that, a.s.,

$$\begin{aligned}
&\sqrt[p]{\mathbb{E}^{\mathcal{F}_a}\left[\sup_{a\leqslant t\in\mathbb{I}} \Phi^p_t\right]}\\
&\leqslant \sqrt[p]{\mathbb{E}^{\mathcal{F}_a}\left[\sup_{a\leqslant t\in\mathbb{I}} (\Phi^0_t)^p \sup_{a\leqslant t\in\mathbb{I}} (\Phi^1_t)^p\right]} \leqslant \sqrt[p_0]{\mathbb{E}^{\mathcal{F}_a}\left[\sup_{a\leqslant t\in\mathbb{I}} (\Phi^0_t)^{p_0}\right]}\sqrt[p_1]{\mathbb{E}^{\mathcal{F}_a}\left[\sup_{a\leqslant t\in\mathbb{I}} (\Phi^1_t)^{p_1}\right]}\\
&\leqslant \|\Phi^0\|_{\mathcal{SM}_{p_0}(\mathbb{I})}\|\Phi^1\|_{\mathcal{SM}_{p_1}(\mathbb{I})}\Phi^0_a\Phi^1_a = \|\Phi^0\|_{\mathcal{SM}_{p_0}(\mathbb{I})}\|\Phi^1\|_{\mathcal{SM}_{p_1}(\mathbb{I})}\Phi_a. \quad \square
\end{aligned}$$

Next we reformulate and extend definitions made so far to be in accordance with [14]:

Definition 8.2. Let $p \in (0,\infty)$, $\mathbb{I} = [0,T)$ or $\mathbb{I} = [0,T]$, and $\Phi \in \mathrm{CL}^+(\mathbb{I})$.

(1) For $Y \in \mathrm{CL}_0(\mathbb{I})$ we let $|Y|_{\mathrm{BMO}^\Phi_p(\mathbb{I})} := \inf c$, where the infimum is taken over all $c \in [0,\infty)$ such that, for all $t \in \mathbb{I}$ and all stopping times $\rho : \Omega \to [0,t]$,

$$\mathbb{E}^{\mathcal{F}_\rho}\left[|Y_t - Y_{\rho-}|^p\right] \leqslant c^p\Phi^p_\rho \text{ a.s.}$$

(2) For $\mathbb{I} = [0,T]$ and $Y \in \mathrm{CL}_0(\mathbb{I})$ we let $|Y|_{\overline{\mathrm{BMO}}^\Phi_p([0,T])} := \inf c$, where the infimum is taken over all $c \in [0,\infty)$ such that, for all stopping times $\rho : \Omega \to [0,T]$,

$$\mathbb{E}^{\mathcal{F}_\rho}\left[|Y_T - Y_{\rho-}|^p\right] \leqslant c^p\Phi^p_\rho \text{ a.s.}$$

(3) For $Y \in \mathrm{CL}_0(\mathbb{I})$ we let $|Y|_{\mathrm{bmo}^\Phi_p(\mathbb{I})} := \inf c$, where the infimum is taken over all $c \in [0,\infty)$ such that, for all $t \in \mathbb{I}$ and all stopping times $\rho : \Omega \to [0,t]$,

$$\mathbb{E}^{\mathcal{F}_\rho}\left[|Y_t - Y_\rho|^p\right] \leqslant c^p\Phi^p_\rho \quad \text{a.s.}$$

(4) If $\Phi_0 \in \mathbf{L}_p$, then we let $|\Phi|_{\mathcal{SM}_p(\mathbb{I})} := \inf c$, where the infimum is taken over all $c \in [1,\infty)$ such that for all stopping times $\rho : \Omega \to \mathbb{I}$ one has

$$\mathbb{E}^{\mathcal{F}_\rho}\left[\sup_{\rho\leqslant t\in\mathbb{I}} \Phi^p_t\right] \leqslant c^p\Phi^p_\rho \quad \text{a.s.}$$

To be in accordance with the above definition we use, for example, the notation $\mathrm{bmo}^\Phi_p([0,T))$ instead of $\mathrm{bmo}^\Phi_p[0,T)$ as we did before. In [14] the definitions $|\cdot|_{\overline{\mathrm{BMO}}^\Phi_p([0,T])}$ and $|\cdot|_{\mathcal{SM}_p([0,T])}$ has been used. We verify that these variants are consistent with the definitions we already introduced:

Proposition 8.3. *Let $p \in (0,\infty)$, $Y \in \mathrm{CL}_0([0,T))$ and $\Phi \in \mathrm{CL}^+([0,T))$.*

(1) If $\Phi_0 \in \mathbf{L}_p$, then one has $|\Phi|_{\mathcal{SM}_p([0,T))} = \|\Phi\|_{\mathcal{SM}_p([0,T))}$.

