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A FIXED POINT THEOREM FOR MONOTONE MULTIVALUED MAPPINGS IN ORDERED METRIC SPACES AND APPLICATION

DAU HONG QUAN[1,2]

ABSTRACT. Let $(X,d,\preceq)$ be a complete ordered metric space. In this work, we present a fixed point existence theorem for monotone multivalued mappings $T : X \to 2^X$ under the assumption of Sadovskiĭ: $\mu(T\Omega) < \mu(\Omega)$ for every bounded subset Ω of X, where μ is a measure of noncompactness on X. As an application, we show the existence of solutions for a specific class of functional integral inclusions.

1. Introduction

The study of the existence of solutions to functional integral inclusions based on an approach involving measures of noncompactness has received much attention in recent years (see [1, 7, 9, 13]). In these theorems, it is necessary to assume that the set-valued function under consideration is either lower semi-continuous (upper semi-continuous) or continuous with respect to to the Hausdorff metric $H(\cdot,\cdot)$ on its domain. Subsequently, the application of Carathéodory's condition for multi-functions is a commonly employed method to prove such existence theorems. More recently, several authors have made noteworthy contributions to fixed point theory for multivalued mappings by using monotonicity instead of continuity (see [3, 5]).

Let (X,d) be a metric space and $T : X \to 2^X$ a multivalued mapping. In 2010, Zhang [14] considered the partial order defined by Caristi's condition and proved that if for any $x \in X$, the set $T(x)$ is a compact subset of X, and the set $\{x \in X : [x,\to) \cap T(x) \neq \emptyset\}$ is nonempty, then T has a fixed point. Afterwards, Taoudi [11] considered a weaker assumption, namely, that $T(C)$ is contained in a compact subset of X for any totally ordered subset C. By using the closedness of order intervals in ordered metric spaces, Taoudi achieved a similar result in the case of single-valued mappings (see Theorem 2.6, [11]). Following Taoudi's approach, we extend in Section 3 the results of Zhang and Taoudi under the assumption of Sadovskiĭ: $\nu(T\Omega) < \nu(\Omega)$ for every bounded subset $\Omega \subseteq X$, where ν is a measure of noncompactness on the ordered metric space $(X,d,\preceq)$. In Section 4 we show the existence of solutions for a functional integral inclusion to illustrate our theorem.

2. Preliminaries

Let (X,d) be a complete metric space. For $x \in X$, $r > 0$, we put $\overline{B}(x,r) := \{z \in X : d(x,z) \leq r\}$. We also put

$$\begin{aligned}\mathbb{B}(X) &:= \{Z \subseteq X : Z \neq \emptyset,\ Z \text{ is bounded}\},\\ \mathbb{CL}(X) &:= \{Z \subseteq X : Z \neq \emptyset,\ Z \text{ is closed}\},\\ \mathbb{CP}(X) &:= \{Z \subseteq X : Z \neq \emptyset,\ Z \text{ is compact}\}.\end{aligned}$$

Definition 2.1. ([12]) Let (X,d) be a complete metric space. A measure of noncompactness (MNCs for short) defined on the set X is a function $\nu : \mathbb{B}(X) \to [0,\infty)$ such that for any $\Omega_1, \Omega_2 \in \mathbb{B}(X)$, we have

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(i) $\nu(\Omega_1)=0 \Leftrightarrow \overline{\Omega}_1 \in \mathbb{CP}(X)$,
(ii) $\nu(\Omega_1)=\nu(\overline{\Omega}_1)$,
(iii) $\nu(\Omega_1 \cup \Omega_2)=\max\{\nu(\Omega_1),\nu(\Omega_2)\}$.

From Definition 2.1, we infer the following properties:

(iv) If $\Omega_1 \subset \Omega_2$ then $\nu(\Omega_1) \le \nu(\Omega_2)$,
(v) $\nu(\Omega_1 \cap \Omega_2) \le \min\{\nu(\Omega_1),\nu(\Omega_2)\}$,
(vi) If $\Omega_1=\{x_1,\dots,x_n\}$ then $\nu(\Omega_1)=0$.

