FIXED POINTS OF G-MONOTONE MAPPINGS IN METRIC AND MODULAR SPACES

DAU HONG QUAN[1,2] AND ANDRZEJ WIŚNICKI[3]

To the memory of Professor Kazimierz Goebel

Abstract. Let C be a bounded, closed and convex subset of a reflexive metric space with a digraph G such that G-intervals along walks are closed and convex. We show in the main theorem that if $T : C \to C$ is a monotone G-nonexpansive mapping and there exists $c \in C$ such that $Tc \in [c, \to)_G$, then T has a fixed point provided for each $a \in C$, $[a, a]_G$ has the fixed point property for nonexpansive mappings. In particular, it gives a wide generalization of the Dehaish-Khamsi theorem concerning partial orders in complete uniformly convex hyperbolic metric spaces. Some counterparts of this result for modular spaces, and for commutative families of mappings are given too.

1. Introduction

In 2004, Ran and Reurings [19] initiated the investigation of an analogue to the classical Banach contraction principle in the context of a complete metric space endowed with a partial order. It was further refined by Nieto and Rodríguez-López [17]. Their fixed point theorems are generalized and extended by numerous authors. In 2008, Jachymski [8] performed a more general approach by using a graph instead of a partial order.

It is natural to study fixed point theorems for monotone nonexpansive mappings. In 2015, Alfuraidan [2] initiated the study of monotone nonexpansive mappings on a Banach space X equipped with a directed graph G. In 2016, Dehaish and Khamsi showed in [4] that if X is a partially ordered uniformly convex hyperbolic metric space, then a monotone nonexpansive map $T : C \to C$ has a fixed point, where C is a nonempty bounded, closed and convex subset of X. This is parallel to Browder-Göhde's fixed point theorem for nonexpansive mappings. Recently, Espínola and Wiśnicki [7] have proved a lemma and applied it to show a fixed point theorem for monotone mappings in a Hausdorff topological space with a partial order $\preceq$. They concluded that a lot of fixed point results related to monotone nonexpansive mappings are a particular case of their theorem. But this approach only works in the case of partially ordered sets. The situation is more difficult in the case of sets endowed with a digraph.

The aim of this paper is two-fold: firstly to present some fixed point theorems in geodesic metric spaces with the UUC property endowed with a digraph, secondly to extend the results of Abdou and Khamsi in [1] for modular spaces with the UUC2 property. The key ingredient in our generalization is the compactness of the order intervals and the fixed point property. Combining with the approach of Espínola and Wiśnicki [7], we show that a monotone G-nonexpansive mapping $T : C \to C$ has a fixed point on a bounded, closed, convex subset C (resp. ρ-bounded, ρ-closed, convex subset C) of a geodesic space with the

2010 *Mathematics Subject Classification.* Primary 47H10; Secondary: 46B20 47J26 47H09 54F05.
Key words and phrases. monotone mapping; nonexpansive mapping; fixed point.

UUC property (resp. a modular space with the UUC2 property). In particular, in the case when digraph is a partial order, Theorem 2.11 gives a wide extension of Theorem 3.1 of Dehaish and Khamsi [4] by dropping both assumptions about hyperbolicity of the space and nonexpansivity of the mapping.

We start this section by recalling some basic notions in graph theory (see [3], [6]).

Definition 1.1. A graph G is a pair $(V(G), E(G))$, where elements of the nonempty set $V(G)$ are called vertices of G, and $E(G)$ is the set of paired vertices called edges. If a direction is imposed on each edge, we call the graph a directed graph or a digraph.

Definition 1.2. Assume that $G = (V(G), E(G))$ is a digraph.

(i) G is reflexive if for each $x \in V(G)$, $(x, x) \in E(G)$.

(ii) G is transitive if for every $x, y, z \in V(G)$ with $(x, y), (y, z) \in E(G)$, we have $(x, z) \in E(G)$.

(iii) We call (V', E') a subgraph of G if $V' \subseteq V(G)$, $E' \subseteq E(G)$, and $x, y \in V'$ whenever $(x, y) \in E'$.

(iv) A (directed) walk (of length k) from x to y in G is a nonempty alternating sequence $v_0 e_0 v_1 e_1 ... e_{k-1} v_k$ of vertices and edges in G such that $v_0 = x$, $v_k = y$ and $e_i = (v_i, v_{i+1})$ for all $i < k$.
A directed path is a directed walk in which all vertices are distinct.

(v) For $a, b \in V(G)$, we define G-intervals along walks as follows:

$$[a, \rightarrow)_G = \{x \in V(G) : \text{there is a walk from } a \text{ to } x\},$$
$$(\leftarrow, b]_G = \{x \in V(G) : \text{there is a walk from } x \text{ to } b\},$$
$$[a, b]_G = [a, \rightarrow)_G \cap (\leftarrow, b]_G.$$

(vi) Let A be a subset of $V(G)$. An element $b \in V(G)$ is called an G-upper (G-lower) bound of A if $a \in (\leftarrow, b]_G$ (resp. $a \in [b, \rightarrow)_G$) for all $a \in A$. A set A is called G-bounded if $A \subset [a, b]_G$.

(vii) A subset J of $V(G)$ is directed if each finite subset of J has a G-upper bound in J.

Definition 1.3 ([2]). Let $(V(G), E(G))$ be a digraph, and A be a nonempty subset of $V(G)$. A mapping $T : A \rightarrow A$ is called G-monotone if $(Tx, Ty) \in E(G)$ for each $x, y \in A$ such that $(x, y) \in E(G)$.

Recall that a nonempty family $\mathcal{A}$ of subsets of a set X is said to satisfy the finite intersection property if the intersection over any finite subfamily of $\mathcal{A}$ is nonempty.

Lemma 1.4 ([18]). *Let $G = (V(G), E(G))$ be a digraph. Assume that any family of G-intervals along walks in $V(G)$ having the finite intersection property has nonempty intersection. If J is a directed subset of $V(G)$, then* $\bigcap_{x \in J} [x, \rightarrow)_G \neq \emptyset$.

Our basic tool is the following result proved in [18]. We sketch the proof for the convenience of the reader, thus making our exposition self-contained.