(2) One has $|Y|_{\mathrm{bmo}^\Phi_p([0,T))} = \|Y\|_{\mathrm{bmo}^\Phi_p([0,T))}$.

(3) One has

$$|Y|_{\overline{\mathrm{BMO}}^\Phi_p([0,T])} \leqslant |Y|_{\mathrm{BMO}^\Phi_p([0,T])} \leqslant 2^{(\frac{1}{p}-1)^+}[1+|\Phi|_{\mathcal{SM}_p([0,T])}]|Y|_{\overline{\mathrm{BMO}}^\Phi_p([0,T])},$$

where $|\Phi|_{\mathcal{SM}_p([0,T])} < \infty$ is assumed for the second inequality.

Proof. (1) Because $\|\Phi\|_{\mathcal{SM}_p([0,T))} \leqslant |\Phi|_{\mathcal{SM}_p([0,T))}$ is evident we assume that $c := \|\Phi\|_{\mathcal{SM}_p([0,T))} < \infty$. Let $\rho : \Omega \to [0,T)$ be a stopping time, $h : [0,T) \to [0,\infty)$ be given by $h(t) := \frac{1}{T-t} - \frac{1}{T}$. For $k, N \in \mathbb{N}_0$ set

$$[a^N_k, b^N_k) := h^{-1}\left(\left[\frac{k}{2^N}, \frac{k+1}{2^N}\right)\right) \subseteq [0,T) \text{ and let } H^N(t) := \sum_{k=0}^\infty \mathbb{1}_{[a^N_k,b^N_k)}(t)b^N_k.$$

Then $H^N(t) \downarrow t$ for all $t \in [0,T)$ and $\rho^N := H^N(\rho) : \Omega \to [0,T)$ is a stopping time as well. Then, a.s.,

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_{\rho^N}}\left[\sup_{\rho^N \leqslant t<T} \Phi_t^p\right] \\
&= \sum_{k=0}^{\infty} \mathbb{E}^{\mathcal{F}_{\rho^N}}\left[\mathbb{1}_{\{\rho^N=b_k^N\}} \sup_{b_k^N \leqslant t<T} \Phi_t^p\right] = \sum_{k=0}^{\infty} \mathbb{1}_{\{\rho^N=b_k^N\}} \mathbb{E}^{\mathcal{F}_{b_k^N}}\left[\mathbb{1}_{\{\rho^N=b_k^N\}} \sup_{b_k^N \leqslant t<T} \Phi_t^p\right] \\
&\leqslant \sum_{k=0}^{\infty} \mathbb{1}_{\{\rho^N=b_k^N\}} c^p \Phi_{b_k^N}^p \leqslant c^p \Phi_{\rho^N}^p .
\end{aligned}$$

This implies that $\mathbb{E}^{\mathcal{F}_\rho}\left[\sup_{\rho^N \leqslant t<T} \Phi_t^p\right] \leqslant c^p \mathbb{E}^{\mathcal{F}_\rho}\left[\Phi_{\rho^N}^p\right]$ a.s. By $N \to \infty$, monotone convergence on the left-hand side and because Φ is càdlàg, and dominated convergence on the right-hand side (Φ is càdlàg and $\mathbb{E}\sup_{t\in[0,T)} \Phi_t^p < \infty$) we obtain the assertion.