Put $I:=[0,1]\subseteq \mathbb{R}$. Let $\mathscr{C}(I,\mathbb{R})$ denote the space of all continuous real-valued functions defined on I. In this paper, we will use the MNCs Ψ_0 defined on $\mathscr{C}(I,\mathbb{R})$ as follows.

Example 1. First, we note that the space $\mathscr{C}(I,\mathbb{R})$ with the maximum norm

$$\|f\| = \max_{x\in I}|f(x)|$$

is a Banach space. Now, we take $\Omega \in \mathbb{B}(\mathscr{C}(I,\mathbb{R}))$, $f\in\Omega$ and $\delta>0$. Put

$$\begin{aligned}\Psi(f,\delta) &= \sup\{|f(x)-f(y)| : x,y\in I, |x-y|\le\delta\},\\ \Psi(\Omega,\delta) &= \sup_{f\in\Omega}\Psi(f,\delta),\\ \Psi_0(\Omega) &= \lim_{\delta\to 0}\Psi(\Omega,\delta).\end{aligned}$$

The function Ψ_0 is a MNCs on $\mathscr{C}(I;\mathbb{R})$ (see [2]).

Next, we need to recall some basic definitions in ordered metric spaces. Let (X,d) be a metric space. Suppose that X is equipped with a partial order $\preceq$. Order intervals are defined as sets of the form

$$\begin{aligned}[a,\to) &:= \{x\in X : a\preceq x\},\\ (\leftarrow,a] &:= \{x\in X : x\preceq a\}\end{aligned}$$

for every $a\in X$.

Definition 2.2. An ordered metric space is a triple $(X,d,\preceq)$ such that in the metric space (X,d), order intervals $[x,\to)$, $(\leftarrow,x]$ are closed for every $x\in X$.

Example 2. On $\mathscr{C}(I,\mathbb{R})$, we consider a partial order $\preceq_{\mathscr{C}}$ defined by

$$f\preceq_{\mathscr{C}} g \text{ if only if } f(t)\le g(t)\ \forall t\in I,$$

for every $f,g\in\mathscr{C}(I,\mathbb{R})$. It is not difficult to prove that $(\mathscr{C}(I,\mathbb{R}),\|.\|,\preceq_{\mathscr{C}})$ is an ordered metric space.

We also need to recall the following basic results in ordered metric spaces.

Proposition 2.3 (see Proposition 1.1.3, [4])**.** *If a nondecreasing (nonincreasing) sequence $(x_n)_n$ in an ordered metric space $(X,d,\preceq)$ has a cluster point a, then $a=\sup_n x_n$ (resp., $a=\inf_n x_n$).*

Lemma 2.4 (see Lemma 1.1.5, [4])**.** *If $(x_n)_n$ is a chain, it has a monotone subsequence.*

Definition 2.5 ([6])**.** A multivalued mapping $T: X\to 2^X\setminus\{\emptyset\}$ is called monotone if for any $x,y\in X$ with $x\preceq y$ and any $x_1\in T(x)$, there exists $y_1\in T(y)$ such that $x_1\preceq y_1$.

If $x\in T(x)$ then the point x is called a fixed point of T. The set of all fixed point of T is denoted by Fix(T).

Example 3. In the ordered space $(\mathscr{C}(I,\mathbb{R}),\preceq_{\mathscr{C}})$, we define the multivalued mappings $T_1,T_2 : \mathscr{C}(I,\mathbb{R}) \to 2^{\mathscr{C}(I,\mathbb{R})} \setminus \{\emptyset\}$ as follows

$$T_1(f) = [f-1,\to) \text{ and } T_2(f) = [f+1,\to),$$

for every $f \in \mathscr{C}(I,\mathbb{R})$. Obviously, T_1,T_2 are monotone and $Fix(T_1) = \mathscr{C}(I,\mathbb{R})$, $Fix(T_2) = \emptyset$.

Example 4. Monotone nonexpansive multivalued mappings in metric spaces provide natural examples of monotone mapping (see [10]).

3. Main results

Before presenting the main results, we establish a lemma that will be used later. This lemma is interesting and may find numerous mathematical applications.