Theorem 1.5. *Let $(V(G), E(G))$ be a digraph. Assume that any family of G-intervals along walks in $V(G)$ having the finite intersection property has nonempty intersection. Let $T : V(G) \rightarrow V(G)$ be a G-monotone mapping such that $Tc \in [c, \rightarrow)_G$ for some $c \in V(G)$. Then there exists $s \in V(G)$ such that $[s, s]_G \neq \emptyset$ and $T([s, s]_G) \subset [s, s]_G$.*

Proof. Set

$$I_0 = \{c, T^n c : n \in \mathbb{N}\}.$$

It is not difficult to prove that I_0 is a directed set and for each $x \in I_0$, $Tx \in I_0$ and $Tx \in [x, \rightarrow)_G$. We note that if $\mathcal{T}$ is a chain of directed subsets of $V(G)$ containing I_0 with the above properties, then $\bigcup \mathcal{T}$ is also a directed subset with the following properties: for each $x \in \bigcup \mathcal{T}$, $Tx \in \bigcup \mathcal{T}$ and $Tx \in [x, \rightarrow)_G$. By Kuratowski-Zorn's lemma, there exists a maximal directed set $I \subset V(G)$ which contains I_0 and satisfies above properties. It follows from Lemma 1.4 that the set $K := \bigcap_{x \in I} [x, \rightarrow)_G$ is nonempty. Choose finite subsets $\{x_1, ..., x_n\}$ of I and $\{y_1, ..., y_n\}$ of K. Since I is directed, $\bigcap_{i=1}^{n} [x_i, y_i]_G$ is nonempty. It deduces that $K_0 := \bigcap_{x \in I, y \in K} [x, y]_G$ is nonempty. Thus there is $s \in K_0$. Clearly, for each $x \in I$, $s \in [x, \rightarrow)_G$, and hence $Ts \in [Tx, \rightarrow)_G$. It yields $Ts \in [x, \rightarrow)_G$ for all $x \in I$, i.e., $Ts \in K$. Hence $Ts \in [s, \rightarrow)_G$. Set

$$I_1 = I \cup \{s, T^n s : n \in \mathbb{N}\}.$$

It is not difficult to see that I_1 is a directed subset of $V(G)$ such that $I_0 \subset I_1, Tx \in I_1$ and $Tx \in [x, \rightarrow)_G$ for each $x \in I_1$. By the maximality of I, $I_1 = I$. It follows that both $s, Ts \in I$. Therefore, $s \in [Ts, \rightarrow)_G$ and $Ts \in [s, \rightarrow)_G$.

Put $H := [s, s]_G \neq \emptyset$. Take $x \in H$. Since $s \in [x, \rightarrow)_G$, we have $Ts \in [Tx, \rightarrow)_G$ and $s \in [Ts, \rightarrow)_G$. Hence $s \in [Tx, \rightarrow)_G$, i.e., $Tx \in (\leftarrow, s]_G$. On the other hand, $Tx \in [Ts, \rightarrow)_G$ since $x \in [s, \rightarrow)_G$. Combining with $Ts \in [s, \rightarrow)_G$ yields $Tx \in [s, \rightarrow)_G$. Therefore, $Tx \in H$ for all $x \in H$, that is, $T(H) \subset H$. □

Remark 1.6. Note that for every $a, b \in K_0$ we have $a \in [b, \rightarrow)_G$ and $b \in [a, \rightarrow)_G$. Indeed, by the above argument, $\{a, b\} \subset I \cap K$. Since $a \in I, b \in K$, we have $b \in [a, \rightarrow)_G$. And $a \in [b, \rightarrow)_G$ follows from $a \in K, b \in I$.

Furthermore, it is clear that $K_0 \subseteq [a, b]_G$. We show that $K_0 = [a, b]_G$. For this purpose, fix $t \in [a, b]$. Since $t \in [a, \rightarrow)_G$ and $a \in [x, \rightarrow)_G$ for each $x \in I$, we have $t \in [x, \rightarrow)_G$ for each $x \in I$. In a similar way, $t \in (\leftarrow, y]_G$ for all $y \in K$. Hence $t \in \bigcap_{x \in I, y \in K} [x, y]_G = K_0$. It implies $[a, b]_G \subseteq K_0$ and thus $K_0 = [a, b]_G$. In particular, $K_0 = [s, s]_G$ and $T(K_0) \subseteq K_0$.

2. Uniformly convex metric spaces

In this section, we obtain some fixed point theorems for monotone G-nonexpansive mappings. The setting are reflexive metric spaces, in particular, uniformly convex metric spaces.

Let (X, d) be a metric space. Recall that X is said to be uniquely geodesic if any two points x, y in X are endpoints of a unique metric segment $[x, y]$ (i.e., $[x, y]$ is an isometric image of the interval $[0, d(x, y)]$). We shall denote by $\alpha x \oplus (1 - \alpha) y$ a unique point z of $[x, y]$ which satisfies

$$d(x, z) = (1 - \alpha) d(x, y) \quad \text{and} \quad d(z, y) = \alpha d(x, y),$$

where $\alpha \in [0, 1]$. A set $C \subset X$ is convex if the metric segment $[x, y] \subset C$ for each $x, y \in C$.

Definition 2.1 ([9]). Let I be a directed set. A complete geodesic metric space X is said to be reflexive if for every nonincreasing family $(C_i)_{i \in I}$ of nonempty, bounded, closed, convex subsets, i.e., $C_i \subset C_j$ whenever $j \le i$, then

$$\bigcap_{i \in I} C_i \neq \emptyset.$$

Lemma 2.2 ([9]). *A space X is reflexive iff any family of nonempty closed bounded convex subsets of X satisfying the finite intersection property has nonempty intersection.*

Our first result is an application of Theorem 1.5 to the case of a reflexive metric space with a partial order $\preceq := E(G)$. Recall that on $(X, \preceq)$, order intervals are sets of the forms $[a, \rightarrow) = \{x \in X : a \preceq x\}$, $(\leftarrow, b] = \{x \in X : x \preceq b\}$ and $[a, b] = [a, \rightarrow) \cap (\leftarrow, b]$ for some $a, b \in X$. A mapping $T : X \to X$ is said to be monotone (or increasing) if $T(x) \preceq T(y)$ whenever $x, y \in X$ such that $x \preceq y$.

Theorem 2.3. *Let $(X, d, \preceq)$ be a reflexive metric space with a partial order $\preceq$, and C be a nonempty bounded closed convex subset of X. Assume that order intervals are closed and convex. Let $T : C \to C$ be a monotone mapping. If there exists $c \in C$ such that $c \preceq Tc$, then T has a fixed point.*

Proof. Let $\mathcal{G}$ be the collection of all subsets of the form $C \cap P$, where P is an order interval in X. By Lemma 2.2, $\mathcal{G}$ satisfies that any subcollection $\mathcal{G}'$ of $\mathcal{G}$ having the finite intersection property, has nonempty intersection. It follows from Theorem 1.5 that there exists $s \in C$ such that $T([s, s]_G) \subset [s, s]_G$. Since $\preceq$ is a partial order, $[s, s]_G$ is a singleton and hence T has a fixed point in C. □

Since nearly uniformly convex metric spaces (in the sense of Kell, see [9, Definition 2.2]) are reflexive, we have the following corollary.

Corollary 2.4. *Let $(X, d, \preceq)$ be a nearly uniformly convex metric space with a partial order $\preceq$, and C be a nonempty bounded closed convex subset of X. Assume that order intervals are closed and convex. Let $T : C \to C$ be a monotone mapping. If there exists $c \in C$ such that $c \preceq Tc$, then T has a fixed point.*

In particular, Corollary 2.4 applies to uniformly ∞-convex spaces as defined in Kell [9]. However, there is another notion of a uniformly convex metric space (see, e.g., [4]) that appears to be not quite comparable to uniformly ∞-convex spaces.