(2) Because $\|Y\|_{\mathrm{bmo}_p^\Phi([0,T))} \leqslant |Y|_{\mathrm{bmo}_p^\Phi([0,T))}$ we assume $c := \|Y\|_{\mathrm{bmo}_p^\Phi([0,T))} < \infty$. For $t \in [0,T)$, a stopping time $\rho : \Omega \to [0,t]$, and $L \in \mathbb{N}_0$ we define the new stopping times $\rho_L(\omega) := \psi_L(\rho(\omega))$ where $\psi_L(0) := 0$ and $\psi_L(s) = s_\ell^L := \ell 2^{-L} t$ when $s \in \left(s_{\ell-1}^L, s_\ell^L\right]$ for $\ell \in \{1,\dots,2^L\}$. By definition, $\rho_L(\omega) \downarrow \rho(\omega)$ for all $\omega \in \Omega$ as $L \to \infty$. Then $\mathbb{E}^{\mathcal{F}_{s_\ell^L}}\left[|Y_t - Y_{s_\ell^L}|^p\right] \leqslant c^p \Phi_{s_\ell^L}^p$ a.s. for $\ell = 0,\dots,2^L$. Multiplying both sides with $\mathbb{1}_{\{\rho_L = s_\ell^L\}}$ and summing over $\ell = 0,\dots,2^L$, we get that

$$\mathbb{E}^{\mathcal{F}_{\rho_L}}\left[|Y_t - Y_{\rho_L}|^p\right] \leqslant c^p \Phi_{\rho_L}^p \text{ a.s.}$$

For any $M > 0$ this implies $\mathbb{E}^{\mathcal{F}_{\rho_L}}\left[|Y_t - Y_{\rho_L}|^p \wedge M\right] \leqslant (c^p \Phi_{\rho_L}^p) \wedge M$ a.s. and

$$\mathbb{E}^{\mathcal{F}_\rho}\left[|Y_t - Y_{\rho_L}|^p \wedge M\right] \leqslant \mathbb{E}^{\mathcal{F}_\rho}\left[(c^p \Phi_{\rho_L}^p) \wedge M\right] \text{ a.s.}$$

The càdlàg properties of Y and Φ imply

$$\mathbb{E}^{\mathcal{F}_\rho}\left[|Y_t - Y_\rho|^p \wedge M\right] \leqslant \mathbb{E}^{\mathcal{F}_\rho}\left[(c^p \Phi_\rho^p) \wedge M\right] \text{ a.s.}$$

By $M \uparrow \infty$ it follows that $|Y|_{\mathrm{bmo}_p^\Phi([0,T))} \leqslant c$ as desired.

(3) The left-hand side inequality is obvious. To check the other inequality we may assume that $c := |Y|_{\overline{\mathrm{BMO}}_p^\Phi([0,T])} < \infty$. We let $t \in [0,T]$ and $\rho : \Omega \to [0,t]$ be a stopping time. Then, a.s.,

$$\begin{aligned}
\left(\mathbb{E}^{\mathcal{F}_\rho}\left[|Y_t - Y_{\rho-}|^p\right]\right)^{\frac{1}{p}} &\leqslant 2^{(\frac{1}{p}-1)^+}\left[\left(\mathbb{E}^{\mathcal{F}_\rho}\left[|Y_T - Y_{\rho-}|^p\right]\right)^{\frac{1}{p}} + \left(\mathbb{E}^{\mathcal{F}_\rho}\left[|Y_T - Y_t|^p\right]\right)^{\frac{1}{p}}\right] \\
&\leqslant 2^{(\frac{1}{p}-1)^+}\left[c\Phi_\rho + \left(\mathbb{E}^{\mathcal{F}_\rho}\left[|Y_T - Y_t|^p\right]\right)^{\frac{1}{p}}\right].
\end{aligned}$$

To estimate the second term we may assume $t \in [0,T)$. We find a sequence $t_n \in (t,T]$ with $t_n \downarrow t$. Using Fatou's Lemma for conditional expectations we get, a.s.,

$$\begin{aligned}
&\left(\mathbb{E}^{\mathcal{F}_\rho}\left[|Y_T - Y_t|^p\right]\right)^{\frac{1}{p}} \leqslant \liminf_n \left(\mathbb{E}^{\mathcal{F}_\rho}\left[|Y_T - Y_{t_n-}|^p\right]\right)^{\frac{1}{p}} \leqslant \liminf_n c\left(\mathbb{E}^{\mathcal{F}_\rho}\left[\Phi_{t_n}^p\right]\right)^{\frac{1}{p}} \\
&\leqslant c|\Phi|_{\mathcal{SM}_p([0,T])}\Phi_\rho. \quad \square
\end{aligned}$$

Finally, we state and verify the main statement of this section:

Theorem 8.4. *Let $0 < p \leqslant q < \infty$, $r \in (0,\infty)$, $\mathbb{I} = [0,T)$ or $\mathbb{I} = [0,T]$, and $\Phi \in \mathrm{CL}^+(\mathbb{I})$.*

(1) If $\Phi \in \mathcal{SM}_q(\mathbb{I})$ with $|\Phi|_{\mathcal{SM}_q(\mathbb{I})} \leqslant d < \infty$, then there is a $c = c(p,q,d) \geqslant 1$ such that $|\cdot|_{\mathrm{BMO}_p^\Phi(\mathbb{I})} \sim_c |\cdot|_{\mathrm{BMO}_q^\Phi(\mathbb{I})}$.