Lemma 3.1. *Let $(x_n)_n$ and $(y_n)_n$ be two sequences in an ordered metric space $(X,d,\preceq)$ that satisfy the following conditions:*

(i) $x_n \preceq x_{n+1}$, and $x_n \preceq y_n$ for every n;

(ii) $\lim\limits_{n\to\infty} x_n = x$, and $\lim\limits_{n\to\infty} y_n = y$.

Then $x \preceq y$.

Proof. Since $\lim\limits_{n\to\infty} x_n = x$ and $(x_n)_n$ is nondecreasing, we infer that $x = \sup\{x_n : n \geq 1\}$. Fix $n \geq 1$. It is not difficult to see that

$$y_m \in [x_n,\to) \text{ for all } m \geq n.$$

Since order intervals are closed, we have $cl(\{y_m : m \geq n\}) \subseteq [x_n,\to)$, where $cl(\{y_m : m \geq n\})$ is the closure of the set $\{y_m : m \geq n\}$. Obviously, $y \in [x_n,\to)$ for every $n \geq 1$. Therefore, $x \preceq y$. □

Theorem 3.2. *Let Y be a nonempty bounded closed subset in a complete ordered metric space $(X,d,\preceq)$ and $\nu : \mathbb{B}(X) \to [0,\infty)$ be a MNCs on X. Let $T : Y \to \mathbb{CL}(Y)$ be a monotone multivalued mapping such that for each $\Omega \subseteq Y$ with $\nu(\Omega) > 0$, we have*

$$\nu(T(\Omega)) < \nu(\Omega),$$

where $T(\Omega) = \bigcup_{x\in\Omega} T(x)$. Assume that $\{x \in Y : [x,\to) \cap T(x) \neq \emptyset\} \neq \emptyset$. Then T has a fixed point.

Proof. We are going to prove that there is a compact subset $A \subseteq Y$ such that $T(A) \subseteq A$. Take any $x_0 \in \{x \in Y : [x,\to) \cap T(x) \neq \emptyset\}$. Put

$$\mathscr{M} = \{M : M \in \mathbb{CL}(Y), x_0 \in M, \text{ and } T(M) \subseteq M\}.$$

Since $Y \in \mathscr{M}$, $\mathscr{M} \neq \emptyset$. We also set

$$A = \bigcap_{M\in\mathscr{M}} M, \quad \text{and } B = \overline{T(A)} \cup \{x_0\}.$$

It is not difficult to show that A belongs to $\mathscr{M}$ and so we have $T : A \to \mathbb{CL}(A)$. Moreover, $A = B$. Indeed, since $x_0 \in A$, $T(A) \subseteq A$, and A is closed, it deduces that $B \subseteq A$. Thus we have

$$T(B) \subseteq T(A) \subseteq B,$$

and so $B \in \mathscr{M}$. Hence $A \subseteq B$. By the properties of ν, we have

$$\nu(A) = \nu(B) = \nu(\overline{T(A)} \cup \{x_0\}) = \nu(\overline{T(A)}) = \nu(T(A)).$$

It deduces that $\nu(A) = 0$. Therefore, A is compact.

Denote

$$U = \{x \in A : T(x) \cap [x,\to) \neq \emptyset\}.$$

Since $x_0 \in U$, U is a nonempty set. Clearly, if $x \in U$ and $x \preceq y$ for some $y \in T(x)$, then $y \in U$. Suppose that Z is a chain in U. We set

$$F_z = [z, \rightarrow) \cap \overline{Z} \quad \text{for each } z \in Z.$$

Clearly, F_z are nonempty closed subsets in A, for all $z \in Z$. Take any $z_1, \ldots, z_n \in Z$. Since Z is a chain, there exists $i_0 \in \{1, \ldots, n\}$ with $z_{i_0} = \max\{z_1, \ldots, z_n\}$. It deduces that $z_{i_0} \in F_{z_i}$ for all $i \in \{1, \ldots, n\}$. Consequently,

$$\bigcap_{i=1}^{n} F_{z_i} \neq \emptyset.$$

This means that the family $(F_z)_{z \in Z}$ has the finite intersection property. It implies that