Definition 2.5. Let (X, d) be a uniquely geodesic metric space. For any $a \in X$, $r > 0$ and $\varepsilon > 0$, define

$$D_a(r, \varepsilon) = \{(x, y) \in X \times X : d(x, a) \le r, d(y, a) \le r, d(x, y) \ge r\varepsilon\},$$

and let

$$\delta(r, \varepsilon) = \inf \Big\{ 1 - \frac{1}{r} d\Big(\frac{1}{2}x \oplus \frac{1}{2}y, a\Big) : (x, y) \in D_a(r, \varepsilon), a \in X \Big\}.$$

In the above, we adopt the convention that $\inf \emptyset = 1$.

(a) We say that X is uniformly convex (UC for short) if $\delta(r, \varepsilon) > 0$ for any $r > 0$ and $\varepsilon > 0$.

(b) We say that X is UUC if for every $s > 0$ and $\varepsilon > 0$, there exists $\eta(s, \varepsilon) > 0$ such that $\delta(r, \varepsilon) > \eta(s, \varepsilon) > 0$ for any $r > s$.

Notice that if a uniquely geodesic metric space X is uniformly convex, then all closed balls $B(a, r) = \{x \in X : d(a, x) \le r\}$ are convex, where $a \in X$, $r > 0$. In fact, we have a stronger conclusion.

Lemma 2.6. *Let (X,d) be a complete uniquely geodesic metric space. Let $r>0$, $a\in X$.*

i) Assume that X is UC. Let $t\in[\alpha,\beta]$, where $0<\alpha\leq\beta<1$. If

$$d(x,a)\leq r, d(y,a)\leq r, d(x,y)\geq r\varepsilon$$

for some $\varepsilon>0$, $x,y\in X$, then there exists $\delta(r,2\varepsilon\min\{\alpha,1-\beta\})\in(0,1)$ such that

$$d(a,(1-t)x\oplus ty)\leq r\Big(1-\delta(r,2\varepsilon\min\{\alpha,1-\beta\})\Big).$$

ii) Assume that $t_n\in[\alpha,\beta]$ for every $n\geq 1$, where $0<\alpha\leq\beta<1$, and $(x_n)_n$, $(y_n)_n$ are two sequences in X such that $\limsup_{n\to\infty} d(a,x_n)\leq r$, $\limsup_{n\to\infty} d(a,y_n)\leq r$, and $\lim_{n\to\infty} d\Big(a,t_nx_n\oplus(1-t_n)y_n\Big)=r$. If X is UCC, then $\lim_{n\to\infty} d(x_n,y_n)=0$.

Proof. (i) Take $\varepsilon>0$, $x,y\in X$. Without loss of generality we may assume that $t<1/2$. Let $z_t=(1-t)x\oplus ty$ and $z_{2t}=(1-2t)x\oplus 2ty$ so that

$$d(x,z_t)=td(x,y),\quad d(y,z_t)=(1-t)d(x,y),$$

and

$$d(x,z_{2t})=2td(x,y),\quad d(y,z_{2t})=(1-2t)d(x,y).$$

Note that $d(z_t,z_{2t})=td(x,y)$. Hence $d(x,z_t)=d(z_t,z_{2t})=td(x,y)=1/2d(x,z_{2t})$. It implies $z_t=\frac{1}{2}x\oplus\frac{1}{2}z_{2t}$. Since $t\geq\min\{\alpha,1-\beta\}$, we have

$$d(x,z_{2t})=2td(x,y)\geq 2r\varepsilon\min\{\alpha,1-\beta\}.$$

Hence

$$d(a,z_t)\leq r(1-\delta(r,2\varepsilon\min\{\alpha,1-\beta\})).$$

(ii) For each $n\geq 1$, define

$$r_n=\max\{d(a,x_n),d(a,y_n)\}.$$

Hence

$$\begin{aligned}\limsup_{n\to\infty} r_n &= \limsup_{n\to\infty}\max\{d(a,x_n),d(a,y_n)\}\\ &= \max\{\limsup_{n\to\infty} d(a,x_n),\limsup_{n\to\infty} d(a,y_n)\}\leq r.\end{aligned}$$

We note that the sequences $(d(a,x_n))_n$ and $(d(a,y_n))_n$ are bounded so that there exists $R>0$ such that $r_n\leq R$ for all $n\geq 1$.

Case 1. If $\limsup_{n\to\infty} r_n=0$, then $\limsup_{n\to\infty} d(a,x_n)=\limsup_{n\to\infty} d(a,y_n)=0$. It is not difficult to see that $\lim_{n\to\infty} d(x_n,y_n)=0$.

Case 2. Let $d=\limsup_{n\to\infty} r_n>0$. Without loss of generality, we assume that $\lim_{n\to\infty} d(x_n,y_n)\neq 0$. Then there exists $\varepsilon>0$ and subsequences $(x_{n_k})_k$, $(y_{n_k})_k$, $(r_{n_k})_k$ such that

$$d(x_{n_k},y_{n_k})\geq\varepsilon\ \text{ and }\ r_{n_k}>d-\varepsilon>0$$

for any $k\geq k_0$. We have

$$d(x_{n_k},y_{n_k})\geq\varepsilon\geq r_{n_k}\frac{\varepsilon}{R}.$$

Since X is UUC, it follows from (i) that there exists $\eta\Big(d-\varepsilon, 2\min\{\alpha, 1-\beta\}\frac{\varepsilon}{R})\Big) \in (0,1)$ such that

$$d\Big(a, t_{n_k}x_{n_k} \oplus (1-t_{n_k})y_{n_k}\Big) \leq r_{n_k}\Big(1-\delta(r_{n_k}, 2\min\{\alpha, 1-\beta\}\frac{\varepsilon}{R})\Big)$$
$$< r_{n_k}\Big(1-\eta(d-\varepsilon, 2\min\{\alpha, 1-\beta\}\frac{\varepsilon}{R})\Big)$$

for any $k \geq k_0$. Taking limsup as $k \to \infty$, we get

$$r \leq d\Big(1-\eta(d-\varepsilon, 2\min\{\alpha, 1-\beta\}\frac{\varepsilon}{R})\Big) < r,$$

which is the desired contradiction. Therefore, $\lim_{n\to\infty} d(x_n, y_n) = 0$. □

Recall that a mapping $T : C \to C$ acting on a subset C of a metric space (X, d) is called nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for any $x, y \in C$. We say that C has the fixed point property for nonexpansive mappings if each nonexpansive mapping $T : C \to C$ has a fixed point in C.

Definition 2.7. Let X be a metric space with a digraph $G = (V(G), E(G)), X = V(G)$, and let C be a nonempty subset of X. A mapping $T : C \to C$ is called monotone G-nonexpansive if T is G-monotone and satisfies

$$d(Tx, Ty) \leq d(x, y)$$

for every $x, y \in C$ such that $x \in [y, \rightarrow)_G$.

In particular, if $E(G)$ is a partial order we obtain the definition of a monotone nonexpansive mapping. Note that a monotone G-nonexpansive map need not to be continuous. Dehaish and Khamsi showed in [4] that if C is a bounded closed and convex subset of a partially ordered uniformly convex *hyperbolic* metric space, then every monotone nonexpansive mapping $T : C \to C$ has a fixed point.