(2) There are $b = b(r) > 0$, $\beta = \beta(r) > 0$, and $c_{(8.2)} = c_{(8.2)}(r,q) > 0$ such that for $Y \in \mathrm{CL}_0(\mathbb{I})$, $0 \leqslant a \leqslant t \in \mathbb{I}$, $D := |(Y_u - Y_a)_{u\in[a,t]}|_{\overline{\mathrm{BMO}}_r^\Phi([a,t])} < \infty$, $\mu \geqslant 1$, and $\nu > 0$ one has, a.s.,

$$\mathbb{P}_{\mathcal{F}_a}\left(\sup_{u\in[a,t]} |Y_u - Y_a| > bD\mu\nu\right) \leqslant \mathrm{e}^{1-\mu} + \beta\mathbb{P}_{\mathcal{F}_a}\left(\sup_{u\in[a,t]} \Phi_u > \nu\right), \tag{8.1}$$

$$\mathbb{E}^{\mathcal{F}_a}\left[\sup_{u\in[a,t]} |Y_u - Y_a|^q\right] \leqslant c_{(8.2)}^q D^q \mathbb{E}^{\mathcal{F}_a}\left[\sup_{u\in[a,t]} \Phi_u^q\right] \quad \text{if} \quad \sup_{u\in[a,t]} \Phi_u \in \mathbf{L}_q. \tag{8.2}$$

Proof (*(1)a*)**.** For $\mathbb{I} = [0,T]$ and $\Phi > 0$ on $[0,T] \times \Omega$ the result follows from [14, Corollary 1(i)], where we use Proposition 8.3 to relate the formally different BMO-definitions to each other and Proposition 8.1 (1).

((1)b) For $\mathbb{I} = [0,T)$ and $\Phi > 0$ on $[0,T) \times \Omega$ this follows from (1a) by considering the restrictions of the processes to $[0,t]$ for $t \in [0,T)$.

((1)c) For $\mathbb{I} = [0,T]$ or $\mathbb{I} = [0,T)$, and $\Phi \geqslant 0$ on $\mathbb{I} \times \Omega$ we proceed as follows: For $\varepsilon > 0$ we consider $\Phi_t^\varepsilon := \Phi_t + \varepsilon$ and observe that $|\Phi^\varepsilon|_{\mathcal{SM}_p(\mathbb{I})} \leqslant c_p |\Phi|_{\mathcal{SM}_p(\mathbb{I})}$ and $\sup_{\varepsilon>0} |\cdot|_{\mathrm{BMO}_p^{\Phi^\varepsilon}(\mathbb{I})} = |\cdot|_{\mathrm{BMO}_p^\Phi(\mathbb{I})}$.

(2) We restrict the stochastic basis to $(A, \mathcal{F}_a \cap A, \mathbb{P}_A, (\mathcal{F}_u \cap A)_{u\in[a,t]})$ with $A \in \mathcal{F}_a$ and $\mathbb{P}(A) > 0$, where $\mathbb{P}_A$ is the normalized restriction of $\mathbb{P}$ to A and $\mathcal{F}_u \cap A$ denotes the trace σ-algebra. So we can assume that $a = 0$ and can replace $\mathbb{P}_{\mathcal{F}_a}$ by $\mathbb{P}$ and $\mathbb{E}^{\mathcal{F}_a}$ by $\mathbb{E}$. Moreover, by replacing Φ_u by Φ_u^ε as above, proving the statement for the new weight, and letting $\varepsilon \downarrow 0$, we may assume that $\Phi > 0$ on $[0,t] \times \Omega$ ($\varepsilon \downarrow 0$ gives $\sup_{u\in[a,t]} \Phi_u \geqslant v$ in (8.1), by adjusting b it can be changed into $>$). Now (8.1) and (8.2) follow from [14, inequalities (5,6) and step (a) of the proof of Corollary 1]. $\square$