$$Z_0 = \bigcap_{z \in Z} F_z \neq \emptyset.$$

Take $v \in Z_0$. Since Z is a chain, we can find a nondecreasing sequence $(z_n)_n$ in Z such that $\lim_n z_n = v$. Since $(z_n)_n \subseteq U$, there exists a sequence $(y_n)_n$ in A such that

$$z_n \preceq y_n \in T(z_n) \text{ for all } n \geq 1.$$

Since $z \preceq v$ for all $z \in Z$, $z_n \preceq v$ for all $n \geq 1$, and it follows from monotonicity of T that there is a sequence $(v_n)_n$ in $T(v)$ such that

$$y_n \preceq v_n \in T(v) \quad \text{for all } n \geq 1.$$

We note that $T(v)$ is compact. Thus we have $\lim_k v_{n_k} = t \in T(v)$ for a subsequence $(v_{n_k})_k$ of $(v_n)_n$.

Now we have

$$z_{n_k} \preceq v_{n_k} \quad \text{for all } k \geq 1.$$

Thus $v \preceq t \in T(v)$. It deduces that v is an upper bound for Z in U. By the Kuratowski-Zorn's lemma we infer that U contains a maximal element u. Then $u \preceq u^*$ for some $u^* \in T(u)$. Since $u^* \in U$, it implies that $u = u^*$. Therefore u is a fixed point of T. □

4. Application: Functional Integral Inclusion

Denote all Lebesgue integrable functions defined on I by $L^1(I, \mathbb{R})$. This space is equipped with the following norm

$$\|g\|_1 = \int_0^1 g d\mu,$$

for every $g \in L^1(I, \mathbb{R})$. Clearly, $(L^1(I, \mathbb{R}), \|.\|_1)$ is a Banach space.

In this section, we prove the existence of solutions to a functional integral inclusion in the following form:

$$(1) \qquad f(x) \in F(x, f(x)) + \int_0^x k(x,s)\mathscr{F}(s, f(s))ds, \quad \text{for every } x \in I,$$

where $F : I \times \mathbb{R} \to \mathbb{R}$, $k : I \times I \to \mathbb{R}$ are continuous and $\mathscr{F} : I \times \mathbb{R} \to \mathbb{CL}(\mathbb{R})$. By solution of (1), we mean a function $f \in \mathscr{C}(I, \mathbb{R})$ such that

$$f(x) = F(x, f(x)) + \int_0^x k(x,s) f_1(s) ds, \quad \text{for every } x \in I,$$

where $f_1(\cdot) \in \mathscr{F}(\cdot, f(\cdot))$ and $f_1 \in L^1(I, \mathbb{R})$.

Firstly, we consider the following partial order $\preceq_1$ on the set $I \times \mathbb{R}$,

$$(x,y) \preceq_1 (x_1, y_1) \Leftrightarrow x \leq x_1 \text{ and } y \leq y_1,$$

for every (x,y), $(x_1,y_1) \in I \times \mathbb{R}$.

Definition 4.1. A multivalued map $\mathscr{F} : I \times \mathbb{R} \to \mathbb{CP}(\mathbb{R})$ is said to be L^1-Carathéodory if

(i) for each $x \in \mathbb{R}$, the mapping $\mathscr{F}(\cdot, x)$ is measurable,
(ii) for almost all $t \in I$, the mapping $\mathscr{F}(t, \cdot)$ is upper semi-continuous,
(iii) for each $\rho > 0$, there exists a function $g_\rho \in L^1(I, \mathbb{R}_+)$ such that

$$|||\mathscr{F}(t,u)||| = \sup\{|v| : v \in \mathscr{F}(t,u)\} \leq g_\rho(t), \quad a.e.\ t \in I,$$

and for all $u \in \mathbb{R}$ with $|u| \leq \rho$.

For any function $f \in \mathscr{C}(I, \mathbb{R})$, consider the selection set

$$S_{\mathscr{F}}(f) = \{f_1 \in L^1(I, \mathbb{R}) : f_1(s) \in \mathscr{F}(s, f(s)),\ \text{a.e. } s \in I\}.$$

In [8], Lasota and Opial showed that if $\mathscr{F}$ is L^1-Carathéodory, then $S_{\mathscr{F}}(f) \neq \emptyset$ for each $f \in \mathscr{C}(I, \mathbb{R})$. They also established the following lemma.