In what follows, we show an analogue of Corollary 2.4 for UUC spaces, thus giving a wide generalization of Dehaish–Khamsi's theorem by dropping both assumptions about hyperbolicity of the space and nonexpansivity of the mapping. We start with the following theorem that has been proved in [12] in the case of hyperbolic UUC spaces.

Theorem 2.8. *Let (X, d) be a complete uniquely geodesic metric space, C a nonempty closed convex subset of X, and $a \in X$. Assume that X is UUC. Let $d(a, C) = \inf\{d(a, y) : y \in C\}$. Then there exists a unique $c \in C$ such that $d(a, C) = d(a, c)$.*

Proof. If $d(a, C) = 0$, then there exists a sequence $(x_n)_{n\geq 1}$ of elements of C that tends to a, and since C is closed, $c := a \in C$. Thus we can assume that $r = d(a, C) > 0$. By definition of infimum, there exists $x_n \in C$ such that $d(a, x_n) \leq (1 + \frac{1}{n})r$ for every $n \geq 1$. We are going to prove that $(x_n)_{n\geq 1}$ is a Cauchy sequence. Assume otherwise that the sequence $(x_n)_{n\geq 1}$ is not Cauchy. Then there exist $\varepsilon_0 > 0$ and two subsequences $(x_{n_k})_{k\geq 1}$ and $(x_{m_k})_{k\geq 1}$ of $(x_n)_n$ such that $n_k > m_k$, $d(x_{n_k}, x_{m_k}) \geq \varepsilon_0$ for any $k \geq 1$. We have

$$d(a, x_{m_k}) \leq (1 + 1/m_k)r, \quad d(a, x_{n_k}) \leq (1 + 1/n_k)r < (1 + 1/m_k)r,$$

and

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon_0 \geq \Big(1 + \frac{1}{m_k}\Big)r\frac{\varepsilon_0}{2r}$$

for any $k \geq 1$. Since X is UUC, there is $\eta(r, \frac{\varepsilon_0}{2r}) < \delta((1+1/m_k)r, \frac{\varepsilon_0}{2r})$ such that

$$d\Big(a, \frac{1}{2}x_{n_k} \oplus \frac{1}{2}x_{m_k}\Big) < \Big(1+\frac{1}{m_k}\Big)r\Big(1-\eta(r, \frac{\varepsilon_0}{2r})\Big).$$

for every $k \geq 1$. We note that $\frac{1}{2}x_{n_k} \oplus \frac{1}{2}x_{m_k} \in C$ since C is convex. Thus for any $k \geq 1$,

$$r < \Big(1+\frac{1}{m_k}\Big)r(1-\eta(r, \frac{\varepsilon_0}{2r})).$$

Letting $k \to \infty$, we obtain a contradiction since $r \leq r(1-\eta(r, \frac{\varepsilon_0}{2r}))$ with $r > 0$ and $\eta(r, \frac{\varepsilon_0}{2r}) \in (0,1)$. Hence $(x_n)_{n\geq 1}$ is a Cauchy sequence. Thus there exists $c \in X$ such that $\lim\limits_{n\to\infty} d(c, x_n) = 0$. It implies that $c \in C$ since C is closed. For each $n \geq 1$, we have

$$\begin{aligned} r = d(a, C) &\leq d(a, c) \leq d(a, x_n) + d(c, x_n) \\ &\leq (1+\frac{1}{n})r + d(c, x_n). \end{aligned}$$

Letting $n \to \infty$, we conclude that $d(a, C) = d(a, c)$.

Next we are going to prove the uniqueness of c. Assume that there exists $c' \in C$ such that $c' \neq c$ and $d(a, c') = r$. Put $r_1 = d(c, c')$ and $\varepsilon = \frac{r_1}{r}$. Since X is UUC, we have

$$d\Big(a, \frac{1}{2}c \oplus \frac{1}{2}c'\Big) \leq r(1-\delta(r, \varepsilon)).$$

Since $\frac{1}{2}x_0 \oplus \frac{1}{2}x_1 \in C$, we have $r \leq r(1-\delta(r,\varepsilon))$. This is a contradiction with $r > 0$ and $\delta(r, \varepsilon) > 0$. Therefore, c is the unique point such that $d(a, c) = d(a, C)$. □

Similarly, the point (i) of the following lemma has been proved in [12] in the case of hyperbolic UUC spaces, and the point (ii) is a counterpart of Proposition 3.5 in [1] for modular spaces.

Lemma 2.9. *Let (X, d) be a complete uniquely geodesic metric space. Assume that X is UUC. Then the following properties hold:*

- *(i) Any nonincreasing sequence $(C_n)_{n\geq 1}$ of nonempty bounded closed convex subsets of X has a nonempty intersection.*
- *(ii) X is reflexive, i.e., any family of nonempty closed bounded convex subsets of X satisfying the finite intersection property has nonempty intersection.*

Proof. (i) Suppose that $(C_n)_{n\geq 1}$ is a nonincreasing sequence of nonempty bounded closed convex subsets of X. If $C_n = X$ for all $n \geq 1$, then we are done. So we assume that $C_{n_0} \neq X$ for some $n_0 > 1$ and take $x \in X \setminus C_{n_0}$. It is not difficult to see that the sequence $(d(x, C_n))_n$ is nondecreasing and bounded. Hence there exists the limit $r = \lim\limits_{n\to\infty} d(x, C_n)$. Clearly, $r \in (0, \infty)$. It follows from Theorem 2.8 that for each $n \geq 1$, there exists $x_n \in C_n$ such that $d(x, C_n) = d(x, x_n)$. Since $(C_n)_n$ is non-increasing, we have that $x_k \in C_n$ for any $k \geq n$. Using a similar argument as in the proof of Theorem 2.8, there exists $x_0 \in X$ such that $\lim\limits_{n\to\infty} d(x_n, x_0) = 0$. Since C_n is closed, $x_0 \in C_n$ for all $n \geq 1$, i.e., $x_0 \in \bigcap\limits_{n\geq 1} C_n$.

(ii) Suppose that $(Y_i)_{i\in I}$ is a family of nonempty bounded closed convex subsets of X such that $\bigcap_{i\in F} Y_i \neq \emptyset$ for any finite subset F of I. We fix $i_0 \in I$, and put $C_i := Y_i \cap Y_{i_0}$ for each $i \in I$. We only need to prove that $\bigcap_{i\in I} C_i \neq \emptyset$. Obviously, $(C_i)_{i\in I}$ is a family of nonempty bounded closed convex subsets of Y_{i_0} satisfying the finite intersection property.

Put

$$\mathcal{J} = \{J \subseteq I : J \text{ is countable}\}.$$

First we are going to prove that if $J \in \mathcal{J}$, then $\bigcap_{j\in J} C_j \neq \emptyset$. Indeed, assume that $J = \{j_1, j_2, ...\}$. For each $n \geq 1$, put $J(n) = \{j_1, ..., j_n\}$. Let $A_n = \bigcap_{j\in J(n)} C_j$ for any $n \geq 1$. It is not difficult to see that $(A_n)_n$ is a decreasing sequence of nonempty bounded closed convex subsets of X. Using (i), we have $C_J = \bigcap_{j\in J} C_j \neq \emptyset$.