8.2. Transition density

The following result, that is taken from [31, p. 263, p. 44], is crucial for estimates on gradients and curvatures on the Wiener space:

Theorem 8.5. *For $\hat{b}, \hat{\sigma} \in C_b^\infty(\mathbb{R})$ with $\hat{\sigma} \geqslant \varepsilon_0 > 0$ there is a jointly continuous transition density $\Gamma_X : (0,T] \times \mathbb{R} \times \mathbb{R} \to (0,\infty)$ such that $\mathbb{P}(X_t^x \in B) = \int_B \Gamma_X(t,x,\xi)\mathrm{d}\xi$ for $t \in (0,T]$ and $B \in \mathcal{B}(\mathbb{R})$, where $(X_t^x)_{t\in[0,T]}$ is the solution to (5.1) starting in $x \in \mathbb{R}$, such that one has:*

(1) One has $\Gamma_X(s,\cdot,\xi) \in C^\infty(\mathbb{R})$ for $(s,\xi) \in (0,T] \times \mathbb{R}$.
(2) For $k \in \mathbb{N}_0$ there is a $c_{(8.3)} = c(k) > 0$ such that for $(s,x,\xi) \in (0,T] \times \mathbb{R} \times \mathbb{R}$ one has that

$$\left| \frac{\partial^k \Gamma_X}{\partial x^k}(s,x,\xi) \right| \leqslant c_{(8.3)} s^{-\frac{k}{2}} \gamma_{c_{(8.3)}s}(x-\xi) \text{ where } \gamma_t(\eta) := \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{\eta^2}{2t}}. \tag{8.3}$$

(3) For $k \in \mathbb{N}$ and $f \in C_X$ (the set C_Y from Section 5 in the case (C1)) one has

$$\frac{\partial^k}{\partial x^k} \int_{\mathbb{R}} \Gamma_X(s,x,\xi) f(\xi)\mathrm{d}\xi = \int_{\mathbb{R}} \frac{\partial^k \Gamma_X}{\partial x^k}(s,x,\xi) f(\xi)\mathrm{d}\xi \quad \text{for} \quad (s,x) \in (0,T] \times \mathbb{R}.$$

8.3. A technical lemma

Lemma 8.6. *For $\theta \in [0,1]$, a function $\varphi : [0,T) \to \mathbb{R}$, and a non-decreasing function $\Psi : [0,T) \to [0,\infty)$ the following assertions are equivalent:*

(1) There is a $c_{(8.4)} > 0$ such that for any $0 \leqslant s \leqslant a < T$ one has

$$|\varphi_a - \varphi_s| \leqslant c_{(8.4)} \frac{(T-s)^{\frac{\theta}{2}}}{(T-a)^{\frac{1}{2}}} \Psi_a. \tag{8.4}$$

(2) (a) $\theta \in [0,1)$: There is a $c_{(8.5)} > 0$ such that for $a \in [0,T)$ one has

$$|\varphi_a - \varphi_0| \leqslant c_{(8.5)} (T-a)^{\frac{\theta-1}{2}} \Psi_a. \tag{8.5}$$

(b) $\theta = 1$: There is a $c_{(8.6)} > 0$ such that for $0 \leqslant s \leqslant a < T$ one has

$$|\varphi_a - \varphi_s| \leqslant c_{(8.6)} \left(1 + \ln \frac{T-s}{T-a}\right) \Psi_a. \tag{8.6}$$

Proof. (1) $\Rightarrow$ (2) We let $t_n := T - \frac{T}{2^n}$ for $n \geqslant 0$. If $s, a \in [t_{n-1}, t_n]$, $n \geqslant 1$, then Eq. (8.4) implies

$$|\varphi_a - \varphi_s| \leqslant c_{(8.4)} \Psi_a T^{\frac{\theta}{2}-\frac{1}{2}} \frac{[1-(1-\frac{1}{2^{n-1}})]^{\frac{\theta}{2}}}{[1-(1-\frac{1}{2^n})]^{\frac{1}{2}}} \leqslant c_{(8.4)} \Psi_a T^{\frac{\theta-1}{2}} (\sqrt{2})^{1+(1-\theta)n}.$$