Lemma 4.2. *Assume that a multivalued map $\mathscr{F}$ satisfies the conditions (i), (ii) of Definition 4.1 with $S_{\mathscr{F}}(f) \neq \emptyset$ for each $f \in \mathscr{C}(I, \mathbb{R})$. Let $\mathscr{G} : L^1(I, \mathbb{R}) \to \mathscr{C}(I, \mathbb{R})$ be a continuous linear mapping. Then $\mathscr{G} \circ S_{\mathscr{F}} : \mathscr{C}(I, \mathbb{R}) \to 2^{\mathscr{C}(I, \mathbb{R})}$ is a closed graph operator on $\mathscr{C}(I, \mathbb{R}) \times \mathscr{C}(I, \mathbb{R})$.*

Now we present the main theorem of this section.

Theorem 4.3. *Assume that the maps in the functional integral inclusion (1) satisfy the following conditions:*

(C1) $F(\cdot, \cdot)$ is continuous on $I \times \mathbb{R}$, and $F(t, \cdot)$ is nondecreasing for every $t \in I$,
(C2) there exists $L \in [0, 1)$ such that

$$|F(x, f) - F(x, g)| \leq L|f - g|, \quad \text{for each } f, g \in \mathbb{R},\ x \in I,$$

(C3) $k(\cdot, \cdot)$ is continuous on $I \times I$,
(C4) $\mathscr{F} : I \times \mathbb{R} \to \mathbb{CP}(\mathbb{R})$ is L^1-Carathéodory,
(C5) $S_{\mathscr{F}}(\cdot)$ is monotone: for any $f, g \in \mathscr{C}(I, \mathbb{R})$ with $f \preceq_{\mathscr{C}} g$ and any $f_1 \in S_{\mathscr{F}}(f)$, there is $g_1 \in S_{\mathscr{F}}(g)$ such that $f_1(s) \leq g_1(s)$ for a.e. $s \in I$,
(C6) there exists a positive number r such that

$$r \geq \frac{\|F(x,0)\| + M\|g_r\|_1}{1 - L},$$

where $M = \max\{|k(x,y)| : (x,y) \in I \times I\}$, and the function g_r satisfies the condition (iii) in Definition 4.1,
(C7) there exists $f_0 \in \mathscr{C}(I, \mathbb{R})$ such that $f_0 \preceq_{\mathscr{C}} h_0$ for some $h_0 \in \mathscr{C}(I, \mathbb{R})$ with

$$h_0(x) \in F(x, f_0(x)) + \int_0^x k(x,s)\mathscr{F}(s, f_0(s))ds, \quad \text{for every } f \in I.$$

Then the integral inclusion (1) has at least one solution in $\mathscr{C}(I, \mathbb{R})$.

Proof. Take $f \in \mathscr{C}(I, \mathbb{R})$ and put

$$\mathscr{T}(f)(x) = F(x, f(x)) + \int_0^x k(x,s)\mathscr{F}(s, f(s))ds, \quad \text{for every } x \in I. \tag{2}$$

Step 1. We recall the following basic result: if $f_1 \in L^1(I, \mathbb{R})$ then the function

$$F_1(x) = \int_0^x k(x,s) f_1(s) ds$$

is continuous on I. It implies that the function

$$F_2(x) = F(x, f(x)) + F_1(x) = F(x, f(x)) + \int_0^x k(x,s) f_1(s) ds$$

is continuous on I for any $f_1 \in S_{\mathscr{F}}(f)$. Hence for each $f \in \mathscr{C}(I,\mathbb{R})$, we have $\mathscr{T}(f) \subseteq \mathscr{C}(I,\mathbb{R})$.