Take $x \in X$. For each $J \in \mathcal{J}$, we put $d_J := d(x, C_J)$ and

$$d_{\mathcal{J}} = \sup\{d_J : J \in \mathcal{J}\}.$$

Clearly, $d_{\mathcal{J}} \in [0, \infty)$. For any $n \geq 1$, there exists a subset $J_n \in \mathcal{J}$ such that

$$d_{\mathcal{J}} - \frac{1}{n} \leq d_{J_n} \leq d_{\mathcal{J}}.$$

For each $n \geq 1$, put $J_n^* = \bigcup_{i=1}^{n} J_i$. Clearly, J_n^* is countable. Thus $\Big(\bigcap_{j\in J_n^*} C_j\Big)_n$ is a decreasing sequence of nonempty bounded closed convex subsets of C_{i_0}. It follows from (i) that $K = \bigcap_{j\in F} C_j \neq \emptyset$, where $F = \bigcup_{n\geq 1} J_n^* = \bigcup_{n\geq 1} J_n$. We note that $\bigcup_{n\geq 1} J_n$ is a countable subset of I, i.e, $\bigcup_{n\geq 1} J_n \in \mathcal{J}$. Hence

$$d_{\mathcal{J}} - \frac{1}{n} \leq d_{J_n} \leq d(x, K) \leq d_{\mathcal{J}}$$

for any $n \geq 1$. It implies $d(x, K) = d_{\mathcal{J}}$. Now Theorem 2.8 yields the existence of a unique $y \in K$ such that $d(x, y) = d(x, K) = d_{\mathcal{J}}$.

Take $i \in I$. Since $F \cup \{i\}$ is countable, $K \cap C_i \neq \emptyset$ and

$$d(x, K) \leq d(x, K \cap C_i) \leq d_{\mathcal{J}}.$$

Hence $d(x, K) = d(x, K \cap C_i) = d_{\mathcal{J}}$, which implies $y \in K \cap C_i$. Thus $y \in C_i$ for every $i \in I$, that is, $y \in \bigcap_{i\in I} C_i$. □

Remark 2.10. We can prove in Lemma 2.9 that (i) is equivalent to (ii). Indeed, we only have to prove that (ii) implies (i). Suppose that (ii) holds and take a decreasing sequence $(C_n)_n$ of nonempty bounded closed bounded convex subsets of X. Then for any finite subset $\{n_1, ..., n_l\} \subset \mathbb{N}$, where $n_1 \leq ... \leq n_l$, we have $\bigcap_{k=1}^{l} C_{n_k} = C_{n_l} \neq \emptyset$. Hence $(C_n)_n$ has the finite intersection property and thus $\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$.

We are now in a position to obtain a counterpart of Corollary 2.4 for UUC spaces.

Theorem 2.11. *Let X be a complete uniquely geodesic metric space with a partial order $\preceq$. Assume that X is UUC, and order intervals are convex and closed. Let C be a nonempty bounded closed convex subset of X and let $T : C \to C$ be a monotone mapping. If there exists $c \in C$ such that $c \preceq Tc$, then T has a fixed point in C.*

Proof. It is enough to notice that from Lemma 2.9 (ii) X is reflexive and then apply Theorem 2.3. □

Before we give our next results concerning metric spaces with a digraph, let us recall definitions of normal structure and uniform normal structure, see e.g., [14].

Definition 2.12. A convexity structure in a metric space X is a family $\mathcal{F}$ of subsets of X such that $\emptyset, X, \{x\} \in \mathcal{F}$ for every $x \in X$, and $\mathcal{F}$ is closed under arbitrary intersections. The structure $\mathcal{F}$ is said to be compact if every subfamily of $\mathcal{F}$ which has the finite intersection property has nonempty intersection.

Given a convexity structure $\mathcal{F}$ in a metric space (X, d), we adopt the following notation: for $D \in \mathcal{F}$ and $x \in X$, set

$$\begin{aligned} r_x(D) &= \sup\{d(x,y) : y \in D\}, \\ r_X(D) &= \inf\{r_x(D) : x \in X\}, \\ r(D) &= \inf\{r_x(D) : x \in D\}. \end{aligned}$$

Definition 2.13. We say that X has normal structure (resp. uniform normal structure) if there exists a convexity structure $\mathcal{F}$ on X such that $r(A) < \operatorname{diam}(A)$ (resp. $r(A) \leq c\operatorname{diam}(A)$ for a fixed constant $c \in (0,1)$) for any nonempty $A \in \mathcal{F}$ which is bounded and not reduced to a single point. We will also say that $\mathcal{F}$ is normal (resp. uniformly normal).

A subset A of a metric space X is said to be admissible if A is the intersection of closed balls centered at points of X. Of particular interest in metric fixed point theory is the convexity structure $\mathcal{A}(X)$ consisting of $\emptyset$, X and all admissible sets in X. Given any bounded set $A \subseteq X$, set

$$cov(A) := \bigcap \{D : D \in \mathcal{A}(X) \text{ and } D \supseteq A\}.$$

Clearly, $cov(A) \in \mathcal{A}(X)$ and thus $A = cov(A) \Leftrightarrow A \in \mathcal{A}(X)$.

Lemma 2.14. *Let X be a complete UUC metric space. Then X has normal structure.*

Proof. Let $\mathcal{F}$ be the family consisting of $\emptyset$, X and all nonempty closed convex bounded subsets of X. Since X is UUC, $\mathcal{F}$ is a compact convexity structure. We are going to prove that $r(A) < \operatorname{diam}(A)$ for any $A \in \mathcal{F}$ which is not reduced to a single point. Assume that $A \in \mathcal{F}$ and A has at least two distinct elements. Denote $d = \operatorname{diam}(A)$, $r = r(A)$. By definition of the diameter of A, we can choose $x, y \in A$ such that $d(x,y) \geq d/2$. Let $w = \frac{1}{2}x \oplus \frac{1}{2}y$. For every $z \in A$, we have $d(z,x) \leq d$, $d(z,y) \leq d$ and $d(x,y) \geq d/2$. Since X is UUC, it follows that

$$d(z,w) \leq d - d\delta(d, 1/2),$$

and so

$$r_w(A) \leq d - d\delta(d, 1/2).$$

Thus

$$r \leq d - d\delta(d, 1/2),$$

and since $\delta(d, 1/2) > 0$, we have $r < d$, i.e., $r(A) < \operatorname{diam}(A)$. Therefore, X has normal structure. □

The following result extends Theorem 2.3 for reflexive metric spaces with digraphs.