We now let $s \in [t_{n-1}, t_n)$ and $a \in [t_{n+m-1}, t_{n+m})$ for $n \geqslant 1$, $m \geqslant 0$ arbitrarily. If $\theta \in [0,1)$, then the triangle inequality and the monotonicity of Ψ give

$$\begin{aligned} |\varphi_a - \varphi_0| &\leqslant c_{(8.4)} \Psi_a T^{\frac{\theta-1}{2}} \sum_{k=1}^{n+m} (\sqrt{2})^{1+(1-\theta)k} \\ &\leqslant c_{(8.4)} c_\theta \Psi_a T^{\frac{\theta-1}{2}} (\sqrt{2})^{(1-\theta)(n+m-1)} \leqslant \frac{c_{(8.4)} c_\theta \Psi_a}{(T-a)^{\frac{1-\theta}{2}}} \end{aligned}$$

for some $c_\theta > 0$ depending on θ only. When $\theta = 1$, similarly as above we get

$$|\varphi_a - \varphi_s| \leqslant c_{(8.4)} \Psi_a \sqrt{2}(1+m) \leqslant 2\sqrt{2} c_{(8.4)} \Psi_a \left(1 + \ln \frac{T-s}{T-a}\right).$$

(2) $\Rightarrow$ (1) If $\theta \in [0,1)$, then Eq. (8.5) implies for any $0 \leqslant s \leqslant a < T$ that

$$\begin{aligned} |\varphi_a - \varphi_s| &\leqslant |\varphi_a - \varphi_0| + |\varphi_s - \varphi_0| \leqslant c_{(8.5)} \left[\Psi_a (T-a)^{\frac{\theta-1}{2}} + \Psi_s (T-s)^{\frac{\theta-1}{2}} \right] \\ &\leqslant c_{(8.5)} \Psi_a \left[\left(\frac{T-a}{T-s}\right)^{\frac{\theta}{2}} + \left(\frac{T-a}{T-s}\right)^{\frac{1}{2}} \right] \frac{(T-s)^{\frac{\theta}{2}}}{(T-a)^{\frac{1}{2}}} \end{aligned}$$

$$\leqslant 2c_{(8.5)}\Psi_a \frac{(T-s)^{\frac{\theta}{2}}}{(T-a)^{\frac{1}{2}}}.$$

The case $\theta = 1$ is derived from the inequality $1 + \ln x \leqslant 2\sqrt{x}$, $x \geqslant 1$. $\square$

8.4. The fine lines in the scale of Hölder spaces $\mathrm{Höl}_{\theta,q}(\mathbb{R})$

To illustrate the scale of Hölder spaces we consider the following example:

Example 8.7. Let

$$h_{\theta,a}(x) := 0 \text{ if } x < 0 \quad \text{and} \quad h_{\theta,a}(x) := \theta \int_0^{1\wedge x} y^{\theta-1}\left(\frac{A}{A-\log y}\right)^a \mathrm{d}y \text{ if } x \geqslant 0$$

for $\theta \in (0,1)$, $A > 0$, and $0 \leqslant a < (1-\theta)A$. In particular, $h_{\theta,0}(x) = (\max\{0,x\})^\theta \wedge 1$. Then we get

$$h_{\theta,0} \in \mathrm{Höl}_{\theta,\infty}(\mathbb{R}) \quad \text{and} \quad h_{\theta,a} \in \mathrm{Höl}_{\theta,q}(\mathbb{R}) \text{ for } a > 1/q \text{ and } q \in [1,\infty).$$

Proof. The case $a = 0$ is obvious as $\mathrm{Höl}_{\theta,\infty}(\mathbb{R})$ are the θ-Hölder continuous functions vanishing at zero. Let $a > 0$. As $K(v, h_{\theta,a}; C_b^0(\mathbb{R}), \mathrm{Höl}_1^0(\mathbb{R})) \leqslant h_{\theta,a}(1)$ for $v \in [1,\infty)$, we only need to check that

$$\left\| v^{-\theta} K(v, h_{\theta,a}; C_b^0(\mathbb{R}), \mathrm{Höl}_1^0(\mathbb{R}))\right\|_{\mathbf{L}_q\left((0,1],\frac{\mathrm{d}v}{v}\right)} < \infty$$

for $a > 1/q$. This follows from

$$v^{-\theta} K(v, h_{\theta,a}; C_b^0(\mathbb{R}), \mathrm{Höl}_1^0(\mathbb{R})) \leqslant (1+\theta)\left(\frac{A}{A-\log v}\right)^a \quad \text{for} \quad v \in (0,1].$$