Next, we are going to show that $\mathscr{T}(f)$ is closed for each $f \in \mathscr{C}(I,\mathbb{R})$. Let (h_n) be a sequence in $\mathscr{T}(f)$ and $h_0 \in \mathscr{C}(I,\mathbb{R})$ such that $\|h_n - h_0\| \to 0$ as $n \to \infty$. We need to show that $h_0 \in \mathscr{T}(f)$. Since $h_n \in \mathscr{T}(f)$, there exists $f_n \in S_{\mathscr{F}}(f)$ such that

$$h_n(x) = F(x, f(x)) + \int_0^x k(x,s) f_n(s) ds, \quad \text{for every } x \in I.$$

We consider the operator $\mathscr{G} : L^1(I,\mathbb{R}) \to \mathscr{C}(I,\mathbb{R})$ defined by

$$\mathscr{G}(f)(x) = \int_0^x k(x,s) f(s) ds, \quad \text{for every } x \in I.$$

Obviously, $\mathscr{G}$ is continuous and linear. By Lemma 4.2, it follows that $\mathscr{G} \circ S_{\mathscr{F}}$ is a closed graph operator on $\mathscr{C}(I,\mathbb{R}) \times \mathscr{C}(I,\mathbb{R})$. Furthermore, since $\max_{x \in I} |(h_n(x) - F(x, f(x)) - (h_0(x) - F(x, f(x))| \to 0$ as $n \to \infty$, and $h_n(x) - F(x, f(x)) \in \mathscr{G} \circ S_{\mathscr{F}}(f)$, we have

$$h_0(x) - F(x, f(x)) \in \mathscr{G} \circ S_{\mathscr{F}}(f).$$

It implies that there is $f_0 \in S_{\mathscr{F}}(f)$ such that

$$h_0(x) - F(x, f(x)) = \int_0^x k(x,s) f_0(s) ds, \quad x \in I.$$

Therefore, $h_0 \in \mathscr{T}(f)$.

Step 2. Next, we are going to prove that $\mathscr{T} : \overline{B}(0,r) \to \mathbb{CL}(\overline{B}(0,r))$. Take $f \in \overline{B}(0,r)$ and $h \in \mathscr{T}(f)$. Then there is $h_1 \in S_{\mathscr{F}}(f)$ such that

$$h(x) = F(x, f(x)) + \int_0^x k(x,s) h_1(s) ds, \quad \text{for every } x \in I.$$

We have

$$\begin{aligned} |h(x)| &\le |F(x, f(x)) - F(x,0)| + |F(x,0)| + \left| \int_0^x k(x,s) h_1(s) ds \right| \\ &\le L|f(x)| + \|F(x,0)\| + \int_0^x |k(x,s)| \| |\mathscr{F}(s, f(s))| \| ds \\ &\le L\|f\| + \|F(x,0)\| + M\|g_r\|_1 \le r \end{aligned}$$

for every $x \in I$. It implies that $h \in \overline{B}(0,r)$. Hence $\mathscr{T}(f) \in \mathbb{CL}(\overline{B}(0,r))$ for every $f \in \overline{B}(0,r)$.

Step 3. Take $f, h \in \overline{B}(0,r)$ such that $f \preceq_{\mathscr{C}} h$. By (C1),

$$F(x, f(x)) \le F(x, h(x)) \text{ for all } x \in I.$$

Furthermore, for each $f_1 \in \mathscr{T}(f)$, there exists $f_2 \in S_{\mathscr{F}}(f)$ such that

$$f_1(x) = F(x, f(x)) + \int_0^x k(x,s) f_2(s) ds, \quad \text{for every } x \in I.$$

By (C5), there is $h_2 \in S_{\mathscr{F}}(h)$ such that $f_2(s) \le h_2(s)$ for a.e. $s \in I$. Put

$$h_1(x) = F(x, h(x)) + \int_0^x k(x,s) h_2(s) ds, \quad \text{for every } x \in I.$$

Clearly, $h_1 \in \mathscr{T}(h)$ and $f_1(x) \le h_1(x)$ for every $x \in I$. Hence $\mathscr{T}$ is monotone on $\overline{B}(0,r)$.