Theorem 2.15. *Let X be a reflexive metric space with a digraph G and let C be a bounded closed convex subset of X. Assume that G-intervals along walks are closed and convex, and for each $a \in C$, $[a,a]_G$ is either empty or has the fixed point property for nonexpansive mappings. If $T : C \to C$ is monotone G-nonexpansive and there exists $c \in C$ such that $Tc \in [c, \to)_G$, then T has a fixed point in C.*

Proof. It follows from Theorem 1.5 that there exists $s \in C$ such that $[s,s]_G \neq \emptyset$, $T([s,s]_G) \subset [s,s]_G$ and T is nonexpansive on $[s,s]_G$ since $x \in [y, \to)_G$ and $y \in [x, \to)_G$ for any $x, y \in [s,s]_G$. By assumption, T has a fixed point in $[s,s]_G$. □

Corollary 2.16. *Let X be a complete UUC metric space with a digraph G. Assume that G-intervals along walks are convex and closed. Let C be a nonempty bounded closed convex subset of X. If $T : C \to C$ is monotone G-nonexpansive and there exists $c \in C$ such that $Tc \in [c, \to)_G$, then T has a fixed point in C.*

Proof. It follows from Lemma 2.9 (ii) that X is reflexive. Without loss of generality we can assume that $V(G) = C$. It is sufficient to prove that each nonempty $[a,a]_G, a \in C$, has the fixed point property for nonexpansive mappings. Fix such $[a,a]_G$, and let $\mathcal{F}$ be the family consisting of $\emptyset$ and all bounded closed convex subsets of $[a,a]_G$. By virtue of Lemma 2.9 (ii), $\mathcal{F}$ is a convexity structure on $[a,a]_G$ and $\mathcal{F}$ is also compact. We invoke Lemma 2.14 to deduce that $[a,a]_G$ has normal structure. Applying Theorem 3.2 in [14] (see also [10, Theorem 8]) we conclude that T has a fixed point in $[a,a]_G$. Now the conclusion follows from Theorem 2.15. □

Remark 2.17. Notice that the set $Fix(T)_{[a,a]_G}$ of fixed points of T in $[a,a]_G$ is closed and convex if X is a complete UUC metric space. Indeed, to show that $Fix(T)_{[a,a]_G}$ is closed, select a sequence $(x_n)_n$ in $Fix(T)_{[a,a]_G}$ which converges to $x \in [a,a]_G$. Then

$$d(x_n, Tx) = d(Tx_n, Tx) \leq d(x_n, x) \text{ for all } n,$$

and hence $(x_n)_{n \geq 1}$ also converges to Tx. By the uniqueness of the limit, $x = Tx$. Thus $x \in Fix(T)_{[a,a]_G}$ and therefore, $Fix(T)_{[a,a]_G}$ is closed.

To show convexity, let $x, y \in Fix(T)_{[a,a]_G}$ with $x \neq y$ and set $2r = d(x,y) > 0$. We prove that $z = \frac{1}{2}x \oplus \frac{1}{2}y \in Fix(T)_{[a,a]_G}$. Assume conversely that $z \neq Tz$ and let $d(z, Tz) = r_0$. Then $d(x,z) = \frac{1}{2}d(x,y) = r$ and

$$d(x, Tz) = d(Tx, Tz) \leq d(x,z), \ d(z, Tz) = r\frac{r_0}{r}.$$

Hence

$$d(x, \frac{1}{2}z \oplus \frac{1}{2}Tz) \leq r\Big(1 - \delta(r, \frac{r_0}{r})\Big),$$

and similarly,

$$d(y, \frac{1}{2}z \oplus \frac{1}{2}Tz) \leq r\Big(1 - \delta(r, \frac{r_0}{r})\Big).$$

By the triangle inequality,

$$2r = d(x,y) \leq 2r - r\Big(\delta(r, \frac{r_0}{r}) + \delta(r, \frac{r_0}{r})\Big) < 2r,$$

and we obtain a contradiction. Therefore, $z = Tz$. This shows that $Fix(T)_{[a,a]_G}$ is convex.

We are thus led to the following theorem.

Theorem 2.18. *Let X be a complete UUC metric space with a digraph G. Assume that G-intervals along walks are convex and closed. Let C be a bounded closed and convex subset of X. Let $T_1, T_2 : C \to C$ be two monotone G-nonexpansive mappings which are commutative. If there exists $c \in C$ such that $T_i c \in [c, \to)_G$ for $i = 1, 2$, then $Fix(T_1) \cap Fix(T_2)$ is nonempty.*

Proof. Without loss of generality we can assume that $V(G) = C$. Arguing in a similar way to the proof of Theorem 1.5 (see also [18]) there exists $s \in C$ such that $T_i([s, s]_G) \subset [s, s]_G$ and T_i are nonexpansive on $[s, s]_G$ for $i = 1, 2$. By Corollary 2.16 and Remark 2.17, $Fix(T_1)_{[s,s]_G}$ and $Fix(T_1)_{[s,s]_G}$ are nonempty, closed and convex. Since T_1, T_2 are commutative, we have $T_2(Fix(T_1)_{[s,s]_G}) \subset Fix(T_1)_{[s,s]_G}$. Hence $T_2 : Fix(T_1)_{[s,s]_G} \to Fix(T_1)_{[s,s]_G}$ has a fixed point in $Fix(T_1)_{[s,s]_G}$ by Corollary 2.16. It implies $Fix(T_1)_{[s,s]_G} \cap Fix(T_2)_{[s,s]_G}$ is nonempty, bounded, closed and convex. Hence $Fix(T_1) \cap Fix(T_2)$ is nonempty. □

Remark 2.19. We note that the conclusion of Theorem 2.18 holds for a finite family of monotone G-nonexpansive mappings which are commutative.

By Remark 2.19 and using the finite intersection property in Lemma 2.9 (ii), we can extend Theorem 2.18 for any commutative family of monotone G-nonexpansive mappings.

Theorem 2.20. *Let X be a complete UUC metric space with a digraph G. Assume that G-intervals along walks are convex and closed. Let C be a nonempty bounded closed convex of X. Let $\mathcal{T}$ be a commutative family of monotone G-nonexpansive mappings from C into C. If there exists $c \in C$ such that $Tc \in [c, \to)_G$ for every $T \in \mathcal{T}$, then $\bigcap_{T \in \mathcal{T}} Fix(T)$ is nonempty.*

3. Modular spaces

In this section we show some fixed point theorems for monotone G_ρ-nonexpansive maps in modular vector spaces. We start with basic definitions concerning modular spaces.

Definition 3.1 ([16]). Let X be a vector space over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). A functional $\rho : X \to [0, \infty]$ is called modular if

(1) $\rho(x) = 0$ if and only if $x = 0$;
(2) $\rho(\alpha x) = \rho(x)$ for $\alpha \in \mathbb{K}$ with $|\alpha| = 1$, for all $x \in X$;
(3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, and $x, y \in X$.

In Definition 3.1 if the condition (3) is replaced by

$$\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y)$$

for any $\alpha, \beta \geq 0$, $\alpha^s + \beta^s = 1$ with $s \in (0, 1]$, then the modular ρ is called an s-convex modular and if $s = 1$, ρ is called a convex modular.

Definition 3.2 ([16]). A modular ρ defines a corresponding modular space, that is, the space X_ρ given by

$$X_\rho = \{x \in X : \lim_{\lambda \to 0} \rho(\lambda x) = 0\}.$$

The Luxemburg norm $\|\cdot\|_\rho : X_\rho \to [0, +\infty)$ is defined by

$$\|x\|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{x}{\alpha}\right) \leq 1\right\}$$

for every $x \in X_\rho$.