To verify this fix $v \in (0,1]$, let $f := h_{\theta,a}$ and $K(y) := \mathbb{1}_{\{0<y\leqslant 1\}}\theta y^{\theta-1}\left(\frac{A}{A-\log y}\right)^a$. For $x \geqslant 0$ we decompose $f(x) = f_1^{(v)}(x) + f_b^{(v)}(x)$ with $f_1^{(v)}(x) := \int_0^{x\wedge 1}(K(y)\wedge K(v))\mathrm{d}y$ and $f_b^{(v)}(x) := f(x) - f_1^{(v)}(x)$ (for $x < 0$ the decomposing functions are defined to be zero). By definition we have $\|f_1^{(v)}\|_{\mathrm{Höl}_1^0(\mathbb{R})} \leqslant K(v)$. Exploiting the monotonicity of K, where we use $a < (1-\theta)A$, we also have $\|f_b^{(v)}\|_{C_b^0(\mathbb{R})} \leqslant \int_0^v K(y)\mathrm{d}y$. Finally, a computation yields $\int_0^v K(y)\mathrm{d}y \leqslant \frac{v}{\theta}K(v)$ so that

$$\begin{aligned} K(v, h_{\theta,a}; C_b^0(\mathbb{R}), \mathrm{Höl}_1^0(\mathbb{R})) &\leqslant \frac{v}{\theta}K(v) + vK(v) = \left(\frac{1}{\theta}+1\right)vK(v) \\ &= (1+\theta)\,v^\theta\left(\frac{A}{A-\log v}\right)^a. \quad \square \end{aligned}$$

8.5. Proof of relation (2.6)

By definition, for $f \in \mathrm{Höl}_{\theta_1,\infty}(\mathbb{R})$ we have

$$K(v, f; C_b^0(\mathbb{R}), \mathrm{Höl}_1^0(\mathbb{R})) \leqslant |f|_{\theta_1,\infty} v^{\theta_1} \quad \text{for} \quad v > 0.$$

Assume $c > |f|_{\theta_1,\infty}$. For $v = 1$ this gives a decomposition $f = f_0 + f_1$ with $\|f_0\|_{C_b^0(\mathbb{R})} + |f_1|_1 \leqslant c$. So it remains to verify that $f_0 \in \mathrm{Höl}_{\theta_0,1}(\mathbb{R})$. For $v \geqslant 1$ we have

$$K(v, f_0; C_b^0(\mathbb{R}), \mathrm{Höl}_1^0(\mathbb{R})) \leqslant \|f_0\|_{C_b^0(\mathbb{R})} \leqslant c,$$

and for $v \in (0,1]$ we have

$$\begin{aligned} K(v, f_0; C_b^0(\mathbb{R}), \mathrm{Höl}_1^0(\mathbb{R})) &\leqslant K(v, f; C_b^0(\mathbb{R}), \mathrm{Höl}_1^0(\mathbb{R})) + K(v, f_1; C_b^0(\mathbb{R}), \mathrm{Höl}_1^0(\mathbb{R})) \\ &\leqslant |f|_{\theta_1,\infty} v^{\theta_1} + |f_1|_1 v \leqslant (|f|_{\theta_1,\infty} + |f_1|_1)v^{\theta_1} \leqslant 2c\, v^{\theta_1}. \end{aligned}$$

Inserting this bound for $K(v, f_0; C_b^0(\mathbb{R}), \mathrm{Höl}_1^0(\mathbb{R}))$ into the definition of the interpolation space $\mathrm{Höl}_{\theta_0,1}(\mathbb{R})$, we derive $|f_0|_{\theta_0,1} + |f_1|_1 \leqslant c_{\theta_0,\theta_1} c + c$ so that

$$|f_0|_{\theta_0,1} + |f_1|_1 \leqslant (c_{\theta_0,\theta_1}+1)|f|_{\theta_1,\infty} =: c_{(8.7)}|f|_{\theta_1,\infty}. \quad \square \tag{8.7}$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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