Step 4. Assume that Ω is a nonempty subset of $\overline{B}(0,r)$ and $f\in\Omega$. Take any function $f_1\in\mathscr{T}(f)$. Then there exists $f_2\in S_{\mathscr{F}}(f)$ such that

$$f_1(x)=F(x,f(x))+\int_0^x k(x,s)f_2(s)ds,\quad \text{for every } x\in I.$$

Fix $\varepsilon>0$ and choose $x,y\in I$ such that $|x-y|\le\varepsilon$, we get

$$\begin{aligned}|f_1(x)-f_1(y)| &\le |F(x,f(x))-F(y,f(y))|+\left|\int_0^x k(x,s)f_2(s)ds-\int_0^y k(y,s)f_2(s)ds\right|\\
&\le |F(x,f(x))-F(x,f(y))|+|F(x,f(y))-F(y,f(y))|\\
&\quad+\left|\int_0^x k(x,s)f_2(s)ds-\int_0^x k(y,s)f_2(s)ds\right|+\left|\int_0^x k(y,s)f_2(s)ds-\int_0^y k(y,s)f_2(s)ds\right|\\
&\le L|f(x)-f(y)|+|F(x,f(y))-F(y,f(y))|\\
&\quad+\int_0^x |k(x,s)-k(y,s)||f_2(s)|ds+\left|\int_x^y |k(y,s)||f_2(s)|ds\right|\\
&\le L|f(x)-f(y)|+|F(x,f(y))-F(y,f(y))|\\
&\quad+\int_0^x |k(x,s)-k(y,s)|g_r(s)ds+M\left|\int_x^y g_r(s)ds\right|\\
&\le L|f(x)-f(y)|+|F(x,f(y))-F(y,f(y))|\\
&\quad+\int_0^1 |k(x,s)-k(y,s)|g_r(s)ds+M|q(x)-q(y)|,\end{aligned}$$

where

$$q(x)=\int_0^x g_r(s)ds.$$

Using given assumptions, we infer that the function $F(z,t)$ is uniformly continuous on $I\times[-r,r]$, and the function $q(x)$ is uniformly continuous on I. Hence when $\varepsilon\to 0$, we have

$$\begin{aligned}\Psi_r(F,\varepsilon)&:=\sup\{|F(x,z)-F(y,z)|:x,y\in I,|x-y|\le\varepsilon,|z|\le r\}\to 0\\
\Psi_r(k,g_r,\varepsilon)&:=\sup\left\{\int_0^1|k(x,s)-k(y,s)|g_r(s)ds:x,y\in I,|x-y|\le\varepsilon\right\}\to 0\\
\overline{\Psi}(q,\varepsilon)&:=\sup\{|q(x)-q(y)|:x,y\in I,|x-y|\le\varepsilon\}\to 0.\end{aligned}$$

Now, from the obtained estimate, we have (see Example 1)

$$\Psi(f_1,\varepsilon)\le L\Psi(f,\varepsilon)+\Psi_r(F,\varepsilon)+\Psi_r(k,g_r,\varepsilon)+M\overline{\Psi}(q,\varepsilon).$$

It yields

$$\begin{aligned}\Psi(\mathscr{T}(\Omega),\varepsilon)=\sup_{f_1\in\mathscr{T}(\Omega)}\Psi(f_1,\varepsilon)&\le L\sup_{f\in\Omega}\Psi(f,\varepsilon)+\Psi_r(F,\varepsilon)+\Psi_r(k,g_r,\varepsilon)+M\overline{\Psi}(q,\varepsilon)\\
&\le L\Psi(\Omega,\varepsilon)+\Psi_r(f,\varepsilon)+\Psi_r(k,g_r,\varepsilon)+M\overline{\Psi}(q,\varepsilon)\end{aligned}$$

and consequently,

$$\Psi_0(\mathscr{T}(\Omega))\le L\Psi_0(\Omega)<\Psi_0(\Omega).$$

It follows that the mapping $\mathscr{T}$ satisfies all conditions of Theorem 3.2. Therefore, the functional integral inclusion (1) admits a solution in $\mathscr{C}(I,\mathbb{R})$. $\square$

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[1]Department of Mathematics, Pedagogical University of Krakow, PL-30-084 Cracow, Poland
E-mail address: dauhongquandhv@gmail.com

[2]Department of Mathematics, Vinh University, 182 Le Duan, Vinh, Nghe An, Vietnam