Definition 3.3 ([16]). Let X_ρ be a modular space.

(a) A sequence $(x_n)_{n\geq 1}$ in X_ρ is said to be ρ-converging to x if $\lim\limits_{n\to\infty} \rho(x_n - x) = 0$.
(b) A sequence $(x_n)_{n\geq 1}$ in X_ρ is said to be ρ-Cauchy if $\lim\limits_{n,m\to\infty} \rho(x_n - x_m) = 0$.
(c) The modular space X_ρ is ρ-complete if any ρ-Cauchy sequence is ρ-convergent.
(d) A subset $B \subset X_\rho$ is said to be ρ-closed if for any sequence $(x_n)_{n\geq 1} \subset B$ with $x_n \to x$, then $x \in B$. We denotes $\overline{B}^\rho$ the closure of B with respect to ρ.
(e) A subset $B \subset X_\rho$ is called ρ-bounded if $\operatorname{diam}_\rho(B) = \sup\{\rho(x - y) : x, y \in B\}$ is finite, $\operatorname{diam}_\rho(B)$ is called the ρ-diameter of B.
(f) A set $B \subset X_\rho$ is called ρ-compact, if for any sequence $(x_n)_{n\geq 1} \subset X_\rho$ there exists a subsequence $(x_{n_k})_{k\geq 1}$ and $x \in B$ such that $(x_{n_k})_{k\geq 1}$ ρ-converges to x.
(g) ρ is said to satisfy the Fatou property if $\rho(x - y) \leq \liminf\limits_{n\to\infty} \rho(x - y_n)$ whenever $(y_n)_{n\geq 1}$ ρ-converges to y for any x, y, y_n in X_ρ. Note that if ρ satisfies the Fatou property, then the ρ-balls
$$B_\rho(x, r) = \{y \in X_\rho : \rho(x - y) \leq r\}$$
with $x \in X_\rho$, $r \geq 0$ are ρ-closed.

Definition 3.4 ([16]). Let ρ be a modular defined on a vector space X. We say that ρ satisfies the $\triangle_2$-type condition if there exists $K > 0$ such that
$$\rho(2x) \leq K\rho(x)$$
for any $x \in X_\rho$. The smallest such constant K will be denoted by $\omega(2)$ [5].

Proposition 3.5 ([16]). *Let ρ be a modular defined in X and let $\|\cdot\|_\rho$ be a norm on X_ρ. Then ρ-convergence follows from norm convergence in X_ρ. Norm convergence and ρ-convergence are equivalent in X_ρ if and only if the following condition holds: for every sequence $(x_n)_{n\geq 1} \subset X_\rho$, if $\lim\limits_{n\to\infty} \rho(x_n) = 0$ then $\lim\limits_{n\to\infty} \rho(2x_n) = 0$.*

Definition 3.6 ([1]). Let ρ be a modular defined on a vector space X and $C \subset X_\rho$. A mapping $T : C \to C$ is called ρ-nonexpansive if for every $x, y \in C$,
$$\rho(T(x) - T(y)) \leq \rho(x - y).$$

Definition 3.7. Let X_ρ be a modular space endowed with a digraph $G = (V(G), E(G))$ and $C \subset X_\rho$. A mapping $T : C \to C$ is called monotone G_ρ-nonexpansive if T is G-monotone and satisfies $\rho(T(x) - T(y)) \leq \rho(x - y)$ for every $x, y \in C$ such that $x \in [y, \to)_G$.

Our first result is an application of compactness.

Theorem 3.8. *Let ρ be a convex modular in X satisfying Δ_2-type condition. Let G be a digraph on X. Assume that X_ρ is ρ-complete, and G-intervals along walks are convex and ρ-closed. Let C be a ρ-compact convex ρ-bounded subset of X_ρ and $T : C \to C$ a monotone G_ρ-nonexpansive mapping. If there exists $c \in C$ such that $Tc \in [c, \to)_G$, then there is $x_0 \in C$ such that $Tx_0 = x_0$.*

Proof. Since ρ satisfies Δ_2-property, there exists $K > 0$ such that
$$\rho(2x) \leq K\rho(x)$$
for any $x \in X_\rho$. Then for each $(x_n)_n \subset X_\rho$ such that $\lim\limits_{n\to\infty} \rho(x_n) = 0$, we have $\lim\limits_{n\to\infty} \rho(2x_n) = 0$. Hence ρ-convergence is equivalent to convergence in the space $(X_\rho, \|\cdot\|_\rho)$. It implies that every ρ-compact subset of X_ρ is compact in $(X_\rho, \|\cdot\|_\rho)$. Now Theorem 1.5 implies

that there exists $s \in C$ such that $[s, s]_G \neq \emptyset$ and $T : [s, s]_G \to [s, s]_G$ is ρ-nonexpansive. It follows from Theorem 4.4 in [1] that T has a fixed point in $[s, s]_G$. $\square$

Definition 3.9 ([13]). Let ρ be a modular, $r > 0$ and $\varepsilon > 0$. Define for $i = 1, 2$,

$$D_i(r, \varepsilon) = \Big\{(x, y) \in (X_\rho)^2 : \rho(x) \leq r, \rho(y) \leq r, \rho\Big(\frac{x - y}{i}\Big) \geq r\varepsilon\Big\}.$$

If $D_i(r, \varepsilon) \neq \emptyset$, let

$$\delta_i(r, \varepsilon) = \inf\Big\{1 - \frac{1}{r}\rho\Big(\frac{x + y}{2}\Big) : (x, y) \in D_i(r, \varepsilon)\Big\}.$$

If $D_i(r, \varepsilon) = \emptyset$, we set $\delta_i(r, \varepsilon) = 1$.

(a) We say that ρ satisfies (UCi) if for each $r > 0$ and $\varepsilon > 0$, we have $\delta_i(r, \varepsilon) > 0$. Note that for each $r > 0$, $D_i(r, \varepsilon) \neq \emptyset$ for $\varepsilon > 0$ small enough.

(b) We say that ρ satisfies (UUCi) if for each $s > 0$ and $\varepsilon > 0$, there exists $\eta_i(s, \varepsilon) > 0$ depending on s and ε such that

$$\delta_i(r, \varepsilon) > \eta_i(s, \varepsilon) > 0$$

for $r > s$.

(c) We say that ρ is strictly convex (SC) if for every $x, y \in X_\rho$ such that $\rho(x) = \rho(y)$ and

$$\rho\Big(\frac{x + y}{2}\Big) = \frac{\rho(x) + \rho(y)}{2},$$

we have $x = y$.

Proposition 3.10 ([13]). *We have the following relations:*

(a) (UUCi) implies (UCi) for i = 1,2.
(b) $\delta_1(r, \varepsilon) \leq \delta_2(r, \varepsilon)$ for $r > 0$ and $\varepsilon > 0$.
(c) (UC1) implies (UC2) implies (SC).
(d) (UUC1) implies (UUC2).

Lemma 3.11 ([1]). *Let ρ be a convex modular satisfying the Fatou property. Assume that X_ρ is ρ-complete and ρ is (UUC2). Then X_ρ has property (R), i.e., every decreasing sequence $(C_n)_{n\in\mathbb{N}}$ of nonempty ρ-closed, convex and ρ-bounded subsets of X_ρ has a nonempty intersection.*

Definition 3.12 ([11]). A modular space X_ρ is said to have ρ-normal structure if for any nonempty ρ-bounded ρ-closed convex subset C of X_ρ not reduced to one point there exists $x \in C$ such that $r_\rho(x, C) < \mathrm{diam}_\rho(C)$, where $r_\rho(x, C) := \sup\{\rho(x - c) : c \in C\}$.

A modular space X_ρ is said to have ρ-uniform normal structure if there exists a constant $c \in (0, 1)$ such that for any subset C as above, there exists $x \in C$ such that $r_\rho(x, C) \leq c\,\mathrm{diam}_\rho(C)$.

Theorem 3.13. *Let ρ be a modular defined in X. If ρ is (UC2), then X_ρ has ρ-normal structure.*

Proof. Assume that $C \subset X_\rho$ is ρ-closed, convex, ρ-bounded, and $\mathrm{diam}_\rho(C) > 0$. Put $\frac{1}{2}C = \{\frac{c}{2} : c \in C\}$. Then $0 < \mathrm{diam}_\rho(\frac{1}{2}C) \leq \mathrm{diam}_\rho(C)$. Define $d_1 = \mathrm{diam}_\rho(C)$ and

$d_2 = \text{diam}_\rho(\frac{1}{2}C)$. Then there are $x, y \in C$ such that $\rho(\frac{x-y}{2}) \geq d_2/2$. For all $z \in C$, we have $\rho(x-z) \leq d_1$ and $\rho(y-z) \leq d_1$. Hence

$$\rho(z-w) \leq d_1 - d_1\delta(d_1, \frac{d_2}{2d_1}),$$

where $w = \frac{x+y}{2}$. Thus

$$r_\rho(w, C) \leq d_1 - d_1\delta(d_1, \frac{d_2}{2d_1}),$$

and since ρ is (UC2), we have $\delta(d_1, \frac{d_2}{2d_1}) > 0$. It follows that $r_\rho(w, C) < d_1$. Therefore, X_ρ has ρ-normal structure. □

In [1, Theorem 4.5], Abdou and Khamsi proved that a nonempty ρ-closed ρ-bounded and convex subset of a complete modular space X_ρ which satisfies the Fatou property and (UUC1) has the fixed point property for nonexpansive mappings. We can apply Theorem 3.13 to obtain a little improvement on the Abdou–Khamsi's result.

Theorem 3.14. *Let ρ be a convex modular satisfying the Fatou property and (UUC2). Assume that X_ρ is ρ-complete. Let C be a nonempty ρ-closed convex ρ-bounded subset of X_ρ and $T : C \to C$ be a ρ-nonexpansive mapping. Then $Fix(T)$ is a nonempty ρ-closed and convex subset of C.*

Proof. It follows from Proposition 3.10 (a) and Theorem 3.13 that X_ρ has ρ-normal structure. Furthermore, by Lemma 3.11, X_ρ has property (R). Now Theorem 4 in [11] yields $Fix(T)$ is nonempty. To prove that $Fix(T)$ is ρ-closed and convex we can argue in the same way as in [1, Theorem 4.5]. □

Having Theorem 3.14 in hand we can prove a fixed point theorem for monotone G_ρ-nonexpansive maps in modular spaces in a similar way to Theorem 2.20.

Theorem 3.15. *Let ρ be a convex modular satisfying the Fatou property and (UUC2). Assume that X_ρ is ρ-complete. Let G be a digraph on X_ρ such that G-intervals along walks are convex and ρ-closed. Let C be a nonempty ρ-bounded ρ-closed convex subset of X_ρ and let $\mathcal{T}$ be a commutative family of monotone G_ρ-nonexpansive mappings from C into C. If there exists $c \in C$ such that $Tc \in [c, \to)_G$ for every $T \in \mathcal{T}$, then $\bigcap_{T\in\mathcal{T}} Fix(T)$ is nonempty.*

References

[1] A. A. N. Abdou, M. A. Khamsi, Fixed point theorems in modular vector spaces, J. Nonlinear Sci. Appl. 10 (2017), 4046-4057.

[2] M. R. Alfuraidan, Fixed points of monotone nonexpansive mappings with a graph, Fixed Point Theory Appl. 49 (2015), 6 pages.

[3] J. Bang-Jensen, G. Gutin, Digraphs Theory, Algorithms and Applications, Springer, London, 2009.

[4] B. A. B. Dehaish, M. A. Khamsi, Browder and Göhde fixed point theorem for monotone nonexpansive mappings, Fixed Point Theory Appl. 20 (2016), 1-9.

[5] T. Dominguez-Benavides, M. A. Khamsi, S. Samadi, Asymptotically regular mappings in modular function spaces, Sci. Math. Jpn. 53 (2001), 295-304.

[6] R. Diestel, Graph Theory, Springer, Berlin, 2017.

[7] R. Espínola, A. Wiśnicki, The Knaster-Tarski theorem versus monotone nonexpansive mappings, Bull. Pol. Acad. Sci. Math. 66 (2018), 1-7.
[8] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Amer. Math. Soc. 136 (2008), 1359-1373.
[9] M. Kell, Uniformly convex metric spaces, Anal. Geom. Metr. Spaces 2 (2014), 359-380.
[10] M. A. Khamsi, On Metric Spaces with Uniform Normal Structure, Proc. Amer. Math. Soc. 106 (1989), 723-726.
[11] M. A. Khamsi, Uniform noncompact convexity, fixed point property in modular spaces, Math. Japonica, 41 (1994), 1-6.
[12] M. A. Khamsi, A. R. Khan, Inequalities in metric spaces with applications, Nonlinear Analysis, 74 (2011) 4036-4045.
[13] M. A. Khamsi, W. M. Kozlowski, Fixed point theory in modular function spaces, with a foreword by W. A. Kirk, Birkhauser/Springer, Cham, 2015.
[14] W. Kirk, N. Shahzad, Fixed Point Theory in Distance Spaces, Springer, Cham, 2014.
[15] K. Menger, Untersuchungen über allgemeine Metrik. Math. Ann. 100 (1928), 75-163.
[16] J. Musielak, Orlicz Spaces and Modular Spaces, Springer, Berlin 1983.
[17] J. J. Nieto, R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order, 3 (2005), 223-239.
[18] D. H. Quan, A. Wiśnicki, Fixed point theorems in Banach spaces endowed with a digraph, to appear.
[19] A. C. M. Ran, M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Am. Math. Soc. 5 (2004), 1435-1443.

[1]Department of Mathematics, Pedagogical University of Krakow, PL-30-084 Cracow, Poland

[2]Department of Mathematics, Vinh University, 182 Le Duan, Vinh, Nghe An, Vietnam
E-mail address: quan.dau@doktorant.up.krakow.pl

[3]Department of Applied Mathematics and Computer Science, University of Life Sciences in Lublin, 20-950 Lublin, Poland
E-mail address: andrzej.wisnicki@up.lublin.pl